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NEW RESULTS ON THE CONVERGENCE OF SOLUTIONS FOR A CERTAIN FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATION

OLUFEMI ADEYINKA ADESINA AND BABATUNDE SUNDAY OGUNDARE

ABSTRACT. This paper is concerned with the convergence of solutions to the fourth order nonlinear differential equation

$$x^{(iv)} + \phi(\dddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

where the nonlinear functions ϕ, f, g, h and p are continuous. By constructing a complete Lyapunov function, some fundamental questions on whether convergence results on the above equation which do not appear to have received any attention in the literature can be proved, are answered.

1. INTRODUCTION

Fourth order nonlinear differential equations have been investigated by several authors in order to study their qualitative behaviour of solutions. See for instance ([1]-[23]). Significant results in this direction have been obtained by Abou-el-ela and Sadek [1], Sadek [15], Sadek and AL-Elaiw [16] on asymptotic stability. Also, Chin [9], Ogundare [11], Ogundare and Okecha [13], Tiryaki and Tunc [19], Tunc [20, 21, 22] and Wu and Xiong [23] made contributions on boundedness and stability properties of solutions. An interesting work on the existence of limiting regime in the sense of Demidovich can be found in Afuwape [6]. All these were done with the aid of Lyapunov's second method. By employing the frequency domain method, Adesina [2] and Afuwape [4, 7, 8] obtained criteria that ensured the existence of periodic, almost periodic, exponential stability and dissipative solutions. It should be noted that the above mentioned papers contain rich bibliography.

However, with respect to our observation in relevant literature, comparatively little have been done on convergence of solutions to fourth order nonlinear differential equations (see for instance Afuwape [3, 5]). The convergence of solutions to nonlinear differential equations is important both theoretically and in practice, since small perturbations from the equilibrium point imply that the trajectory will return to it when time goes to infinity.

It is well known that the problem of the qualitative behaviour of solutions of the equation

$$x^{(iv)} + \phi(\dddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.1}$$

in which the nonlinear functions $\phi, f, g,\ h$ and p are continuous and depend (at most) only on the arguments displayed explicitly, has so far remained intractable due to (i) the number of the nonlinear terms ϕ, f, g and h simultaneously involved

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and (ii) the form of the functions ϕ and f (see for instance Tejumola [17]). Here and elsewhere, all the solutions considered and all the functions which appear are supposed real. The dots indicate differentiation with respect to t. Furthermore, we shall say that two solutions $x_1(t)$, $x_2(t)$ of the equation (1.1) converge (to each other) if $x_2 - x_1 \to 0$, $\dot{x}_2 - \dot{x}_1 \to 0$, $\ddot{x}_2 - \ddot{x}_1$, $\dddot{x}_2 - \dddot{x}_1 \to 0$ as $t \to \infty$.

An interesting but simple case of the problem arises if $p = 0$, and ϕ, f, g, h are such that the equation (1.1) is linear, i.e. if (1.1) is of the form

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0, \tag{1.2}$$

where a, b, c and d are constants. In this case, all solutions tend to the trivial solution, as $t \to \infty$, provided that the Routh-Hurwitz criteria

$$a > 0,\ ab - c > 0,\ (ab - c)c - a^2 d > 0, d > 0 \tag{1.3}$$

are satisfied. Generalization of this result as well as conditions (1.3) to quite different nonlinear equations in which the functions are assumed differentiable, at times twice differentiable, and also in some cases multiplied with signum functions have been obtained in the literature. See for instance Reissig *et al.*[14].

Unfortunately, the corresponding situation when all the four nonlinear functions are not necessarily differentiable, has not been as fully investigated and all available results in this direction have been obtained on equations that are special forms of the equation (1.1). Specifically, in this regard, the equation (1.1) has not attracted the interest of any author. Also most of the existing results are characterized with incomplete Lyapunov functions. These we find too weak. On the one hand, the purpose of this paper is to investigate whether convergence of solutions to the fourth order nonlinear differential equation (1.1) are provable under non differentiability conditions on all the nonlinear functions involved. While on the other hand, by constructing a complete Lyapunov function, which is a difficult task, the paper circumvents the usual heavy restrictions on ϕ, f, f', g, g', h and h' which dominate the results in some of the above mentioned works. Our results generalize previous results of Afuwape [3, 5] and Ogundare and Okecha [12]. They also complement and improve existing results on fourth order nonlinear differential equations in the literature.

This paper is organized as follows. In Section 2, we present basic assumptions and main results. Section 3 is devoted to some preliminary results while in Section 4, we give the proofs of the main results.

2. Assumptions and Main Results

Assumptions:

(1) The function $p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ is equal to $q(t) + r(t, x, \dot{x}, \ddot{x}, \dddot{x})$ with $r(t, 0, 0, 0, 0) = 0$ for all t;
(2) For some positive constants a, b, c, d , $(ab-c)c-a^2d > 0$ and $(ab-c) > 0$;
(3) For some positive constants a, b, c, d, D, Δ_0, Δ_1 K_0 and K_1, the intervals

$$I_0 \equiv \Big[\Delta_0, K_0\big[\frac{[(ab-c)c]}{a^2}\big]\Big], \tag{2.1}$$

$$I_1 \equiv \Big[\Delta_1, K_1\big[\frac{[(a^2d+c^2)]}{ac}\big]\Big] \tag{2.2}$$

are in the Routh-Hurwitz interval.

The following results are proved.

Theorem 2.1. *In addition to the basic assumptions and 1–3 above, suppose that* $\phi(0) = f(0) = g(0) = h(0) = 0$ *and that*

(i) *there are positive constants* a, a_0, b, b_0, c *and* c_0 *such that*

$$a \leq \frac{\phi(w_2) - \phi(w_1)}{w_2 - w_1} \leq a_0, \quad w_2 \neq w_1, \tag{2.3}$$

$$b \leq \frac{f(z_2) - f(z_1)}{z_2 - z_1} \leq b_0, \quad z_2 \neq z_1, \tag{2.4}$$

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq c_0, \quad y_2 \neq y_1; \tag{2.5}$$

(ii) *for any* ζ, η, $(\eta \neq 0)$, *the increment ratios for* h *and* g *satisfy*

$$\frac{h(\zeta + \eta) - h(\zeta)}{\eta} \in I_0, \tag{2.6}$$

$$\frac{g(\zeta + \eta) - (\zeta)}{\eta} \in I_1; \tag{2.7}$$

(iii) *there is a continuous function* $\vartheta(t)$ *such that*

$$\begin{aligned} &|r(t, x_2, y_2, z_2, w_2) - r(t, x_1, y_1, z_1, w_1)| \\ &\leq \vartheta(t)(|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|) \end{aligned} \tag{2.8}$$

holds for arbitrary $t, x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2$.

Then there exists a constant $D_1 > 0$ *such that if*

$$\int_0^t \vartheta^{\varrho}(\tau) d\tau \leq D_1 \tag{2.9}$$

for some ϱ *with* $1 \leq \varrho \leq 2$, *then all solutions of* (1.1) *converge.*

Theorem 2.2. *Suppose that all the conditions of the Theorem 2.1 are satisfied. Let* $x_1(t)$, $x_2(t)$ *be any two solutions of the equation* (1.1). *Then for each fixed* ϱ *in the range* $1 \leq \varrho \leq 2$, *there exist positive constants* D_2, D_3 *and* D_4 *such that for* $t_2 \geq t_1$,

$$S(t_2) \leq D_2 S(t_1) \exp\big\{ - D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \vartheta^{\varrho}(\tau) d\tau \big\}, \tag{2.10}$$

where

$$\begin{aligned} S(t) &= (x_2(t) - x_1(t))^2 + (\dot{x}_2(t) - \dot{x}_1(t))^2 + (\ddot{x}_2(t) - \ddot{x}_1(t))^2 \\ &\quad + (\dddot{x}_2(t) - \dddot{x}_1(t))^2. \end{aligned} \tag{2.11}$$

Remark 2.3. *If* $p = 0$ *and assumption 3 with hypotheses (i) and (ii) of the Theorem 2.1 hold for arbitrary* $\eta \neq 0$, *then the trivial solution of equation* (1.1) *is exponentially stable.*

Remark 2.4. *If* $p = 0$ *and assumption 3 with hypotheses (i) and (ii) of the Theorem 2.1 hold for arbitrary* $\eta \neq 0$, *and* $\zeta = 0$, *then there exists a constant* $D_5 > 0$ *such that every solution* $x(t)$ *of* (1.1) *satisfies*

$$|x(t)| \leq D_5; \quad |\dot{x}(t)| \leq D_5; \quad |\ddot{x}(t)| \leq D_5; \quad |\dddot{x}(t)| \leq D_5.$$

For the rest of this article, $D_1, D_2, D_3, \ldots$ stand for positive constants. Their identities are preserved throughout this paper.

3. Preliminary Result

Let $Q(t) = \int_0^t q(\tau)d\tau$. By setting $\dot{x} = y$, $\dot{y} = z$, $\dot{z} = w$, it is convenient to consider the equation (1.1) as the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -\phi(w) - f(z) - g(y) - h(x) + r(t,x,y,z,w) + Q(t). \end{aligned} \tag{3.1}$$

Let $x_i(t), y_i(t), z_i(t), w_i(t)$, $(i = 1, 2)$, be two solutions of the system (3.1), such that inequalities (2.3)-(2.4) and

$$\Delta_0 \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq K_0\big[\frac{[(ab-c)c]}{a^2}\big], \tag{3.2}$$

$$\Delta_1 \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq K_1\big[\frac{[a^2d + c^2]}{ac}\big] \tag{3.3}$$

are satisfied, where the constants remain as defined as in section two.

The main tool in the proofs of the convergence theorems will be the following function:

$$\begin{aligned} 2V = \; & [\beta(1-\epsilon)x + \gamma y + \delta z + w]^2 + [(1-\epsilon)D - 1](\delta z + w)^2 \\ & + \beta D[\epsilon + (1-\epsilon)D - 1]y^2 + \gamma(D-1)z^2 + \epsilon D w^2 + \beta^2\epsilon(1-\epsilon)x^2 \\ & + 2\gamma\delta[(1-\epsilon)^2 D - 1]yz, \end{aligned} \tag{3.4}$$

where $0 < \epsilon < 1 - \epsilon < 1$, $\frac{\gamma\delta}{\beta} > 1 - \epsilon$, β, γ, δ are positive real numbers and $D = 1 + \frac{\beta(1-\epsilon)[\gamma\delta - \beta(1-\epsilon)]}{\gamma\delta - \beta\epsilon}$ with $D > \frac{1}{(1-\epsilon)^2}$ always.

Lemma 3.1. *The function V defined in equation* (3.4) *is positive definite and moreover, there exist constants D_6 and D_7 such that*

$$D_6(x^2 + y^2 + z^2 + w^2) \leq V \leq D_7(x^2 + y^2 + z^2 + w^2).$$

Proof. Indeed we can rearrange $2V$ as

$$2V = 2V_1 + 2V_2, \tag{3.5}$$

where

$$\begin{aligned} 2V_1 &= [\beta(1-\epsilon)x + \gamma y + \delta z + w]^2 + [(1-\epsilon)D - 1](\delta z + w)^2 \\ &\quad + \epsilon D w^2 + \beta^2\epsilon(1-\epsilon)x^2 + \epsilon\beta D y^2; \\ 2V_2 &= \beta D[(1-\epsilon)D - 1]y^2 + 2\gamma\delta[(1-\epsilon)^2 D - 1]yz + \gamma(D-1)z^2. \end{aligned}$$

Since $0 < \epsilon < 1$, $\frac{\gamma\delta}{\beta} > 1 - \epsilon$, $\beta, \gamma, \delta > 0$ and D can be chosen such that $D > \frac{1}{1-\epsilon}$, it follows that V_1 is positive definite. Also V_2, regarded as a quadratic form in y and z, is always positive since the conditions on all parameters involved are satisfied. Hence V is positive definite. Therefore, a constant $D_6 > 0$ can be found such that

$$D_6(x^2 + y^2 + z^2 + w^2) \leq V. \tag{3.6}$$

Furthermore, by using the Schwartz inequality $|y||z| \leq \frac{1}{2}(y^2 + z^2)$, it follows that $|2V_2| \leq D^*(y^2 + z^2)$ for some $D^* = D^*(\beta, \gamma, \delta, D, \epsilon)$. Thus there exists D_7 such that

$$V \leq D_7(x^2 + y^2 + z^2 + w^2). \tag{3.7}$$

On combining inequalities (3.7) and (3.7), we have

$$D_6(x^2 + y^2 + z^2 + w^2) \leq V \leq D_7(x^2 + y^2 + z^2 + w^2). \tag{3.8}$$

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□

Lemma 3.2. *Let the hypotheses of the Theorem 2.1 hold, and let the function* $W(t) = W(x_2 - x_1, y_2 - y_1, z_2 - z_1, w_2 - w_1)$ *be defined by*

$$\begin{aligned} 2W = \; & [\beta(1-\epsilon)(x_2-x_1) + \gamma(y_2-y_1) + \delta(z_2-z_1) + (w_2-w_1)]^2 \\ & + [(1-\epsilon)D - 1](\delta(z_2-z_1) + (w_2-w_1))^2 + \beta D[\epsilon + (1-\epsilon)D - 1](y_2-y_1)^2 \\ & + \gamma(D-1)(z_2-z_1)^2 + \epsilon D(w_2-w_1)^2 + \beta^2\epsilon(1-\epsilon)(x_2-x_1)^2 \\ & + 2\gamma\delta[(1-\epsilon)^2 D - 1](y_2-y_1)(z_2-z_1), \end{aligned}$$

then there exist positive constants D_8 *and* D_9 *such that*

$$\frac{dW}{dt} \leq -2D_8 S + D_9 S^{1/2}|\theta|, \tag{3.9}$$

where $\theta = r(t, x_2, y_2, z_2, w_2) - r(t, x_1, y_1, z_1, w_1)$.

Proof. On using the system (3.1), a direct computation of $\frac{dW}{dt}$ gives after simplifications

$$\frac{dW}{dt} = -U_1 + U_2, \tag{3.10}$$

where

$$\begin{aligned} U_1 = \; & \beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2 + \gamma[G(y_2,y_1) - \beta(1-\epsilon)](y_2-y_1)^2 \\ & + D\delta(1-\epsilon)[F(z_2,z_1) - \gamma(1-\epsilon)](z_2-z_1)^2 + D[\Phi(w_2,w_1) - \delta(1-\epsilon)](w_2-w_1)^2 \\ & - \beta(1-\epsilon)[G(y_2,y_1) - \beta](x_2-x_1)(y_2-y_1) - \beta(1-\epsilon)[F(z_2,z_1) - \gamma](x_2-x_1)(z_2-z_1) \\ & - \beta(1-\epsilon)[\Phi(w_2,w_1) - \delta](x_2-x_1)(w_2-w_1) \\ & - [D\delta(1-\epsilon)G(y_2,y_1) + \gamma(F(z_2,z_1) - \gamma) - D\beta\gamma](y_2-y_1)(z_2-z_1) \\ & - [DG(y_2,y_1) + \gamma\Phi(w_2,w_1) - \beta(1-\epsilon) - D\gamma\delta(1-\epsilon)^2](y_2-y_1)(w_2-w_1) \\ & - [D(F(z_2,z_1) + D\delta\Phi(w_2,w_1) + D(\delta^2(1-\epsilon) + \gamma)](z_2-z_1)(w_2-w_1) \end{aligned} \tag{3.11}$$

and

$$U_2 = \theta[\beta(1-\epsilon)(x_2-x_1) + \gamma(y_2-y_1) + D\delta(1-\epsilon)(z_2-z_1) + D(w_2-w_1)], \tag{3.12}$$

with

$$\begin{aligned} H(x_2,x_1) = \; & \tfrac{h(x_2)-h(x_1)}{x_2-x_1},\ x_2 \neq x_1; \\ G(y_2,y_1) = \; & \tfrac{g(y_2)-g(y_1)}{y_2-y_1},\ y_2 \neq y_1; \\ F(z_2,z_1) = \; & \tfrac{f(z_2)-f(z_1)}{z_2-z_1},\ z_2 \neq z_1; \\ \Phi(w_2,w_1) = \; & \tfrac{\phi(w_2)-\phi(w_1)}{w_2-w_1},\ w_2 \neq w_1. \end{aligned}$$

Furthermore, let $\chi_1 = G(y_2,y_1) - \beta,\ y_2 \neq y_1,\ \chi_2 = F(z_2,z_1) - \gamma,\ z_2 \neq z_1$ and $\chi_3 = \Phi(w_2,w_1) - \delta,\ w_2 \neq w_1$. Also let $\lambda_1 = G(y_2,y_1) - \beta(1-\epsilon),\ y_2 \neq y_1$, $\lambda_2 = F(z_2,z_1) - \gamma(1-\epsilon),\ z_2 \neq z_1$ and $\lambda_3 = \Phi(w_2,w_1) - \delta(1-\epsilon),\ w_2 \neq w_1$. Define

$$\sum_{i=1}^{3}\mu_i = 1; \quad \sum_{i=1}^{7}\nu_i = 1; \quad \sum_{i=1}^{7}\kappa_i = 1; \quad \sum_{i=1}^{7}\tau_i = 1,$$

with $\mu_i > 0$, $\nu_i > 0$, $\kappa_i > 0$ and $\tau_i > 0$. Then we can rearrange U_1 as

$$U_1 = W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10} + W_{11} + W_{12},$$

where

$$
\begin{aligned}
W_1 = \;& \mu_1\beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2+\beta(1-\epsilon)\chi_1(x_2-x_1)(y_2-y_1)+\nu_1\gamma\lambda_1(y_2-y_1)^2;\\
W_2 = \;& \mu_1\beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2+\beta(1-\epsilon)\chi_2(x_2-x_1)(z_2-z_1)\\
& +\kappa_1 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2;\\
W_3 = \;& \mu_1\beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2+\beta(1-\epsilon)\chi_3(x_2-x_1)(w_2-w_1)+\tau_1 D\lambda_3(w_2-w_1)^2;\\
W_4 = \;& \nu_2\gamma\lambda_1(y_2-y_1)^2+D\delta(1-\epsilon)\lambda_1(y_2-y_1)(z_2-z_1)+\kappa_2 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2;\\
W_5 = \;& \nu_3\gamma\lambda_1(y_2-y_1)^2+\gamma\chi_2(y_2-y_1)(z_2-z_1)+\kappa_3 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2;\\
W_6 = \;& \nu_4\gamma\lambda_1(y_2-y_1)^2+D\beta\delta((1-\epsilon)^2-1)(y_2-y_1)(z_2-z_1)+\kappa_4 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2;\\
W_7 = \;& \nu_5\gamma\lambda_1(y_2-y_1)^2+D\chi_1(y_2-y_1)(w_2-w_1)+\tau_2 D\lambda_3(w_2-w_1)^2;\\
W_8 = \;& \nu_6\gamma\lambda_1(y_2-y_1)^2+\gamma\chi_3(y_2-y_1)(w_2-w_1)+\tau_3 D\lambda_3(w_2-w_1)^2;\\
W_9 = \;& \nu_7\gamma\lambda_1(y_2-y_1)^2+[D(\beta-\gamma\delta(1-\epsilon)^2)+\gamma\delta-\beta(1-\epsilon)](y_2-y_1)(w_2-w_1)\\
& +\tau_4 D\lambda_3(w_2-w_1)^2;\\
W_{10} = \;& \kappa_5 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2+D\chi_2(z_2-z_1)(w_2-w_1)+\tau_5 D\lambda_3(w_2-w_1)^2;\\
W_{11} = \;& \kappa_6 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2+D\delta\chi_3(z_2-z_1)(w_2-w_1)+\tau_6 D\lambda_3(w_2-w_1)^2;\\
W_{12} = \;& \kappa_7 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2+D[\delta^2(1-\epsilon)+2\gamma+\delta^2](z_2-z_1)(w_2-w_1)\\
& +\tau_7 D\lambda_3(w_2-w_1)^2.
\end{aligned}
$$

It is not difficult to see that each of $W_1, W_2, ..., W_{12}$ is quadratic in its respective variables. Thus we can use the fact that any quadratic of the form $AX^2+BXY+CY^2$ is non negative if $4AC-B^2\geq 0$ to obtain the following inequalities:

$$
\begin{aligned}
&W_1\geq 0, && \text{if } {\chi_1}^2\leq \tfrac{4\mu_1\nu_1\gamma H(x_2,x_1)\lambda_1}{\beta(1-\epsilon)},\ \text{for } x_2\neq x_1;\\
&W_2\geq 0, && \text{if } {\chi_2}^2\leq \tfrac{4D\mu_2\kappa_1\delta H(x_2,x_1)\lambda_2}{\beta},\ \text{for } x_2\neq x_1;\\
&W_3\geq 0, && \text{if } {\chi_3}^2\leq \tfrac{4D\mu_3\tau_1 H(x_2,x_1)\lambda_3}{\beta(1-\epsilon)},\ \text{for } x_2\neq x_1;\\
&W_4\geq 0, && \text{if } \lambda_1\leq \tfrac{4\nu_2\kappa_2\gamma\lambda_2}{\beta(1-\epsilon)},\ \text{for } z_2\neq z_1\\
&W_5\geq 0, && \text{if } {\chi_2}^2\leq 4\nu_3\kappa_3 D\delta(1-\epsilon)\lambda_1\lambda_2,\ \text{for } y_2\neq y_1,\ z_2\neq z_1;\\
&W_6\geq 0, && \text{if } \lambda_2\geq \tfrac{D\delta\beta^2((1-\epsilon)^2-1)^2}{4\nu_4\gamma\kappa_4(1-\epsilon)\lambda_1},\ \text{for } y_2\neq y_1,\ z_2\neq z_1\\
&W_7\geq 0, && \text{if } {\chi_1}^2\leq \tfrac{4\nu_5\tau_2\lambda_1\lambda_3}{D},\ \text{for } y_2\neq y_1,\ w_2\neq w_1;\\
&W_8\geq 0, && \text{if } {\chi_3}^2\leq \tfrac{4D\nu_6\tau_3\lambda_1\lambda_3}{\gamma},\ \text{for } y_2\neq y_1,\ w_2\neq w_1;\\
&W_9\geq 0, && \text{if } \lambda_3\geq \tfrac{[D(\beta-\gamma\delta(1-\epsilon)^2)+\gamma\delta-\beta(1-\epsilon)]^2}{4\nu_7\gamma\tau_4 D\lambda_1},\ \text{for } y_2\neq y_1,\ w_2\neq w_1;\\
&W_{10}\geq 0, && \text{if } {\chi_2}^2\leq 4\tau_5\kappa_5\delta(1-\epsilon)\lambda_2\lambda_3,\ \text{for } z_2\neq z_1,\ w_2\neq w_1;\\
&W_{11}\geq 0, && \text{if } {\chi_3}^2\leq \tfrac{4\kappa_6\tau_6(1-\epsilon)\lambda_2\lambda_3}{\gamma},\ \text{for } z_2\neq z_1,\ w_2\neq w_1;\\
&W_{12}\geq 0, && \text{if } \lambda_3\geq \tfrac{[\delta^2(1-\epsilon)^2)+2\gamma+\delta^2]^2}{4\kappa_7\tau_7\delta(1-\epsilon)\lambda_2},\ \text{for } z_2\neq z_1,\ w_2\neq w_1.
\end{aligned}
$$

Thus $U_1\geq W_1$, provided that

$$
\begin{aligned}
0\leq {\chi_1}^2\leq\;& 4\min\Big\{\tfrac{\mu_1\nu_1\gamma H(x_2,x_1)\lambda_1}{\beta(1-\epsilon)};\tfrac{\nu_5\tau_2\lambda_1\lambda_3}{D}\Big\};\\
0\leq {\chi_2}^2\leq\;& 4\min\Big\{\tfrac{D\mu_2\kappa_1\delta H(x_2,x_1)\lambda_2}{\beta};\nu_3\kappa_3 D\delta(1-\epsilon)\lambda_1\lambda_2;\tau_5\kappa_5\delta(1-\epsilon)\lambda_2\lambda_3\Big\};\\
0\leq {\chi_3}^2\leq\;& 4\min\Big\{\tfrac{4D\mu_3\tau_1 H(x_2,x_1)\lambda_3}{\beta(1-\epsilon)};\tfrac{4D\nu_6\tau_3\lambda_1\lambda_3}{\gamma};\tfrac{4\kappa_6\tau_6(1-\epsilon)\lambda_2\lambda_3}{\gamma}\Big\};\\
&\lambda_1\leq \tfrac{4\nu_2\kappa_2\gamma\lambda_2}{\beta(1-\epsilon)};\\
&\lambda_2\geq \tfrac{D\delta\beta^2((1-\epsilon)^2-1)^2}{4\nu_4\gamma\kappa_4(1-\epsilon)\lambda_1};\\
0\leq {\lambda_3}^2\leq\;& \max\Big\{\tfrac{[D(\beta-\gamma\delta(1-\epsilon)^2)+\gamma\delta-\beta(1-\epsilon)]^2}{4\nu_7\gamma\tau_4 D\lambda_1};\tfrac{[\delta^2(1-\epsilon)^2)+2\gamma+\delta^2]^2}{4\kappa_7\tau_7\delta(1-\epsilon)\lambda_2}\Big\},
\end{aligned}
$$

where H and G lie respectively in

$$I_0\equiv\Big[\Delta_0, K_0\big[\frac{[(ab-c)c]}{a^2}\big]\Big],$$

$$I_1 \equiv \Big[\Delta_1, K_1\big[\frac{[(a^2d+c^2)]}{ac}\big]\Big]$$

with $K_0 < 1$, $K_1 < 1$, $\Delta_0 > 0$ and $\Delta_1 > 0$ constants.

By choosing

$$2D_8 = \min\big\{\beta(1-\epsilon)\Delta_0; \gamma\lambda_1; D\delta(1-\epsilon)\lambda_2; D\lambda_3\big\},$$

it follows that

$$U_1 \geq W_1 \geq 2D_8 S. \tag{3.13}$$

Furthermore, if we choose

$$D_9 = \max\big\{\beta(1-\epsilon; \gamma; D\delta(1-\epsilon); D(w_2-w_1)\big\}, \tag{3.14}$$

we have

$$U_2 \leq D_9 S^{\frac{1}{2}}|\theta|.$$

On combining (3.13) and (3.14) in (3.10), we obtain (3.9). This completes the proof of Lemma 2. □

4. Proof of Main Results

Proof of Theorem 2.2. We shall first give the proof of the Theorem 2.2. For this, let ϱ be any constant in the range $1 \leq \varrho \leq 2$ and set $\sigma = 1 - \frac{\varrho}{2}$ so that $0 \leq \sigma \leq \frac{1}{2}$. We re-write inequality (3.9) in the form

$$\frac{dW}{dt} + D_8 S \leq D_9 S^{1/2}|\theta| - D_8 S, \tag{4.1}$$

from which

$$\frac{dW}{dt} + D_8 S \equiv D_{10} S^{\sigma} W^*,$$

where

$$W^* = S^{(\frac{1}{2}-\sigma)}[|\theta| - D_{11}S^{1/2}], \tag{4.2}$$

with $D_{11} = \frac{D_8}{D_{10}}$. Considering the two cases (i) $|\theta| \leq D_{11}S^{1/2}$, then $W^* < 0$; and (ii) $|\theta| \geq D_{11}S^{1/2}$, then the definition of W^* in the equation (4.2) gives at least

$$W^* \leq S^{(\frac{1}{2}-\sigma)}|\theta|$$

and also $S^{1/2} \leq \frac{|\theta|}{D_{11}}$. Thus

$$S^{\frac{1}{2}(1-2\sigma)} \leq [\frac{|\theta|}{D_{11}}]^{(1-2\sigma)},$$

and from this together with W^* follows

$$W^* \leq D_{12}|\theta|^{2(1-\sigma)},$$

where $D_{12} = {D_{11}}^{(\sigma-1)}$. On using the estimate on W^* in inequality (4.1), we obtain

$$\frac{dW}{dt} + D_8 S \leq D_{10}D_{12}S^{\sigma}|\theta|^{2(1-\sigma)} \leq D_{13}S^{\sigma}\vartheta^{2(1-\sigma)}S^{(1-\sigma)} \tag{4.3}$$

which follows from

$$\begin{aligned}&|r(t,x_2,y_2,z_2,w_2) - r(t,x_1,y_1,z_1,w_1)|\\ &\leq \vartheta(t)(|x_2-x_1| + |y_2-y_1| + |z_2-z_1| + |w_2-w_1|).\end{aligned}$$

In view of the fact that $\varrho = 2(1-\sigma)$, we obtain

$$\frac{dW}{dt} \leq -D_8 S + D_{13}\vartheta^{\varrho}S,$$

and on using inequalities (3.9), we have

$$\frac{dW}{dt} + [D_{14} - D_{15}\vartheta^{\varrho}(t)]W \leq 0, \tag{4.4}$$

for some constants D_{14} and D_{15}. On integrating the estimate (4.4) from t_1 to t_2 $(t_1 \leq t_2)$, we obtain

$$W(t_2) \leq W(t_1) \exp\big\{ - D_{14}(t_2 - t_1) + D_{15} \int_{t_1}^{t_2} \vartheta^{\varrho}(\tau) d\tau \big\}. \tag{4.5}$$

On using Lemma 3.1, we obtain inequality (2.10), with $D_2 = \frac{D_7}{D_6}$; $D_3 = D_{14}$ and $D_4 = D_{15}$. This completes the proof of the Theorem 2.2. □

Proof of Theorem 2.1. The proof follows from the inequality (2.10) and the condition (2.9) on ϑ. On choosing $D_2 = \frac{D_3}{D_4}$ in inequality (2.10), then as $t = (t_2 - t_1) \to \infty$, $S(t) \to 0$ which proves that $x_2 - x_1 \to 0$, $y_2 - y_1 \to 0$, $z_2 - z_1 \to 0$, $w_2 - w_1 \to 0$, as $t \to \infty$. This completes the proof of Theorem 2.1. □

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Applications of an Ostrowski-type Inequality

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We extend an Ostrowski-type inequality established by the first author and A. Acu. This extension is used in order to investigate the limit of the iterates of some positive linear operators. The rate of convergence of the iterates is described in terms of $\widetilde{\omega}$, the least concave majorant of the first order modulus of continuity. The same $\widetilde{\omega}$ is used in order to estimate the difference of some classical positive linear operators.

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1 Introduction

One of Ostrowski's classical inequalities deals with the most primitive form of a quadrature rule. It was published in 1938 in Switzerland (see [17]) and reads in its original form as follows.

Theorem 1.1. *Es sei $h(x)$ im Intervall $J : a < x < b$ stetig und differentiierbar, und es sei in J durchweg*

$$|h'(x)| \leq m, \ m > 0.$$

Dann gilt für jedes x aus J:

$$\left| h(x) - \frac{1}{b-a}\int_a^b h(x)dx \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)m.$$

A simplified form can be found, for example, in G. Anastassiou's 1995 article [5] and reads as follows.

Theorem 1.2. *Let f be in $C^1[a,b]$, $x \in [a,b]$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \cdot \|f'\|_\infty .$$

The characteristic feature of Ostrowski's approach is thus to approximate an integral by a single value of the function in question and to estimate the difference assuming differentiability of the function in question. The latter is a dispensable assumption as was observed by A. Acu and the first author in the next result (see [1]).

Theorem 1.3. *Let $L : C[a,b] \to C[a,b]$ be non-zero, linear and bounded, and such that $L : C^1[a,b] \to C^1[a,b]$ with $\|(Lg)'\| \leq c_L \cdot \|g'\|$ for all $g \in C^1[a,b]$. Then for all $f \in C[a,b]$ and $x \in [a,b]$ we have*

$$\left| Lf(x) - \frac{1}{b-a}\int_a^b Lf(t)dt \right| \leq \|L\| \cdot \widetilde{\omega}\left(f; \frac{c_L}{\|L\|} \cdot \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right).$$

The right hand side in the latter inequality is given in terms of the least concave majorant of the first order modulus of continuity of an arbitrary $f \in C[a,b]$ and thus generalizes Ostrowski's inequality in the form given by Anastassiou. For the reader's convenience we recall the following (see [1]).

Definition 1.4. *Let $f \in C[a,b]$. If for $t \in [0,\infty)$ the quantity*

$$\omega(f;t) = \sup\{|f(x) - f(y)|, |x-y| \leq t\}$$

is the usual modulus of continuity, its least concave majorant is given by

$$\widetilde{\omega}(f;t) = \sup\left\{ \frac{(t-x)\omega(f;y) + (y-t)\omega(f;x)}{y-x}; 0 \leq x \leq t \leq y \leq b-a, x \neq y \right\}.$$

It is important to note that for all $f \in C[a,b]$ the following equality holds:

$$\inf_{g \in C^1[a,b]} \left(\|f-g\|_\infty + \frac{t}{2} \|g'\|_\infty \right) = \frac{1}{2} \widetilde{\omega}(f;t), \ t \geq 0.$$

The latter representation can be found in an article of Mitjagin and Semenov [22].

We remark that - according to our knowledge - B. and I. Gavrea in [7] were the first to observe the possibility of using "omega-tilde" in this context.

Ostrowski-type inequalities have attracted a most remarkable amount of attention in the past. The reader should consult Ch. XV on "Integral inequalities involving functions with bounded derivatives" in the book by D.S. Mitrinović et al. [16] and Chapters 2 - 9 in the recent monography of G. Anastassiou [4].

In our present note we generalize the above-mentioned result of Acu et al. for integrals w.r.t. probability measures λ and apply the new estimates to iterates of certain positive linear operators and to differences of such mappings.

2 On Ostrowski's inequality

We first consider the space

$$Lip[0,1] := \left\{ f \in C[0,1] | \, |f|_{Lip} := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty \right\}.$$

Let $M_1^+[0,1]$ be the set of all probability Borel measures on $[0,1]$ and $\lambda \in M_1^+[0,1]$ a given measure. Then the inequality

$$\begin{aligned} \left| f(x) - \int_0^1 f(t) d\lambda(t) \right| &= \left| \int_0^1 (f(x)-f(t)) d\lambda(t) \right| \\ &\leq \int_0^1 |f(x)-f(t)| \, d\lambda(t) \\ &\leq |f|_{Lip} \int_0^1 |t-x| \, d\lambda(t) \end{aligned}$$

holds, for $f \in Lip[0,1]$ and $x \in [0,1]$. Let also

$$w_\lambda(x) := \int_0^1 |t-x| \, d\lambda(t).$$

We have obtained a general form of Ostrowski's inequality:

$$\left| f(x) - \int_0^1 f(t) d\lambda(t) \right| \leq |f|_{Lip} \, w_\lambda(x), \tag{2.1}$$

for all $f \in \mathrm{Lip}[0,1]$, $\lambda \in M_1^+[0,1]$ and $x \in [0,1]$.

3 A result of A. Acu and H. Gonska

We shall present a slight modification of the inequality established by A. Acu and H. Gonska in [1].

Theorem 3.1. *Let $L : C[0,1] \to C[0,1]$ be non-zero, linear and bounded. Suppose that $L(Lip[0,1]) \subset Lip[0,1]$ and there exists $c_L > 0$ such that*

$$|Lg|_{Lip} \leq c_L \, |g|_{Lip} \, ,$$

for all $g \in Lip[0,1]$. Then for all $f \in C[0,1]$, $\lambda \in M_1^+[0,1]$ and $x \in [0,1]$ we have

$$\left| Lf(x) - \int_0^1 Lf(t) d\lambda(t) \right| \leq \|L\| \, \widetilde{\omega} \left(f, \frac{c_L}{\|L\|} w_\lambda(x) \right). \tag{3.1}$$

Proof. Let $A_x : C[0,1] \to \mathbb{R}$ be defined by

$$A_x(f) := f(x) - \int_0^1 f(t) d\lambda(t).$$

Then A_x is a bounded linear functional with $\|A_x\| \leq 2$. We have

$$\begin{aligned} |A_x(Lf)| &\leq |Lf(x)| + \int_0^1 |Lf(t)| \, d\lambda(t) \\ &\leq 2 \, \|L\| \, \|f\|_\infty \, , \end{aligned}$$

for $f \in C[0,1]$. Let $g \in Lip[0,1]$. By using (2.1) we get

$$|A_x(Lg)| = \left| Lg(x) - \int_0^1 Lg(t) d\lambda(t) \right| \leq |Lg|_{Lip} \, w_\lambda(x) \leq c_L \, |g|_{Lip} \, w_\lambda(x).$$

Consequently,

$$\begin{aligned} |A_x(Lf)| &= |(A_x \circ L)(f - g + g)| \\ &\leq |(A_x \circ L)(f - g)| + |A_x(Lg)| \\ &\leq 2 \, \|L\| \, \|f - g\|_\infty + c_L \, |g|_{Lip} \, w_\lambda(x). \end{aligned}$$

Passing to the infimum over $g \in Lip[0,1]$ we get

$$\begin{aligned} |A_x(Lf)| &\leq 2 \, \|L\| \inf_{g \in Lip[0,1]} \left\{ \|f - g\|_\infty + \frac{c_L}{2 \, \|L\|} w_\lambda(x) \, |g|_{Lip} \right\} \\ &= \|L\| \, \widetilde{\omega} \left(f, \frac{c_L}{\|L\|} w_\lambda(x) \right). \end{aligned}$$

□

Corollary 3.2. *In the setting of Theorem 3.1 suppose that, moreover, L is a positive linear operator reproducing the constant functions. Then*

$$\left|Lf(x) - \int_0^1 Lf(t)d\lambda(t)\right| \leq \widetilde{\omega}\left(f, c_L w_\lambda(x)\right) \tag{3.2}$$

holds, for all $f \in C[0,1]$, $\lambda \in M_1^+[0,1]$ and $x \in [0,1]$.

Let $e_0(x) := 1$ for $x \in [0,1]$. It is well known (see, e.g., [13], p.178) that if L is a positive linear operator and $Le_0 = e_0$, then L has at least one *invariant measure* μ, i.e., there exists $\mu \in M_1^+[0,1]$ such that

$$\int_0^1 Lf(t)d\mu(t) = \int_0^1 f(t)d\mu(t), \tag{3.3}$$

for $f \in C[0,1]$.

Now from Corollary 3.2 we obtain

Corollary 3.3. *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator with $Le_0 = e_0$, and μ an invariant measure for L. Suppose that $L(Lip[0,1]) \subset Lip[0,1]$ and there exists $c_L > 0$ such that $|Lg|_{Lip} \leq c_L |g|_{Lip}$, $g \in Lip[0,1]$. Then the inequality*

$$\left|Lf(x) - \int_0^1 f(t)d\mu(t)\right| \leq \widetilde{\omega}\left(f, c_L w_\mu(x)\right) \tag{3.4}$$

holds, for all $f \in C[0,1]$, $x \in [0,1]$.

Under the hypothesis of Corollary 3.3, let $m \geq 1$ be an integer. Then $L^m e_0 = e_0$, $|L^m g|_{Lip} \leq c_L^m |g|_{Lip}$, $g \in Lip[0,1]$, and μ is an invariant measure for the iterate L^m. Consequently, we can state the following result.

Corollary 3.4. *In the setting of Corollary 3.3 we have*

$$\left|L^m f(x) - \int_0^1 f(t)d\mu(t)\right| \leq \widetilde{\omega}\left(f, c_L^m w_\mu(x)\right), \tag{3.5}$$

for all $f \in C[0,1]$, $x \in [0,1]$, $m \geq 1$. Moreover, if $c_L < 1$, then

$$\lim_{m\to\infty} L^m f = \left(\int_0^1 f(t)d\mu(t)\right) e_0, \text{ uniformly on } [0,1], \tag{3.6}$$

and, consequently, L has exactly one invariant measure $\mu \in M_1^+[0,1]$.

Related results, in a more general context, can be found in [3].

4 Examples involving iterates of positive linear operators

Example 4.1.

Let $n \geq 1$, $p = [n/2]$, $0 \leq k \leq p$, $0 \leq x \leq 1$. Consider the polynomials

$$w_{n,k}(x) := \frac{n+1-2p+2k}{(n+1)2^{n+1}x}\binom{n+1}{p-k}\times \\ \times \left((1-x)^{p-k}(1+x)^{n+1-p+k} - (1-x)^{n+1-p+k}(1+x)^{p-k}\right).$$

The operators $\beta_n : C[0,1] \to C[0,1]$, defined by

$$\beta_n f(x) := \sum_{k=0}^{p} f\left(\frac{n-2p+2k}{n}\right) w_{n,k}(x)$$

were introduced in [23] (see also [18], [19]). They are positive linear operators with $\beta_n e_0 = e_0$. According to the results of [18], $c_{\beta_n} = \frac{n-1}{n}$ and the probability measure concentrated on 1 is invariant for β_n. Now Corollary 3.4 entails

$$|\beta_n^m f(x) - f(1)| \leq \widetilde{\omega}\left(f, \left(\frac{n-1}{n}\right)^m (1-x)\right), \tag{4.1}$$

for all $m, n > 1$, $f \in C[0,1]$, $x \in [0,1]$. This result supplements the qualitative results presented in ([20], Ex. 5.7.).

Example 4.2.

For $n \geq 1$ and $j \in \{0, 1, \ldots, n\}$ let

$$b_{nj}(x) := \binom{n}{j} x^j (1-x)^{n-j},\ x \in [0,1]. \tag{4.2}$$

Let $0 \leq \alpha \leq \beta$, $\beta > 0$. Consider the Stancu operators $L_n : C[0,1] \to C[0,1]$ given by

$$L_n f(x) := \sum_{j=0}^{n} b_{nj}(x) f\left(\frac{j+\alpha}{n+\beta}\right).$$

It is easy to verify that $c_{L_n} = \frac{n}{n+\beta} < 1$. According to Corollary 3.4, L_n has a unique invariant measure $\mu_n \in M_1^+[0,1]$; in fact, μ_n was already determined in [11] and [21]. The quantitative result derived from (3.5) accompanies the qualitative results of [11] and [21]. In particular, we see that the rate of convergence, generally expressed by c_L^m, is expressed here by $\left(\frac{n}{n+\beta}\right)^m$.

Example 4.3

Consider the Bernstein-Durrmeyer operators with Jacobi weights $\alpha, \beta > -1$, defined by

$$M_n^{\alpha,\beta} f(x) := \sum_{j=0}^{n} b_{nj}(x) \left(\int_0^1 t^{j+\alpha}(1-t)^{n-j+\beta} f(t)dt \right) \Big/ \left(\int_0^1 t^{j+\alpha}(1-t)^{n-j+\beta} dt \right),$$

for $f \in C[0,1]$, $x \in [0,1]$, $n \geq 1$.

According to the results of [3],

$$c_{M_n^{\alpha,\beta}} = \frac{n}{n+\alpha+\beta+2} < 1,$$

and the invariant measure μ is described by

$$\int_0^1 f(t)d\mu(t) = \left(\int_0^1 t^{\alpha}(1-t)^{\beta} f(t)dt \right) \Big/ \left(\int_0^1 t^{\alpha}(1-t)^{\beta} dt \right), \text{ with } f \in C[0,1].$$

Using the Cauchy-Schwarz inequality we get

$$\begin{aligned} w_\mu(x) &= \int_0^1 |t-x| \, d\mu(t) \leq \left(\int_0^1 (t-x)^2 d\mu(t) \right)^{\frac{1}{2}} \\ &= \left(\left(x - \frac{\alpha+1}{\alpha+\beta+2} \right)^2 + \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)} \right)^{\frac{1}{2}}. \end{aligned}$$

Now Corollary 3.4 entails

$$\begin{aligned} &\left| (M_n^{\alpha,\beta})^m f(x) - \left(\int_0^1 t^{\alpha}(1-t)^{\beta} f(t)dt \right) \Big/ \left(\int_0^1 t^{\alpha}(1-t)^{\beta} dt \right) \right| \\ &\leq \widetilde{\omega} \left(f, \left(\frac{n}{n+\alpha+\beta+2} \right)^m \left(\left(x - \frac{\alpha+1}{\alpha+\beta+2} \right)^2 + \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)} \right)^{\frac{1}{2}} \right) \end{aligned}$$

This is a quantitative companion to the results of ([3], Section 3.2.).

Example 4.4.

For each $n \geq 1$, let $\vartheta_n \in L^1[0,1]$, $\vartheta_n \geq 0$, be a periodic function with period $\frac{1}{n+1}$, such that

$$\int_0^{\frac{1}{n+1}} \vartheta_n(t)dt = 1.$$

Consider the generalized Kantorovich operators $K_n : C[0,1] \to C[0,1]$ defined by

$$K_n f(x) := \sum_{j=0}^{n} b_{nj}(x) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(t)\vartheta_n(t)dt.$$

It is easy to verify that $K_n(Lip[0,1]) \subset Lip[0,1]$ and $c_{K_n} = \frac{n}{n+1}$, $n \geq 1$.

We want to determine the invariant measure. Consider the matrix

$$T_n := \left[\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} b_{ni}(t)\vartheta_n(t)dt\right]_{i,j=0,1,\ldots,n}$$

and let $a = (a_0, a_1, \ldots, a_n)^t \in \mathbb{R}^{n+1}$. T_n is the transpose of a regular stochastic matrix, so that the system $T_n a = a$ has a unique solution with $a_i \geq 0$, $i = 0, 1, \ldots, n$, and $a_0 + \ldots + a_n = 1$. For this solution, we have

$$a_i = \sum_{j=0}^{n} a_j \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} b_{ni}(t)\vartheta_n(t)dt,$$

for $i = 0, \ldots, n$. Let $\varphi_n(t) := a_j\vartheta_n(t)$, $t \in \left(\frac{j}{n+1}, \frac{j+1}{n+1}\right)$, $j = 0, \ldots, n$. Define $\mu_n \in M_1^+[0,1]$ by $d\mu_n(t) = \varphi_n(t)dt$. Then μ_n is the invariant measure of K_n. Indeed,

$$\begin{aligned}
\int_0^1 K_n f(t)d\mu_n(t) &= \sum_{i=0}^{n} \int_0^1 b_{ni}(t)d\mu_n(t) \cdot \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\vartheta_n(t)dt \\
&= \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} b_{ni}(t)a_j\vartheta_n(t)dt\right) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\vartheta_n(t)dt \\
&= \sum_{i=0}^{n} a_i \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\vartheta_n(t)dt \\
&= \sum_{i=0}^{n} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\varphi_n(t)dt \\
&= \int_0^1 f(t)d\mu_n(t).
\end{aligned}$$

i) As a particular case, let $\alpha,\ \beta > -1$ and

$$\vartheta_n(t) = \left(t - \frac{j}{n+1}\right)^{\alpha} \left(\frac{j+1}{n+1} - t\right)^{\beta} \Big/ \left(\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} \left(s - \frac{j}{n+1}\right)^{\alpha} \left(\frac{j+1}{n+1} - s\right)^{\beta} ds\right),$$

for all $t \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$, $j = 0, 1, \ldots, n$. Denote the corresponding operators K_n by $K_n^{\alpha,\beta}$; it is easy to see that they can be expressed also as

$$K_n^{\alpha,\beta} f(x) = \frac{1}{B(\alpha+1, \beta+1)} \sum_{j=0}^{n} b_{nj}(x) \int_0^1 s^{\alpha}(1-s)^{\beta} f\left(\frac{s+j}{n+1}\right) ds.$$

In fact, these are the operators introduced in ([15], (1.5)).

ii) More particularly,

$$K_n^{0,0} f(x) = (n+1) \sum_{j=0}^{n} b_{nj}(x) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(t)dt$$
$$= \sum_{j=0}^{n} b_{nj}(x) \int_0^1 f\left(\frac{s+j}{n+1}\right) ds$$

are the classical Kantorovich operators. For them, as in the above general case, the parameter c_{K_n} is $\frac{n}{n+1}$; moreover, the invariant measure μ is the Lebesgue measure on $[0,1]$. Thus

$$w_\mu(x) = \int_0^1 |t-x|\, dt = \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}.$$

From Corollary 3.4 we infer

$$\left|(K_n^{0,0})^m f(x) - \int_0^1 f(t)dt\right| \leq \widetilde{\omega}\left(f, \left(\frac{n}{n+1}\right)^m \left(\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}\right)\right).$$

Related results, for multivariate Kantorovich operators, can be found in ([3], Section 3.1).

Remark 4.1. *Results of this type are especially significant in case of sequences* (L_n) *for which the strong limit*

$$T(t) := \lim_{n\to\infty} L_n^{[nt]}$$

exists for all $t \geq 0$. *In such a case* $(T(t))_{t\geq 0}$ *is a* C_0*-semigroup of operators and the above results can be used in order to study its asymptotic behaviour from a quantitative point of view. Details can be found in [3].*

5 Examples involving differences of positive linear operators

In the preceding section the main tool was Corollary 3.4 which makes use of the invariant measure. Now we present applications of Corollary 3.2.

Let again $L : C[0,1] \to C[0,1]$ be a positive linear operator with $Le_0 = e_0$. Suppose that $L(Lip[0,1]) \subset Lip[0,1]$ and there exists $c_L > 0$ such that $|Lg|_{Lip} \leq c_L |g|_{Lip}$, for all $g \in Lip[0,1]$. Let $A : C[0,1] \to C[0,1]$ be a positive linear

operator with $Ae_0 = e_0$. For each $x \in [0,1]$ consider the measure $\lambda_x \in M_1^+[0,1]$ defined by

$$\int_0^1 f(t)d\lambda_x(t) = Af(x),\ f \in C[0,1].$$

Then we have

$$\int_0^1 Lf(t)d\lambda_x(t) = A(Lf)(x) = (A \circ L)f(x),\ f \in C[0,1].$$

Now Corollary 3.2 implies

Proposition 5.1. *With the above notation we have*

$$\begin{aligned}|(A \circ L)f(x) - Lf(x)| &\leq \widetilde{\omega}\left(f, c_L A(|t-x|, x)\right)\\ &\leq \widetilde{\omega}\left(f, c_L(A((t-x)^2, x))^{\frac{1}{2}}\right),\end{aligned}$$

for all $f \in C[0,1]$ and $x \in [0,1]$.

Example 5.2

Let $L = \overline{\mathbb{B}}_n$, where (see [14], [2] and the references therein)

$$\overline{\mathbb{B}}_n f(x) := \begin{cases} f(0) & \text{, if } x = 0,\\ f(1) & \text{, if } x = 1,\\ \frac{\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}f(t)dt}{B(nx,n(1-x))} & \text{, if } 0 < x < 1,\end{cases}$$

for all $f \in C[0,1]$, $x \in [0,1]$. Then $\overline{\mathbb{B}}_n e_0 = e_0$.

In [2] it was proved using probabilistic methods that $\overline{\mathbb{B}}_n f$ is increasing whenever f is increasing.

Remark 5.2. *This shape-preserving property can be proved, as in ([6], Ex.3.1), using analytical tools involving total positivity, not only for $\overline{\mathbb{B}}_n$ but also for the Beta operators $\mathcal{B}_n^{\alpha,\beta}$ which will be described in Example 5.4.*

Let $g \in Lip[0,1]$. Then $|g|_{Lip} e_0 \pm g$ are increasing functions, so that $|g|_{Lip} e_0 \pm \overline{\mathbb{B}}_n g$ are also increasing. It follows that $\left|\overline{\mathbb{B}}_n g\right|_{Lip} \leq |g|_{Lip}$, and so $c_{\overline{\mathbb{B}}_n} = 1$.

Take $A = B_n$, the classical Bernstein operator. Then $A \circ L = U_n$, the genuine Bernstein-Durrmeyer operator (see, e.g., [8]). From Proposition 5.1 we get

$$\left|U_n f(x) - \overline{\mathbb{B}}_n f(x)\right| \leq \widetilde{\omega}\left(f, \left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}\right).$$

Example 5.3

Let $L = B_n$ and $A = \overline{\mathbb{B}}_n$. Then $A \circ L = S_n$ is a Stancu-type operator investigated in [14]. We infer that

$$|S_n f(x) - B_n f(x)| \leq \widetilde{\omega}\left(f, \left(\frac{x(1-x)}{n+1}\right)^{\frac{1}{2}}\right).$$

Example 5.4

For $\alpha, \beta > -1$, let $\mathcal{B}_n^{\alpha,\beta} : C[0,1] \to C[0,1]$ be defined by

$$\mathcal{B}_n^{\alpha,\beta} f(x) := \left(\int_0^1 t^{nx+\alpha}(1-t)^{n(1-x)+\beta} f(t)dt\right) \Big/ \left(\int_0^1 t^{nx+\alpha}(1-t)^{n(1-x)+\beta} dt\right).$$

As in Example 5.2 (see also Remark 5.2), it can be proved that the corresponding parameter $c_{\mathcal{B}_n^{\alpha,\beta}}$ is $\frac{n}{n+\alpha+\beta+2}$.

If we take $L = \mathcal{B}_n^{\alpha,\beta}$ and $A = B_n$, then $A \circ L = M_n^{\alpha,\beta}$, the Bernstein-Durrmeyer operator with Jacobi weights discussed in Example 4.3. From Proposition 5.1 we obtain

$$\left|M_n^{\alpha,\beta} f(x) - \mathcal{B}_n^{\alpha,\beta} f(x)\right| \leq \widetilde{\omega}\left(f, \frac{n}{n+\alpha+\beta+2}\left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}\right).$$

In particular, we can see what happens when $\alpha \to \infty$ and/or $\beta \to \infty$.

Example 5.5

Let $L = B_{n+1}$ and $A = B_n$. Then $A \circ L = D_n$, an operator which was investigated in [8]. In this case we have

$$|D_n f(x) - B_{n+1} f(x)| \leq \widetilde{\omega}\left(f, \left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}\right).$$

Remark 5.3. *Other kind of results concerning differences of positive linear operators can be found in [8], [9], [10], [12] and the references therein.*

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COMPACT DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS ON $H^\infty(B_N)$

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ABSTRACT. In this paper, the compactness of differences of two weighted composition operators on the space of bounded analytic functions on the open unit ball in the uniform operator topology is characterized. A sufficient condition when two weighted composition operators lie in the same component of the space of nonzero weighted composition operators on $H^\infty(B_N)$, which is denoted by $\mathcal{C}_\omega(H^\infty(B_N))$, is established as well.

1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^N$ and $H(\Omega)$ a space of analytic functions on Ω. Let $\phi(z) = (\phi_1(z), \cdots, \phi_n(z))$ be an analytic self-map of Ω and $u \in H(\Omega)$. The composition operator C_ϕ induced by ϕ is defined by

$$(C_\phi f)(z) = f(\phi(z));$$

the multiplication operator induced by u is defined by

$$M_u f(z) = u(z)f(z);$$

and the weighted composition operator uC_ϕ induced by ϕ and u is defined by

$$(uC_\phi f)(z) = u(z)f(\phi(z))$$

for $z \in \Omega$ and $f \in H(\Omega)$. If let $u \equiv 1$, then $uC_\phi = C_\phi$; if let ϕ be an identity mapping, then $uC_\phi = M_u$. So we can regard weighted composition operator as a generalization of a multiplication operator and a composition operator. These operators are linear. By $H^\infty(\Omega)$ denote the Banach space of bounded analytic functions on Ω.

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When X is a Banach space of analytic functions, we write $C(X)$ for the space of composition operators on X under the operator norm topology. The investigation of the topological structure of $C(H^2)$ was initiated by Berkson [3] in 1981, who focused attention on topological structure with his isolation results on composition operators acting on the Hardy space. Readers interested in this topic can refer to the following papers: [1, 4, 6, 9, 10, 11, 12, 13, 15, 18, 19, 20, 21, 25, 26, 27].

Building on these results, this paper answers the question of compact difference for two weighted composition operators acting on $H^\infty(B_N)$, and gives a sufficient condition when two weighted composition operators lie in the same component of the space of nonzero weighted composition operators on $H^\infty(B_N)$.

2. Some preliminary lemmas and definitions

Recall that $H^\infty(B_N)$ be the set of all bounded analytic functions on the open unit ball B_N of $\mathbb{C}^N$. Then $H^\infty(B_N)$ is the Banach algebra with the supremum norm

$$\|f\|_\infty := sup\left\{|f(z)| : z \in \mathbb{C}^N\right\}.$$

We denote by $B(H^\infty(B_N))$ the closed unit ball of $H^\infty(B_N)$. For $z, w \in B_N$, we denote by Φ_a the involution which interchanges the origin and point a. The pseudohyperbolic distance between z and w is given by

$$\beta(z,w) := \sup\left\{|f(z)| : \|f\|_\infty = 1, f(w) = 0\right\}.$$

The *induced distance* between z and w is defined as

$$d_\infty(z,w) := \sup_{\|f\|_\infty=1} |f(z) - f(w)|.$$

It was shown that in [2]

$$d_\infty(z,w) = \frac{2 - 2\sqrt{1-\beta(z,w)^2}}{\beta(z,w)}. \tag{2.1}$$

Put $\gamma(t) = \frac{2(1-\sqrt{1-t^2})}{t}, 0 \le t \le 1$, then $d_\infty(z,w) = \gamma(\beta(z,w))$. It is easy to check that γ is an increasing function on $[0,1]$ and $0 \le \gamma \le 2$.

We state a lemma which was given in [26].

Lemma 2.1. *For any z in B_N, we have*
(a) $\beta(z,w) = |\Phi_z(w)|$ *for any* $w \in B_N$;
(b) $\{w : \beta(z,w) < \lambda\} = \Phi_z(\lambda B_N)$.

Let $\mathcal{S}(B_N)$ denote the set of holomorphic maps from B_N to B_N, and $\mathcal{C}(H^\infty(B_N))$ the space of composition operators on $H^\infty(B_N)$. Let uC_ϕ be a weighted composition operator on $H^\infty(B_N)$. Put $T_t = tuC_\phi$ for

$0 \le t \le 1$. Then we can easily obtain that uC_ϕ and 0 are in the same path component. So it is enough for us to consider only the space of nonzero weighted composition operators on $H^\infty(B_N)$, which is denoted by $\mathcal{C}_\omega(H^\infty(B_N))$.

There are several useful lemmas which are used to get the main results of this paper. The following Lemma is important, which was presented by Berndtsson in [5].

Lemma 2.2. *Let $\{x_i\}$ be a sequence in the ball B_N satisfying*

$$\prod_{j:j\neq k} |\Phi_{x_j}(x_k)| \ge d > 0 \quad \text{for any} \quad k. \tag{2.2}$$

Then there exists a number $M = M(d) < \infty$ and a sequence of functions $h_k \in H^\infty(B_N)$ such that

$$(a)\ \ h_k(x_j) = \delta_{kj}; \quad (b)\ \ \sum_k |h_k(z)| \le M \quad \text{for} \quad |z| < 1. \tag{2.3}$$

(The symbol δ_{kj} is equal to 1 if $k = j$ and 0 otherwise.)

Next two lemmas were proved by Carl Toews in [26].

Lemma 2.3. *Let $\{z_n\} \subset B_N$ be a sequence with $|z_n| \to 1$ as $n \to \infty$. Then for any given $d \in (0, 1)$ there is a subsequence such that $\{x_i\} := \{z_{n_i}\}$ satisfies (2.2).*

From this lemma, we can find that there is a subsequence which satisfying (2.2) for every sequence converging to the boundary of the ball, and Lemma 2.2 holds for this subsequence.

Lemma 2.4. *Let $\{h_k\}$ be a sequence of $H^\infty(B_N)$ functions such that $\sum_k |h_k(z)| \le M < \infty$ for all $z \in B_N$. Then $h_k \to 0$ weakly.*

Let T be a bounded linear operator on a Banach space. Recall that T is said to be *compact* if T maps every bounded set into relatively compact one, and that T is said to be *completely continuous* if T maps every weakly convergent sequence into a norm convergent one. In general, every compact operators is completely continuous, but the converse is not always true.

Lemma 2.5. *(Proposition 3.11 of [8]) Let $\phi, \psi \in \mathcal{S}(B_N)$, and $u, v \in H^\infty(B_N)$, then $uC_\phi - vC_\psi$ is compact on $H^\infty(B_N)$ if and only if $\|(uC_\phi - vC_\psi)f_n\|_\infty \to 0$ for every bounded sequence $\{f_n\}$ in $H^\infty(B_n)$ such that $f_n \to 0$ uniformly on every compact subset of B_N.*

Let $H(B_N, \mathbb{C}^N)$ be the collection of all the holomorphic maps from B_N to $\mathbb{C}^N$, and $H^\infty(B_N, \mathbb{C}^N)$ the subspace defined by

$$H^\infty(B_N, \mathbb{C}^N) := \{F = (f_1, ..., f_N) : f_j \in H^\infty(B_N), j = 1, ..., N\}.$$

It is clear that $H^\infty(B_N, \mathbb{C}^N)$ equipped with

$$\|F\|_{H^\infty(B_N,\mathbb{C}^N)} = \|(f_1, ..., f_N)\|_{H^\infty(B_N,\mathbb{C}^N)} := \sum_{j=1}^{N} \sup_{z\in B_N} |f_j(z)|$$

as its norm becomes a Banach space.

Now, if $F(z) = (f_1(z), ..., f_N(z)) \in H^\infty(B_N, \mathbb{C}^N), u(z) \in H^\infty(B_N)$ and $\phi \in \mathcal{S}(B_N)$, we define $u \cdot F := (uf_1, ..., uf_N)$, and $uC_\phi(F) := (uf_1 \circ \phi, ..., uf_N \circ \phi)$. Thus every weighted composition operator acting on $H^\infty(B_N, \mathbb{C}^N)$ is bounded.

3. Compact differences of two weighted composition operators

In this section, we shall state a necessary and sufficient condition for the difference of two weighted composition operators to be compact on $H^\infty(B_N)$, and give its proof in detail.

Theorem 3.1. *Let $uC_\phi, vC_\psi \in \mathcal{C}_\omega(H^\infty(B_N))$, then $uC_\phi - vC\psi$ is compact on $H^\infty(B_N)$ if and only if the following three conditions hold:*

(a) If $\{z_n\} \subset B_N, |\phi(z_n)| \to 1$ and $\liminf_{n\to\infty} \beta(\phi(z_n), \psi(z_n)) > 0$, then $u(z_n) \to 0$;

(b) If $\{z_n\} \subset B_N, |\psi(z_n)| \to 1$ and $\liminf_{n\to\infty} \beta(\phi(z_n), \psi(z_n)) > 0$, then $v(z_n) \to 0$;

(c) If $\{z_n\} \subset B_N, |\phi(z_n)| \to 1, |\psi(z_n)| \to 1$, then $u(z_n) - v(z_n) \to 0$.

Proof. The proof of the necessity. Suppose $uC_\phi - vC_\psi$ is compact on $H^\infty(B_N)$. Let $\{z_n\} \subset B_N$ with $|\phi(z_n)| \to 1$, and

$$\liminf_{n\to\infty} \beta(\phi(z_n), \psi(z_n)) = \sigma > 0. \tag{3.1}$$

To prove (a). Suppose not, then we may assume that $\limsup_{n\to\infty} |u(z_n)| = \delta_0 > 0$. So we can assume that there exists a $\delta \in (0, \delta_0]$ and a subsequence $\{z_{n_j}\} \subset \{z_n\}$ such that $\lim_{n\to\infty} |u(z_{n_j})| = \delta$. To simplify the sign, we now write the subsequence $\{z_n\}$ for $\{z_{n_j}\}$. So

$$\lim_{n\to\infty} |u(z_n)| = \delta > 0. \tag{3.2}$$

Then we conclude from Lemma 2.3 that there is a subsequence of $\phi(z_n)$ such that for a given $d \in (0, 1)$, $\{x_j\} := \{\phi(z_{n_j})\}$ satisfies

$\prod_{j:j\neq k} |\Phi_{x_j}(x_k)| \geq d > 0$ for all fixed k. So by Lemma 2.2, there exists a number $M = M(d) < \infty$ and a sequence of functions $f_k \in H^\infty(B_N)$ such that

$$(3.3)\quad (i)\ \ f_k(\phi(z_{n_j})) = \delta_{kj}; \quad (ii)\ \ \sum_{k=1}^{\infty} |f_k(\phi(z))| \leq M \quad \text{for} \quad |z| < 1.$$

It follows from Lemma 2.4 that $f_n \to 0$ weakly in $H^\infty(B_N)$.

Set $g_n(z) = \Phi_{\psi(z_n)}(z) f_n(z)$, then $g_n(z) \in H^\infty(B_N, \mathbb{C}^N)$, and

$$|g_n(z)| = \sqrt{\sum_{i=1}^{N} |g_n^i(z)|^2} = |\Phi_{\psi(z_n)}(z)||f_n(z)| \leq |f_n(z)|,$$

where $g_n^i(z)$ denote the i^{th} coordinate component of $g_n(z)$. So

$$\sum_n |g_n^i(z)| \leq \sum_n |f_n(z)| \leq M.$$

By Lemma 2.4, it follows that $g_n^i \to 0$ weakly in $H^\infty(B_N)$ for every $1 \leq i \leq N$. Because $uC_\phi - vC_\psi$ is compact operator acting on $H^\infty(B_N)$, it takes every weakly convergent sequence into a norm convergent one, and then

$$(3.4)\qquad \|(uC_\phi - vC_\psi)g_n\|_{H^\infty(B_N,\mathbb{C}^N)} = \sum_{i=1}^{N} \|(uC_\phi - vC_\psi)g_n^i\|_\infty \to 0.$$

On the other hand,

$$\|(uC_\phi - vC_\psi)g_n\|_{H^\infty(B_N,\mathbb{C}^N)} \geq |u(z_n)g_n(\phi(z_n)) - v(z_n)g_n(\psi(z_n))|$$

$$= |u(z_n)||\Phi_{\psi(z_n)}(\phi(z_n))||f_n(\phi(z_n))| = |u(z_n)|\beta(\phi(z_n), \psi(z_n)).$$

By (3.1) and (3.2), $\lim_{n\to\infty} \|(uC_\phi - vC_\psi)g_n\|_{H^\infty(B_N,\mathbb{C}^N)} \geq \delta\sigma > 0$. This contradicts (3.4).

The proof of (b) is the same as that of (a).

Next, let $\{z_n\} \subset B_N$ with $|\phi(z_n)| \to 1, |\psi(z_n)| \to 1$ as $n \to \infty$. To prove (c), similarly as the analysis in the proof of (a), we may assume that

$$(3.5)\qquad u(z_n) - v(z_n) \to \alpha \neq 0.$$

If

$$(3.6)\qquad \beta(\phi(z_n), \psi(z_n)) \to 0,$$

by Lemma 2.4, we may also find a sequence of function $\{f_k\}$ for $\{\phi(z_n)\}$ with the properties (3.3). From Lemma 2.5, the compactness of $uC_\phi -$

vC_ψ implies $\|(uC_\phi - vC_\psi)f_k\|_\infty \to 0$. Hence

$$u(z_n)f_n(\phi(z_n)) - v(z_n)f_n(\psi(z_n)) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

On the other hand, by (2.1), (3.3) and (3.6),

$$\left| \frac{f_n}{\|f_n\|_\infty}(\phi(z_n)) - \frac{f_n}{\|f_n\|_\infty}(\psi(z_n)) \right| \leq \gamma(\beta(\phi(z_n), \psi(z_n))),$$

so there is a $M > 0$ such that

$$|f_n(\phi(z_n)) - f_n(\psi(z_n))| \leq M\gamma(\beta(\phi(z_n), \psi(z_n))) \to 0.$$

Note that $f_n(\phi(z_n)) = 1$, it follows that $f_n(\psi(z_n)) \to 1$ as $n \to \infty$. Then by (3.7), we have

$$\begin{aligned} & |u(z_n) - v(z_n)| \\ = \; & |u(z_n)f_n(\phi(z_n)) - v(z_n)f_n(\psi(z_n)) + v(z_n)(f_n(\psi(z_n)) - 1)| \\ \leq \; & |u(z_n)f_n(\phi(z_n)) - v(z_n)f_n(\psi(z_n))| + |v(z_n)|f_n(\psi(z_n)) - 1| \to 0. \end{aligned}$$

It contradicts (3.5).

Now we should discuss the situation when

$$\limsup \beta(\phi(z_n), \psi(z_n)) = \eta_0 > 0.$$

If there is an arbitrary $\eta \in (0, \eta_0]$ such that $\lim \beta(\phi(z_{n_j}), \psi(z_{n_j})) = \eta$ for any arbitrary subsequence of $\{z_{n_j}\} \subset \{z_n\}$, then by (a) and (b), it follows that $u(z_{n_j}) \to 0, v(z_{n_j}) \to 0$, so $u(z_{n_j}) - v(z_{n_j}) \to 0$.

If not, then there is subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that

$$\beta(\phi(z_{n_j}), \psi(z_{n_j})) \to 0.$$

This case is already discussed above, so $u(z_{n_j}) - v(z_{n_j}) \to 0$, and it also contradicts (3.5). The necessity is proved.

The proof of the sufficiency.

Let $\{f_n\}$ be an arbitrary sequence of $H^\infty(B_N)$ such that $\|f_n\|_\infty \leq 1$ and $f_n \to 0$ uniformly on every compact subset of B_N. By Lemma 2.5, it is enough to prove $\|uf_n \circ \phi - vf_n \circ \psi\|_\infty \to 0$. Suppose not, we may assume that for some $\varepsilon > 0$, $\|uf_n \circ \phi - vf_n \circ \psi\|_\infty > \varepsilon$ for every n. Then there exists a sequence $\{z_n\}_n \subset B_N$ such that

$$|u(z_n)f_n(\phi(z_n)) - v(z_n)f_n(\psi(z_n))| > \varepsilon \tag{3.8}$$

for any n. From which we claim that $max\{|\phi(z_n)|, |\psi(z_n)|\} \to 1$ as $n \to \infty$. In fact, if not, then we may assume that $\phi(z_n) \to \xi_1, \psi(z_n) \to \xi_2$, where both ξ_1 and ξ_2 are inner point of B_N. Since $f_n \to 0$ on every compact subset of B_N, we have $f_n(\phi(z_n)) \to 0, f_n(\psi(z_n)) \to 0$. This contradicts (3.8).

So we may assume that $|\phi(z_n)| \to 1$ and $\psi(z_n) \to \omega_0$ for some ω_0 such that $|\omega_0| \leq 1$. Moreover we assume

$$\beta(\phi(z_n), \psi(z_n)) \to r \quad \text{as} \quad n \to \infty. \tag{3.9}$$

Suppose $r > 0$. When $|\omega_0| = 1$, we conclude from (a) and (b) that $u(z_n) \to 0, v(z_n) \to 0$, which contradicts (3.8). When $|\omega_0| < 1$, since $f_n \to 0$ uniformly on every compact subset of B_N, $(f_n \circ \psi)(z_n) \to 0$. Note that $u(z_n) \to 0$ as the consequence of (a), these also contradict (3.8). Hence we have $r = 0$.

Then by (3.9), we have $|\psi(z_n)| \to 1$, and by (2.1) it follows that

$$|f_n(\phi(z_n)) - f_n(\psi(z_n))| \leq \gamma(\beta(\phi(z_n), \psi(z_n))) \to 0.$$

This shows that $f_n(\phi(z_n)) - f_n(\psi(z_n)) \to 0$. By (c), we also have $u(z_n) - v(z_n) \to 0$. So $u(z_n)f_n(\phi(z_n)) - v(z_n)f_n(\psi(z_n)) \to 0$, it also contradicts (3.8). This completes the proof of the theorem. □

4. Path connected subset in $\mathcal{C}_\omega(H^\infty(B_N))$

For $uC_\phi, vC_\psi \in \mathcal{C}_\omega(H^\infty(B_N))$, we write $uC_\phi \sim vC_\psi$ if uC_ϕ and vC_ψ are in the some path component of $\mathcal{C}_\omega(H^\infty(B_N))$.

The following theorem gives a sufficient condition when two weighted composition operators lies in the same component of $\mathcal{C}_\omega(H^\infty(B_N))$. The similar sufficient and necessary condition of composition operator without weights appears in [26] (that is, Theorem 1 in section 3, [26]).

Theorem 4.1. *Let ϕ and ψ be analytic self maps of B_N with $d_\beta(\phi, \psi) < 1$, and nonzero functions $u(z), v(z) \in H^\infty(B_N)$, then $uC_\phi \sim vC_\psi$.*

We postpone the proof until we have established a number rather technical lemmas.

Lemma 4.2. *Let nonzero functions $u, v \in H^\infty(B_N)$, and $\phi \in \mathcal{S}(B_N)$. Then $uC_\phi \sim vC_\phi$.*

Proof. For every t, $0 \leq t \leq 1$, put $T_t = (tu + (1-t)v)C_\phi$. Then $T_0 = vC_\phi$, $T_1 = uC_\phi$, $T_t \neq 0$, and $\|T_t - T_{t'}\| \leq |t - t'|\|C_\phi\|\|u - v\|_\infty$. Hence the mapping $t \mapsto T_t \in \mathcal{C}_\omega(H^\infty(B_N))$, $0 \leq t \leq 1$, is continuous. □

Let $\phi, \psi \in \mathcal{S}(B_N)$, and $d_\beta(\phi, \psi) := \sup_{z \in B_N} \beta(\phi(z), \psi(z))$. This defines a $[0, 1]$-valued metric on $\mathcal{S}(B_N)$. Denote the ensuing topological space by $\mathcal{S}(B_N, d_\beta)$. The next lemma, whose composition operator without weights appears in [26], shows that there is certain connection between $\mathcal{S}(B_N)$ and $\mathcal{C}(H^\infty(B_N))$. And that may be useful for us to find a path connected subset in $\mathcal{C}_\omega(H^\infty(B_N))$.

Lemma 4.3. *Let $\phi, \psi : B_N \to B_N$ and nonzero $u(z) \in H^\infty(B_N)$. Then*

$$\|uC_\phi - uC_\psi\| \leq \frac{2 - 2\sqrt{1 - d_\beta(\phi,\psi)^2}}{d_\beta(\phi,\psi)} \|u\|_\infty.$$

Proof.

$$\begin{aligned}
\|uC_\phi - uC_\psi\| &= \sup_{\|f\|_\infty=1} \sup_{z\in B_N} |u(z)f(\phi(z)) - u(z)f(\psi(z))| \\
&\leq \|u\|_\infty \sup_{\|f\|_\infty=1} \sup_{z\in B_N} |f(\phi(z)) - f(\psi(z))| \\
&= \|u\|_\infty \sup_{z\in B_N} \sup_{\|f\|_\infty=1} |f(\phi(z)) - f(\psi(z))| \\
&= \|u\|_\infty \sup_{z\in B_N} d_\infty(\phi(z), \psi(z)) \\
&= \|u\|_\infty \sup_{z\in B_N} \frac{2 - 2\sqrt{1 - \beta(\phi(z),\psi(z))^2}}{\beta(\phi(z),\psi(z))} \\
&= \frac{2 - 2\sqrt{1 - d_\beta(\phi,\psi)^2}}{d_\beta(\phi,\psi)} \|u\|_\infty.
\end{aligned}$$

□

The former lemma follows from the observation that the function

$$f(x) := \frac{2 - 2\sqrt{1 - x^2}}{x}$$

maps $(0, 1]$ continuously onto $(0, 2]$ and increases monotonically in x.

Now we begin by introducing some abusive but convenient notation. Given two points z and w in B_N define

$$z_t := (1 - t)z + tw. \tag{4.1}$$

Analogously, given two self maps of the ball ϕ and ψ, we define

$$\phi_t(z) := (1 - t)\phi(z) + t\psi(z). \tag{4.2}$$

Note that $z_0 = z, z_1 = w$, and z_t lies on the straight line connecting z and w; in particular, any convex set that contains z and w also contains z_t.

Next lemma was proved by Carl Toews in [26]:

Lemma 4.4. *Let ϕ and ψ be analytic self maps of B_N satisfying $d_\beta(\phi, \psi) \leq \lambda < 1$, and let ϕ_t be as in (4.2). Then, for $t \in [0, 1]$ and ε such that $t + \varepsilon \in [0, 1]$, we have*

$$\lim_{|\varepsilon|\to 0} d_\beta(\phi_t, \phi_{t+\varepsilon}) = 0.$$

Now using the previous lemmas, we turn our attention back to the proof of Theorem 4.1.

Proof. Given ϕ and ψ in $\mathcal{S}(B_N)$ with $d_\beta(\phi,\psi) < 1$, define ϕ_t as in (4.2). It follows from Lemma 4.4 that the map $t \mapsto \phi_t$ is a continuous path from ϕ to ψ in $\mathcal{S}(B_N, d_\beta)$ and from Lemma 4.3 that the map $t \mapsto uC_{\phi_t}$ is continuous from uC_ϕ to uC_ψ in $\mathcal{C}_\omega(H^\infty(B_N))$. And by Lemma 4.2 we know uC_ϕ and vC_ψ are in the same path component in $\mathcal{C}_\omega(H^\infty(B_N))$. □

It follows from Lemma 2 in [26] that $d_\beta(\phi,\psi) < 1$ if and only if $\|C_\phi - C_\psi\| < 2$, so we have an equivalent condition.

Theorem 4.5. *Let ϕ and ψ be analytic self maps of B_N with $\|C_\phi - C_\psi\| < 2$, and nonzero functions $u(z), v(z) \in H^\infty(B_N)$, then $uC_\phi \sim vC_\psi$.*

Remark. Carl Toews [26] gave a sufficient and necessary condition when two composition operator without weights lie in the same component. But it is somewhat different from the main theorem of this section, because equation in Lemma 4.3 is not equal and the method of [26] may be invalid.

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RIEMANN-STIELTJES OPERATOR BETWEEN ITERATED LOGARITHMIC BLOCH SPACE AND MIXED-NORM SPACE ON THE UNIT BALL

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Abstract. In this paper, we discuss the boundedness and compactness of Riemann-Stieltjes operator between iterated logarithmic Bloch space and mixed-norm space on the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{B}$ be the unit ball of the complex space $\mathbb{C}^n$, and $S = \partial\mathbb{B}$ the boundary of the unit ball. Denote by $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$ and $S(\mathbb{B})$ the collection of all the holpmorphic self mappings of $\mathbb{B}$. $d\nu(z)$ is the normalized Lebesgue volume measure on $\mathbb{B}$ and $d\sigma(\zeta)$ is the normalized surface measure on S. Let $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ be the points in $\mathbb{C}^n$ and $\langle z, w\rangle = \sum_{k=1}^n z_k\bar{w_k}$. For $f \in H(\mathbb{B})$, let

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of f.

Denote $\mathbb{N} = \{1, 2, \cdots, n, \cdots\}$, for a fixed $k \in \mathbb{N}$, $f \in H(\mathbb{B}))$ is said to belong to the weighted-type space, denoted by $H_{log_k} = H_{log_k}(\mathbb{B})$ (see, e.g. [22]), if

$$\|f\|_{H_{log_k}} = \sup_{z\in\mathbb{B}}(1 - |z|^2)\left(\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1 - |z|^2}\right)|f(z)| < \infty, \tag{1}$$

where $e^{[k]}$ is defined inductively by $e^{[1]} = e, e^{[k]} = e^{e^{[k-1]}}$ and $\ln^{[j]} z = \underbrace{\ln \cdots \ln}_{j\text{times}} z$. It is easy to show that H_{log_k} is a Banach space with the norm $\|.\|_{H_{log_k}}$.

We recall that the α-Bloch space $\mathcal{B}^\alpha$ $(\alpha \geq 0)$, consists of all $f \in H(\mathbb{B})$ such that

$$b_\alpha(f) = \sup_{z\in\mathbb{B}}(1 - |z|^2)^\alpha |\Re f(z)| < \infty.$$

The expression $b_\alpha(f)$ defines a semi-norm while the natural norm is given by $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$. This norm makes $\mathcal{B}^\alpha$ into a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the classical Bloch space.

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The logarithmic Bloch space $\mathcal{B}_{log} = \mathcal{B}_{log}(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that (see, e.g. [14])

$$\|f\|_{b_{log}} = \sup_{z\in B}(1-|z|^2)(\ln\frac{e}{1-|z|^2})|\Re f(z)| < \infty.$$

With the norm $\|f\|_{\mathcal{B}_{log}} = |f(0)| + \|f\|_{b_{log}}$ it becomes a Banach space.

Let $k \in \mathbb{N}$, the iterated logarithmic Bloch space $\mathcal{B}_{log_k} = \mathcal{B}_{log_k}(\mathbb{B})$, consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{b_{log_k}} = \sup_{z\in\mathbb{B}}(1-|z|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)|\Re f(z)| < \infty,$$

where $e^{[k]}$ is defined inductively by $e^{[1]} = e, e^{[k]} = e^{e^{[k-1]}}$ and $\ln^{[j]} z = \underbrace{\ln ... \ln}_{jtimes} z$.

The norm in $\mathcal{B}_{log_k}$ is given by

$$\|f\|_{\mathcal{B}_{log_k}} = |f(0)| + \|f\|_{b_{log_k}}. \tag{2}$$

Then the iterated logarithmic Bloch space $\mathcal{B}_{log_k}$ is a Banach space with the norm given above.

When $k = 1$, it is the logarithmic Bloch space $\mathcal{B}_{log_1} = \mathcal{B}_{log}$. For the case of the unit disk, the logarithmic Bloch space has appeared for the first time in characterizing the multipliers of the Bloch space [9]. For the case of the unit ball, see Theorem 3.21 in [35].

The little iterated logarithmic Bloch space $\mathcal{B}_{log_k,0} = \mathcal{B}_{log_k,0}(\mathbb{B})$, consists of all $f \in \mathcal{B}_{log_k}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)|\Re f(z)| = 0.$$

Moreover, the set of all polynomials is dense in $\mathcal{B}_{log_k,0}$ and is the closed subspace of the iterated logarithmic Bloch space. See [16, 22]) for more information of the iterated logarithmic Bloch space on $\mathbb{B}$.

A positive continuous function ϕ on $[0,1)$ is called normal [32], if there exist three positive constants $0 \le \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta, 1)$

$$\frac{\phi(r)}{(1-r)^a} \downarrow 0, \quad \frac{\phi(r)}{(1-r)^b} \uparrow \infty \tag{3}$$

as $r \to 1$. In the rest of this paper we always assume that ϕ is normal on $[0,1)$.

From now on if we say that a function $\phi : \mathbb{B} \to [0,\infty)$ is normal, we will also assume that it is radial on $\mathbb{B}$, that is, $\phi(z) = \phi(|z|), z \in \mathbb{B}$.

For $0 < p, q < \infty$ and ϕ is normal, the mixed-norm space $H(p,q,\phi) = H(p,q,\phi)(\mathbb{B})$, consists of all functions $f \in H(\mathbb{B})$ such that (see, e.g. [15, 29])

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{1/p} < \infty,$$

where

$$M_q(f,r) = \left(\int_S |f(r\zeta)|^q d\sigma(\zeta)\right)^{1/q}.$$

For $p=q$, $\alpha>-1$ and $\phi(r)=(1-r^2)^{(\alpha+1)/p}$, the mixed-norm space is equivalent with the weighted Bergman space $A^p_\alpha(\mathbb{B})=A^p_\alpha$.

As we all know, the Riemann-Stieltjes operator was introduced by Li and Stević, for $h\in H(\mathbb{B})$, then

$$L_hf(z)=\int_0^1 \Re f(tz)h(tz)\frac{dt}{t},\ \ f\in H(\mathbb{B}),\ z\in\mathbb{B}. \tag{4}$$

and studied, for example, in [1, 2, 3, 17, 18, 19, 20, 21, 30].

In this paper, we will characterize the boundedness and compactness of Riemann-Stieltjes operator between iterated logarithmic Bloch space and mixed-norm space on the unit ball of $\mathbb{C}^n$.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A\asymp B$ means that there is a positive constant C such that $B/C\le A\le CB$.

2. Auxiliary results

Now let us introduce several auxiliary results, which are used in the proofs of the main results.

Lemma 1. *For every $f,h\in H(\mathbb{B})$ and $\varphi\in S(\mathbb{B})$. Then*

$$\Re[L_hf](z)=\Re f(z)h(z)$$

Proof. The proof of this Lemma follows by standard arguments (see [17, 18, 20]). □

The following consequence of lemma appears in ([22]).

Lemma 2. *There exists $N=N(n)$ and functions $f_1,...,f_N\in H_{log_k}(\mathbb{B})$ such that*

$$\sum_{m=1}^N |f_m(z)|\ge\frac{C}{(1-|z|^2)\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|z|^2}},\ \ z\in\mathbb{B}$$

for some positive constant $C>0$.

Remark: For $m=1,\cdots,N$, and f_m as above, let

$$F_m(z)=\int_0^1 (f_m(zt)-f_m(0))\frac{dt}{t},\ \ m=1,...,N. \tag{5}$$

It is clear that $\Re F_m(z)=f_m(z)-f_m(0)$ and by Lemma2, $\Re F_m(z)\in H_{log_k}(\mathbb{B})$ since $f_m\in H_{log_k}(\mathbb{B})$. That is

$$\sup_{z\in\mathbb{B}}(1-|z|^2)\Big(\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)|\Re F_j(z)|<\infty.$$

Therefore $F_m\in\mathcal{B}_{log_k}(\mathbb{B})$ and

$$\sum_{m=1}^N |\Re F_m(z)+f_m(0)|\ge\frac{C}{(1-|z|^2)\prod_{j=1}^k \ln^{[j]}\frac{e^{[k]}}{1-|z|^2}},\ \ z\in\mathbb{B} \tag{6}$$

for some positive constant $C>0$.

Lemma 3. [15, 29] *Assume that $0 < p, q < \infty$, ϕ is normal and $m \in \mathbb{N}$. Then the following asymptotic relationship holds*

$$\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr \asymp |f(0)|^p + \int_0^1 M_q^p(\Re^m f,r)(1-r)^{mp}\frac{\phi^p(r)}{1-r}dr.$$

Lemma 4. *Assume that $0 < p, q > -1, k \in \mathbb{N}, \phi$ is normal and $h \in H(\mathbb{B})$. Let X, Y be one of the following spaces $\mathcal{B}_{log_k}, H(p,q,\phi)$. Then the operator $L_h : X \to Y$ is compact if and only if $L_h : X \to Y$ is bounded and for every bounded sequence $\{f_k\}_{k\in N}$ in X which converges to zero uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$, we have $\|L_h f_k\|_Y \to 0$ as $k \to \infty$.*

Proof. This Lemma can be proved in a standard way (see, e.g. the proofs of the corresponding lemmas in [18]), hence we omit its proof. □

Lemma 5. *Assume that $0 < p, q < \infty$, and ϕ is normal. Then there is a positive constant C independent of f such that*

$$|\Re f(z)| \le C\frac{\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{\frac{n}{q}+1}}, \quad z \in \mathbb{B}$$

for any $f \in H(p,q,\phi)$.

Proof. By Lemma 3, (3) and using the Theorem 7.2.5 in [24] we get that

$$\begin{aligned}
\|f\|^p_{H(p,q,\phi)} &\ge \int_{\frac{1+|z|}{2}}^{\frac{3+|z|}{4}} M_p^q(f,r)\frac{\phi^p(r)}{1-r}dr \\
&\ge C\int_{\frac{1+|z|}{2}}^{\frac{3+|z|}{4}} M_p^q(\Re f,r)\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&\ge CM_p^q(\Re f,|z|)\int_{\frac{1+|z|}{2}}^{\frac{3+|z|}{4}} \frac{\phi^p(r)}{(1-r)^{1-p}}dr \\
&= CM_p^q(\Re f,|z|)\frac{\phi^p(z)}{(1-|z|)^{bp}}\int_{\frac{1+|z|}{2}}^{\frac{3+|z|}{4}} (1-r)^{pb+p-1}dr \\
&\ge C|\Re f(z)|^p(1-|z|^2)^{np/q}\phi^p(z)(1-|z|)^p \\
&\ge C|\Re f(z)|^p\phi^p(z)(1-|z|^2)^{p+np/q}.
\end{aligned}$$

So

$$|\Re f(z)| \le C\frac{\|f\|_{H_{p,q,\phi}}}{\phi(z)(1-|z|^2)^{n/q+1}}.$$

□

The next lemma was obtained in [25].

Lemma 6. *If $a > 0, b > 0$, then the elementary inequality holds,*

$$(a+b)^p \le \begin{cases} a^p + b^p, & 0 < p < 1; \\ 2^{b-1}(a^p+b^p), & p \ge 1. \end{cases}$$

It is obvious that Lemma 5 holds for the sum of finite number k , that is

$$(a_1 + ... + a_k)^p \le C(a_1^p + ... + a_k^p).$$

Where $a_1, \cdots, a_k >$, C is a positive constant.

Lemma 7. *A closed set E in $\mathcal{B}_{log_k,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1}\sup_{f\in E}(1-|z|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)|\Re f(z)|=0$$

Proof. The proof of this lemma follows the same idea as the proof of Lemma 1 in [23] with minor modifications, we omit the detail. □

3. The boundedness and compactness of the operator $L_h:\mathcal{B}_{log_k}\to H(p,q,\phi)$

Theorem 1. *Assume that $0<p,q<\infty, k\in\mathbb{N}$ and suppose that ϕ is normal and $h\in H(\mathbb{B})$. Then $L_h:\mathcal{B}_{log_k}\to H(p,q,\phi)$ is bounded if and only if*

$$M:=\int_0^1\Big(\int_S\frac{|h(r\zeta)|^q d\sigma(\zeta)}{(1-r^2)^q\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-r^2}\Big)^q}\Big)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr<\infty. \tag{7}$$

Proof. First, we suppose that (7) holds. For any $f\in\mathcal{B}_{log_k}$, note that $|L_hf(0)|=0$, then using Lemma 1 and Lemma 3, it follows that

$$\begin{aligned}
\|L_hf\|^p_{H(p,q,\phi)} &= \int_0^1 M_q^p(L_hf,r)\frac{\phi^p(r)}{1-r}dr\\
&\asymp \int_0^1 M_q^p(\Re(L_hf),r)\frac{\phi^p(r)}{(1-r)^{1-p}}dr\\
&= \int_0^1\Big(\int_S|\Re f(r\zeta)h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr\\
&\le C\int_0^1\Big(\int_S\frac{|h(r\zeta)|^q d\sigma(\zeta)}{(1-|r\zeta|^2)^q\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|r\zeta|^2}\Big)^q}\Big)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr\|f\|^p_{\mathcal{B}_{log_k}}\\
&= CM\|f\|^p_{\mathcal{B}_{log_k}}<\infty.
\end{aligned}\tag{8}$$

From which we know that $L_h:\mathcal{B}_{log_k}\to H(p,q,\phi)$ is bounded.

For the converse direction, we suppose that $L_h:\mathcal{B}_{log_k}\to H(p,q,\phi)$ is bounded. Let $f_1,...,f_N\in H_{log_k}(\mathbb{B})$ be the functions which satisfy Lemma 2, then F_m satisfy (5) and (6). By Lemma 1 and Lemma 3 we get

$$\begin{aligned}
\|L_hF_m\|^p_{H(p,q,\phi)} &= \int_0^1 M_q^p(L_hF_m,r)\frac{\phi^p(r)}{1-r}dr\\
&\asymp \int_0^1\Big(\int_S|\Re F_m(r\zeta)h(r\zeta)|^q d\sigma(\zeta)\Big)^{\frac{p}{q}}\frac{\phi^p(r)}{1-r}dr\\
&= \int_0^1\Big(\int_S|f_m(r\zeta)-f_m(0)|^q|h(r\zeta)|^q d\sigma(\zeta)\Big)^{\frac{p}{q}}\frac{\phi^p(r)}{1-r}dr\\
&\ge \int_0^1\Big(\int_S|f_m(r\zeta)|^q|h(r\zeta)|^q d\sigma(\zeta)\Big)^{\frac{p}{q}}\frac{\phi^p(r)}{1-r}dr\\
&- \int_0^1\Big(\int_S|f_m(0)|^q|h(r\zeta)|^q d\sigma(\zeta)\Big)^{\frac{p}{q}}\frac{\phi^p(r)}{1-r}dr.
\end{aligned}\tag{9}$$

Setting the test functions $f_l^{(m)}(z) = z_l^m$ for fixed $l \in \{1, \cdots, n\}$. It is easy to check that $\|f_l^{(m)}\|_{\mathcal{B}_{log_k}} \leq C$, and $\Re f_l^{(m)} = z_l$. Thus we have

$$\begin{aligned}\|L_h f_l^{(m)}\|^p_{H(p,q,\phi)} &= \int_0^1 M_q^p(L_h f_l^{(m)}, r)\frac{\phi^p(r)}{1-r}dr \\ &\asymp \int_0^1 \Big(\int_S |r\zeta_l|^{mq}|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr < \infty.\end{aligned}$$

By the above inequality we obtain that

$$\begin{aligned}&\int_{\frac{1}{2}}^1 \Big(\int_S |\zeta_l|^{mq}|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr \\ &\leq 2^{mq}\int_0^1 \Big(\int_S |r\zeta_l|^{mq}|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr < \infty. \qquad (10)\end{aligned}$$

On the other hand, it is obvious that

$$\int_0^{\frac{1}{2}} \Big(\int_S |\zeta_l|^{mq}|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr < \infty. \qquad (11)$$

Combing (10)and (11), it follows that

$$\int_0^1 \Big(\int_S |\zeta_l|^{mq}|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr < \infty \qquad (12)$$

Thus using Lemma 6 we obtain

$$\begin{aligned}&\int_0^1 \Big(\int_S |\zeta|^{mq}|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr \\ &\leq C\int_0^1 \Big(\int_S (\sum_{l=1}^n |\zeta|^{mq})|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr < \infty.\end{aligned}$$

That is

$$\int_0^1 \Big(\int_S |h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr < \infty. \qquad (13)$$

Then from (9) we get

$$\int_0^1 \Big(\int_S |f_m(r\zeta)|^q|h(r\zeta)|^q d\sigma(\zeta)\Big)^{p/q}\frac{\phi^p(r)}{1-r}dr < \infty. \qquad (14)$$

By Lemma 2 and Lemma 6, it follows that

$$\begin{aligned}&\int_0^1 \Big(\int_S \frac{|h(r\zeta)|^q d\sigma(\zeta)}{(1-r^2)^q\Big(\prod_{j=1}^k ln^{[j]}\frac{e^{[k]}}{1-r^2}\Big)^q}\Big)^{p/q}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\ &\leq \int_0^1 \Big(\int_S (\sum_{j=1}^N |f_j(r\zeta)|)^q|h(r\zeta)|^q d\sigma(\zeta)\Big)^{\frac{p}{q}}\frac{\phi^p(r)}{(1-r)^{1-p}}dr \\ &\leq C\sum_{j=1}^N \int_0^1 \Big(\int_S |f_m(r\zeta)|^q|h(r\zeta)|^q d\sigma(\zeta)\Big)^{\frac{p}{q}}\frac{\phi^p(r)}{1-r}dr < \infty.\end{aligned}$$

The desired result (7) follows. The proof of this theorem is completed. □

Theorem 2. *Assume that $0 < p, q < \infty, k \in \mathbb{N}$ and suppose that ϕ is normal and $h \in H(\mathbb{B})$. Then $L_h : \mathcal{B}_{log_k} \to H(p, q, \phi)$ is compact if and only if (7) holds.*

Proof. Assume that $L_h : \mathcal{B}_{log_k} \to H(p,q,\phi)$ is compact, it follows that $L_h : \mathcal{B}_{log_k} \to H(p,q,\phi)$ is bounded. By Theorem 1 we obtain that (7) holds.

Conversely, suppose that (7) holds, then by Theorem 1, it follows that $L_h : \mathcal{B}_{log_k} \to H(p,q,\phi)$ is bounded. From the proof of Theorem 1 we know that

$$M_1 := \int_0^1 \Big(\int_S |h(r\zeta)|^q d\sigma(\zeta) \Big)^{p/q} \frac{\phi^p(r)}{1-r} dr < \infty.$$

Moreover from (7) we know for any $\varepsilon > 0$, there is $0 < \delta < 1$ such that

$$\int_\delta^1 \Big(\int_S \frac{|h(r\zeta)|^q d\sigma(\zeta)}{(1-r^2)^q \Big(\prod_{j=1}^k ln^{[j]} \frac{e^{[k]}}{1-r^2} \Big)^q} \Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \varepsilon. \tag{15}$$

Assume $\{f_k\}_{k\in N}$ is a bounded sequence in $\mathcal{B}_{log_k}$ converging to zero uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. Denote $\sup_{k\in N} \|f_k\|_{\mathcal{B}_{log_k}} \leq L$.

Then by previous calculation we get

$$\begin{aligned} \|L_h f_k\|^p_{H(p,q,\phi)} &= \int_0^1 M_q^p(L_h f_k, r) \frac{\phi^p(r)}{1-r} dr \\ &\asymp \int_0^1 \left(\int_S |\Re f_k(r\zeta) h(r\zeta)|^q d\sigma(\zeta) \right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\ &= I_1 + I_2, \end{aligned} \tag{16}$$

where

$$I_1 := \int_0^\delta \left(\int_S |\Re f_k(r\zeta) h(r\zeta)|^q d\sigma(\zeta) \right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr$$

and

$$I_2 := \int_\delta^1 \left(\int_S |\Re f_k(r\zeta) h(r\zeta)|^q d\sigma(\zeta) \right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr.$$

Let $K = \{w \in \mathbb{B}, |w| \leq \delta\}$, it is a compact subset of $\mathbb{B}$. Then by Cauchy inequality we have $|\Re f_k(w)| \to 0, k \to \infty$ on the compact subsets of $\mathbb{B}$. Hence

$$\begin{aligned} I_1 &\leq \sup_{w\in K} |\Re f_k(w)| \int_0^\delta \left(\int_S |h(r\zeta)|^q d\sigma(\zeta) \right)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\ &\leq M_1 \sup_{w\in K} |\Re f_k(w)| \to 0, k \to \infty. \end{aligned} \tag{17}$$

On the other hand, by Lemma 3, (9), (10) and $\sup_{k\in N} \|f_k\|_{\mathcal{B}log_k} \leq L$ we obtain

$$I_2 \leq C \int_\delta^1 \Big(\int_S \frac{|h(r\zeta)|^q d\sigma(\zeta)}{(1-r^2)^q \Big(\prod_{j=1}^k ln^{[j]} \frac{e^{[k]}}{1-r^2} \Big)^q} \Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \|f\|^p_{\mathcal{B}log_k} \leq CL^p\varepsilon. \tag{18}$$

Thus from (16-18), we get

$$\lim_{k\to\infty} \|L_h f_k\|_{H(p,q,\phi)} = 0.$$

By Lemma 4, it follows that $L_h : \mathcal{B}_{log_k} \to H(p,q,\phi)$ is compact. The proof of this theorem is completed. □

4. The boundedness of the operator $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ (or $\mathcal{B}_{log_k,0}$)

Theorem 3. *Assume that $0 < p, q < \infty, k \in N$ and suppose that ϕ is normal and $h \in H(\mathbb{B})$. Then $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ is bounded if and only if*

$$M_3 := \sup_{z\in\mathbb{B}} \left(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2} \right) \frac{|h(z)|}{\phi(z)(1-|z|^2)^{n/q}} < \infty. \tag{19}$$

Proof. Suppose that (19) holds. Then for every $f \in H(p,q,\phi)$, by Lemma 1, Lemma 5 and $L_h f(0) = 0$, we have

$$\begin{aligned}
\|L_h f\|_{\mathcal{B}_{log_k}} &= |L_h f(0)| + \sup_{z\in\mathbb{B}} (1-|z|^2) \left(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2} \right) |\Re(L_h f)(z)| \\
&= \sup_{z\in\mathbb{B}} (1-|z|^2) \left(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2} \right) |\Re f(z) h(z)| \\
&\le C \sup_{z\in\mathbb{B}} (1-|z|^2) \Big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2} \Big) \frac{|h(z)| \|f\|_{H(p,q,\phi)}}{\phi(z)(1-|z|^2)^{n/q+1}} \\
&\le C M_3 \|f\|_{H(p,q,\phi)} < \infty.
\end{aligned}$$

Thus we get $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ is bounded.

Conversely, we assume that $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ is bounded. Then using the function

$$f_w(z) = \frac{(1-|w|^2)^\beta}{\phi(w)(1-\langle z,w\rangle)^{n/q+\beta}}, \quad w \in \mathbb{B}$$

where $w \in \mathbb{B}$ and $\beta > b, b$ is the element in (3). By [15] we have $\sup_{w\in\mathbb{B}} \|f_w\|_{H(p,q,\phi)} \le C$. Moreover,

$$\Re f_w(z) = (n/q+\beta) \frac{(1-|w|^2)^\beta \langle z,w\rangle}{\phi(w)(1-\langle z,w\rangle)^{n/q+\beta+1}}.$$

Thus we have

$$\begin{aligned}
\infty &> C\|L_h\|_{H(p,q,\phi)\to\mathcal{B}_{log_k}} \ge \|L_h f_w\|_{\mathcal{B}_{log_k}} \\
&= \sup_{z\in\mathbb{B}} (1-|z|^2) \left(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2} \right) |\Re f_w(z) h(z)| \\
&\ge (n/q+\beta)(1-|w|^2) \Big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|w|^2} \Big) \frac{|h(w)||w|^2}{\phi(w)(1-|w|^2)^{n/q+1}} \\
&= (n/q+\beta) \Big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|w|^2} \Big) \frac{|h(w)||w|^2}{\phi(w)(1-|w|^2)^{n/q}}.
\end{aligned}$$

Consequently,

$$\sup_{z\in\mathbb{B}} \Big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2} \Big) \frac{|h(z)||z|^2}{\phi(z)(1-|z|^2)^{n/q}} < \infty \tag{20}$$

since w is an arbitrary element in $\mathbb{B}$.
Therefore from (20) we obtain

$$\sup_{|z|>\frac{1}{2}}\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)\frac{|h(z)|}{\phi(z)(1-|z|^2)^{n/q}}$$
$$\leq 2^2\sup_{z\in\mathbb{B}}\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)\frac{|h(z)||z|^2}{\phi(z)(1-|z|^2)^{n/q}}<\infty. \tag{21}$$

On the other hand, it is obvious that

$$\sup_{|z|\leq\frac{1}{2}}\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)\frac{|h(z)|}{\phi(z)(1-|z|^2)^{n/q}}<\infty. \tag{22}$$

Combing (21) and (22), the desired result (19) follows. The proof of this theorem is completed. □

Theorem 4. *Assume that $0<p,q<\infty, k\in N$ and suppose that ϕ is normal and $h\in H(\mathbb{B})$. Then $L_h: H(p,q,\phi)\to\mathcal{B}_{log_k,0}$ is bounded if and only if $L_h: H(p,q,\phi)\to\mathcal{B}_{log_k}$ is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)|h(z)|=0. \tag{23}$$

Proof. Suppose $L_h: H(p,q,\phi)\to\mathcal{B}_{log_k,0}$ is bounded, it is obvious that $L_h: H(p,q,\phi)\to\mathcal{B}_{log_k}$ is bounded. Taking the function $f_a(z)=\frac{\langle z,a\rangle}{|a|^2}\in H(p,q,\phi)$ for fixed $a\in\mathbb{B}$. It is easy to show that $\Re f_a(a)=1$, thus

$$(1-|a|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|a|^2}\right)|\Re(L_hf_a)(a)|$$
$$=\quad(1-|a|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|a|^2}\right)|h(a)|\to 0,\quad |a|\to 1.$$

The desired result (23) follows.

Conversely, assume that $L_h: H(p,q,\phi)\to\mathcal{B}_{log_k}$ is bounded and (23) holds. For every polynomial p, since $\sup_{z\in\mathbb{B}}\|\Re p(z)\|<\infty$ we have

$$(1-|z|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)|\Re p(z)h(z)|$$
$$\leq(1-|z|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)|h(z)|\|\Re p\|_\infty\to 0,$$

as $|z|\to 1$. From which it follows that $L_hp\in\mathcal{B}_{log_k,0}$. As we all know, the set of all polynomials is dense in $H(p,q,\phi)$, we have that for every $f\in H(p,q,\phi)$, there is a sequence of polynomials$(p_m)_{m\in N}$ such that

$$\lim_{m\to\infty}\|f-p_m\|_{H(p,q,\phi)}=0.$$

From this and the boundedness of $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ is bounded, we obtain that

$$\|L_h f - L_h p_m\|_{B_{log_k}} \leq \|L_h\|_{H(p,q,\phi)\to B_{log_k}} \|f - p_m\|_{H(p,q,\phi)} \to 0,$$

as $m \to \infty$. Hence $L_h(H(p,q,\phi)) \subseteq B_{log_k,0}$, and the operator $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k,0}$ is bounded. This completes the proof of this theorem. □

5. The compactness of the operator $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ (or $\mathcal{B}_{log_k,0}$)

Theorem 5. *Assume that $0 < p,q < \infty, k \in N$ and suppose that ϕ is normal and $h \in H(\mathbb{B})$. Then the following statements are equivalent.*

(i) $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ is compact.

(ii) $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k,0}$ is compact.

(iii)

$$\lim_{|z|\to 1} \left(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2} \right) \frac{|h(z)|}{\phi(z)(1-|z|^2)^{n/q}} = 0. \tag{24}$$

Proof. (ii) ⇒(i). This implication is obvious.

(i) ⇒(iii) Assume that $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ is compact. Then it is clearly that $L_h : H(p,q,\phi) \to \mathcal{B}_{log_k}$ is bounded. Let $\{z_m\}_{m\in N}$ be a sequence in $\mathbb{B}$ such that $|z_m| \to 1$ as $m \to \infty$. Setting

$$f_m = \frac{(1-|z_m|^2)^\beta}{\phi(z_m)(1-\langle z, z_m\rangle)^{n/q+\beta}}, \quad m \in N$$

where $w \in \mathbb{B}$ and $\beta > b, b$ is the element in (3), and $\sup_{k\in N} \|f_m\|_{H(p,q,\phi)} \leq C$.

Moreover, since $\beta > b$, from (3) we have that

$$\lim_{m\to\infty} \frac{(1-|z_m|^2)^\beta}{\phi(z_m)} = 0,$$

From which it follows that $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$.

Using Lemma 4 we obtain

$$\lim_{m\to\infty} \|L_h f_m\|_{\mathcal{B}_{log_k}} = 0.$$

On the other hand we have

$$\|L_h f_m\|_{\mathcal{B}_{log_k}} = \sup_{z\in\mathbb{B}} (1-|z|^2)\Big(\prod_{j=1}^{k} ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\Big)|\Re f_m(z)h(z)|$$

$$\geq (1-|z_m|^2) \left(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z_m|^2} \right) \frac{|h(z_m)||z_m|^2}{\phi(z_m)(1-|z_m|^2)^{n/q+1}}.$$

The desired result (24) follows from letting $m \to \infty$ in the above inequality.

(iii)⇒(ii). Suppose (24) holds. Then

$$
\begin{aligned}
&(1-|z|^2)\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)|\Re(L_hf)(z)|\\
&=(1-|z|^2)\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)|\Re f(z)h(z)|\\
&\le C(1-|z|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)\frac{|h(z)|}{\phi(z)(1-|z|^2)^{n/q+1}}\|f\|_{H(p,q,\phi)}\\
&=C\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)\frac{|h(z)|}{\phi(z)(1-|z|^2)^{n/q}}\|f\|_{H(p,q,\phi)}.
\end{aligned}
$$

Taking the supremum in the above inequality over the set $\|f\|_{H(p,q,\phi)}\le 1$, then letting $|z|\to 1$, and using (24) we obtain

$$
\lim_{|z|\to 1}\sup_{\|f\|_{H(p,q,\phi)}\le 1}(1-|z|^2)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)|\Re(L_hf)(z)|=0. \tag{25}
$$

From (25) and Lemma 7 we obtain $T_hC_\varphi : H(p,q,\phi)\to\mathcal{B}_{log_k,0}$ is compact. The proof of this theorem is completed. □

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Method of Fundamental Solutions With an Optimal Regularization Technique for the Cauchy Problem of the Modified Helmholtz Equation

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Abstract In this paper, we propose a high-accuracy and fast numerical method for the Cauchy problem associated with the two-dimensional modified Helmholtz equation defined in a complex region. The problem is solved using the method of fundamental solution (MFS) in conjunction with the Tikhonov regularization method, whose side constraint is a Sobolev norm. A suitable regularization parameter is determined by the L-curve method. Numerical results are presented for problems with given noisy Cauchy data. Two error criteria are employed and we compare the numerical results with those from other methods for these problems.

Keyword Cauchy problem; Modified Helmholtz equation; Tikhonov regularization; Method of fundamental solutions; L-curve

1 INTRODUCTION

The Helmholtz equation is often used to describe acoustics [3], heat conduction in fins [17], wave propagation and scattering [4], and the vibration of a structure [1], etc. The forward problem for the Helmholtz equation, where boundary conditions are specified on the entire boundary of the domain and the so-called wave number κ that characterizes the equation is known, has been studied extensively. Unfortunately, in many engineering problems, either the boundary conditions are incomplete due to technical difficulties, or the geometry of the domain under investigation is not completely known, or the wave number κ is unknown; these lead to inverse problems for the Helmholtz equation. As a particular case, Cauchy problem is a classical case of an inverse problem. In this case, on a part of the boundary, both Dirichlet and Neumann conditions are specified, while on the remaining boundary, no boundary conditions are specified. The problem is ill-posed in the sense of Hadamard [6] and suffers from instability in the sense that small perturbation in the data may result in an enormous deviation in the solution.

Over the last decade, several numerical methods and theoretical analysis have been proposed for the Cauchy problem associated with the Helmholtz equation. In [12, 13], the alternating iterative algorithm, which reduces the Cauchy problem to solving a sequence of forward boundary value problems, was implemented numerically by using the boundary element method (BEM). The method is extremely time-consuming. Some regularization methods, such as Tikhonov and SVD, in conjunction with boundary knot method or MFS (Method of fundamental solutions) were proposed to solve the problem in the papers [9, 15]. In this paper, we propose a high-accuracy and fast numerical method to solve the Cauchy problem.

2 MATHEMATICAL FORMULATION OF THE PROBLEM SETTING

Let $\Omega \subset R^2$ be an open bounded domain with the boundary curve $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \varnothing, \Gamma_2 \neq \varnothing$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. We consider the modified Helmholtz equation in Ω, namely

$$\Delta u = \kappa^2 u \quad in\ \Omega, \tag{1}$$

where the wave number $\kappa > 0$ and Δ is the Laplace operator. Let $\mathbf{n}(x, y)$ be the unit outward normal vector at the point (x, y) on the boundary $\partial\Omega$, and we prescribe Dirichlet and Neumann boundary conditions simultaneously on Γ_1 as follows:

$$u = f \quad on \quad \Gamma_1, \tag{2}$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \quad on \quad \Gamma_1, \tag{3}$$

where $f,\ g$ denote given continuous functions, which are called the Cauchy data. We describe our Cauchy problem as:

Problem 1. For the given Cauchy data $f,\ g \in C(\Gamma_1)$, find $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that (1) and (2) hold.

The Cauchy problem is a well known ill-posed problem, the solution is unstable with respect to small perturbations of the data on Γ_1. For example, we take the wave number $\kappa = n$, the integer $k > 0$ and

$$\Omega = \{(x, y) : 0 < x < 1,\ 0 < y < 1\},$$

$$\Gamma_1 = \{(x, 0) : 0 \leq x \leq 1\},$$

$$f(x, 0) = \frac{1}{n^k}\cosh(nx),\ g(x, 0) = -1,\ \frac{\partial u}{\partial \mathbf{n}} = g\ on \Gamma_1,$$

the solution is $u(x, y) = \frac{1}{n^k}(\cosh(nx) + \sinh(ny))$. We can see that

$$\sup_{(x,y)\in\Gamma_1} |f(x, 0)| \to 0, \quad \sup_{(x,y)\in\bar{\Omega}} |u(x, y)| \to \infty \quad (n \to \infty).$$

This indicates that the Dirichlet and Neumann data on boundary Γ_1 tend to zero, the solution u in the whole domain $\bar{\Omega}$ is unbounded. In other words , the solution u of the Cauchy problem does not depend continuously on the Cauchy data $f,\ g$.

3 METHOD OF FUNDAMENTAL SOLUTION AND ITS DISCRETIZATION

The method of fundamental solutions (MFS) is a truly meshless method and a powerful technique for solving linear partial differential equations (PDE) easily and rapidly. Some mathematical theories on the method are discussed in papers [2, 11]. The method is applicable when a fundamental solution of a PDE is explicitly known and approximates the solution by a linear combination of fundamental solutions with source points located on a fictitious boundary outside of the solution domain. The MFS does not require any treatment for the singularity of the fundamental solution and can easily be extended to higher dimensional cases. It has been successfully applied to a large variety of physical problems since it was introduced in late 1970s by Mathon and Johnston [16].

The fundamental solution of the modified Helmholtz equation at point P in two dimensions is defined as (see [14])

$$G^*(P) = \frac{1}{2\pi} K_0(\kappa \parallel r \parallel),\ P \in R^2,$$

where K_0 is the modified Bessel function of the second kind of order zero, and $r = \parallel P \parallel$, $\parallel \cdot \parallel$ is the Euclidean norm in R^2. Note that the fundamental solution to the differential operator is not unique.

We locate the source points $\{Q_j\}_{j=1}^{n_s}$ on a "pseudo-boundary" outside the solution domain and define the basis functions with respect to n_s source points $\{Q_j\}_{j=1}^{n_s}$ as

$$G_j(P) = G^*(|P - Q_j|),\quad j = 1, \ldots, n_s. \tag{4}$$

The main idea of the MFS consists in approximating u by a linear combination of fundamental functions G_j by

$$u(P) \approx u_N(\mathbf{C}, P, Q) := \sum_{j=1}^{n_s} c_j G_j(P),\ P \in \bar{\Omega}, \tag{5}$$

and $\mathbf{C} = [c_1, c_2, \ldots, c_{n_s}]^T$ are expansion coefficients to be determined below, T denotes the matrix transposition unless specified otherwise.

The normal heat flux g can be approximated Γ_1 by

$$g(P) \approx g_N(\mathbf{C}, P, Q) = \sum_{j=1}^{n_s} c_j H_j(P),\ P \in \Gamma_1, \tag{6}$$

where

$$H(P) = -\frac{\kappa}{2\pi} K_1(\kappa \parallel P - Q \parallel)[\frac{P - Q}{\parallel P - Q \parallel} \cdot \mathbf{n}]. \tag{7}$$

Let the collocation points $\{P_i\}_{i=1}^{n_a}$ and $\{P_i\}_{i=n_a+1}^{n_a+n_b}$ be on the boundary Γ_1 where the Dirichlet data and Neumann data are prescribed respectively. By substituting (5) into boundary condition (2) at points $\{P_i\}_{i=1}^{n_a}$ and (6) into boundary condition (3) at points $\{P_i\}_{i=n_a+1}^{n_a+n_b}$, we have

$$f(P_i) = \sum_{j=1}^{n_s} c_j G_j(P_i),\quad i = 1, 2, \ldots, n_a, \tag{8}$$

$$g(P_i) = \sum_{j=1}^{n_s} c_j G_j(P_i), \quad i = n_a + 1, n_a + 2, \ldots, n_a + n_b. \tag{9}$$

In the matrix form:

$$\mathbf{AC} = \mathbf{b}, \tag{10}$$

where $\mathbf{b} = [f(p_1), f(P_2), \ldots, f(P_{n_a}), g(P_{n_a+1}), g(P_{n_a+2}), \ldots, g(P_{n_a+n_b})]^T$ is a data vector, the matrix $\mathbf{A} = (A_{ij}) \in R^{(n_a+n_b)\times n_s}$ is an interpolation matrix, and its entries are defined by

$$A_{ij} = \begin{cases} G_j(P_i) & i = 1, 2, \ldots, n_a, \\ H_j(P_i) & i = n_a + 1, n_a + 2, \ldots, n_a + n_b. \end{cases} \tag{11}$$

The matrix $\mathbf{A}$ in equation (10) is often severely ill-conditioned. In the case of noisy boundary data, standard numerical methods, such as Gaussian elimination, the LU factorization and the least squares method, fail to produce an acceptable solution. How to obtain physically meaningful numerical solutions will be discussed in detail in the next section.

4 REGULARIZATION TECHNIQUES FOR DISCRETE ILL-POSED PROBLEMS

4.1 Tikhonov regularization

When the Cauchy data f, g contain some noise, we consider the following equation instead of (10):

$$\mathbf{AC} = \mathbf{b}^\delta, \ \mathbf{b}^\delta = \mathbf{b} + \mathbf{e} \tag{12}$$

with the noise $\mathbf{e} \in R^n$. For the system (12), we define its approximate solution $\mathbf{C}_\lambda^\delta$ as the minimizer of the following functional with a regularization parameter $\lambda > 0$ and an initial estimate $\mathbf{C}^*$ according to the Tikhonov regularization:

$$\mathbf{C}_\lambda^\delta = argmin\{\| \mathbf{AC} - \mathbf{b}^\delta \|_2^2 + \lambda^2 \| \mathbf{L}(\mathbf{C} - \mathbf{C}^*) \|_2^2\}, \tag{13}$$

Based on GSVD, for $\mathbf{C}^* = \mathbf{0}$, the Tikhonov regularization solution can be written as

$$\mathbf{C}_\lambda^\delta = \sum_{i=1}^{n_s} f_i \frac{\mathbf{u}_i^T \mathbf{b}^\delta}{\sigma_i} \mathbf{v}_i, \ (\mathbf{L} = \mathbf{I}_{n_s})$$

or

$$\mathbf{C}_\lambda^\delta = \sum_{i=1}^{m} f_i \frac{\mathbf{u}_i^T \mathbf{b}^\delta}{\sigma_i} \mathbf{X}_i + \sum_{i=m+1}^{n_s} (\mathbf{u}_i^T \mathbf{b}^\delta) \mathbf{X}_i, \ (\mathbf{L} \neq \mathbf{I}_{n_s})$$

Substituting $\mathbf{C}^\delta$ into (5), we shall find the approximate temperature u_N in $\bar{\Omega}$.

4.2 L-curve

The performance of regularization methods depends on suitable choice of the regularization parameter. The most convenient graphical tool for analysis of discrete ill-posed problems is the so-called L-curve, which is a curve defined as $\mathcal{L} = \{(\log \parallel \mathbf{AC}_\lambda^\delta - \mathbf{b}^\delta \parallel_2, \log \parallel \mathbf{LC}_\lambda^\delta \parallel_2)\}$ in [7]. The curve displays the compromise between the residual norm $\parallel \mathbf{AC}_\lambda^\delta - \mathbf{b}^\delta \parallel_2$ and the solution norm $\parallel \mathbf{LC}_\lambda^\delta \parallel_2$. A suitable regularization parameter is given by the one corresponding to a regularized solution near the "corner" of the L-curve.

5 NUMERICAL EXPERIMENTS

In this section, assuming the exact $u(x, y)$ is unknown and identify the value of control points on the rest of the boundary Γ_2, we present the numerical results when we apply the MFS in conjunction with the Tikhonov regularization method, whose side constraint is a Sobolev norm , i.e, in (13), $\mathbf{L}$ equal to the Cholesky factor of the matrix $\alpha_0^2\mathbf{I_{n_s}} + \sum_{i=1}^{2} \alpha_i^2 \mathbf{L_i}^T \mathbf{L_i}$, where $\mathbf{L_i}$ approximates the ith derivative operator. In this paper, the parameters $\alpha_0,\ \alpha_1,\ \alpha_2$ are obtained by trial. We recover the usual Tikhonov regularization method when $\alpha_0 = 1,\ \alpha_1 = 0,\ \alpha_2 = 0$.

Two different error criteria are employed on the boundary, which is sufficient due to the maximum principle, using global average error $e_G = \frac{\sqrt{\sum_{i=1}^{n_p}(u_e^i - u_a^i)^2}}{n_p \max(u_e^i)}$ and local relative error $e_L = \frac{\max(|u_e^i - u_a^i|)}{\max|u_e^i|}$, respectively. Here u_e^i is the exact value of the solution at the $i-$th control point on the boundary Γ_2 of the solution and u_a is its approximate value at the same point, and n_p is the number of equidistant control points on Γ_2.

We consider only the perturbation of $\mathbf{b}$ in the system (10) and use simulated noisy data generated by the formula $\bar{b_i} = b_i(1 + \epsilon\xi),\ (i = 1, \ldots, n_s)$, where ξ is generated by using Matlab function $rand()$ and ϵ is the level of data noise. Cauchy data $f,\ g$ are available at the same collocation points on the boundary Γ_1. All computational codes are written in MATLAB and the tool package developed by P.C Hansen [8].

Example 1 and Example 2 are taken from [10].

5.1 The circular region

Example 1. In this example, the region $\Omega = \{(x, y)|x^2 + y^2 < 1\}$, the analytical solution $u(x, y)$ is taken to be $u(x, y) = exp(\frac{x}{2} + \frac{\sqrt{3}y}{2}),\ (x, y) \in \Omega$, with the wave number $\kappa = 1$. The error level $\epsilon = 0.01$. Figure 1 shows distributions of the collocation and source points.

5.2 The complicated region

When applying MFS to solve the boundary value problems defined in a irregular region, there are two ways of designation of the shape of 'Pseudo-boundary' on which the source points are placed [5]. First, the fictitious boundary is a circle (Case 1) and second the fictitious boundary is a contour geometrically similar to the boundary contour of the region under consideration (Case 2).

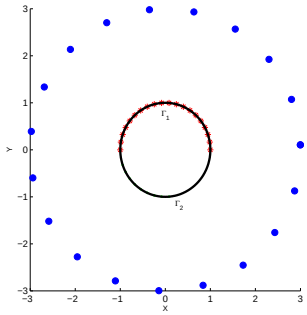
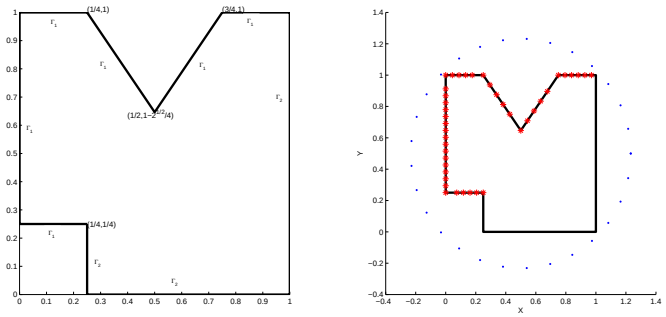


Figure 1: Distributions of collocation(red '*') and source(blue '.') points $(R, n_a, n_s) = (3, 20, 20)$).

The authors carried out numerical experiments and pointed out it is difficult to recommend the parameters R-the radius of the circle in Case 1 and S-the distance between similar contour in Case 2.

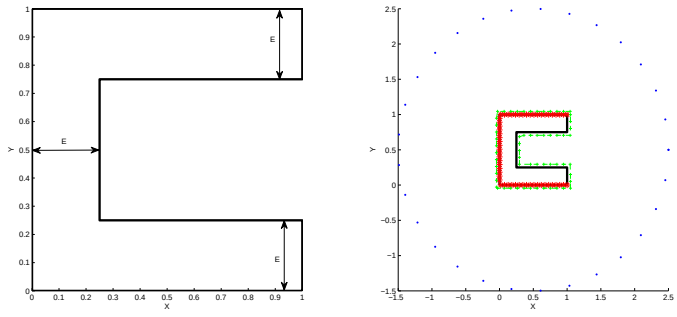
Example 2. The domain Ω is showed in Figure 2(a), and the analytical solution is taken as $u(x, y) = exp(x + y), \quad (x, y) \in \Omega$, with the wave number $\kappa = \sqrt{2}$. The error level $\epsilon = 0.02$. Figure 2(b) shows the distributions of the collocation and source points.



(a) (b)

Figure 2: (a) The configuration of the region in example 2. (b) Distributions of collocation(red '*') and source(blue '.') points $(R, n_a, n_s) = (0.8, 40, 30)$).

Example 3. The region Ω is concave [5], as showed in Figure 3(a). The analytical solution $u(x, y) = exp(\frac{x}{2} + \frac{\sqrt{3}y}{2}), \quad (x, y) \in \Omega$, with the wave number $\kappa = 1$. Figure 3(b) shows distributions of the source points in the pseudo-boundaries for both Case 1 and Case 2. In the former, the radius of the circle is R and in the latter, the distance between the concave boundary and the similar contour is S.



(a) (b)

Figure 3: (a) The configuration of the region in example 3 where $E = 0.25$. (b) Distributions of collocation(red '*') and source(blue '.' in Case 1 and green '+' in Case 2) points $(R, n_a, n_s, S) = (2, 30, 30, 0.05)$).

5.3 Numerical results

We use Tikhonov regularization method to solve the ill-posed linear system (12) and compare the results with those from the truncated singular value decomposition (TSVD) [10]. In all examples, we try the parameters $\alpha_i = \frac{1}{3},\ i = 0, 1, 2$ or $\alpha_0 = 1, \alpha_1 = 0, \alpha_2 = 0$, and denote the two cases TR1 or TR2 respectively in the following text.

In Example 1, we distribute uniformly the source points ($n_s = 20$) on a concentric circle ($R = 2$) and the control points ($n_p = 20$) on the partial boundary Γ_2. Table 1 shows the local relative error and the global average error against the number of the collocation points on Γ_1. We can see that the results of the three methods are nearly of the same accuracy. In [10], the authors pointed out that similar results can be obtained by TSVD and TR2. When dealing with the complex region, however, different methods provide rather different results. In

n_a	e_L	e_G	e_L	e_G	e_L	e_G
10	5.18E-02	5.34E-01	5.18E-02	5.31E-01	5.18E-02	5.34E-01
12	5.69E-02	5.81E-01	5.68E-02	5.82E-01	5.69E-02	5.81E-01
14	5.18E-02	5.32E-01	5.18E-02	5.32E-01	5.18E-02	5.32E-01
16	5.77E-02	5.02E-01	5.77E-02	5.02E-01	5.77E-02	5.02E-01
18	6.42E-02	4.94E-01	6.42E-02	4.95E-01	6.42E-02	4.94E-01
20	6.35E-02	4.83E-01	6.34E-02	4.82E-01	6.31E-02	4.81E-01

Table 1: The global average error e_G and local relative error e_L of the TR1, TSVD and TR2 against the number of collocation points in Example 1 ($(R, n_s) = (3, 20)$).

Example 2, the region is complex, the source points are distributed uniformly on a circle with its radius $R = 2$ and center at $(0, 0)$. We consider the local relative error of the numerical solution $u(x, y)$ of control points ($n_p = 30$) against the

numbers of collocation points. The results are showed in Figure 4. Our experience indicates a solution becomes more accurate with more observation data. We can see that the TR1 and the TSVD coincide with this and the numerical result of the former is better than the latter. The optimal radius of the circle

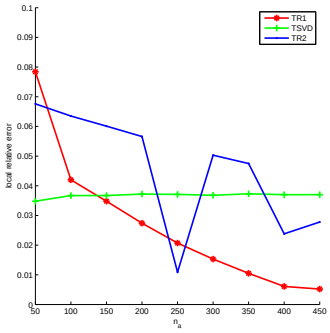
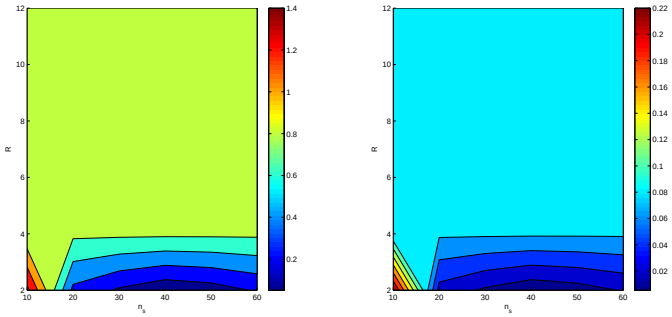


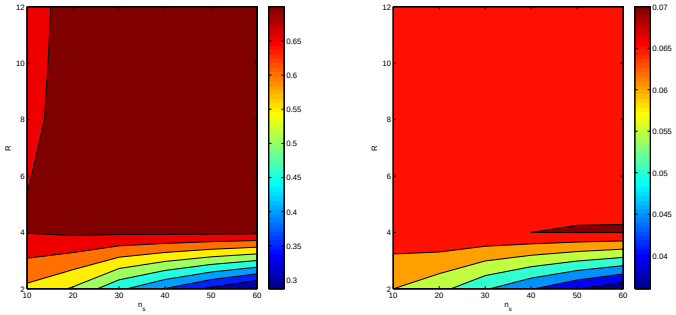
Figure 4: The local relative errors of the numerical results of Example 2.

contour is important in MFS and it is usually difficult to decide. In Figure 5-Figure 7, we show the contour lines of the Contour line of the global average error and the local relative error against (n_s, R) and a fixed $n_a = 300$ for three methods. We find that the TR1 and the TR2 can tell the information of the optimal radius. While the TSVD yields accurate numerical results in a wider range of radius R, a large value of the radius R is generally not advised [10]. Through the result in the TR1, we observe that both the global average error and the local relative error are roughly independent of the number of source points n_s for the fixed radius R of the circle where source points are distributed. This observation coincides with the reality. But the TR2 and the TSVD can't offer the same information. From Figure 5, we know that the optimal radius $R \cong 2$ and $n_s \geq 30$ will yield a better solution. In Example 3, we consider two cases of 'pseudo-boundary', i.e, a circle outside the domain (Case 1) and a contour geometrically similar to the boundary contour of the domain (Case 2) under consideration. The optimal parameter S in Case 2 is also difficult to choose. We take $S = 0.045$, a value used in [5] when dealing with numerical experiments in the application of the MFS to the elastic torsion of prismatic rods. Table 2 shows the numerical results of the local relative error and the global average error against the radius R and the distance S where $(n_a, n_S) = (40, 30)$. We see that better numerical results can be obtained by using the source contour in a shape of circle. A better regularization solution whose error is approximately 2% can be obtained by taking appropriate radius R and large n_a. This observation is significant when the shape of region considered is not known in advance. In Figure 8-Figure 10, we show the Contour line of the global average error and the local relative error against (n_s, R) for the fixed number of the collocation points $n_a = 800$ and their L-curve for three methods in example 3. We see that for TR1, the local relative error are roughly independent of the number of source points n_s for the fixed radius R of the circle where source points are distributed, while the other two are not. As we can observe in Figure 8 (b), the corner of the L-curve is located at the point $(\| \mathbf{AC}_\lambda^\delta - \mathbf{b}^\delta \|_2, \| \mathbf{LC}_\lambda^\delta \|_2)$ with



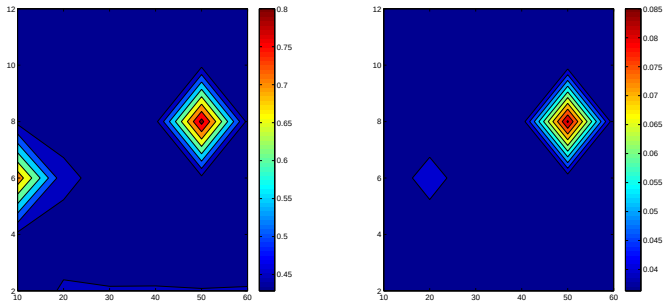
(a) (b)

Figure 5: The contour line of the global average error (a) and the local relative error (b) for TR1 method.



(a) (b)

Figure 6: The global average error (a) and the local relative error (b) for TR2 method.



(a) (b)

Figure 7: The contour line of global average error (a) and the local relative error (b) for TSVD method.

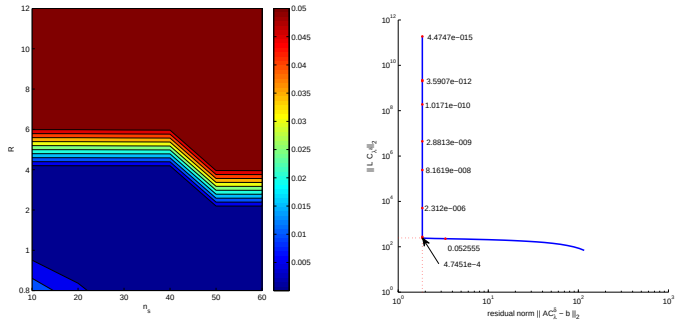
n_a	e_L	e_G	e_L	e_G	e_L	e_G
2	3.04E-05	4.97E-04	3.49E-05	5.00E-04	1.95E-04	3.20E-03
3	2.74E-04	3.71E-03	5.05E-02	5.99E-01	5.05E-02	5.96E-01
4	1.21E-04	1.59E-03	5.05E-02	6.03E-01	5.05E-02	6.00E-01
5	1.75E-04	2.29E-03	5.05E-02	6.04E-01	5.05E-02	6.03E-01
6	1.07E-04	1.43E-03	5.06E-02	6.06E-01	5.06E-02	6.04E-01
7	4.84E-05	5.92E-04	5.06E-02	6.08E-01	5.06E-02	6.06E-01
8	2.13E-04	2.00E-03	5.07E-02	6.09E-01	5.06E-02	6.06E-01

Table 2: The local relative error e_L and the global average error e_G of the TR1, TSVD and TR2 against the radius R of the circular pseudo-boundary and the distance between similar contour $s = 0.045$ for example 3. $(n_a, n_S) = (40, 30)$.

the regularization parameter $\lambda = 4.7451e-4$ for $n_s = 30,\ R = 4$. In Figure 9 (b) and Figure 10 (b), we show the regularization parameter with respect to the corner of L-curve locate, $\lambda = 7.0070e-3$ for $n_s = 60,\ R = 4$ in the TR2 and $\lambda = 11$ for $n_s = 30,\ R = 4$ in the TSVD.

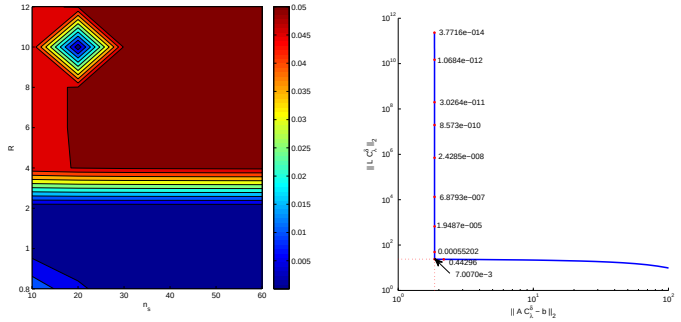
5.4 Conclusions

We use the MFS in conjunction with Tikhonov regularization method, whose side constraint is a Sobolev norm, to solve the Cauchy problem of the modified Helmholtz equation defined in a complex region. The L-curve method gives a suitable regularization parameter. It is difficult to determine the optimal value of the radius R or the distance S when using the MFS method. Two error criteria are employed. Numerical results show that we can obtain a high-accuracy numerical solution when the 'Pseudo-boundary' is a circle, though the author of [5] concluded that the error is smaller when the source contour is geometrically similar to the boundary of the region in comparison with the circular source contour. We plot the error contour lines and obtain the optimal parameters in our method. The results show that the proposed scheme is more



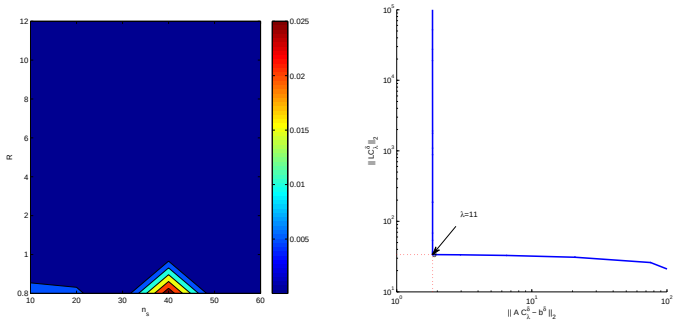
(a) (b)

Figure 8: The contour line of the global average error for the TR1 (a) and the L-curve (The corner is located at the point $\lambda = 4.7451e-4$) (b)in example 3.



(a) (b)

Figure 9: The contour line of the global average error for the TR2 (a) and the L-curve (The corner is located at the point $\lambda = 7.0070e-3$) (b)in Example 3.



(a) (b)

Figure 10: The contour line of the global average error for the TSVD (a) and the L-curve (The corner is located at the point $p = 11$) (b)in Example 3.

competitive than the other methods for these problems. The scheme is readily extended to the three dimensional case and the Cauchy problem associated with the Helmholtz equation. Through the Sobolev norm, we can control the smoothness of the regularized solution and we can apply the TR1 to solve a problem in a complex region with the Cauchy data that contains noises with large size.

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STATISTICAL APPROXIMATION BY MEANS OF OPERATORS CONSTRUCTED BY q-LAGRANGE POLYNOMIALS

ESRA ERKUŞ-DUMAN

ABSTRACT. In this paper, using q-Lagrange polynomials, a sequence of positive linear operators are constructed. Approximation properties of these q-type operators are established via A-statistical convergence. Furthermore, we compute rates of A-statistical convergence by means of modulus of continuity, the elements of the Lipschitz class, Peetre's K-functional and the Lipschitz type maximal function, respectively. This statistical approximation process enables us more powerful results than the classical aspects.

1. INTRODUCTION

The theory of q-calculus has wide-ranging applications in analytic number theory and theoretical physics (see [1]). Especially, it has found many important applications in quantum group theory (see, for instance, [14, 16, 17, 19]). In the present paper, we mainly discuss its use in approximation theory and summability theory. So, applying the techniques of q-calculus and using the notion of A-statistical convergence, where $A = (a_{jn})$ is non-negative regular summability matrix, we get various Korovkin-type statistical approximation results for the q-generalizations of positive linear operators constructed by means of certain polynomials in several variables.

We first recall some basic definitions used in the paper.

Recently, Altın, Erkuş and Taşdelen [3] have introduced the following q-Lagrange polynomials generated by

$$\prod_{i=1}^{r}\frac{1}{(x_i t;q)_{\alpha_i}}=\sum_{m=0}^{\infty}g_{m,q}^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)t^m, \tag{1.1}$$

where $|t|<\min\left\{|x_1|^{-1},...,|x_r|^{-1}\right\}$. In this case, (1.1) yields the following explicit representation

$$g_{m,q}^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)=\sum_{k_1+...+k_r=m}\left(\prod_{i=1}^{r}(q^{\alpha_i},q)_{k_i}\frac{x_i^{k_i}}{(q,q)_{k_i}}\right), \tag{1.2}$$

where, for a complex number q with $|q|<1$, and also for arbitrary parameters λ and μ,

$$(\lambda;q)_\mu:=\frac{(\lambda;q)_\infty}{(\lambda q^\mu;q)_\infty}\text{ and }(\lambda;q)_\infty:=\prod_{k=0}^{\infty}(1-\lambda q^k).$$

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The case for which $\mu = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ reduces to the following

$$(\lambda;q)_n := \begin{cases} 1 & , \text{ if } \quad n = 0 \\ (1-\lambda)(1-\lambda q)...(1-\lambda q^{n-1}) & , \text{ if } \quad n = 1,2,3,... \end{cases}$$

We should note that the further properties of the above statements may be found in [20]. Notice that q-Lagrange polynomials given by (1.2) are q-extension of the Chan-Chyan-Srivastava multivariable polynomials [5].

In [9], some Korovkin type approximation results have been obtained for positive linear operators constructed by means of the Chan-Chyan-Srivastava polynomials in the case of $q = 1$. In the present paper our primary interest is to investigate the approximation properties of positive linear operators constructed by q-Lagrange polynomials for $0 < q < 1$ via the concept of statistical convergence.

With this terminology, we present our operators as follows.

Let $u^{(i)} := \left\{u_n^{(i)}\right\}_{n\in\mathbb{N}}$ $(i = 1,2,...,r)$ be sequences of real numbers such that $0 < u_n^{(i)} < 1$ for each $n \in \mathbb{N}$, and let $0 < q < 1$. For simplicity, we write $\beta(r) := (u^{(1)}, u^{(2)}, ..., u^{(r)})$. Then by using (1.2) we introduce the following positive linear operators on $C[0,1]$, the space of all continuous real valued functions on $[0,1]$:

$$\begin{aligned} L_n^{\beta(r)}(f;q;x) &= \left\{\prod_{j=1}^{r}(u_n^{(j)}x;q)_n\right\}\sum_{m=0}^{\infty}\sum_{k_1+...+k_r=m} \\ &\times\left\{f\left(\frac{1-q^{k_r}}{1-q^{n+k_r-1}}\right)\prod_{i=1}^{r}\frac{(u_n^{(i)})^{k_i}}{(q;q)_{k_i}}(q^n;q)_{k_i}\right\}x^m \end{aligned} \tag{1.3}$$

where $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$. Observe that our operators in (1.3) are positive and linear for $0 < q < 1$.

We now recall the concept of statistical convergence.

Let $A := (a_{jn})$ be a non-negative regular summability matrix. Then, a sequence (x_n) is said to be A-statistically convergent to a number L if, for every $\varepsilon > 0$, $\lim_{j\to\infty}\sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0$. This limit is denoted by $st_A - \lim_n x_n = L$ [11]. Replacing the matrix A by the identity matrix, A-statistical convergence reduces to the ordinary convergence. It is well-known that every convergent sequence is A-statistically convergent to the same value for any non-negative regular summability matrix A, but its converse is not always true. Especially, Kolk [15] proved that if a non-negative regular matrix $A = (a_{jn})$ satisfies the condition $\lim_j \max_n\{a_{jn}\} = 0$, then A-statistical convergence is stronger than the ordinary convergence. So, in this case, one can construct a sequence such that it is A-statistically convergent but not convergent in the ordinary sense. Furthermore, choosing $A = C_1$, the Cesáro matrix of order one, A-statistical convergence coincides with the concept of statistical convergence (see [10, 12]).

Because of the reasons above, using the matrix summability methods in the approximation theory gains more powerful results than the classical aspects (see, for instance, [7, 8, 9, 13]). Especially, in recent years another form of regular (non-matrix) summability transformation, the so-called "*statistical convergence*" has shown to be quite effective in summing non-convergent sequences [12, 15].

This paper is organized as follows: In the second section, we obtain a Korovkin type result for the operators (1.3) by means of A-statistical convergence. In the third section, we compute the rates of A-statistical convergence of the operators

with the help of modulus of continuity, the elements of Lipschitz class, Peetre's $K-$functional and Lipschitz type maximal function, respectively.

2. Statistical Korovkin-type result

In this section, we give a statistical Korovkin-type approximation theorem for the operators $L_n^{\beta(r)}(f;q;x)$ given by (1.3).

We first consider the case of $r = 2$. Throughout the paper we use the test functions e_i defined by $e_i(x) = x^i$ $(i = 0, 1, 2)$.

To get our main theorem we need the following lemmas.

Lemma 2.1. *For each $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

$$\left\| L_n^{\beta(2)}(e_0;q;\cdot) - e_0 \right\| = 0,$$

where $\beta(2) := (u^{(1)}, u^{(2)})$ as stated before, and the symbol $\|\cdot\|$ denote the usual supremum norm on the space $C[0,1]$.

Proof. By a direct computation, we may write, for $0 < q < 1$, that

$$\begin{aligned} L_n^{\beta(2)}(e_0;q;x) &= (u_n^{(1)}x;q)_n (u_n^{(2)}x;q)_n \sum_{m=0}^{\infty} g_{m,q}^{(n,n)}\left(u_n^{(1)}, u_n^{(2)}\right) x^m \\ &= 1, \end{aligned}$$

whence the result. □

Lemma 2.2. *For every $n \in \mathbb{N}$, we have*

$$\left\| L_n^{\beta(2)}(e_1;q;\cdot) - e_1 \right\| = 1 - u_n^{(2)}.$$

Proof. Let $x \in [0,1]$. Then we get

$$\begin{aligned} L_n^{\beta(2)}(e_1;q;x) &= (u_n^{(1)}x;q)_n (u_n^{(2)}x;q)_n \sum_{m=0}^{\infty} \sum_{k_1+k_2=m} \\ &\quad \times \left\{ \frac{1-q^{k_2}}{1-q^{n+k_2-1}} \prod_{i=1}^{2} \frac{(u_n^{(i)})^{k_i}(q^n;q)_{k_i}}{(q;q)_{k_i}} \right\} x^m, \end{aligned}$$

or equivalently

$$\begin{aligned} L_n^{\beta(2)}(e_1;q;x) &= (u_n^{(1)}x;q)_n (u_n^{(2)}x;q)_n \sum_{m=0}^{\infty} \sum_{k=0}^{m} \\ &\quad \times \left\{ \frac{1-q^k}{1-q^{n+k-1}} \frac{\left(u_n^{(1)}\right)^{m-k} \left(u_n^{(2)}\right)^k (q^n;q)_{m-k} (q^n;q)_k}{(q;q)_{m-k} (q;q)_k} \right\} x^m, \end{aligned}$$

and hence

$$\begin{aligned} L_n^{\beta(2)}(e_1;q;x) &= x u_n^{(2)} (u_n^{(1)}x;q)_n (u_n^{(2)}x;q)_n \sum_{m=1}^{\infty} \sum_{k=1}^{m} \\ &\quad \times \left\{ \frac{\left(u_n^{(1)}\right)^{m-k} \left(u_n^{(2)}\right)^{k-1} (q^n;q)_{m-k} (q^n;q)_{k-1}}{(q;q)_{m-k} (q;q)_{k-1}} \right\} x^{m-1}. \end{aligned}$$

Then we obtain that

$$L_n^{\beta(2)}(e_1;q;x) = xu_n^{(2)}(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \times \left\{ \frac{\left(u_n^{(1)}\right)^{(m-1)-k}\left(u_n^{(2)}\right)^{k}(q^n;q)_{(m-1)-k}(q^n;q)_k}{(q;q)_{(m-1)-k}(q;q)_k} \right\} x^{m-1},$$

which implies

$$\begin{aligned} L_n^{u^{(1)},u^{(2)}}(e_1;q;x) &= xu_n^{(2)}(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=1}^{\infty}\sum_{k_1+k_2=m-1} \\ &\quad \times \left\{ \frac{\left(u_n^{(1)}\right)^{k_1}\left(u_n^{(2)}\right)^{k_2}(q^n;q)_{k_1}(q^n;q)_{k_2}}{(q;q)_{k_1}(q;q)_{k_2}} \right\} x^{m-1} \\ &= xu_n^{(2)}(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=1}^{\infty} g_{m-1,q}^{(n,n)}\left(u_n^{(1)},u_n^{(2)}\right) x^{m-1}. \end{aligned}$$

Now, for the case of $r=2$ in (1.2), taking $\alpha_1=\alpha_2=n$ and $x_1=u^{(1)}$, $x_2=u^{(2)}$ we have

$$L_n^{u^{(1)},u^{(2)}}(e_1;q;x) = xu_n^{(2)} \tag{2.1}$$

Therefore, the proof follows from (2.1) immediately. □

Lemma 2.3. *For every $n\in\mathbb{N}$, we have*

$$\left\| L_n^{\beta(2)}(e_2;q;\cdot) - e_2 \right\| \le 2\left(1-\left(u_n^{(2)}\right)^2\right) + u_n^{(2)}\left(\frac{1-q}{1-q^n}\right).$$

Proof. Let $x\in[0,1]$ be fixed. Then we get

$$\begin{aligned} L_n^{\beta(2)}(e_2;q;x) &= (u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=0}^{\infty}\sum_{k_1+k_2=m} \\ &\quad \times \left\{ \left(\frac{1-q^{k_2}}{1-q^{n+k_2-1}}\right)^2 \frac{\left(u_n^{(1)}\right)^{k_1}\left(u_n^{(2)}\right)^{k_2}(q^n;q)_{k_1}(q^n;q)_{k_2}}{(q;q)_{k_1}(q;q)_{k_2}} \right\} x^m, \end{aligned}$$

and hence

$$\begin{aligned} L_n^{\beta(2)}(e_2;q;x) &= (u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=0}^{\infty}\sum_{k=0}^{m} \\ &\quad \times \left\{ \left(\frac{1-q^{k}}{1-q^{n+k-1}}\right)^2 \frac{\left(u_n^{(1)}\right)^{m-k}\left(u_n^{(2)}\right)^{k}(q^n;q)_{m-k}(q^n;q)_{k}}{(q;q)_{m-k}(q;q)_{k}} \right\} x^m, \end{aligned}$$

which yields that

$$L_n^{\beta(2)}(e_2;q;x) = xu_n^{(2)}(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=1}^{\infty}\sum_{k=1}^{m} \times \left\{ \frac{1-q^k}{1-q^{n+k-1}} \frac{\left(u_n^{(1)}\right)^{m-k}\left(u_n^{(2)}\right)^{k-1}(q^n;q)_{m-k}(q^n;q)_{k-1}}{(q;q)_{m-k}(q;q)_{k-1}} \right\} x^{m-1}.$$

This gives that

$$\begin{aligned} L_n^{\beta(2)}(e_2;q;x) &= x^2\left(u_n^{(2)}\right)^2(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=2}^{\infty}\sum_{k=2}^{m} \\ &\times \left\{ \frac{1-q^{n+k-2}}{1-q^{n+k-1}} \frac{\left(u_n^{(1)}\right)^{m-k}\left(u_n^{(2)}\right)^{k-2}(q^n;q)_{m-k}(q^n;q)_{k-2}}{(q;q)_{m-k}(q;q)_{k-2}} \right\} x^{m-2} \\ &+xu_n^{(2)}(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=1}^{\infty}\sum_{k=1}^{m} \\ &\times \left\{ \frac{q^{k-1}-q^k}{1-q^{n+k-1}} \frac{\left(u_n^{(1)}\right)^{m-k}\left(u_n^{(2)}\right)^{k-1}(q^n;q)_{m-k}(q^n;q)_{k-1}}{(q;q)_{m-k}(q;q)_{k-1}} \right\} x^{m-1}. \end{aligned}$$

Using this we get

$$\begin{aligned} L_n^{\beta(2)}(e_2;q;x) &\le x^2\left(u_n^{(2)}\right)^2(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=2}^{\infty} g_{m-2,q}^{(n,n)}\left(u_n^{(1)},u_n^{(2)}\right)x^{m-2} \\ &+xu_n^{(2)}\frac{1-q}{1-q^n}(u_n^{(1)}x;q)_n(u_n^{(2)}x;q)_n \sum_{m=1}^{\infty} g_{m-1,q}^{(n,n)}\left(u_n^{(1)},u_n^{(2)}\right)x^{m-1} \\ &= x^2\left(u_n^{(2)}\right)^2 + xu_n^{(2)}\left(\frac{1-q}{1-q^n}\right), \end{aligned}$$

or

$$L_n^{\beta(2)}(e_2;q;x) - e_2(x) \le x^2\left(\left(u_n^{(2)}\right)^2 - 1\right) + xu_n^{(2)}\left(\frac{1-q}{1-q^n}\right). \tag{2.2}$$

On the other hand, since

$$0 \le L_n^{\beta(2)}((y-x)^2;q;x) = L_n^{\beta(2)}(e_2;q;x) - 2xL_n^{\beta(2)}(e_1;q;x) + x^2,$$

it follows from Lemmas 2.1 and 2.2 that

$$L_n^{\beta(2)}(e_2;q;x) - e_2(x) \ge 2x^2\left(u_n^{(2)} - 1\right). \tag{2.3}$$

Combining (2.2) with (2.3) we immediately have

$$\left|L_n^{\beta(2)}(e_2;q;x) - e_2(x)\right| \le 2x^2\left(1 - \left(u_n^{(2)}\right)^2\right) + xu_n^{(2)}\left(\frac{1-q}{1-q^n}\right),$$

whence the result. □

Now let $A=(a_{jn})$ be a non-negative regular summability matrix. Then replacing q in the definition (1.3) by a sequence $(q_n)_{n\in\mathbb{N}}$ lying in the interval $(0,1)$ so that

$$st_A-\lim_n \frac{1-q_n}{1-q_n^n}=0 \tag{2.4}$$

hold. Indeed, one can construct a sequence $(q_n)_{n\in\mathbb{N}}$ satisfying (2.4). For example, take $A=C_1$, the Cesáro matrix of order one, and define the sequence $(q_n)_{n\in\mathbb{N}}$ by

$$q_n=\begin{cases} 1/2; & if\ \ n=m^2\,, \\ 1-\frac{1}{n}; & if\ \ n\neq m^2\,. \end{cases}\quad (m=1,2,3,...) \tag{2.5}$$

Observe that this sequence satisfies the conditions in (2.4), however it is non-convergent in the ordinary sequence.

We are now ready to give a Korovkin type approximation result for the operators $L_n^{\beta(2)}$ by means of A-statistical convergence.

Theorem 2.4. *Let $A=(a_{jn})$ be a non-negative regular summability matrix and let $(q_n)_{n\in\mathbb{N}}$ be a sequence satisfying* (2.4)*. Then, for all $f\in C[0,1]$,*

$$st_A-\lim_n \left\|L_n^{\beta(2)}(f;q_n;.)-f\right\|=0. \tag{2.6}$$

if and only if

$$st_A-\lim_n u_n^{(2)}=1 \tag{2.7}$$

holds.

Proof. Assume first that (2.6) holds for all $f\in C[0,1]$. Then, in particular, since $e_1\in C[0,1]$, we may write that

$$st_A-\lim_n \left\|L_n^{\beta(2)}(e_1;q_n;.)-e_1\right\|=0.$$

By Lemma 2.2, the proof of the implication (2.6)$\Rightarrow$(2.7) is clear. Now assume that $u^{(2)}$ is a sequence satisfying (2.7). So, by Lemma 2.1, it is not hard to see that

$$st_A-\lim_n \left\|L_n^{\beta(2)}(e_0;q_n;.)-e_0\right\|=0 \tag{2.8}$$

since every convergent sequence is also A-statistically convergent provided that A is a non-negative regular summability matrix. Also, it immediately follows from Lemma 2.2 and condition (2.7) that

$$st_A-\lim_n \left\|L_n^{\beta(2)}(e_1;q_n;.)-e_1\right\|=0 \tag{2.9}$$

holds. Furthermore, we claim that

$$st_A-\lim_n \left\|L_n^{\beta(2)}(e_2;q_n;.)-e_2\right\|=0. \tag{2.10}$$

To see this, for a given $\varepsilon>0$, define the following sets:

$$\begin{aligned} D &:= \left\{n:\left\|L_n^{\beta(2)}(e_2;q_n;.)-e_2\right\|\geq\varepsilon\right\}, \\ D_1 &:= \left\{n:1-\left(u_n^{(2)}\right)^2\geq\frac{\varepsilon}{4}\right\}, \\ D_2 &:= \left\{n:u_n^{(2)}\left(\frac{1-q_n}{1-q_n^n}\right)\geq\frac{\varepsilon}{2}\right\}. \end{aligned}$$

So, by Lemma 2.3, we easily see that $D \subseteq D_1 \cup D_2$, which implies, for each $j \in \mathbb{N}$, that

$$\sum_{n\in D} a_{jn} \leq \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn}. \tag{2.11}$$

Then, observe that (2.7) implies

$$st_A - \lim_n \left(1 - \left(u_n^{(2)}\right)^2\right) = 0 \text{ and } st_A - \lim_n u_n^{(2)} \left(\frac{1-q_n}{1-q_n^n}\right) = 0.$$

Using these facts and also taking limit as $n \to \infty$ in (2.11) we conclude that

$$\lim_j \sum_{n\in D} a_{jn} = 0,$$

which gives (2.10). Now, combining (2.8), (2.9), (2.10) and using the Korovkin type approximation theorem in statistical sense proved by Gadjiev and Orhan [13] (see also [7, 8, 9]) the proof is completed. □

In a similar manner one can extend Theorem 2.4 to the $r-$dimensional case for the operators $L_n^{\beta(r)}(f;q_n;x)$ given by (1.3) as follows:

Theorem 2.5. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $(q_n)_{n\in\mathbb{N}}$ be a sequence satisfying* (2.4)*. Then, for all $f \in C[0,1]$,*

$$st_A - \lim_n \left\| L_n^{\beta(r)}(f;q_n;.) - f \right\| = 0$$

if and only if

$$st_A - \lim_n u_n^{(r)} = 1.$$

If one replaces the non-negative regular matrix $A = (a_{jn})$ in Theorem 2.5 with the identity matrix, then the following result is obtained, which is a classical case of Theorem 2.5.

Theorem 2.6. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of real number such that $0 < q_n < 1$ for each $n \in \mathbb{N}$ and $\lim_n \left(\frac{1-q_n}{1-q_n^n}\right) = 0$. Then, for all $f \in C[0,1]$, the sequence $\left\{L_n^{\beta(r)}(f;q_n;.)\right\}$ is uniformly convergent to f on the interval $[0,1]$ if and only if $\lim_n u_n^{(r)} = 1$ holds (in the ordinary sense).*

However, the following example shows that our Theorem 2.5 is stronger than its classical case (Theorem 2.6).

Example. Consider $A = C_1 := (c_{jn})$, the Cesáro matrix of order one. Note that, under this choice, A-statistical convergence reduces to the statistical convergence. In this case we use the notation st instead of st_{C_1}. Let $\beta(r) := (u^{(1)}, u^{(2)}, ..., u^{(r)})$ such that the sequence $u^{(r)} := \left\{u_n^{(r)}\right\}_{n\in\mathbb{N}}$ is defined by

$$u_n^{(r)} = \begin{cases} \dfrac{1}{2}, & \text{if } n = m^2 \, (m = 1, 2, ...) \\ \dfrac{n}{n+r}, & \text{otherwise.} \end{cases} \tag{2.12}$$

Also consider the sequence $(q_n)_{n\in\mathbb{N}}$ given by (2.5). Then observe that $0 < u_n^{(r)} < 1$ and $0 < q_n < 1$ for each $n \in \mathbb{N}$ and also that

$$st - \lim_n u_n^{(r)} = 1 \text{ and } st - \lim_n \frac{1-q_n}{1-q_n^n} = 0$$

Therefore, by Theorem 2.5 we obtain, for all $f \in C[0,1]$, that

$$st - \lim_n \left\| L_n^{\beta(r)}(f;q_n;.) - f \right\| = 0.$$

However, since neither the sequence $u^{(r)}$ in (2.12) nor $(q_n)_{n\in\mathbb{N}}$ in (2.5) are convergent (in the ordinary sense), it is easy to check that Theorem 2.6 does not work.

3. Rates of A-statistical convergence

In this section, we study the rates of A-statistical convergence in Theorem 2.5 by the means of the modulus of continuity, the elements of Lipschitz class, Peetre's K-functional and the Lipschitz type maximal functions, respectively.

As usual, the modulus of continuity of a function $f \in C[0,1]$ is defined by

$$w(f,\delta) = \sup_{|y-x|\leq\delta,\ x,y\in[0,1]} |f(y) - f(x)| \quad (\delta > 0).$$

Now we have the following result.

Theorem 3.1. *Let (q_n) be a sequence such that $0 < q_n < 1$ for each n. For each $n \in \mathbb{N}$ and for all $f \in C[0,1]$, we have*

$$\left\| L_n^{\beta(r)}(f;q_n;.) - f \right\| \leq 2w(f,\delta_n), \tag{3.1}$$

where

$$\delta_n := \left\{ 4\left(1 - \left(u_n^{(r)}\right)^2\right) + u_n^{(r)}\left(\frac{1-q_n}{1-q_n^n}\right) \right\}^{1/2}. \tag{3.2}$$

Proof. By the definition of the modulus of continuity, it is easy to see, for any $\delta > 0$, $x \in [0,1]$ and $n \in \mathbb{N}$, that

$$\begin{aligned} \left| L_n^{\beta(r)}(f;q_n;x) - f(x) \right| &\leq L_n^{\beta(r)}\left(|f(y) - f(x)|;q_n;x\right) \\ &\leq w(f,\delta)\left\{ 1 + \frac{1}{\delta} L_n^{\beta(r)}\left(|y-x|;q_n;x\right) \right\}. \end{aligned}$$

Now, applying the Cauchy-Schwarz inequality for positive linear operators we have

$$\begin{aligned} \left| L_n^{\beta(r)}(f;q_n;x) - f(x) \right| &\leq w(f,\delta)\left\{ 1 + \frac{1}{\delta}\left(L_n^{\beta(r)}((y-x)^2;q_n;x) \right)^{1/2} \right\} \\ &\leq w(f,\delta)\left\{ 1 + \frac{1}{\delta}\Big(\left| L_n^{\beta(r)}(e_2;q_n;x) - e_2(x) \right| \right. \\ &\quad \left. + 2x\left| L_n^{\beta(r)}(e_1;q_n;x) - e_1(x) \right| \Big)^{\frac{1}{2}} \right\}, \end{aligned}$$

which gives

$$\begin{aligned} \left\| L_n^{\beta(r)}(f;q_n;\cdot) - f \right\| &\leq w(f,\delta)\left\{ 1 + \frac{1}{\delta}\Big(\left\| L_n^{\beta(r)}(e_2;q_n;\cdot) - e_2 \right\| \right. \\ &\quad \left. + 2\left\| L_n^{\beta(r)}(e_1;q_n;\cdot) - e_1 \right\| \Big)^{\frac{1}{2}} \right\} \end{aligned}$$

Then it follows from Lemmas 2.2 and 2.3 that

(3.3)
$$\left\|L_n^{\beta(r)}(f;q_n;\cdot)-f\right\| \leq w\left(f,\delta\right)\left\{1+\frac{1}{\delta}\left[4\left(1-\left(u_n^{(r)}\right)^2\right)+u_n^{(r)}\left(\frac{1-q_n}{1-q_n^n}\right)\right]^{1/2}\right\}$$

Hence, choosing $\delta := \delta_n$ given as (3.2), the inequality (3.1) follows from (3.3) at once. □

We will now study the rate of convergence of the positive linear operators $L_n^{\beta(r)}(f;q_n;x)$ given by (1.3) by means of the elements of the Lipschitz class $Lip_M(\alpha)$, for $0<\alpha\leq 1$ and $M>0$. We recall that a function $f\in C[0,1]$ belongs to $Lip_M(\alpha)$ if the inequality

$$|f(y)-f(x)| \leq M\left|y-x\right|^{\alpha},\ \ (x,y\in[0,1]) \tag{3.4}$$

holds.

Theorem 3.2. *Let (q_n) be a sequence such that $0<q_n<1$ for each n. Then, for each $n\in\mathbb{N}$ and for all $f\in Lip_M(\alpha)$, we have*

$$\left\|L_n^{\beta(r)}(f;q_n;.)-f\right\| \leq M\delta_n^{\alpha}, \tag{3.5}$$

where δ_n is the same as in (3.2).

Proof. Let $f\in Lip_M(\alpha)$ and $x\in[0,1]$. From (3.4), we obtain

$$\begin{aligned}\left|L_n^{\beta(r)}(f;q_n;x)-f(x)\right| &\leq L_n^{\beta(r)}\left(\left|f(y)-f(x)\right|;q_n;x\right)\\ &\leq ML_n^{\beta(r)}\left(\left|y-x\right|^{\alpha};q_n;x\right).\end{aligned}$$

Now, applying the Hölder inequality with $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$ we get

$$\left|L_n^{\beta(r)}(f;q_n;x)-f(x)\right| \leq M\left\{L_n^{\beta(r)}((y-x)^2;q_n;x)\right\}^{\alpha/2}. \tag{3.6}$$

As in the proof of Theorem 3.1, we can write from (3.6) that

$$\left\|L_n^{\beta(r)}(f;q_n;\cdot)-f\right\| \leq M\left\{4\left(1-\left(u_n^{(r)}\right)^2\right)+u_n^{(r)}\left(\frac{1-q_n}{1-q_n^n}\right)\right\}^{\alpha/2}. \tag{3.7}$$

If we take δ_n as in (3.2), then (3.5) is obtained from (3.7). □

Now we recall the following space and norm: $C^2[0,1]$, the space of all functions f such that f, f', $f''\in C[0,1]$. The norm on the space $C^2[0,1]$ can be defined as

$$\|f\|_{C^2[0,1]} := \|f\|+\|f'\|+\|f''\|.$$

We consider the following Peetre's K-functional (similarly as in [4])

$$K(f;\delta) := \inf_{g\in C^2[0,1]}\left\{\|f-g\|+\delta\|g\|_{C^2[0,1]}\right\}.$$

It is clear that if $f\in C[0,1]$, then we have $\lim_{\delta\to 0}K(f;\delta)=0$. Some further results on the Peetre's K-functional may be found in [6].

Theorem 3.3. *Let (q_n) be a sequence such that $0<q_n<1$ for each n. Then for all $f\in C[0,1]$, we have*

$$\left\|L_n^{\beta(r)}(f;q_n;.)-f\right\| \leq 2K(f;\delta_n^2), \tag{3.8}$$

where δ_n is the same as in (3.2).

Proof. Let $g \in C^2[0,1]$ and $x \in [0,1]$. From the Taylor expansion we have

$$\begin{aligned}\left|L_n^{\beta(r)}(g;q_n;x)-g(x)\right| &\leq \left|L_n^{\beta(r)}(y-x;q_n;x)\right||g'| \\ &\quad +\frac{1}{2}\left|L_n^{\beta(r)}\left((y-x)^2;q_n;x\right)\right||g''|.\end{aligned}$$

Then, by Lemmas 2.1-2.3, we obtain that

$$\begin{aligned}\left\|L_n^{\beta(r)}(g;q_n;.)-g\right\| &\leq \left(1-u_n^{(r)}\right)\|g'\| \\ &\quad +\frac{1}{2}\left\{4\left(1-\left(u_n^{(r)}\right)^2\right)+u_n^{(r)}\left(\frac{1-q_n}{1-q_n^n}\right)\right\}\|g''\| \\ &\leq \left\{4\left(1-\left(u_n^{(r)}\right)^2\right)+u_n^{(r)}\left(\frac{1-q_n}{1-q_n^n}\right)\right\}\|g\|_{C^2[0,1]}.\end{aligned}$$

Since for all $f \in C[0,1]$

$$\left\|L_n^{\beta(r)}(f;q_n;.)-f\right\| \leq 2\|f-g\|+\left\|L_n^{\beta(r)}(g;q_n;.)-g\right\|,$$

we can write that

$$\left\|L_n^{\beta(r)}(f;q_n;.)-f\right\| \leq 2\left\{\|f-g\|+\delta_n^2\|g\|_{C^2[0,1]}\right\}, \tag{3.9}$$

where δ_n is the same as in (3.2). Now, by taking infimum over $g \in C^2[0,1]$ on both sides of (3.9) we obtain (3.8). □

Finally, by using the similar technique as in Theorem 3.9 of Agratini [2], we may estimate the following rates of the operators $L_n^{\beta(r)}(f;q_n;x)$ by means of the Lipschitz type maximal function of order α. Here we recall that the Lipschitz type maximal function of order α introduced by B. Lenze in [18] as follows:

$$\widetilde{w}_\alpha(f) := \sup_{y\neq x;\ x,y\in[0,1]} \frac{|f(y)-f(x)|}{|y-x|^\alpha}, \quad \alpha \in (0,1].$$

Of course, the boundedness of $\widetilde{w}_\alpha(f)$ is equivalent to $f \in Lip_M(\alpha)$.

The we have the following

Theorem 3.4. *Let (q_n) be a sequence such that $0 < q_n < 1$ for each n. Then, we have*

$$\left|L_n^{\beta(r)}(f;q_n;x)-f(x)\right| \leq \delta_n^\alpha\ \widetilde{w}_\alpha(f),$$

where δ_n is the same as in (3.2).

Concluding remarks. Let $A=(a_{jn})$ be a non-negative regular summability matrix. If the conditions

$$st_A-\lim_n u_n^{(r)}=1 \text{ and } st_A-\lim\frac{1-q_n}{1-q_n^n}=0$$

hold, then observe that $st_A-\lim_n \delta_n=0$, where δ_n is given by (3.2). In this case, we get

$$st_A-\lim_n w(f,\delta_n)=st_A-\lim K(f,\delta_n^2)=0$$

for all $f \in C[0,1]$. So, Theorems 3.1-3.3 give us the rates of A-statistical convergence in Theorem 2.5. Of course, if we replace $A=(a_{jn})$ by the identity matrix, then we get the ordinary rates of convergence.

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A bivariate blending interpolator and the properties

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Abstract

In this paper a new weighted bivariate blending interpolator is constructed based on function values and partial derivatives of a function. The interpolator has a simple and explicit mathematical representation, and is C^1 for any positive parameters in the whole interpolating region. Also, some properties of the interpolator are derived, such as the properties of the basis function, the bounded property and the error estimates.

Keywords: Rational spline; bivariate blending interpolation; weighted interpolation; error estimates; computer-aided geometric design.

1 Introduction

The construction method of curve and surface and the mathematical description of them is a key issue in computer-aided geometric design. There are many ways to tackle this problem [1-3,9,12-14,17,18], for example, the polynomial spline method, the NURBS method and the Bézier method. These methods are effective and applied widely in shape design of industrial products. Generally speaking, most of the polynomial spline methods are the interpolating methods. However, one of the disadvantages of the polynomial spline method is that it can not be used to modify the local shape of the interpolating surfaces for unchanged given data. The NURBS and Bézier methods are the so-called "no-interpolating type" methods; this means that the constructed curve and surface do not pass through the given data, and the given points play the role of the control points.

In recent years, Motivated by the univariate rational spline interpolation [4,5,10,11,15], the bivariate rational interpolation with parameters which has simple and explicit mathematical representation has been studied [6-8]. Since the parameters in the interpolator are selective according to the control need, the constrained control of the shape becomes possible. However, for this kind of bivariate rational interpolator, only when the parameters of y-direction satisfy certain condition can it be C^1 in the interpolating region [6,7]. It is inconvenient to modify the shape of surfaces. To overcome the problem, in this paper, a new weighted bivariate blending interpolator will be constructed based on function values and derivatives of a function, and its properties is also discussed.

This paper is arranged as follows. In Section 2, a weighted bivariate blending rational interpolator is constructed. In Section 3, the basis of this interpolator is derived. Section 4 is about

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some properties of the interpolator, including the properties of the basis function and the bounded property. Sections 5 deals with the error estimates of the interpolator. Some examples are given in Section 6, which show that this interpolator is with good approximation to the interpolated function.

2 Interpolation

Let $\Omega : [a, b; c, d]$ be the plane region, and $\{(x_i, y_i, f_{i,j}, e_{i,j}, d_{i,j}), i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a given set of data points, where $a = x_1 < x_2 < \cdots < x_n = b$ and $c = y_1 < y_2 < \cdots < y_m = d$ are the knot spacings, $f_{i,j}$, $e_{i,j}$, $d_{i,j}$ represent $f(x_i, y_j)$, $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$ at the point (x_i, y_j) respectively. Let $h_i = x_{i+1} - x_i$, and $l_j = y_{j+1} - y_j$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the xy-plane, Let $\theta = (x - x_i)/h_i$ and $\eta = (y - y_j)/l_j$. First, for each $y = y_j, j = 1, 2, \cdots, m$, construct the x-direction interpolating curve [5]; this is given by

$$p^*_{i,j}(x) = \frac{(1-\theta)^3\alpha_{i,j}f_{i,j} + \theta(1-\theta)^2V^*_{i,j} + \theta^2(1-\theta)W^*_{i,j} + \theta^3 f_{i+1,j}}{(1-\theta)\alpha_{i,j} + \theta}, \ i = 1, 2, \cdots, n-1, \tag{1}$$

where

$$V^*_{i,j} = (2\alpha_{i,j} + 1)f_{i,j} + \alpha_{i,j}h_i e_{i,j},$$
$$W^*_{i,j} = (\alpha_{i,j} + 2)f_{i+1,j} - h_i e_{i+1,j},$$

with $\alpha_{i,j} > 0$. This interpolation is called the rational cubic interpolator based on function values and derivatives which satisfies

$$p^*_{i,j}(x_i) = f_{i,j}, \quad p^*_{i,j}(x_{i+1}) = f_{i+1,j}, \quad {p^*_{i,j}}'(x_i) = e_{i,j}, \quad {p^*_{i,j}}'(x_{i+1}) = e_{i+1,j}.$$

For each pair of $(i, j), i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, using the x-direction interpolation $p^*_{i,j}(x)$, define the interpolation function $P^*(x, y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$P^*_{i,j}(x, y) = (1-\eta)^3 p^*_{i,j}(x) + \eta(1-\eta)^2 V_{i,j} + \eta^2(1-\eta)W_{i,j} + \eta^3 p^*_{i,j+1}(x),$$
$$i = 1, 2, \cdots, n-1; \quad j = 1, 2, \cdots, m-1, \tag{2}$$

where

$$V_{i,j} = 3p^*_{i,j}(x) + l_j((1-\theta)^2(3\theta+1)d_{i,j} + \theta^2(2-\theta)d_{i+1,j}),$$
$$W_{i,j} = 3p^*_{i,j+1}(x) - l_j((1-\theta)^2(3\theta+1)d_{i,j+1} + \theta^2(2-\theta)d_{i+1,j+1}).$$

The term $P^*_{i,j}(x, y)$ is called the bivariate blending rational Hermite interpolation based on function values and partial derivatives which satisfies

$$P^*_{i,j}(x_r, y_s) = f(x_r, y_s), \ \frac{\partial P^*_{i,j}(x_r, y_s)}{\partial x} = e_{r,s}, \ \frac{\partial P^*_{i,j}(x_r, y_s)}{\partial y} = d_{r,s}, \ r = i, i+1, \ s = j, j+1.$$

If the knots are equally spaced for variable x, namely, $h_i = (b-a)/n$, it is easy to test that the interpolation function $P^*_{i,j}(x, y)$ is C^1 in the whole interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\alpha_{i,j}, \alpha_{i,j+1}$ might be [6]. Furthermore, when $\alpha_{i,j} = \alpha_{i,j+1} = 1$, the interpolator is a bivariate standard Hermite interpolation.

The interpolating scheme above begins in x-direction first. Now, let the interpolation begins with y-direction first. For each $x = x_i, i = 1, 2, \cdots, n$, denote the y-direction interpolation in $[y_j, y_{j+1}]$ by

$$q^*_{i,j}(y) = \frac{(1-\eta)^3\beta_{i,j}f_{i,j} + \eta(1-\eta)^2T^*_{i,j} + \eta^2(1-\eta)U^*_{i,j} + \eta^3 f_{i,j+1}}{(1-\eta)\beta_{i,j} + \eta},\ i = 1, 2, \cdots, m-1, \tag{3}$$

where

$$\begin{aligned} T^*_{i,j} &= (2\beta_{i,j} + 1)f_{i,j} + \beta_{i,j}l_j d_{i,j}, \\ U^*_{i,j} &= (\beta_{i,j} + 2)f_{i,j+1} - l_j d_{i,j+1}, \end{aligned}$$

with $\beta_{i,j} > 0$.

For each pair $(i, j), i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, using the y-direction interpolation $q^*_{i,j}(y)$, define the interpolation function $Q^*_{i,j}(x, y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$\begin{aligned} Q^*_{i,j}(x,y) = (1-\theta)^3 q^*_{i,j}(y) + \theta(1-\theta)^2 T_{i,j} + \theta^2(1-\theta)U_{i,j} + \theta^3 q^*_{i+1,j}(y), \\ i = 1, 2, \cdots, n-1; \quad j = 1, 2, \cdots, m-1, \end{aligned} \tag{4}$$

where

$$\begin{aligned} T_{i,j} &= 3q^*_{i,j}(y) + h_i((1-\eta)^2(3\eta+1)e_{i,j} + \eta^2(2-\eta)e_{i,j+1}), \\ U_{i,j} &= 3q^*_{i+1,j}(y) - h_i((1-\eta)^2(3\eta+1)e_{i+1,j} + \eta^2(2-\eta)e_{i+1,j+1}). \end{aligned}$$

The interpolation function $Q^*_{i,j}(x, y)$ satisfies

$$Q^*_{i,j}(x_r, y_s) = f(x_r, y_s), \frac{\partial Q^*_{i,j}(x_r, y_s)}{\partial x} = e_{r,s}, \frac{\partial Q^*_{i,j}(x_r, y_s)}{\partial y} = d_{r,s},\ r = i, i+1,\ s = j, j+1.$$

Similarly, if the knots are equally spaced for variable y, namely, $l_j = (d-c)/m$, the interpolation $Q^*_{i,j}(x, y)$ is C^1 in the whole interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\beta_{i,j}, \beta_{i+1,j}$ might be.

The weighted bivariate blending rational interpolator will be constructed by using the two kinds of interpolator described above. Let

$$P_{i,j}(x,y) = \lambda_{i,j}P^*_{i,j}(x,y) + (1-\lambda_{i,j})Q^*_{i,j}(x,y), \tag{5}$$

with the weight coefficient $\lambda_{i,j} \in [0, 1]$, then $P_{i,j}(x, y)$ satisfies

$$P_{i,j}(x_r, y_s) = f(x_r, y_s),\ \frac{\partial P_{i,j}(x_r, y_s)}{\partial x} = e_{r,s},\ \frac{\partial P_{i,j}(x_r, y_s)}{\partial y} = d_{r,s},\ r = i, i+1,\ s = j, j+1,$$

and it is C^1 in the whole interpolating region $[x_1, x_n; y_1, y_m]$ if the knots are equally spaced for variable x and y, no matter what the parameters $\alpha_{i,j}, \alpha_{i,j+1}$ and $\beta_{i,j}, \beta_{i+1,j}$ might be. In the following of this paper, consider the interpolator defined in (5).

3 The basis of the interpolation

From (1)-(5), the interpolation function $P_{i,j}(x,y)$ defined in (5) can be written as follows:

$$P_{i,j}(x,y)=\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}[a_{r,s}(\theta,\eta)f_{r,s}+b_{r,s}(\theta,\eta)h_ie_{r,s}+c_{r,s}(\theta,\eta)l_jd_{r,s}], \tag{6}$$

where

$$a_{i,j}(\theta,\eta)=(1-\theta)^2(1-\theta)^2[\frac{(\theta+(1+\theta)\alpha_{i,j})(1+2\eta)\lambda_{i,j}}{(1-\theta)\alpha_{i,j}+\theta}+\frac{(1+2\theta)(\eta+(1+\eta)\beta_{i,j})(1-\lambda_{i,j})}{(1-\eta)\beta_{i,j}+\eta}],$$

$$a_{i,j+1}(\theta,\eta)=(1-\theta)^2\eta^2[\frac{(\theta+(1+\theta)\alpha_{i,j+1})(3-2\eta)\lambda_{i,j}}{(1-\theta)\alpha_{i,j+1}+\theta}+\frac{(1+2\theta)(2-\eta+(1-\eta)\beta_{i,j})(1-\lambda_{i,j})}{(1-\eta)\beta_{i,j}+\eta}],$$

$$a_{i+1,j}(\theta,\eta)=\theta^2(1-\eta)^2[\frac{(2-\theta+(1-\theta)\alpha_{i,j})(1+2\eta)\lambda_{i,j}}{(1-\theta)\alpha_{i,j}+\theta}+\frac{(3-2\theta)(\eta+(1+\eta)\beta_{i+1,j})(1-\lambda_{i,j})}{(1-\eta)\beta_{i+1,j}+\eta}],$$

$$a_{i+1,j+1}(\theta,\eta)=\theta^2\eta^2[\frac{(2-\theta+(1-\theta)\alpha_{i,j+1})(3-2\eta)\lambda_{i,j}}{(1-\theta)\alpha_{i,j+1}+\theta}+\frac{(3-2\theta)(2-\eta+(1-\eta)\beta_{i+1,j})(1-\lambda_{i,j})}{(1-\eta)\beta_{i+1,j}+\eta}],$$

$$b_{i,j}(\theta,\eta)=\theta(1-\theta)^2(1-\eta)^2[\frac{(1+2\eta)\lambda_{i,j}\alpha_{i,j}}{(1-\theta)\alpha_{i,j}+\theta}+(1+3\eta)(1-\lambda_{i,j})],$$

$$b_{i,j+1}(\theta,\eta)=\theta(1-\theta)^2\eta^2[\frac{(3-2\eta)\lambda_{i,j}\alpha_{i,j+1}}{(1-\theta)\alpha_{i,j+1}+\theta}+(2-\eta)(1-\lambda_{i,j})],$$

$$b_{i+1,j}(\theta,\eta)=-\theta^2(1-\theta)(1-\eta)^2[\frac{(1+2\eta)\lambda_{i,j}}{(1-\theta)\alpha_{i,j}+\theta}+(1+3\eta)(1-\lambda_{i,j})],$$

$$b_{i+1,j+1}(\theta,\eta)=-\theta^2(1-\theta)\eta^2[\frac{(3-2\eta)\lambda_{i,j}}{(1-\theta)\alpha_{i,j+1}+\theta}+(2-\eta)(1-\lambda_{i,j})],$$

$$c_{i,j}(\theta,\eta)=(1-\theta)^2\eta(1-\eta)^2[\frac{(1+2\theta)(1-\lambda_{i,j})\beta_{i,j}}{(1-\eta)\beta_{i,j}+\eta}+(1+3\theta)\lambda_{i,j}],$$

$$c_{i,j+1}(\theta,\eta)=-(1-\theta)^2\eta^2(1-\eta)[\frac{(1+2\theta)(1-\lambda_{i,j})}{(1-\eta)\beta_{i,j}+\eta}+(1+3\theta)\lambda_{i,j}],$$

$$c_{i+1,j}(\theta,\eta)=\theta^2\eta(1-\eta)^2[\frac{(3-2\theta)(1-\lambda_{i,j})\beta_{i+1,j}}{(1-\eta)\beta_{i+1,j}+\eta}+(2-\theta)\lambda_{i,j}],$$

$$c_{i+1,j+1}(\theta,\eta)=-\theta^2\eta^2(1-\eta)[\frac{(3-2\theta)(1-\lambda_{i,j})}{(1-\eta)\beta_{i+1,j}+\eta}+(2-\theta)\lambda_{i,j}].$$

Terms $a_{r,s}(\theta,\eta), b_{r,s}(\theta,\eta), c_{r,s}(\theta,\eta), (r=,i,i+1; s=j,j+1)$ are called the basis of the bivariate interpolator defined by (5), and satisfy following equations and inequalities:

$$a_{i,j}(\theta,\eta)+a_{i,j+1}(\theta,\eta)+a_{i+1,j}(\theta,\eta)+a_{i+1,j+1}(\theta,\eta)=1, \tag{7}$$

$$b_{i,j}(\theta,\eta)+b_{i,j+1}(\theta,\eta)-b_{i+1,j}(\theta,\eta)-b_{i+1,j+1}(\theta,\eta)$$
$$=\theta(1-\theta)(1+\eta(1-3\eta+2\eta^2)(1-\lambda_{i,j})), \tag{8}$$

$$c_{i,j}(\theta,\eta)-c_{i,j+1}(\theta,\eta)+c_{i+1,j}(\theta,\eta)-c_{i+1,j+1}(\theta,\eta)$$
$$=\eta(1-\eta)(1+\theta(1-3\theta+2\theta^2)\lambda_{i,j}), \tag{9}$$

$$a_{r,s}(\theta,\eta) > 0,\ r = i, i+1;\quad s = j, j+1, \tag{10}$$
$$b_{i,s}(\theta,\eta) > 0,\ b_{i+1,s}(\theta,\eta) < 0,\quad s = j, j+1, \tag{11}$$
$$c_{r,j}(\theta,\eta) > 0,\ c_{r,j+1}(\theta,\eta) < 0,\quad r = i, i+1. \tag{12}$$

4 Some properties of the interpolation

For the interpolator defined by (5), we use Eq.(7) to arrive at the following unity property.

Property 1. *Let $f(x,y) \equiv 1$, $\forall (x,y) \in \Omega$, and $P_{i,j}(x,y)$ is its interpolation function over $[x_i, x_{i+1}; y_j, y_{j+1}]$ defined in (5). No matter what positive number the parameters $\alpha_{i,s}, \beta_{r,j}$ take, the unity property holds, namely*

$$\int\int_D P_{i,j}(x,y)dxdy = h_i l_j,$$

where D denotes the subregion $[x_i, x_{i+1}; y_j, y_{j+1}]$.

Use (6) to define

$$\int\int_D P_{i,j}(x,y)dxdy = h_i l_j \sum_{r=i}^{i+1}\sum_{s=j}^{j+1}(a^*_{r,s} f_{r,s} + b^*_{r,s} h_i e_{r,s} + c^*_{r,s} l_j d_{r,s}),$$

where

$$a^*_{r,s} = \int\int_{[0,1;0,1]} a_{r,s}(\theta,\eta)d\theta d\eta,\quad r = i, i+1,\ s = j, j+1,$$
$$b^*_{r,s} = \int\int_{[0,1;0,1]} b_{r,s}(\theta,\eta)d\theta d\eta,\quad r = i, i+1,\ s = j, j+1,$$
$$c^*_{r,s} = \int\int_{[0,1;0,1]} c_{r,s}(\theta,\eta)d\theta d\eta,\quad r = i, i+1,\ s = j, j+1.$$

$a^*_{r,s}, b^*_{r,s}$ and $c^*_{r,s}$ are called the integral weights coefficients of the interpolator defined in (5). Thus, we use Eq.(7)-(9) to yield the following property.

Property 2. *Let $P_{i,j}(x,y)$ is the interpolation function over $[x_i, x_{i+1}; y_j, y_{j+1}]$ defined in (5). No matter what positive number the parameters $\alpha_{i,s}$ and $\beta_{r,j}$ take, the integral weights coefficients of interpolation $P_{i,j}(x,y)$ satisfy the following equations:*

$$\sum_{r=0}^{1}\sum_{s=0}^{1} a^*_{i+r,j+s} = 1,$$
$$\sum_{r=0}^{1}\sum_{s=0}^{1}(-1)^r b^*_{i+r,j+s} = \frac{1}{6},$$
$$\sum_{r=0}^{1}\sum_{s=0}^{1}(-1)^s c^*_{i+r,j+s} = \frac{1}{6}.$$

From (10)-(12), it is easy to see that the following inequalities hold:

$$a^*_{r,s} > 0,\ r = i, i+1,\ s = j, j+1, \tag{13}$$
$$b^*_{i,s} > 0,\ b^*_{i+1,s} < 0,\ s = j, j+1, \tag{14}$$
$$c^*_{r,j} > 0,\ c^*_{r,j+1} < 0,\ r = i, i+1. \tag{15}$$

From Property 2 and the inequalities (13)-(15), the following bounded property can be obtained.

Property 3. *Let $P_{i,j}(x,y)$ be the interpolation function over $[x_i, x_{i+1}; y_j, y_{j+1}]$ defined in (5). Denoting*

$$\begin{aligned} M_1 &= \max\{|f_{r,s}|, \quad r = i, i+1; \quad s = j, j+1\} \\ M_2 &= \max\{h_i|e_{r,s}|, \quad r = i, i+1; \quad s = j, j+1\}, \\ M_3 &= \max\{l_j|d_{r,s}|, \quad r = i, i+1; \quad s = j, j+1\}. \end{aligned}$$

No matter what positive number the parameters $\alpha_{i,s}$ and $\beta_{r,j}$ take, the integral of interpolation function $P_{i,j}(x,y)$ satisfy the following inequality:

$$|\int\int_D P_{i,j}(x,y)dxdy| \le (M_1 + \frac{M_2 + M_3}{6})h_i l_j,$$

where D denotes the subregion $[x_i, x_{i+1}; y_j, y_{j+1}]$.

Furthermore, when $\lambda_{i,j} = 1$, the interpolation function $P_{i,j}(x,y)$ becomes $P^*_{i,j}(x,y)$ defined in (2). Denoting

$$\begin{aligned} u_{r,s} &= \int_0^1 a_{r,s}(\theta,\eta)d\eta, \ r = i, i+1, \ s = j, j+1, \\ v_{r,s} &= \int_0^1 b_{r,s}(\theta,\eta)d\eta, \ r = i, i+1, \ s = j, j+1, \\ w_{r,s} &= \int_0^1 c_{r,s}(\theta,\eta)d\eta, \ r = i, i+1, \ s = j, j+1. \end{aligned}$$

Then the following conclusions hold for any parameters $\alpha_{i,s} > 0$,

$$\begin{aligned} &0 \le u_{r,s} \le \frac{1}{2}, \quad r = i, i+1, \ s = j, j+1; \\ &0 \le v_{i,s} \le \frac{1}{8}, \ -\frac{1}{8} \le v_{i+1,s} \le 0, \ s = j, j+1; \\ &w_{i,j} = \frac{(1-\theta)^2(1+3\theta)}{12}, \ w_{i,j+1} = -\frac{(1-\theta)^2(1+3\theta)}{12}, \\ &w_{i+1,j} = \frac{\theta^2(2-\theta)}{12}, \ w_{i+1,j+1} = -\frac{\theta^2(2-\theta)}{12}. \end{aligned}$$

Thus, the following property can be concluded.

Property 4. *For the bivariate blending interpolation $P^*_{i,j}(x,y)$ defined in (2), the integral weights coefficients of the interpolation satisfy:*

$$\begin{aligned} &0 < a^*_{r,s} < \frac{1}{2}, \quad r = i, i+1, \ s = j, j+1; \\ &0 < b^*_{i,s} < \frac{1}{8}, \ -\frac{1}{8} < b^*_{i+1,s} < 0, \ s = j, j+1; \\ &c^*_{i,j} = \frac{7}{144}, \ c^*_{i,j+1} = -\frac{7}{144}, \ c^*_{i+1,j} = \frac{5}{144}, \ c^*_{i+1,j+1} = -\frac{5}{144}. \end{aligned}$$

Similarly, for the interpolation function $Q^*_{i,j}(x,y)$ defined in (4), there is the following property.

Property 5. *For the bivariate blending interpolation $Q^*_{i,j}(x,y)$ defined in (4), the integral weights coefficients of interpolation satisfy:*

$$\begin{aligned}
&0 < a^*_{r,s} < \frac{1}{2}, \quad r = i, i+1, \ s = j, j+1;\\
&b^*_{i,j} = \frac{7}{144}, \ b^*_{i,j+1} = \frac{5}{144}, \ b^*_{i+1,j} = -\frac{7}{144}, \ b^*_{i+1,j} = -\frac{5}{144};\\
&0 < c^*_{r,j} < \frac{1}{8}, \quad -\frac{1}{8} < c^*_{r,j+1} < 0, \ r = i, i+1.
\end{aligned}$$

From (5) and Property 4 and Property 5, it is easy to derive the following property.

Property 6. *For the interpolation $P_{i,j}(x,y)$ defined in (5), no matter what positive number the parameters $\alpha_{i,s}, \beta_{r,j}$ take, the integral weights coefficients of interpolation satisfy:*

$$\begin{aligned}
&0 < a^*_{r,s} < \frac{1}{2}, \quad r = i, i+1, \ s = j, j+1,\\
&0 < b^*_{i,s} < \frac{1}{8}, \quad -\frac{1}{8} < b^*_{i+1,s} < 0, \ s = j, j+1,\\
&0 < c^*_{r,j} < \frac{1}{8}, \quad -\frac{1}{8} < c^*_{r,j+1} < 0, \ r = i, i+1.
\end{aligned}$$

5 Error estimates of the interpolation

For the error estimation of the interpolator defined in (5), note that the interpolator is local, without loss of generality, it is only necessary to consider the interpolating region $[x_i, x_{i+1}; y_j, y_{j+1}]$ in order to process its error estimates. Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ is the interpolation function defined by (5) over $[x_i, x_{i+1}; y_j, y_{j+1}]$.

Denoting

$$\|\frac{\partial f}{\partial y}\| = \max_{(x,y)\in D} |\frac{\partial f(x,y)}{\partial y}|, \quad \|\frac{\partial P}{\partial y}\| = \max_{(x,y)\in D} |\frac{\partial P_{i,j}(x,y)}{\partial y}|,$$

where $D = [x_i, x_{i+1}; y_j, y_{j+1}]$. By the Taylor expansion and the Peano-Kernel Theorem [16] gives the following:

$$\begin{aligned}
|f(x,y) - P_{i,j}(x,y)| &\le |f(x,y) - f(x,y_j)| + |P_{i,j}(x,y_j) - P_{i,j}(x,y)| + |f(x,y_j) - P_{i,j}(x,y_j)|\\
&\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + |\int_{x_i}^{x_{i+1}} \frac{\partial^2 f(\tau, y_j)}{\partial x^2} R_x[(x-\tau)_+]d\tau|\\
&\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + \|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| \int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+]|d\tau, \qquad (16)
\end{aligned}$$

where $\|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| = \max_{x\in[x_i,x_{i+1}]} |\frac{\partial^2 f(x,y_j)}{\partial x^2}|$, and

$$\begin{aligned}
R_x[(x-\tau)_+] &= \begin{cases} (x-\tau) - a_{i+1,j}(\theta,0)(x_{i+1}-\tau) - b_{i+1,j}(\theta,0)h_i, & x_i < \tau < x;\\ -a_{i+1,j}(\theta,0)(x_{i+1}-\tau) - b_{i+1,j}(\theta,0)h_i, & x < \tau < x_{i+1}, \end{cases}\\
&= \begin{cases} r(\tau), & x_i < \tau < x;\\ t(\tau), & x < \tau < x_{i+1}. \end{cases}
\end{aligned}$$

Thus, by simple integral calculation, it can be derived that

$$\int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+]|d\tau = h_i^2 \frac{B_1(\theta,\alpha_{i,j},\lambda_{i,j})}{B_2(\theta,\alpha_{i,j},\lambda_{i,j})} = h_i^2 B(\theta,\alpha_{i,j},\lambda_{i,j}), \tag{17}$$

where

$$\begin{aligned} B_1(\theta,\alpha_{i,j},\lambda_{i,j}) &= \theta^2(1-\theta)^2[2\theta+(1-2\theta)\lambda_{i,j}+\alpha_{i,j}(2-2\theta-(1-2\theta)\lambda_{i,j})]^2, \\ B_2(\theta,\alpha_{i,j},\lambda_{i,j}) &= [\theta(3-2\theta)+2(1-\theta)^2\lambda_{i,j}+\alpha_{i,j}(1-\theta)(3-2\theta-2(1-\theta)\lambda_{i,j})] \\ &\quad [\theta(1+2\theta-2\theta\lambda_{i,j})+\alpha_{i,j}(1+\theta-2\theta^2+2\theta^2\lambda_{i,j})]. \end{aligned}$$

For the fixed $\alpha_{i,j}$ and $\lambda_{i,j}$, let

$$B_{i,j}^{(x)} = \max_{\theta\in[0,1]} B(\theta,\alpha_{i,j},\lambda_{i,j}). \tag{18}$$

This leads to the following theorem.

Theorem 1. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolator defined by (5) in $[x_i,x_{i+1};y_j,y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s},\beta_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y)-P_{i,j}(x,y)| \le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + h_i^2\|\frac{\partial^2 f(x,y_j)}{\partial x^2}\|B_{i,j}^{(x)},$$

where $B_{i,j}^{(x)}$ defined by (18).

Similarly, denoting $\|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\| = \max_{x\in[x_i,x_{i+1}]}|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}|$, then the following theorem holds.

Theorem 2. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolation function defined by (5) in $[x_i,x_{i+1};y_j,y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s},\beta_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y)-P_{i,j}(x,y)| \le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + h_i^2\|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\|B_{i,j+1}^{(x)},$$

where $B_{i,j+1}^{(x)} = \max_{\theta\in[0,1]} B(\theta,\alpha_{i,j+1},\lambda_{i,j})$, and $B(\theta,\alpha_{i,j},\lambda_{i,j})$ defined by (17).

Note that the interpolator defined by (5) is symmetric about two variables x and y, we denote $\|\frac{\partial f}{\partial x}\| = \max_{(x,y)\in D}|\frac{\partial f(x,y)}{\partial x}|$, $\|\frac{\partial P}{\partial x}\| = \max_{(x,y)\in D}|\frac{\partial P_{i,j}(x,y)}{\partial x}|$, and $\|\frac{\partial^2 f(x_r,y)}{\partial y^2}\| = \max_{y\in[y_j,y_{j+1}]}|\frac{\partial^2 f(x_r,y)}{\partial y^2}|$, $r=1,2$, then the following theorem can be obtained.

Theorem 3. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolator defined by (5) in $[x_i,x_{i+1};y_j,y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s},\beta_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y)-P_{i,j}(x,y)| \le h_i(\|\frac{\partial f}{\partial x}\| + \|\frac{\partial P}{\partial x}\|) + l_j^2\|\frac{\partial^2 f(x_r,y)}{\partial y^2}\|B_{r,j}^{(y)},\ r=1,2,$$

where $B_{r,j}^{(y)} = \max_{\eta\in[0,1]} B(\eta,\beta_{r,j},\lambda_{i,j})$, and $B(\theta,\alpha_{i,j},\lambda_{i,j})$ defined by (17).

Furthermore, for $B_{i,j}^{(x)}$, when the parameter $\alpha_{i,j} > 0$ and the weight coefficient $\lambda_{i,j} \in [0,1]$,

$$B_2(\theta,\alpha_{i,j},\lambda_{i,j}) \ge \theta(1-\theta)[2\theta+(1-2\theta)\lambda_{i,j}+\alpha_{i,j}(2-2\theta-(1-2\theta)\lambda_{i,j})]^2,$$

so it is easy to derive that

$$0 \leq B_{i,j}^{(x)} \leq \frac{1}{4}.$$

Thus, we can conclude the following theorem.

Theorem 4. *For any positive parameters $\alpha_{i,s}, \beta_{r,j}$ and weight coefficient $\lambda_{i,j} \in [0,1]$, $B_{i,s}^{(x)}$ and $B_{r,j}^{(y)}$ are bounded, and*

$$0 \leq B_{i,s}^{(x)} = B_{r,j}^{(y)} \leq \frac{1}{4}, \quad r = 1,2; j = 1,2.$$

6 Numerical examples

Example 1. Let the interpolated function be $f(x,y) = \sin(5(x^2 - y)/2)$, $(x,y) \in [0, 0.75; 0, 0.75]$, and let $h_i = l_j = 0.25$, then $x_i = 0.25i, y_j = 0.25j, i = 0,1,2,3, j = 0,1,2,3$. Also let $\alpha_{i,j} = 0.9 + 0.2i - 0.2j, \beta_{i,j} = 1.1 - 0.2i + 0.3j, \lambda_{i,j} = 0.8, i = 0,1,2,3, j = 0,1,2,3$. Figure 1 shows the graph of the interpolated function $f(x,y)$. Figure 2 shows the graph of the interpolation function $P(x,y)$ defined by (5). Figure 3 shows the surface of the error $f(x,y) - P(x,y)$. From Figure 3, it is easy to see that the interpolator defined by (5) gives a good approximation to the interpolated function.

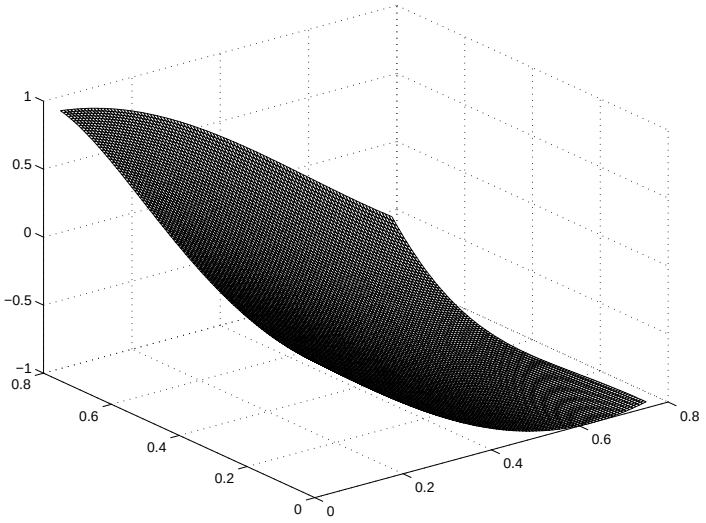


Figure 1: Graph of surface $f(x,y)$.

Furthermore, for the same parameters $\alpha_{i,j}$ and $\beta_{i,j}$, the interpolation function $P_{i,j}(x,y)$ defined in (5) can give a better approximation to the interpolated function by selecting suitable weight coefficient than the interpolator discussed in [6]. In order to show the case, an example is given in the following.

Example 2. Let the interpolated function be $f(x,y) = e^{x^2-y}$. Note that the interpolator is local, we only consider a small interpolating region: $[0.5, 0.7; 0.5, 0.7]$. Let $h_i = l_j = 0.2$.

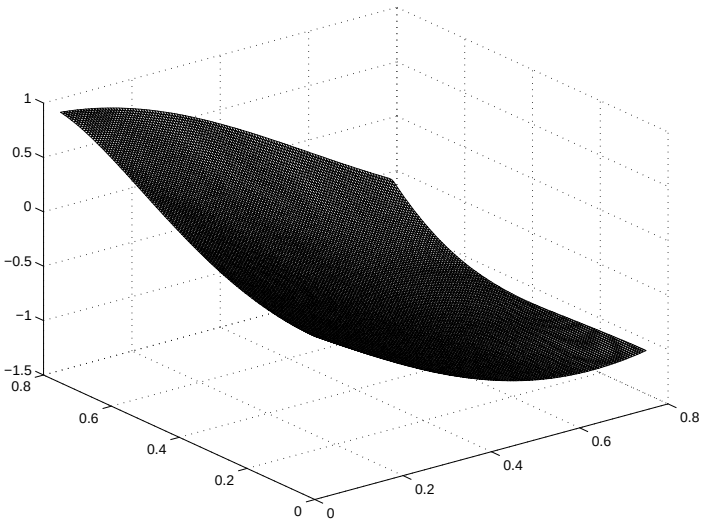


Figure 2: Graph of surface $P(x, y)$.

$P_{i,j}(x, y)$ is the interpolation function defined by (5), $Q_{i,j}(x, y)$ is the interpolation function defined in [6], and denote $R[f, Q] = |f(x, y) - Q_{i,j}(x, y)|, R[f, P] = |f(x, y) - P_{i,j}(x, y)|$. For the same parameters $\alpha_{i,s} = 1.15$ and $\beta_{r,j} = 0.99$, we take $\lambda_{i,j} = 0.95$. Table 1 gives some values of $f(x, y), P_{i,j}(x, y), Q_{i,j}(x, y), R[f, Q]$ and $R[f, P]$. It is easy to see that the values of $P_{i,j}(x, y)$ are closer to $f(x, y)$ than that of $Q_{i,j}(x, y)$ in the interpolating region $[0.5, 0.7; 0.5, 0.7]$.

Table 1. Values of $f(x, y), Q_{i,j}(x, y), P_{i,j}(x, y), R[f, Q]$ and $R[f, P]$.

x	y	$f(x, y)$	$Q_{i,j}(x, y)$	$P_{i,j}(x, y)$	$R[f, Q]$	$R[f, P]$
.55	.50	.82078	.82077	.82077	.00001	.00001
.55	.55	.78075	.77964	.77974	.00111	.00101
.55	.60	.74267	.74243	.74244	.00024	.00023
.55	.65	.70645	.70719	.70710	.00074	.00065
.55	.70	.67200	.67199	.67199	.00001	.00001
.65	.50	.92543	.92541	.92541	.00002	.00002
.65	.55	.88029	.88172	.88146	.00143	.00117
.65	.60	.83736	.83765	.83761	.00029	.00025
.65	.65	.79652	.79553	.79571	.00099	.00081
.65	.70	.75768	.75766	.75766	.00002	.00002

7 Concluding remarks

- This paper gives the explicit expression of a weighted bivariate blending rational interpolator. In the interpolator, there are four parameters $\alpha_{i,j}, \alpha_{i,j+1}, \beta_{i,j}, \beta_{i+1,j}$ and a weight coefficient $\lambda_{i,j}$, and it is C^1 in the whole interpolating region $[x_1, x_n; y_1, y_m]$ for any parameters $\alpha_{i,j}, \beta_{i,j}$ and weight coefficient $\lambda_{i,j} \in [0, 1]$. Positive parameters and weight coefficient can be freely selected according to the needs of practical design.

- For each pitch of the interpolating surface, the value of the interpolation function depends on

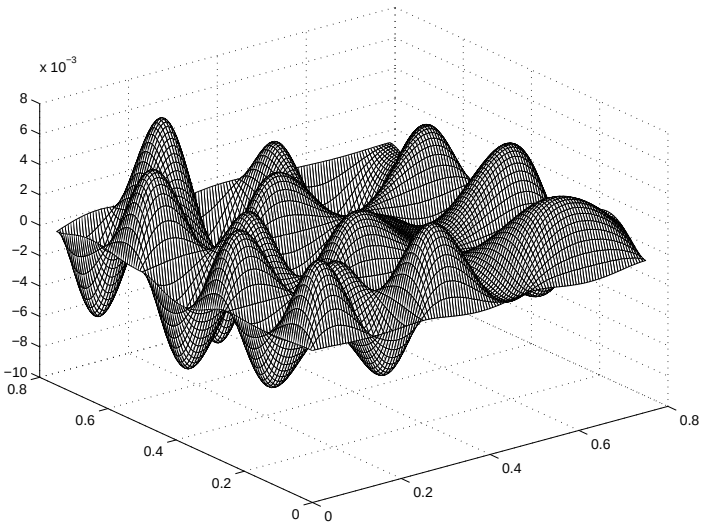


Figure 3: Graph of surface $f(x, y) - P(x, y)$.

the interpolating data. Property 3 shows that the interpolation is stable. Properties 4–6 show that the function values and the partial derivatives play not the same roles in the interpolator. That is, the magnitude of the integral weight coefficient values describe that of "function" of the interpolating knots and data. Also, the error estimation formula of this interpolator is obtained.

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A blending rational spline for value control of curves with minimal energy

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Abstract

A weighted blending rational spline is constructed using only the function values of a function. The interpolation can approximate the interpolated function very well by selecting suitable parameter and weight coefficient. And what is more, a new method of value control is employed to control the shape of curves, and the optimal solution with minimal strain energy for the value control equation is derived. Also when the weight coefficient is in $[0, 1]$, The error estimation formula of this interpolation function is obtained.

Key words: Rational spline; weighted blending interpolation; minimal strain energy; value control; error estimation

1 Introduction

Spline interpolation is a useful and powerful tool in curve and surface design, such as the outer shape of a ship, car or aeroplane. Many authors have studied several kinds of spline for curve and surface control [2-4,9,14,15,19]. In recent years, the rational spline and its application to shape preserving and shape control have received attention. For the univariate rational spline interpolation with parameters, in References [13,16,17], positivity preserving, monotonic preserving and convexity preserving of the interpolating curves have been discussed. In Reference [11], a method to construct conic blending arcs from constraints using rational parametric representation that combines the separate cases of blending parallel and non-parallel edges have developed. In References [6-8], the region control and convexity control of the interpolating curves have been studied. Since the parameters in the interpolation function are selective according to the control need, the constrained control of the shape becomes possible.

In fact, some practical designs in CAGD need only local control, but there are reports on only a few methods for local control. In Reference [10], a local control method of interval tension using weighted splines was given. In References [1,5], a local control method of the rational cubic spline with parameters based on function values was discussed, including value control, inflection-point and convex control. However, for the local shape control of the interpolating curve at a given point, only when certain condition is satisfied can the shape of the interpolating curve at the point be

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modified. This is to say that there are still some cases for which there are no parameters that can be used in order to control the shape of the curve. To overcome this problem, in this paper, a weighted blending rational interpolator is constructed using a rational cubic spline and a polynomial spline. Moreover, a method to control the value of the interpolation function at a prescribed parameter is also developed.

This paper is structured as follows. In Section 2, a C^1 continuous weighted blending interpolation function is constructed using only function values. Section 3 deals with a method to control the value of the interpolation function at a prescribed parameter. Moreover, the minimal strain energy spline interpolating an additional point is computed and numerical examples are presented to show the performance of the method. In Section 4, when the interpolated function $f(t)$ is in $C^2[a,b]$ and weight coefficient λ_i is in $[0,1]$, the error estimation formula of the interpolation is obtained.

2 Interpolation

Let $\{(t_i, f_i); i = 1, \cdots, n\}$ be a given set of data points, with $a = t_1 < \cdots < t_n = b$ the knot sequence and f_i the values of the interpolated function $f(t)$ at the knots t_i $(i = 1, 2, \cdots, n)$. Denote by $h_i = t_{i+1} - t_i, \theta = (t - t_i)/h_i$, and let α_i be positive parameter. Then a C^1-continuous, piecewise rational cubic spline with linear denominator is defined on an interpolating interval as follows:

$$P^*(t)\big|_{[t_i,t_{i+1}]} = \frac{p_i(t)}{q_i(t)}, \quad i = 1, 2, \cdots, n-1, \tag{1}$$

where

$$\begin{aligned} p_i(t) &= (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3 f_{i+1}, \\ q_i(t) &= (1-\theta)\alpha_i + \theta, \\ V_i &= (2\alpha_i + 1)f_i + \alpha_i h_i d_i, \\ W_i &= (\alpha_i + 2)f_{i+1} - h_i d_{i+1}, \end{aligned}$$

with $d_1 = \Delta_1, d_n = \Delta_{n-1}, d_i = (h_i\Delta_{i-1} + h_{i-1}\Delta_i)/(h_{i-1} + h_i), i = 2, 3, \cdots, n-1$, and $\Delta_i = (f_{i+1} - f_i)/h_i$. This rational cubic spline $P^*(t)$ satisfies

$$P^*(t_i) = f_i, \quad P^{*\prime}(t_i) = d_i, \quad i = 1, \ldots, n,$$

and it exists and is unique for the given data and the positive parameter α_i.

We consider another interpolation function: the cubic polynomial spline based on function values and given by

$$P_*(t)\big|_{[t_i,t_{i+1}]} = (1-\theta)^3 f_i + \theta(1-\theta)^2(3f_i + h_i d_i) + \theta^2(1-\theta)(3f_{i+1} - h_i d_{i+1}) + \theta^3 f_{i+1}, \tag{2}$$

where $d_i, i = 1, 2, \cdots, n$ are defined as in (1). Then $P_*(t)$ satisfies

$$P_*(t_i) = f_i, \quad P_*{}'(t_i) = d_i, \quad i = 1, \ldots, n,$$

A weighted blending rational interpolation will be constructed using the two kinds of interpolation described above. Let

$$P(t)|_{[t_i,t_{i+1}]} = \lambda_i P^*(t)|_{[t_i,t_{i+1}]} + (1-\lambda_i)P_*(t)|_{[t_i,t_{i+1}]}, \quad i = 1, 2, \cdots, n-1, \tag{3}$$

with the weight coefficient $\lambda_i \in R$. The interpolation function based on function values in $[a, b]$ defined by (3) is denoted by $P(t)$. It is obvious that $P(t)$ is a C^1-continuous, piecewise rational interpolation called *the weighted blending rational spline*, and satisfies

$$P(t_i) = f_i, \quad P'(t_i) = d_i, \quad i = 1, 2, \ldots, n.$$

Example 1. Let the interpolated function be $f(t) = \ln(1 + 8t)$, $t \in [0, 1.5]$ with interpolating knots at $t = ih, i = 0, 1, 2, 3, h = 0.5$. Denote the corresponding C^1-continuous interpolation functions defined by $(1) - (3)$ in $[0, 1.5]$, by $P^*(t), P_*(t)$ and $P(t)$, respectively. Consider the section of the interpolation functions corresponding to the interval $[0.5, 1.0]$ with $\alpha_i = 0.6, \lambda_i = 1.8$. Fig.1 shows the graphs of the curves $f(t), P^*(t), P_*(t)$ and $P(t)$ in the interval $[0.5, 1.0]$. It is easy to see that $P(t)$ gives a better approximation to $f(t)$ than the others in the interval $[0.5, 1.0]$.

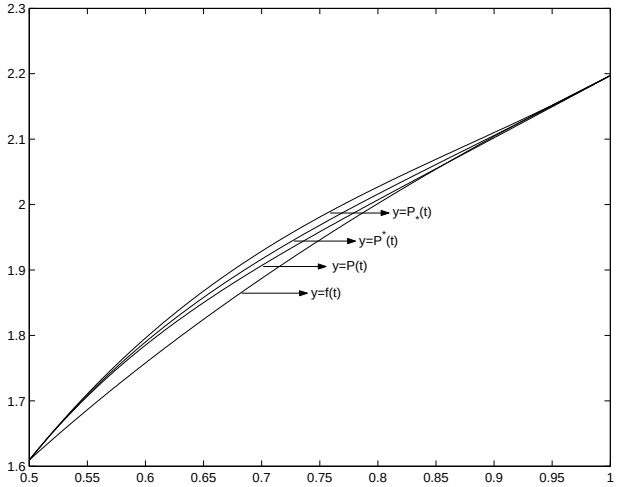


Figure 1: Graphs of the curves $f(t), P^*(t), P_*(t)$ and $P(t)$.

3 Value control with minimal strain energy

The shape of interpolating curves on an interpolating interval depends on the interpolating data. Generally speaking, when the interpolating data are given, the shape of the interpolating curve is fixed. However, note that there are the independent parameter α_i and the weight coefficient λ_i in (3), with them varying, the interpolation function can be changed for unchanged interpolating data. Thus, the shape of the interpolating curve can be modified by selecting the parameter and the weight coefficient appropriately. In what follows in this paper, we consider the case of equally spaced knots, namely, $h_i = h_j$ for all $i, j \in \{1, 2, \cdots, n-1\}$.

Consider the following constraint control issue. For a point t^* in the interpolating interval $[t_i, t_{i+1}]$, let $\theta^* = (t^* - t_i)/h_i$ be its local co-ordinate. If the practical design requires the function value of the interpolation at the point t^* to be equal to a real number M, and $M \in (\min\{f_i, f_{i+1}\}, \max\{f_i, f_{i+1}\})$, how it can be achieved. This kind of control is called the value control of the interpolation at a point. Denoting

$$M = uf_i + (1 - u)f_{i+1}, \tag{4}$$

with $u \in (0,1)$. Then the equation $P(t^*) = M$ can be written as

$$\lambda_i p_i(t^*) + (1-\lambda_i) q_i(t^*) P_*(t^*) - [u f_i + (1-u) f_{i+1}] q_i(t^*) = 0. \tag{5}$$

Eq. (5) is equivalent to

$$B_1 \alpha_i + B_2 \lambda_i (\alpha_i - 1) + B_3 = 0, \tag{6}$$

where

$$\begin{aligned} B_1 &= (1-\theta^*)B, \\ B_2 &= \theta^{*2}(1-\theta^*)^2[2(f_i - f_{i+1}) + (d_i + d_{i+1})h_i], \\ B_3 &= \theta^* B, \end{aligned}$$

with

$$B = [(1-\theta^*)^2(1+2\theta^*) - u]f_i + [\theta^{*2}(3-2\theta^*) - 1 + u]f_{i+1} + \theta^*(1-\theta^*)[(1-\theta^*)d_i - \theta^* d_{i+1}]h_i.$$

From Eq. (6), it is easy to see that there must exist a positive parameter α_i and a weight coefficient λ_i satisfying Eq. (5). Therefore we have the following theorem.

Theorem 1. *For the given interpolation data $\{(t_i, f_i); i = 1, \cdots, n\}$, where the knots are equally spaced, let $P(t)$ be the interpolation function over $[t_i, t_{i+1}]$ defined by (3), let t^* be a point in $[t_i, t_{i+1}]$, M is a real number, and $M \in (\min\{f_i, f_{i+1}\}, \max\{f_i, f_{i+1}\})$. Then there must exist a positive parameter α_i and a weight coefficient λ_i to satisfy $P(t^*) = M$.*

Th.1 shows that the value control problem can be always achieved. However, from Eq. (6), it is evident that the value control equation (5) has many solutions. For each solution, an interpolating curve can be obtained respectively. This mean that the uniqueness of the interpolating curve for the value control problem cannot be guaranteed. Hence, it is an important problem that how to choose the best solution in all solutions. In the following, we will present a solution to this problem.

First of all, we consider the strain energy of the interpolating curve. The strain energy of a (piecewise) C^2-continuous curve $f(t)$ defined on $[a, b]$ is defined as follows:

$$\int_a^b [f''(t)]^2 dt$$

where $f''(t)$ is the second derivative of $f(t)$. Note that the interpolation function $P(t)$ defined in (3) is local, without loss of generality, it is only necessary to consider the strain energy of the interpolating curve $P(t)$ in the subinterval $[t_i, t_{i+1}]$.

We denote the strain energy of the interpolating curve $P(t)$ in the subinterval $[t_i, t_{i+1}]$ by $E(\alpha_i, \lambda_i)$, then

$$E(\alpha_i, \lambda_i) = \int_{t_i}^{t_{i+1}} [P''(t)]^2 dt = \frac{A_1 \alpha_i + A_2 (1-\alpha_i)^2 \lambda_i^2}{\alpha_i}, \tag{7}$$

where

$$\begin{aligned} A_1 &= \frac{4}{h_i^3}[3(f_i - f_{i+1})^2 + 3h_i(d_i + d_{i+1})(f_i - f_{i+1}) + (d_i^2 + d_i d_{i+1} + d_{i+1}^2)h_i^2], \\ A_2 &= \frac{4}{5h_i^3}[2(f_i - f_{i+1}) + (d_i + d_{i+1})h_i]^2. \end{aligned}$$

It is evident that the strain energy $E(\alpha_i, \lambda_i)$ of the interpolating curve $P(t)$ defined by (3) in the subinterval $[t_i, t_{i+1}]$ depends only on the parameter α_i and the weight coefficient λ_i, as shown in (7). From (7), when $\lambda_i = 0$ or $\alpha_i = 1$, the strain energy of the interpolation curve $P(t)$ defined by (3) is minimal. In this case, the interpolating curve $P(t)$ defined by (3) is the polynomial curve defined in (2), and its strain energy is A_1.

Definition. *The solution (α_i, λ_i) of Eq. (5) is called* ***the optimal solution****, if (α_i, λ_i) is the minimal value point of the strain energy $E(\alpha_i, \lambda_i)$ defined in (7) under constraint condition (6); And if the optimal solution (α_i, λ_i) exists, then the value control is called* ***the value control with minimal strain energy.***

In order to obtain the optimal solution for the value control, we let

$$\begin{aligned} L(\alpha_i, \lambda_i, \mu) &= E(\alpha_i, \lambda_i) - \mu(B_1\alpha_i + B_2\lambda_i(\alpha_i - 1) + B_3) \\ &= \frac{A_1\alpha_i + A_2(1-\alpha_i)^2\lambda_i^2}{\alpha_i} - \mu(B_1\alpha_i + B_2\lambda_i(\alpha_i - 1) + B_3). \end{aligned} \tag{8}$$

From the following system of equations

$$\begin{cases} \frac{\partial L(\alpha_i,\lambda_i,\mu)}{\partial \alpha_i} = \frac{A_2\lambda_i^2(\alpha_i^2-1)}{\alpha_i^2} - B_1\mu - B_2\lambda_i\mu = 0, \\ \frac{\partial L(\alpha_i,\lambda_i,\mu)}{\partial \lambda_i} = \frac{2A_2\lambda_i(1-\alpha_i)^2}{\alpha_i} - B_2(\alpha_i - 1)\mu = 0, \\ \frac{\partial L(\alpha_i,\lambda_i,\mu)}{\partial \mu} = -B_1\alpha_i - B_2(\alpha_i - 1)\lambda_i - B_3 = 0, \end{cases}$$

it is easy to conclude that the stationary point $(\alpha_i, \lambda_i, \mu)$ of the function $L(\alpha_i, \lambda_i, \mu)$ defined by (8) is given by

$$\alpha_i = \frac{\theta^*}{1-\theta^*}, \quad \lambda_i = \frac{2B_1B_3}{B_2(B_1 - B_3)}, \quad \mu = -\frac{4A_2B_1}{B_2^2}, \tag{9}$$

where B_1, B_2, B_3 are defined in Eq. (6), A_2 is defined in (7). It is easy to prove that (α_i, λ_i) is just the optimal solution for the value control. This can be stated in the following theorem for value control with minimal strain energy.

Theorem 2. *For the given interpolation data $\{(t_i, f_i); i = 1, \cdots, n\}$, where the knots are equally spaced, let $P(t)$ be the interpolation function over $[t_i, t_{i+1}]$ defined by (3), let t^* be a point in $[t_i, t_{i+1}]$, M is a real number, and $M \in (\min\{f_i, f_{i+1}\}, \max\{f_i, f_{i+1}\})$. Then the optimal solution of Eq. (5) is given by*

$$\alpha_i = \frac{\theta^*}{1-\theta^*}, \quad \lambda_i = \frac{2B_1B_3}{B_2(B_1 - B_3)}.$$

Remark: In the optimal solution (α_i, λ_i), when $\theta^* = \frac{1}{2}$, $\alpha_i = 1$. In this case, the interpolation function $P(t)$ defined in (3) is the polynomial interpolation defined by (2), its shape is fixed, the function value can not be modified with minimal strain energy.

Example 2.Given the interpolating data shown in Table 1, let $P(t)$ be the interpolation function defined by (3) in the interpolating interval $[0.5, 4]$. Since the interpolation function is local it is possible to compute directly its expression in the interval $[1, 2]$ only.

Table 1. The interpolating data

t	0.5	1.0	2.0	2.4	3.0	4.0
$f(t)$	3.0	5.0	3.0	3.0	4.0	6.5

This example will show how the value control with minimal strain energy of the interpolating curves can be obtained by selecting a optimal solution under the condition that the interpolating data are not changed.

Taking $\alpha_i = 2$ and $\lambda_i = 0.4$ the interpolation function in the interval $[1, 2]$ is given by

$$P_1(t) = \frac{941 - 3403t + 3012t^2 - 1014t^3 + 114t^4}{35t - 105}. \tag{10}$$

It can be shown that $P_1(1.4) = \frac{20637}{4375} = 4.71703$ by (10). For the given interpolating data, if the design requires $P(1.4) = 4.6$, then $u = 0.8$ by (4). Thus $\alpha_i = \frac{2}{3}, \lambda_i = \frac{17}{57}$ by (9), that is to say that $(\frac{2}{3}, \frac{17}{57})$ is the optimal solution of the value control problem. We denote the interpolation function in $[1, 2]$ by $P_2(t)$.

$$P_2(t) = \frac{-385 + 939t - 202t^2 - 222t^3 + 80t^4}{21t + 21}.$$

Furthermore, if the design requires $P(1.4) = 4.8$, then $u = 0.9$ by (4). Thus $\alpha_i = \frac{2}{3}, \lambda_i = -\frac{47}{38}$ by (9), $(\frac{2}{3}, -\frac{47}{38})$ is the optimal solution of the value control problem. We denote the interpolation function in $[1, 2]$ by $P_3(t)$.

$$P_3(t) = \frac{105 - 387t + 691t^2 - 424t^3 + 85t^4}{7t + 7}.$$

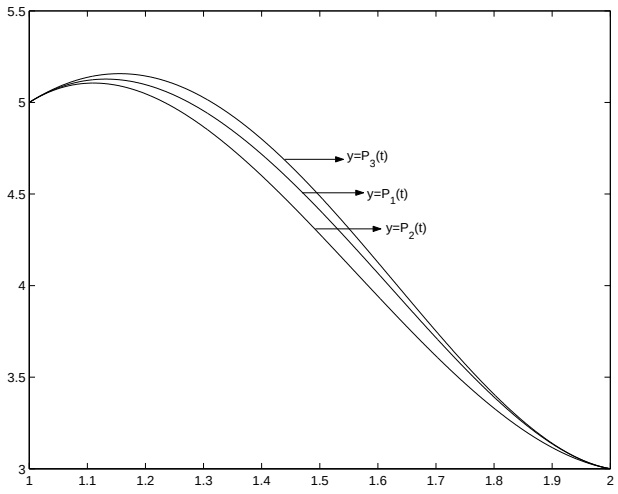


Figure 2: Graphs of the curves $P_1(t), P_2(t)$ and $P_3(t)$.

Fig. 2 shows the graphs of the curves $P_1(t), P_2(t)$ and $P_3(t)$, it can be examined that $P_2(1.4) = 4.6$, $P_3(1.4) = 4.8$.

4 Error estimation

To estimate the error estimation of the weighted blending rational interpolation function defined in (3), we assume that the weight coefficient λ_i is in $[0,1]$ and the knots are equally spaced, namely $h_i = h_j$ for all $i, j \in \{1, 2, \cdots, n-1\}$. Since the interpolation is local, it is necessary to consider just the subinterval $[t_i, t_{i+1}]$. Assuming that $f(t) \in C^2[a,b]$ and $P(t)$ is the rational cubic interpolation function of $f(t)$ in $[t_i, t_{i+1}]$, we discuss the following three cases.

Case 1. $i = 2, 3, \cdots, n-2$.

In this case, after some rearrangement, (3) can be rewritten as

$$P(t) = \omega_1(\theta, \alpha_i, \lambda_i) f_{i-1} + \omega_2(\theta, \alpha_i, \lambda_i) f_i + \omega_3(\theta, \alpha_i, \lambda_i) f_{i+1} + \omega_4(\theta, \alpha_i, \lambda_i) f_{i+2}, \tag{11}$$

where

$$\begin{aligned}
\omega_1(\theta, \alpha_i, \lambda_i) &= -\frac{\theta(1-\theta)^2[\alpha_i + \theta(1-\alpha_i)(1-\lambda_i)]}{2((1-\theta)\alpha_i + \theta)}, \\
\omega_2(\theta, \alpha_i, \lambda_i) &= \frac{(1-\theta)[\theta(2-\theta) + 2(1-\theta^2)\alpha_i + 3\theta^2(1-\theta)(1-\alpha_i)(1-\lambda_i)]}{2((1-\theta)\alpha_i + \theta)}, \\
\omega_3(\theta, \alpha_i, \lambda_i) &= \frac{\theta[2\theta(2-\theta) + (1-\theta^2)\alpha_i - 3\theta(1-2\theta+3\theta^2)(1-\alpha_i)(1-\lambda_i)]}{2((1-\theta)\alpha_i + \theta)}, \\
\omega_4(\theta, \alpha_i, \lambda_i) &= -\frac{\theta^2(1-\theta)[\theta + (1-\theta)((1-\lambda_i)\alpha_i + \lambda_i)]}{2((1-\theta)\alpha_i + \theta)}.
\end{aligned}$$

Using the Peano-Kernel Theorem [18] we obtain

$$R[f] = f(t) - P(t) = \int_{t_i}^{t_{i+2}} f''(\tau) R_t[(t-\tau)_+] \, d\tau, \tag{12}$$

where

$$R_t[(t-\tau)_+] = \begin{cases}
p(\tau) = (t-\tau) + \dfrac{\theta^2(1-\theta)[\theta + (1-\theta)((1-\lambda_i)\alpha_i + \lambda_i)](t_{i+2} - \tau)}{2((1-\theta)\alpha_i + \theta)} \\
\qquad - \dfrac{\theta[2\theta(2-\theta) + (1-\theta^2)\alpha_i - 3\theta(1-2\theta+3\theta^2)(1-\alpha_i)(1-\lambda_i)](t_{i+1} - \tau)}{2((1-\theta)\alpha_i + \theta)}, \\
\qquad\qquad\qquad\qquad t_i < \tau < t \ ; \\
q(\tau) = -\dfrac{\theta[2\theta(2-\theta) + (1-\theta^2)\alpha_i - 3\theta(1-2\theta+3\theta^2)(1-\alpha_i)(1-\lambda_i)](t_{i+1} - \tau)}{2((1-\theta)\alpha_i + \theta)} \\
\qquad + \dfrac{\theta^2(1-\theta)[\theta + (1-\theta)((1-\lambda_i)\alpha_i + \lambda_i)](t_{i+2} - \tau)}{2((1-\theta)\alpha_i + \theta)}, \qquad t < \tau < t_{i+1}; \\
r(\tau) = \dfrac{\theta^2(1-\theta)[\theta + (1-\theta)((1-\lambda_i)\alpha_i + \lambda_i)](t_{i+2} - \tau)}{2((1-\theta)\alpha_i + \theta)}, \qquad t_{i+1} < \tau < t_{i+2}.
\end{cases}$$

Now, we consider the properties of the kernel function $R_t[(t-\tau)_+]$ of the variable $\tau \in [t_i, t_{i+1}]$. It is easy to see that $r(\tau) \geq 0$ for all $t \in [t_{i+1}, t_{i+2}]$.

For $q(\tau)$, since

$$q(t) = \frac{\theta(1-\theta)[(1-\theta)(1+\theta+\theta(1-\lambda_i)(1-2\theta))\alpha_i + \theta(1+(1-\theta)\lambda_i + 2\theta(1-\theta)(1-\lambda_i))]h_i}{-2((1-\theta)\alpha_i + \theta)} \leq 0,$$

and

$$q(t_{i+1}) = \frac{\theta^2(1-\theta)[\theta + (1-\theta)((1-\lambda_i)\alpha_i + \lambda_i)]h_i}{2((1-\theta)\alpha_i + \theta)} \geq 0,$$

the root τ^* of $q(\tau)$ is

$$\tau^* = t_{i+1} - \frac{\theta(1-\theta)[\theta + (1-\theta)((1-\lambda_i)\alpha_i + \lambda_i)]h_i}{(1-\theta)(1+\theta(3-2\lambda_i) - 2\theta^2(1-\lambda_i))\alpha_i + \theta(1+3\theta-2\theta^2+2(1-\theta)^2\lambda_i)}.$$

Hence, $q(\tau) \leq 0$ for $t < \tau < \tau^*$ and $q(\tau) \geq 0$ for $\tau^* < \tau < t_{i+1}$.

Also, since

$$p(t_i) = \frac{\theta(1-\theta)^2[\alpha_i + \theta(1-\alpha_i)(1-\lambda_i)]h_i}{2((1-\theta)\alpha_i + \theta)} \geq 0, \quad p(t) = q(t) \leq 0,$$

the root τ_* of $p(\tau)$ is

$$\tau_* = t_i + \frac{\theta(1-\theta)[\alpha_i + \theta(1-\alpha_i)(1-\lambda_i)]h_i}{(2+\theta-2\theta^2(1-\lambda_i))(\alpha_i(1-\theta)+\theta) - 2\theta^2\lambda_i}.$$

Hence, $p(\tau) \geq 0$ for $t_i < \tau < \tau_*$ and $p(\tau) \leq 0$ for $\tau_* < \tau < t$.

Thus, we use (12) to arrive at

$$\begin{aligned}
\|R[f]\| &= \|f(t) - P(t)\| = |\int_{t_i}^{t_{i+2}} f''(\tau) R_t[(t-\tau)_+]\, d\tau| \\
&\leq \|f''(t)\|[\int_{t_i}^{\tau_*} p(\tau)d\tau - \int_{\tau_*}^{t} p(\tau)d\tau - \int_{t}^{\tau^*} q(\tau)d\tau + \int_{\tau^*}^{t_{i+1}} q(\tau)d\tau + \int_{t_{i+1}}^{t_{i+2}} r(\tau)d\tau] \\
&= h_i^2 \|f''(t)\| U_i(\theta, \alpha_i, \lambda_i), \qquad (13)
\end{aligned}$$

where

$$U_i(\theta, \alpha_i, \lambda_i) = \frac{w_i(\theta, \alpha_i, \lambda_i)}{v_i(\theta, \alpha_i, \lambda_i)},$$

with

$$\begin{aligned}
w_i(\theta, \alpha_i, \lambda_i) &= \theta^3[2 + 11\theta + 10\theta^2 - 27\theta^3 + 4\theta^4 + 4\theta^5 + (6 + 9\theta - 58\theta^2 + 47\theta^3 + 8\theta^4 - 12\theta^5)\lambda_i \\
&\quad +4(1-\theta)^3(2 - 2\theta - 3\theta^2)\lambda_i^2 - 4\theta(1-\theta)^4\lambda_i^3] + \alpha_i\theta^2(1-\theta)[12\theta^3(1-\theta)^3\lambda_i^3 \\
&\quad +4(1-\theta)^2(2 - 8\theta + 3\theta^2 + 9\theta^3)\lambda_i^2 + (12 + 12\theta - 131\theta^2 + 137\theta^3 + 12\theta^4 - 36\theta^5)\lambda_i \\
&\quad +6 + 33\theta + 3\theta^2 - 81\theta^3 + 12\theta^4 + 12\theta^5] + \alpha_i^2\theta(1-\theta)^2[4\theta(1-\theta)(4 - 10\theta + 9\theta^3)\lambda_i^2 \\
&\quad +(6 - 3\theta - 88\theta^2 + 133\theta^3 - 36\theta^5)\lambda_i + 6 + 33\theta + 30\theta^2 - 81\theta^3 + 12\theta^4 + 12\theta^5 \\
&\quad -12\theta^3(1-\theta)^2\lambda_i^3] + \alpha_i^3(1-\theta)^3[2 + 11\theta + 10\theta^2 - 27\theta^3 + 4\theta^4 + 5\theta^5 + 4\theta^4(1-\theta)\lambda_i^3 \\
&\quad +4\theta^2(2 - 4\theta - \theta^2 + 3\theta^3)\lambda_i^2 - \theta(6 + 15\theta - 43\theta^2 + 4\theta^3 + 12\theta^4)\lambda_i], \\
v_i(\theta, \alpha_i, \lambda_i) &= 4[(1-\theta)(1+\theta(3-2\lambda_i) - 2\theta^2(1-\lambda_i))\alpha_i + \theta(1+3\theta-2\theta^2+2(1-\theta)^2\lambda_i)] \\
&\quad [(1-\theta)\alpha_i + \theta][(2+\theta-2\theta^2(1-\lambda_i))(\alpha_i(1-\theta)+\theta) - 2\theta^2\lambda_i].
\end{aligned}$$

Case 2. $i = 1$. In this case, similar to Case 1, we obtain

$$\|R[f]\| = |\int_{t_1}^{t_3} f''(\tau)R_t[(t-\tau)_+]\ d\tau| \le h_i^2\|f''(t)\|U_1(\theta,\alpha_1,\lambda_1), \tag{14}$$

where

$$U_1(\theta,\alpha_1,\lambda_1) = \frac{w_1(\theta,\alpha_1,\lambda_1)}{v_1(\theta,\alpha_1,\lambda_1)},$$

with

$$\begin{aligned} w_1(\theta,\alpha_1,\lambda_1) &= \theta(1-\theta)[\theta^2(2+\theta-\theta^3+(1-\theta)^2(1+2\theta)\lambda_1+(1-\theta)^3\lambda_1^2) \\ &\quad +\alpha_1\theta(1-\theta)(4+2\theta-2\theta^2+(1-\theta-4\theta^2+4\theta^3)\lambda_1-2\theta(1-\theta)^2\lambda_1^2) \\ &\quad +\alpha_1^2(1-\theta)^2(2+\theta-\theta^3-\theta(1+\theta-2\theta^2)\lambda_1+\theta^2(1-\theta)\lambda_1^2)], \\ v_1(\theta,\alpha_1,\lambda_1) &= 2(\alpha_1(1-\theta)+\theta)[(2+\theta(1-\theta)(1-\lambda_1))(\alpha_1(1-\theta)+\theta)+\theta(1-\theta)\lambda_1]. \end{aligned}$$

Case 3. When $i = n-1$, we conclude

$$\|R[f]\| = |\int_{t_{n-2}}^{t_n} f''(\tau)R_t[(t-\tau)_+]\ d\tau| \le h_i^2\|f''(t)\|U_{n-1}(\theta,\alpha_{n-1},\lambda_{n-1}), \tag{15}$$

where

$$U_{n-1}(\theta,\alpha_{n-1},\lambda_{n-1}) = \frac{w_{n-1}(\theta,\alpha_{n-1},\lambda_{n-1})}{v_{n-1}(\theta,\alpha_{n-1},\lambda_{n-1})},$$

with

$$\begin{aligned} w_{n-1}(\theta,\alpha_{n-1},\lambda_{n-1}) &= \theta(1-\theta)[\theta^2(2+5\theta-4\theta^2+\theta^3+2\lambda_{n-1}(1-\theta)^3+\theta(1-\theta)^2\lambda_{n-1}^2) \\ &\quad +2\alpha_{n-1}\theta(1-\theta)(2+5\theta-4\theta^2+\theta^3+\lambda_{n-1}(1-\theta)^2(1-2\theta) \\ &\quad -\theta^2(1-\theta)\lambda_{n-1}^2)+\alpha_{n-1}^2(1-\theta)^2(2+5\theta-4\theta^2+\theta^3 \\ &\quad -2\theta(1-\theta)^2\lambda_{n-1}+\theta^3\lambda_{n-1}^2)], \\ v_{n-1}(\theta,\alpha_{n-1},\lambda_{n-1}) &= 4[2(\theta+\alpha_{n-1})-\alpha_{n-1}\theta(1-\theta)+\theta^2(1-\theta)(1-\alpha_{n-1})(1-\lambda_{n-1})]\cdot \\ &\quad (\alpha_{n-1}(1-\theta)+\theta). \end{aligned}$$

For the fixed parameter α_i and weight coefficient λ_i, let

$$c_i = \max_{0\le\theta\le1} U_i(\theta,\alpha_i,\lambda_i), \quad i = 1,2,\cdots,n-1. \tag{16}$$

This leads to the following theorem.

Theorem 3. *Let $f(t) \in C^2[a,b]$, $P(t)$ is the weighted blending rational interpolation function of $f(t)$ in $[t_i,t_{i+1}]$ defined by (3). When the knots are equally spaced, for the parameter α_i and the weight coefficient $\lambda_i \in [0,1]$,*

$$\|R[f]\| \le h_i^2\|f''(t)\|c_i,$$

where c_i is defined by (16).

It is obvious that the optimal error constant c_i does not depend on the subinterval $[t_i,t_{i+1}]$, but just on the parameter α_i and the weight coefficient λ_i, as shown in (16). Some values c_i for different α_i and λ_i are given in Table 2.

Table 2. Values c_i for various values α_i and λ_i.

α_1	λ_1	c_1	α_i	λ_i	c_i	α_{n-1}	λ_{n-1}	c_{n-1}
1.0	0.88	0.13203	1.0	0.95	0.11128	1.0	0.92	0.10205
1.1	0.88	0.13149	1.1	0.95	0.10996	1.1	0.92	0.10128
1.2	0.88	0.13102	1.2	0.95	0.10877	1.2	0.92	0.10059
1.5	0.88	0.12988	1.5	0.95	0.10585	1.5	0.92	0.09890
2.0	0.88	0.12864	2.0	0.95	0.10244	2.0	0.92	0.09695
3.0	0.88	0.12732	3.0	0.95	0.09844	3.0	0.92	0.09469
5.0	0.88	0.12628	5.0	0.95	0.09483	5.0	0.92	0.09264
10.0	0.88	0.12559	10.0	0.95	0.09194	10.0	0.92	0.09097
20.0	0.88	0.12562	20.0	0.95	0.09051	20.0	0.92	0.09011
50.0	0.88	0.12518	50.0	0.95	0.08968	50.0	0.92	0.08958
100.0	0.88	0.12514	100.0	0.95	0.08942	100.0	0.92	0.08940
1000.0	0.88	0.12511	1000.0	0.95	0.08914	1000.0	0.92	0.08924

5 Conclusion

In this paper, a C^1-continuous, piecewise weighted blending interpolation based only on function values is constructed. This interpolation function has a simple and explicit mathematical representation, and approximates the interpolated function very well by selecting suitable parameter and weight coefficient. Also, a value control method of interpolating curves is developed. When the given data are not changed, in view of this method, the uniqueness of the solution for the value control equation not only can be guaranteed, but the value control with minimal strain energy can be achieved only by selecting suitable parameter α_i and weight coefficient λ_i.

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The Construction and Approximation of a Class of Neural Networks Operators with Ramp Functions*

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Abstract

Single hidden layer feedforward neural networks with ramp sigmoidal activation functions are constructed to approximate two-variable functions defined on compact interval. With the help of modulus of continuity and analysis techniques, two Jackson-type theorems are given respectively when the objective functions being continuous and having continuous partial derivatives of order N.

Keywords neural network operators; modulus of continuity; order of approximation

AMS(2000) Subject Classification 41A20; 41A25

1 Introduction

Essentially, artificial neural networks are nonlinear parametric expressions representing multivariate numerical functions. In connection with such paradigms there arise mainly three problems: the density, the complexity and the algorithm. The density deals with the question: which functions can be approximated in a suitable sense. This problem has been satisfactorily solved (see [8, 9, 12, 13, 14, 15, 16]).

The complexity which deals with the relationship between the size of an expression (i.e., the number of neurons) and its approximation capacity is a key issue. Usually, to solve the complexity problem some neural network operators are constructed and used (for example, see [1, 2, 3, 4, 5, 7, 10]).

A function $s : R \to R$ is called sigmoidal if

$$\lim_{x\to+\infty} s(x) = 1, \quad \lim_{x\to-\infty} s(x) = 0.$$

Cheang Gerald [6] introduced the following so called ramp sigmoidal function

$$\phi_v(x) = \begin{cases} 0, & x \in (-\infty, 0), \\ vx, & x \in [0, \frac{1}{v}], \\ 1, & x \in (\frac{1}{v}, +\infty) \end{cases}$$

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and investigated the approximation by the single hidden layer feedforward neural networks with the ramp sigmoidal activation function. In this paper we will study the approximation of 2-dimensional continuous functions defined on $[-1,1]\times[-1,1]$ by constructing single hidden layer feedforward neural network operators with ramp sigmoidal activation function.

Now we introduce some notations. Let $\tilde{\phi}_v(x) := \phi_v(x+\frac{1}{v}) - \phi_v(x)$, and $\phi(x) := \tilde{\phi}_v(\frac{x}{v})$. Obviously $\phi(x)$ has support set $[-1,1]$, and is nondecreasing on $[-1,0]$ and nonincreasing on $[0,1]$. Furthermore, we assume that $\int_{-1}^{1}\phi(x)dx = 1$.

Let $b(x_1,x_2) := \phi(x_1)\phi(x_2)$, then $b(x_1,x_2)$ has support $[-1,1]\times[-1,1]$, and satisfies

$$\int_{-1}^{1}\int_{-1}^{1} b(x_1,x_2)dx_1dx_2 = 1,$$

which is called bell-shaped function (see[3]).

For the continuous function $f(x_1,x_2)$ defined on $[-1,1]\times[-1,1] := [-1,1]^2$, we modify the multivariate neural network operators introduced by [3, 5] as follows:

$$F_n(f)(x_1,x_2) = \sum_{\lceil -n-n^\alpha\rceil}^{[n+n^\alpha]}\sum_{\lceil -n-n^\alpha\rceil}^{[n+n^\alpha]} \frac{f(\frac{k_1}{n+n^\alpha},\frac{k_2}{n+n^\alpha})}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),(n^{1-\alpha}(x_2-\frac{k_2}{n})\right),$$

where $0<\alpha<1$, $[a]$ denotes the integral part of a number a, and $\lceil a\rceil$ the ceiling. Clearly, F_n are single hidden layer feedforward neural network operators. We will give some estimations based on the procedures of [3].

2 Some Lemmas

To prove our results, we shall give some lemmas.

Lemma 1

(1) If $t_i \le t_i' \le 0 (i=1,2)$, then $b(t_1,t_2) \le b(t_1',t_2')$.

(2) If $0 \le t_i \le t_i' (i=1,2)$, then $b(t_1,t_2) \ge b(t_1',t_2')$.

Proof Noting that $b(t_1,t_2) \le b(t_1',t_2) \le b(t_1',t_2')$, the proof of (i) can be finished easily. The proof of (2) is similar.

Lemma 2

(i) For $k_i \in Z : \lceil nx_i\rceil + 1 \le k_i \le [nx_i+n^\alpha], i=1,2$, we have

$$\begin{aligned}
&\int_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]+1}\int_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]+1} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2\\
&\le \sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)\\
&\le \int_{\lceil nx_1\rceil}^{[nx_1+n^\alpha]}\int_{\lceil nx_2\rceil}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2.
\end{aligned}$$

(ii) For $k_i \in Z : \lceil nx_i - n^\alpha\rceil \le k_i \le [nx_i]-1, i=1,2$, we have

$$\begin{aligned}
&\int_{\lceil nx_1-n^\alpha\rceil-1}^{[nx_1]-1}\int_{\lceil nx_2-n^\alpha\rceil-1}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2\\
&\le \sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]-1}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)
\end{aligned}$$

$$\leq \int_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1]} \int_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2]} b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) dt_1 dt_2.$$

(iii) For $\lceil nx_1 \rceil + 1 \leq k_1 \leq [nx_1 + n^\alpha], \lceil nx_2 - n^\alpha \rceil \leq k_2 \leq [nx_2] - 1$, we have

$$\begin{aligned}
&\int_{\lceil nx_1 \rceil + 1}^{[nx_1 + n^\alpha]+1} \int_{\lceil nx_2 - n^\alpha \rceil - 1}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) dt_1 dt_2 \\
&\leq \sum_{\lceil nx_1 \rceil + 1}^{[nx_1 + n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) \\
&\leq \int_{\lceil nx_1 \rceil}^{[nx_1 + n^\alpha]} \int_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2]} b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) dt_1 dt_2.
\end{aligned}$$

(iv) For $\lceil nx_1 - n^\alpha \rceil \leq k_1 \leq \lceil nx_1 \rceil - 1, \lceil nx_2 \rceil + 1 \leq k_2 \leq [nx_2 + n^\alpha]$, we have

$$\begin{aligned}
&\int_{\lceil nx_1 - n^\alpha \rceil - 1}^{[nx_1]-1} \int_{\lceil nx_2 \rceil + 1}^{[nx_1 + n^\alpha]+1} b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) dt_1 dt_2 \\
&\leq \sum_{\lceil nx - n^\alpha \rceil}^{[nx_1]-1} \sum_{\lceil nx_2 \rceil + 1}^{[nx_2 + n^\alpha]} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) \\
&\leq \int_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1]} \int_{\lceil nx_2 \rceil}^{[nx_2 + n^\alpha]} b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) dt_1 dt_2.
\end{aligned}$$

Proof (i) Let t_i satisfy $nx_i \leq k_i - 1 \leq t_i \leq k_i$. Then

$$x_i - \frac{k_i}{n} \leq x_i - \frac{t_i}{n} \leq x_i - \frac{k_i - 1}{n} \leq 0, \quad i = 1, 2.$$

By Lemma 1 we know

$$\begin{aligned}
b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), (n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) &\leq b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) \\
&\leq b\left(n^{1-\alpha}(x_1 - \frac{k_1 - 1}{n}), n^{1-\alpha}(x_2 - \frac{k_2 - 1}{n})\right).
\end{aligned}$$

Hence

$$\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) \leq \int_{k_1 - 1}^{k_1} \int_{k_2 - 1}^{k_2} b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) dt_1 dt_2.$$

Let t_i satisfy $nx_i < k_i \leq t_i \leq k_i + 1$. Then

$$x_i - \frac{k_i + 1}{n} \leq x_i - \frac{t_i}{n} \leq x_i - \frac{k_i}{n} \leq 0, \quad i = 1, 2.$$

Again, by Lemma 1 we obtain

$$\begin{aligned}
b\left(n^{1-\alpha}(x_1 - \frac{k_1 + 1}{n}), n^{1-\alpha}(x_2 - \frac{k_2 + 1}{n})\right) &\leq b\left(n^{1-\alpha}(x_1 - \frac{t_1}{n}), n^{1-\alpha}(x_2 - \frac{t_2}{n})\right) \\
&\leq b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right).
\end{aligned}$$

Consequently

$$\int_{k_1}^{k_1+1}\int_{k_2}^{k_2+1} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2$$
$$\leq b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)$$

Therefore,

$$\int_{k_1}^{k_1+1}\int_{k_2}^{k_2+1} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2$$
$$\leq b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)$$
$$\leq \int_{k_1-1}^{k_1}\int_{k_2-1}^{k_2} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2.$$

We thus get

$$\int_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]+1}\int_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]+1} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2$$
$$\leq \sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)$$
$$\leq \int_{\lceil nx_1\rceil}^{[nx_1+n^\alpha]}\int_{\lceil nx_2\rceil}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2.$$

(ii) If t_i satisfy $k_i-1\leq t_i\leq k_i\leq nx_i$, then

$$x_i-\frac{k_i-1}{n}\geq x_i-\frac{t_i}{n}\geq x_i-\frac{k_i}{n}\geq 0.$$

By Lemma 1, we obtain that

$$\begin{aligned} b\left(n^{1-\alpha}(x_1-\frac{k_1-1}{n}),n^{1-\alpha}(x_2-\frac{k_2-1}{n})\right) &\leq b\left(n^{1-\alpha}(x-\frac{t_1}{n}),n^{1-\alpha}(x-\frac{t_2}{n})\right)\\ &\leq b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right).\end{aligned}$$

Consequently

$$\int_{k_1-1}^{k_1}\int_{k_2-1}^{k_2} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2$$
$$\leq b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right).$$

Now let t_i satisfy $k_i\leq t_i\leq k_i+1\leq nx_i$, then

$$x_i-\frac{k_i}{n}\geq x_i-\frac{t_i}{n}\geq x_i-\frac{k_i+1}{n}\geq 0.$$

So from Lemma 1, we obtain

$$\begin{aligned} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right) &\leq b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)\\ &\leq b\left(n^{1-\alpha}(x_1-\frac{k_1+1}{n}),n^{1-\alpha}(x_2-\frac{k_2+1}{n})\right),\end{aligned}$$

and

$$b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)\leq\int_{k_1}^{k_1+1}\int_{k_2}^{k_2+1}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2.$$

We have established that

$$\begin{aligned}&\int_{k_1-1}^{k_1}\int_{k_2-1}^{k_2}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_1-\frac{t_2}{n})\right)dt_1dt_2\\&\leq b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)\\&\leq\int_{k_1}^{k_1+1}\int_{k_2}^{k_2+1}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2\end{aligned}$$

which shows that

$$\begin{aligned}&\int_{\lceil nx_1-n^\alpha\rceil-1}^{[nx_1]-1}\int_{\lceil nx_2-n^\alpha\rceil-1}^{[nx_2]-1}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2\\&\leq\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]-1}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]-1}b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)\\&\leq\int_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]}\int_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2.\end{aligned}$$

(iii) First, it is easy to see that when $t_i\leq t_i'\leq 0,\quad b(t_1,t_2)\leq b(t_1',t_2)$ and $b(t_1,t_2)\leq b(t_1,t_2')$, and when $t_i'\geq t_i\geq 0, b(t_1,t_2)\geq b(t_1,t_2')$ and $b(t_1,t_2)\leq b(t_1',t_2)$. Therefore,

$$\begin{aligned}&\int_{k_1}^{k_1+1}\int_{k_2-1}^{k_2}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2\\&\leq\int_{k_1}^{k_1+1}\left[\int_{k_2-1}^{k_2}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)dt_2\right]dt_1\\&=\int_{k_1}^{k_1+1}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)dt_1\\&\leq\int_{k_1}^{k_1+1}b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)dt_1\\&=b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)\\&\leq\int_{k_2}^{k_2+1}b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_2\\&\leq\int_{k_1-1}^{k_1}\int_{k_2}^{k_2+1}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]+1}\int_{\lceil nx_2-n^\alpha\rceil-1}^{[nx_2]-1}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2\\&\leq\sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]-1}b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)\\&\leq\int_{\lceil nx_1\rceil}^{[nx_1+n^\alpha]}\int_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]}b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2.\end{aligned}$$

The proof of (iv) is similar to (iii).

Lemma 3 We have

$$\text{(i)}\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)-\int_{-1}^{0}\int_{-1}^{0}b(x_1,x_2)dx_1dx_2\right|\le\frac{4}{n^\alpha}.$$

$$\text{(ii)}\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]-1}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)-\int_{0}^{1}\int_{0}^{1}b(x_1,x_2)dx_1dx_2\right|\le\frac{4}{n^\alpha}.$$

$$\text{(iii)}\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)-\int_{-1}^{0}\int_{0}^{1}b(x_1,x_2)dx_1dx_2\right|\le\frac{4}{n^\alpha}.$$

$$\text{(iv)}\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]-1}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)-\int_{0}^{1}\int_{-1}^{0}b(x_1,x_2)dx_1dx_2\right|\le\frac{4}{n^\alpha}.$$

Proof (i) Since

$$\begin{aligned}
&\int_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]+1}\int_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]+1} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2\\
&\le\sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)\\
&\le\int_{\lceil nx_1\rceil}^{[nx_1+n^\alpha]}\int_{\lceil nx_2\rceil}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{t_1}{n}),n^{1-\alpha}(x_2-\frac{t_2}{n})\right)dt_1dt_2
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{\frac{nx_1-[nx_1+n^\alpha]-1}{n^\alpha}}^{\frac{nx_1-\lceil nx_1\rceil-1}{n^\alpha}}\int_{\frac{nx_2-[nx_2+n^\alpha]-1}{n^\alpha}}^{\frac{nx_2-\lceil nx_2\rceil-1}{n^\alpha}} b(x_1,x_2)dx_1dx_2-\int_{-1}^{0}\int_{-1}^{0}b(x_1,x_2)dx_1dx_2\\
&\le\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)dt_1dt_2-\int_{-1}^{0}\int_{-1}^{0}b(x_1,x_2)dx_1dx_2\\
&\int_{\frac{nx_1-[nx_1+n^\alpha]}{n^\alpha}}^{\frac{nx_1-\lceil nx_1\rceil}{n^\alpha}}\int_{\frac{nx_2-[nx_2+n^\alpha]}{n^\alpha}}^{\frac{nx_2-\lceil nx_2\rceil}{n^\alpha}} b(x_1,x_2)dx_1dx_2-\int_{-1}^{0}\int_{-1}^{0}b(x_1,x_2)dx_1dx_2.
\end{aligned}$$

Writing the left side of above inequality as $\triangle_1$, and letting

$$-1<nx-\lceil nx\rceil\le 0,\quad -n^\alpha\le nx-[nx+n^\alpha]<-n^\alpha+1,$$

one has

$$\begin{aligned}
|\triangle_1| &= \left|\int_{\frac{nx_1-[nx_1+n^\alpha]-1}{n^\alpha}}^{\frac{nx_1-\lceil nx_1\rceil-1}{n^\alpha}}\int_{\frac{nx_2-[nx_2+n^\alpha]-1}{n^\alpha}}^{\frac{nx_2-\lceil nx_2\rceil-1}{n^\alpha}} b(x_1,x_2)dx_1dx_2-\int_{-1}^{0}\int_{-1}^{0}b(x_1,x_2)dx_1dx_2\right|\\
&\le \left|\int_{\frac{nx_1-\lceil nx_1\rceil-1}{n^\alpha}}^{0}\int_{-1}^{0}b(x_1,x_2)dx_1dx_2+\int_{-1}^{0}\int_{\frac{nx_2-\lceil nx_2\rceil-1}{n^\alpha}}^{0}b(x_1,x_2)dx_1dx_2\right|\le\frac{4}{n^\alpha}.
\end{aligned}$$

We write the right side as $\triangle_2$ and get

$$\begin{aligned}|\triangle_2| &= \left|\int_{\frac{nx_1-[nx_1+n^\alpha]}{n^\alpha}}^{\frac{nx_1-\lceil nx_1\rceil}{n^\alpha}}\int_{\frac{nx_2-[nx_2+n^\alpha]}{n^\alpha}}^{\frac{nx_2-\lceil nx_2\rceil}{n^\alpha}} b(x_1,x_2)dx_1dx_2 - \int_{-1}^{0}\int_{-1}^{0} b(x_1,x_2)dx_1dx_2\right| \\ &\le \left|\int_{-1}^{0}\int_{-1}^{\frac{nx_2-[nx_2+n^\alpha]}{n^\alpha}} b(x_1,x_2)dx_1dx_2 + \int_{-1}^{0}\int_{\frac{nx_2-\lceil nx_2\rceil}{n^\alpha}}^{0} b(x_1,x_2)dx_1dx_2\right| \\ &+ \left|\int_{-1}^{\frac{nx_1-[nx_1+n^\alpha]}{n^\alpha}}\int_{-1}^{0} b(x_1,x_2)dx_1dx_2 + \int_{\frac{nx_1-\lceil nx_1\rceil}{n^\alpha}}^{0}\int_{-1}^{0} b(x_1,x_2)dx_1dx_2\right| \le \frac{4}{n^\alpha}.\end{aligned}$$

Therefore,

$$\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)dt_1dt_2 - \int_{-1}^{0}\int_{-1}^{0}\right| \le \frac{4}{n^\alpha}.$$

Similar calculations to (i), we use Lemma 2 and obtain that

$$\begin{aligned}&\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]-1}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right) - \int_0^1\int_0^1 b(x_1,x_2)dx_1dx_2\right| \\ &\le \max\{|\triangle_3|,|\triangle_4|\} \le \frac{4}{n^\alpha},\end{aligned}$$

where

$$\begin{aligned}|\triangle_3| &= \left|\int_{\frac{nx_1-[nx_1]}{n^\alpha}}^{\frac{nx_1-\lceil nx_1-n^\alpha\rceil}{n^\alpha}}\int_{\frac{nx_2-[nx_2]}{n^\alpha}}^{\frac{nx_2-\lceil nx_2-n^\alpha\rceil}{n^\alpha}} b(x_1,x_2)dx_1dx_2 - \int_0^1\int_0^1 b(x_1,x_2)dx_1dx_2\right|, \\ |\triangle_4| &= \left|\int_{\frac{nx_1-[nx_1]+1}{n^\alpha}}^{\frac{nx_1-\lceil nx_1-n^\alpha\rceil+1}{n^\alpha}}\int_{\frac{nx_2-[nx_2]+1}{n^\alpha}}^{\frac{nx_2-\lceil nx_2-n^\alpha\rceil+1}{n^\alpha}} b(x_1,x_2)dx_1dx_2 - \int_0^1\int_0^1 b(x_1,x_2)dx_1dx_2\right|.\end{aligned}$$

This finishes the proof of (ii).

Next we prove (iii) and (iV). It is clear that

$$\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1\rceil+1}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2]-1} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right) - \int_{-1}^{0}\int_0^1 b(x_1,x_2)dx_1dx_2\right| \le \frac{4}{n^\alpha}$$

and

$$\left|\frac{1}{n^{2\alpha}}\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]-1}\sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right) - \int_0^1\int_{-1}^{0} b(x_1,x_2)dx_1dx_2\right| \le \frac{4}{n^\alpha}.$$

The proof of Lemma 3 is completed.

3 The Main Results

We shall give following Jackson type theorems.

Theorem 1 Let $f \in C([-1,1]^2), 0<\alpha<1$. Then

$$|F_n(f)(x_1,x_2) - f(x_1,x_2)| \le \left(9\omega_1(f,\frac{2}{n^{1-\alpha}}) + \frac{52}{n^\alpha}\|f\|_\infty\right),$$

for any $(x_1, x_2) \in [0,1]^2$, where $\omega_1(f,h)$ is modulus of continuity of function f (see[17]), and

$$\|f\|_\infty = \sup_{(x_1,x_2)\in[-1,1]^2} |f(x_1,x_2)|.$$

Proof Since $|n^{1-\alpha}(x_i - \frac{k_i}{n})| \le 1 (i=1,2)$ iff $nx_i - n^\alpha \le k_i \le nx_i + n^\alpha (i=1,2)$, when $k_i > [nx_i + n^\alpha]$ or $k_i < \lceil nx_i - n^\alpha \rceil$, we have $b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) = 0$. Therefore, a simple calculation implies

$$\begin{aligned}
&|F_n(f)(x_1,x_2) - f(x_1,x_2)| \\
&= \left| \sum_{\lceil -n-n^\alpha \rceil}^{[n+n^\alpha]} \sum_{\lceil -n-n^\alpha \rceil}^{[n+n^\alpha]} \frac{f(\frac{k_1}{n+n^\alpha}, \frac{k_2}{n+n^\alpha})}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), (n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - f(x_1,x_2) \right| \\
&= \left| \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{f(\frac{k_1}{n+n^\alpha}, \frac{k_2}{n+n^\alpha})}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - f(x_1,x_2) \right| \\
&\le \left| \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{f(\frac{k_1}{n+n^\alpha}, \frac{k_2}{n+n^\alpha}) - f(x_1,x_2)}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) \right| \\
&+ |f(x_1,x_2)| \left| \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - 1 \right| \\
&\le \omega_1(f, \frac{2}{n^{1-\alpha}}) \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) \\
&+ \|f\|_\infty \left| \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - 1 \right| \\
&\le 9\omega_1(f, \frac{2}{n^{1-\alpha}}) + \|f\|_\infty E_n,
\end{aligned}$$

where

$$E_n = \left| \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - 1 \right|.$$

Now we estimate E_n.

$$\begin{aligned}
E_n &= \left| \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - \int_{-1}^{1}\int_{-1}^{1} b(x_1,x_2)dx_1dx_2 \right| \\
&\le \left| \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1]-1} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2]-1} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - \int_{0}^{1}\int_{0}^{1} b(x_1,x_2)dx_1dx_2 \right| \\
&+ \left| \sum_{\lceil nx_1 \rceil + 1}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 \rceil + 1}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - \int_{-1}^{0}\int_{-1}^{0} b(x_1,x_2)dx_1dx_2 \right| \\
&+ \left| \sum_{\lceil nx_1 \rceil + 1}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2]-1} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1 - \frac{k_1}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n})\right) - \int_{-1}^{0}\int_{0}^{1} b(x_1,x_2)dx_1dx_2 \right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left| \sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1]-1} \sum_{\lceil nx_2\rceil+1}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}), n^{1-\alpha}(x_2-\frac{k_2}{n})\right) - \int_0^1\int_{-1}^0 b(x_1,x_2)dx_1dx_2 \right| \\
&+ \sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1-\frac{[nx_1]}{n}), n^{1-\alpha}(x_2-\frac{k_2}{n})\right) \\
&+ \sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1-\frac{\lceil nx_1\rceil}{n}), n^{1-\alpha}(x_2-\frac{k_2}{n})\right) \\
&+ \sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}), n^{1-\alpha}(x_2-\frac{[nx_2]}{n})\right) \\
&+ \sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_1+n^\alpha]} \frac{1}{n^{2\alpha}} b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}), n^{1-\alpha}(x_2-\frac{\lceil nx_2\rceil}{n})\right) \\
&:= I_1+I_2+\cdots+I_8.
\end{aligned}
$$

By Lemma 3, we have $I_1+I_2+I_3+I_4 \le \frac{16}{n^\alpha}$. On the other hand, we have $I_5+I_6+I_7+I_8 \le \frac{36}{n^\alpha}$. Thus, we obtain

$$|F_n(f)(x_1,x_2)-f(x_1,x_2)| \le \left(9\omega_1(f,\frac{2}{n^{1-\alpha}})+\frac{52}{n^\alpha}\|f\|_\infty\right).$$

This finishes the proof of Theorem 1.

Theorem 2 Let $f \in C^N([-1,1]^2)$ (the set of function f having continuous partial derivatives of order N on $[-1,1]^2$), $0<\alpha<1$. Then

$$|F_n(f)(x_1,x_2)-f(x_1,x_2)| \le \left(\frac{52}{n^\alpha}\|f\|_\infty+\frac{36M}{n^{1-\alpha}}+\frac{9\cdot 4^N}{N!n^{N(1-\alpha)}}\max_{|\vec\beta|=N}\omega_1(f_{\vec\beta},\frac{2}{n^{1-\alpha}})\right).$$

for any $(x_1,x_2)\in[0,1]^2$, where

$$f_{\vec\beta}=\frac{\partial^{\vec\beta}f}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}},\quad \vec\beta=(\beta_1,\beta_2),\quad |\vec\beta|=\beta_1+\beta_2.$$

Proof Put

$$g_{\frac{\vec k}{n}}(t)=f\left(x_1+t(\frac{k_1}{n+n^\alpha}-x_1),x_2+t(\frac{k_2}{n+n^\alpha}-x_2)\right),\quad 0\le t\le 1,$$

then

$$g^{(j)}_{\frac{\vec k}{n}}(t)=\left\{(\sum_{i=1}^2(\frac{k_i}{n+n^\alpha}-x_i)\frac{\partial}{\partial x_i})^j f\right\}\left(x_1+t(\frac{k_1}{n+n^\alpha}-x_1),x_2+t(\frac{k_2}{n+n^\alpha}-x_2)\right),$$

and $g_{\frac{\vec k}{n}}(0)=f(x_1,x_2)$. By Taylor's formula we obtain

$$
\begin{aligned}
f(\frac{k_1}{n+n^\alpha},\frac{k_2}{n+n^\alpha}) &= g_{\frac{\vec k}{n}}(1) \\
&= \sum_{j=0}^N \frac{g^{(j)}_{\frac{\vec k}{n}}(0)}{j!}+\int_0^1\int_0^{t_1}\cdots\int_0^{t_{N-1}}(g^{(N)}_{\frac{\vec k}{n}}(t_N)-g^{(N)}_{\frac{\vec k}{n}}(0))dt_N\cdots dt_1 \\
&= \sum_{j=0}^N \frac{g^{(j)}_{\frac{\vec k}{n}}(0)}{j!}+R_N(\frac{\vec k}{n},0).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&F_n(f)(x_1,x_2)-f(x_1,x_2)\\
&=\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]}\frac{f(\frac{k_1}{n+n^\alpha},\frac{k_2}{n+n^\alpha})}{n^{2\alpha}}b\left(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n})\right)-f(x_1,x_2)\\
&=-f(x_1,x_2)+\sum_{j=0}^{N}\frac{1}{j!}\left(\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]}g_{\frac{\vec{k}}{n}}^{(j)}(0)\frac{b(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n}))}{n^{2\alpha}}\right)+R^*,
\end{aligned}$$

where

$$R^*=\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]}\frac{b(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n}))}{n^{2\alpha}}R_N(\frac{\vec{k}}{n},0).$$

Thus

$$\begin{aligned}
&F_n(f)(x_1,x_2)-f(x_1,x_2)\\
&=f(x_1,x_2)\left(\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]}\frac{1}{n^{2\alpha}}b(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n}))-1\right)\\
&+\sum_{j=1}^{N}\frac{1}{j!}\left(\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]}g_{\frac{\vec{k}}{n}}^{(j)}(0)\frac{b(n^{1-\alpha}(x_1-\frac{k_1}{n}),n^{1-\alpha}(x_2-\frac{k_2}{n}))}{n^{2\alpha}}\right)+R^*,\\
&=I_9+I_{10}+R^*.
\end{aligned}$$

In view of Theorem 1, we have $|I_9|\le\frac{52}{n^\alpha}\|f\|_\infty$.

Next we estimate I_{10}.

$$\begin{aligned}
|I_{10}| &\le \frac{1}{n^{2\alpha}}\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]}\left(\sum_{j=1}^{N}\frac{|g_{\frac{\vec{k}}{n}}^{(j)}(0)|}{j!}\right)\\
&\le \frac{1}{n^{2\alpha}}\sum_{\lceil nx_1-n^\alpha\rceil}^{[nx_1+n^\alpha]}\sum_{\lceil nx_2-n^\alpha\rceil}^{[nx_2+n^\alpha]}\left(\sum_{j=1}^{N}\frac{2^j}{j!n^{(1-\alpha)j}}(|\frac{\partial f}{\partial x_1}|+|\frac{\partial f}{\partial x_2}|)^j\right)
\end{aligned}$$

where $|\frac{\partial f}{\partial x_1}|^j$ means $|\frac{\partial^j f}{\partial x_1^j}|$. Therefore,

$$|I_{10}|\le 9\sum_{j=1}^{N}\frac{2^j}{j!n^{(1-\alpha)j}}(|\frac{\partial f}{\partial x_1}|+|\frac{\partial f}{\partial x_2}|)^j.$$

Let

$$M=\max\left\{|\frac{\partial f}{\partial x_1}|+|\frac{\partial f}{\partial x_2}|,\cdots,(|\frac{\partial f}{\partial x_1}|+|\frac{\partial f|}{\partial x_2}|)^N\right\},$$

then

$$|I_{10}|\le\frac{36M}{n^{1-\alpha}}.$$

Finally, we estimate R^*. We observe $(0\le t_N\le 1)$

$$\left|g_{\frac{\vec{k}}{n}}^{(N)}(t_N)-g_{\frac{\vec{k}}{n}}^{(N)}(0)\right|$$

$$= \left| (\sum_{i=1}^{2} (\frac{k_i}{n+n^\alpha} - x_i) \frac{\partial}{\partial x_i})^N f(\vec{x} + t_N(\frac{\vec{k}}{n+n^\alpha} - \vec{x})) - (\sum_{i=1}^{2} (\frac{k_i}{n+n^\alpha} - x_i) \frac{\partial}{\partial x_i})^N f(\vec{x}) \right|$$
$$\leq \frac{4^N}{n^{N(1-\alpha)}} \max_{|\vec{\beta}|=N} \omega_1(f_{\vec{\beta}}, \frac{2}{n^{1-\alpha}}).$$

Hence

$$\begin{aligned} \left| R_N(\frac{\vec{k}}{n}, 0) \right| &\leq \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{N-1}} \left| g^N_{\frac{\vec{k}}{n}}(t_N) - g^N_{\frac{\vec{k}}{n}}(0) \right| dt_1 dt_2 \cdots dt_N \\ &\leq \frac{4^N}{N! n^{N(1-\alpha)}} \max_{|\vec{\beta}|=N} \omega_1(f_{\vec{\beta}}, \frac{2}{n^{1-\alpha}}), \end{aligned}$$

Consequently

$$\begin{aligned} |R^*| &\leq \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \frac{b(n^{1-\alpha}(x_1 - \frac{[nx_1]}{n}), n^{1-\alpha}(x_2 - \frac{k_2}{n}))}{n^{2\alpha}} \left| R_N(\frac{\vec{k}}{n}, 0) \right| \\ &\leq \frac{1}{n^{2\alpha}} \sum_{\lceil nx_1 - n^\alpha \rceil}^{[nx_1+n^\alpha]} \sum_{\lceil nx_2 - n^\alpha \rceil}^{[nx_2+n^\alpha]} \left| R_N(\frac{\vec{k}}{n}, 0) \right| \\ &\leq \frac{4^N}{N! n^{N(1-\alpha)}} 9 \max_{|\vec{\beta}|=N} \omega_1(f_{\vec{\beta}}, \frac{2}{n^{1-\alpha}}). \end{aligned}$$

Therefore,

$$|F_n(f)(x_1, x_2) - f(x_1, x_2)| \leq \left(\frac{52}{n^\alpha} \|f\|_\infty + \frac{36M}{n^{1-\alpha}} + \frac{9 \cdot 4^N}{N! n^{N(1-\alpha)}} \max_{|\vec{\beta}|=N} \omega_1(f_{\vec{\beta}}, \frac{2}{n^{1-\alpha}}) \right).$$

This finishes the proof of Theorem 2.

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A New Stability Analysis for Solving Biharmonic Equations by Finite Difference Methods*

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Abstract

The paper presents a new stability analysis for the finite difference method (FDM) for biharmonic equations, based on the effective condition number. Upper bounds of the effective condition number and the simplest effective condition number are derived. In general cases, the bound of the effective condition number is $O(h^{-3.5})$, which are smaller than Cond$_\infty = O(h^{-4})$, where h is the minimal meshspace of the difference grids. Surprisingly, the bounds of the effective condition number are only $O(1)$ for homogeneous boundary conditions. Namely, these new stability analysis are more valid than previous stability analysis. Numerical experiments are provided to verify the stability analysis.

Keyword: Stability analysis; Condition number; Finite difference method; Biharmonic equation; Effective condition number.

1 Introduction

The biharmonic equation([9], [16], [19]), which is one of the most important partial differential equations is applied in all subject areas of fundamental importance in the engineering sciences such as the elasticity, mechanics of elastic plates and fracture mechanics. In general, we can use finite difference method,

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radial basis function method and method of fundamental solution to solve biharmonic equation with the mixed type of boundary value problems. It is proved that the numerical solution is of high accuracy while the traditional condition number(Cond_) is very large, which results in a worse or even invalid numerical stability. Fortunately, in practical applications, we deal with only a certain vector $\mathbf{b}$, and the true relative errors may be much smaller than the worst Cond_[1], which is called the effective condition number in Chan and Foulser [4] and Christiansen and Hansen [5]. For this reason, the new stability analysis in this paper by means of effective condition number may reevaluate the existing numerical methods. Recently, the new computational formulae of the effective condition number have been studied and applied to symmetric and positive-definite matrices which are obtained from Poisson's equation by the Finite difference method(FDM) in [11].

The matrix $\mathbf{A} \in R^{n\times n}$ is symmetric and positive definite. The eigenvalues are arranged in a descending order, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$, and the eigenvectors $\mathbf{u_i}$ satisfy $\mathbf{A}\mathbf{u_i} = \lambda_\mathbf{i}\mathbf{u_i}$, where $\mathbf{u_i}$ are orthogonal. Since the eigenvectors $\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}\}$ are complete, $\mathbf{b}$ can be written as $\sum_{i=1}^{n} \beta_i \mathbf{u_i}$ with the coefficients $\beta_i = \mathbf{u_i^T b}$. The effective condition number is defined as [11]

$$\text{Cond_eff}=\frac{||\mathbf{b}||}{\lambda_n||\mathbf{x}||} = \frac{1}{\lambda_n}\frac{||\mathbf{b}||}{\sqrt{\sum_{i=1}^{n}\frac{\beta_i^2}{\lambda_i^2}}}, \tag{1}$$

which can be simplified to

$$\text{Cond_E}=\frac{||\mathbf{b}||}{\sqrt{(||\mathbf{b}||^2-\beta_n^2)\frac{\lambda_n^2}{\lambda_1^2}+\beta_n^2}}, \tag{2}$$

when all λ_i equal to λ_1. In particular, when $\beta_n \neq 0$, the simplest effective condition number from (2) is

$$\text{Cond_EE}=\frac{||\mathbf{b}||}{|\beta_n|}, \beta_n \neq 0. \tag{3}$$

In this paper, the effective condition number is applied to the standard 13-point finite difference equation for biharmonic equations on the unit square, and the bounds of effective condition number are derived. When the mixed types of the clamped and simply support boundary conditions are subjected to the solution boundary, it is proved in this paper that the bounds of the effective condition number are $O(h^{-3.5})$ while in general cases they are smaller than $O(h^{-4})$, where h is the minimal meshspace of the difference grids. Surprisingly, the bounds of the effective condition number are only $O(1)$ for homogeneous boundary conditions. Since the stability is a severe issue for numerical biharmonic equations, the reduction of effective condition number is significant. So

[1]The definition of the traditional condition number was given in Wilkinson [1] and then used in many books and papers ([2], [3], [8], [18], [19], [20]).

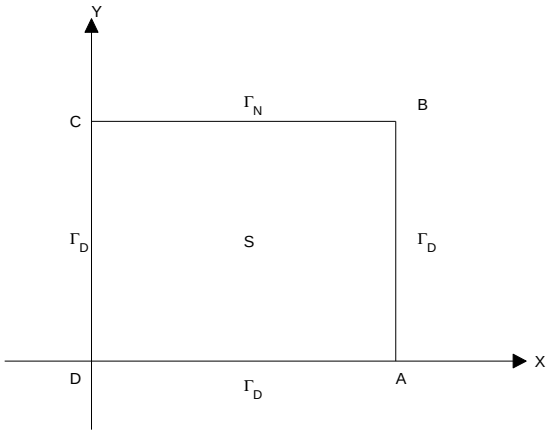


Figure 1: The unit square domain with the mixed boundary conditions.

we may employ a transformation to convert the biharmonic equation with non-homogeneous boundary conditions to that with homogeneous boundary conditions. Hence, the $O(1)$ effective condition number may be achieved, and we yield an excellent stability of numerical biharmonic equations.

This paper is organized as follows. In the next section, upper bounds of the effective condition number and the simplest effective condition number are derived for biharmonic equations in the case of two dimension (2D). In Section 3, a uniform view is given for the assumptions needed for deriving the bounds of the effective condition number. Finally, in Section 4, some numerical examples are provided to illustrate the features of the methods discussed in this paper and a few remarks are made.

2 A New Stability analysis of numerical biharmonic Equations in 2D

In this paper, we consider the biharmonic equation on the unit square $[0,1]^2$ with the mixed type of the clamped and support boundary conditions, see Figure 1

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f, \text{ in } S, \tag{4}$$

$$u = g, \text{ on } \Gamma, \tag{5}$$

$$\frac{\partial u}{\partial n} = g^* \text{ on } \Gamma_D, \frac{\partial^2 u}{\partial n^2} = g^{**} \text{ on } \Gamma_N, \tag{6}$$

where $\Gamma = \partial S = AB \cup BC \cup CD \cup DA$, $\Gamma_D = AB \cup CD \cup DA$ and $\Gamma_N = BC$. The functions in (4)-(6) are supposed to be bounded

$$||f||_{0,S} \leq C,\ ||g||_{0,\Gamma} \leq C,\ ||g^*||_{0,\Gamma_D} \leq C,\ ||g^{**}||_{0,\Gamma_N} \leq C, \tag{7}$$

where C is a bounded constant and

$$||f||_{0,S} = \sqrt{\int\int_s f^2 ds}, ||g||_{0,\Gamma} = \sqrt{\int_\Gamma g^2 dl}. \tag{8}$$

Divide S by the uniform difference grids $x_i = ih, y_j = jh$ with $h = 1/N$. Denote $u_{i,j} = u(x_i, y_j) = u(ih, jh)$. We use the standard 13-point difference equation (DE) to solve (4-6). This approach yields the following linear system

$$Ax = b, \tag{9}$$

where the matrix $A \in R^{(N-1)^2 \times (N-1)^2}$ is symmetric and positive definite, and the unknown vector x consists of $u_{i,j}$, $1 \leq i, j \leq N-1$, b is the known vector.

2.1 Preliminary Lemmas

Denote

$$\mathbf{b} = \mathbf{b_f} + \mathbf{b_g} + \mathbf{b_{g^*}} + \mathbf{b_{g^{**}}}, \tag{10}$$

where $\mathbf{b_f}, \mathbf{b_g}, \mathbf{b_{g^*}}$ and $\mathbf{b_{g^{**}}}$ result from the functions $f, g, g*$ and g^{**}, respectively. From (9), we have

$$\mathbf{b_f} = \{\cdots, h^4 f_{i,j}, \cdots\}, \tag{11}$$

$$\begin{aligned}\mathbf{b_{g^*}} = &-2h\{\langle\cdots, g^*_{i,0}, \cdots\rangle, 0, \cdots, 0, \langle\cdots, g^*_{N,j}, \cdots\rangle, 0, \cdots, 0,\\ &\langle\cdots, g^*_{0,j}, \cdots\rangle, 0, \cdots, 0\}^T,\end{aligned} \tag{12}$$

$$\mathbf{b_{g^{**}}} = -h^2\{0, \cdots, 0, \langle\cdots, g^{**}_{i,n}, \cdots\rangle, 0, \cdots, 0\}^T, \tag{13}$$

where $\langle\cdots, g^{**}_{i,n}, \cdots\rangle = \langle g^{**}_{1,n}, g^{**}_{2,n}, \cdots, g^{**}_{N-1,n}\rangle$. The vector $\mathbf{b_g} = (\mathbf{b_g})_{\mathbf{i,j}}$ is more complicated, whose entries are given as follows

$$\begin{aligned}&(\mathbf{b_g})_{i,j} = 0, 2 \leq i, j \leq N-2,\\ &(\mathbf{b_g})_{N-2,j} = -g_{n,j}, 1 \leq j \leq n-1,\\ &(\mathbf{b_g})_{N-1,j} = 8g_{N,j} - 2(g_{n,j+1} + g_{N,j-1}), 1 \leq j \leq n-1,\\ &(\mathbf{b_g})_{2,j} = -g_{0,j}, 1 \leq j \leq n-1,\\ &(\mathbf{b_g})_{1,j} = 8g_{0,j} - 2(g_{0,j+1} + g_{0,j-1}), 1 \leq j \leq n-1,\\ &(\mathbf{b_g})_{i,2} = -g_{i,0}, 1 \leq i \leq n-1,\\ &(\mathbf{b_g})_{i,1} = 8g_{i,0} - 2(g_{i+1,0} + g_{i-1,0}), 1 \leq i \leq n-1,\\ &(\mathbf{b_g})_{i,N-2} = -g_{i,n}, 1 \leq i \leq n-1,\\ &(\mathbf{b_g})_{i,N-1} = 6g_{i,N} - 2(g_{i+1,N} + g_{i-1,N}), 1 \leq i \leq n-1.\end{aligned} \tag{14}$$

Lemma 2.1. *Suppose that* $||f||_{0,S} \le C$, $||g||_{0,\Gamma} \le C$, $||g^*||_{0,\Gamma_D} \le C$ *and* $||g^{**}||_{0,\Gamma_N} \le C$ *with* $||f||_{0,S} = \sqrt{\int\int_s f^2 ds}$ *and* $||g||_{0,\Gamma} = \sqrt{\int_\Gamma g^2 dl}$. *Then there exist the bounds,*

$$||b|| \le C\{h^3||f||_{0,S} + h^{-0.5}||g||_{0,\Gamma} + h^{0.5}||g^*||_{0,\Gamma_D} + h^{1.5}||g^{**}||_{0,\Gamma_N}\}, \tag{15}$$

where C *is a bounded constant independent of* h.

Proof. From (11) we have

$$\begin{aligned} ||\mathbf{b_f}|| &= \sqrt{\sum_{i=1}^{n-1}\sum_{i=1}^{n-1}(h^4 f_{ij})^2} = h^3\sqrt{\sum_{i=1}^{n-1}\sum_{i=1}^{n-1}(h f_{ij})^2} \\ &\le Ch^3\sqrt{\int\int_s f^2 ds} = Ch^3||f||_{0,S}, \end{aligned} \tag{16}$$

where C is a bounded constant. Next, from (12) and (13), there exist the bounds

$$\begin{aligned} ||b_{g^*}|| &= \{\sum_{i=1}^{n-1}(2hg^*_{i,0})^2 + \sum_{i=1}^{n-1}[(2hg^*_{0,j})^2 + (2hg^*_{N,j})^2]\}^{1/2} \\ &= 2h^{1/2}\{\sum_{i=1}^{n-1} h(g^*_{i,0})^2 + \sum_{i=1}^{n-1} h[(g^*_{0,j})^2 + (g^*_{N,j})^2]\}^{1/2} \\ &\le Ch^{1/2}\sqrt{\int_{Ab\cup CD\cup DA}(g^*)^2 dl} = Ch^{1/2}||g^*||_{0,\Gamma_D}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} ||\mathbf{b_{g^{**}}}|| &= \sqrt{\sum_{i=1}^{n-1}(h^2 g^{**}_{i,N})^2} = h^{3/2}\sqrt{\sum_{i=1}^{n-1} h(g^{**}_{i,N})^2} \\ &\le Ch^{3/2}\sqrt{\int_{BC}(g^{**})^2 dl} = Ch^{3/2}||g^{**}||_{0,\Gamma_N}. \end{aligned} \tag{18}$$

Finally, we obtain from (14)

$$\begin{aligned} ||\mathbf{b_g}||^2 =& \sum_{i=1}^{n-1}\{(g_{n,j})^2 + (8g_{N,j} - 2(g_{n,j+1} + g_{N,j-1}))^2\} \\ &+ \sum_{i=1}^{n-1}\{(g_{0,j})^2 + (8g_{0,j} - 2(g_{0,j+1} + g_{0,j-1}))^2\} \\ &+ \sum_{i=1}^{n-1}\{(g_{i,0})^2 + (8g_{i,0} - 2(g_{i+1,0} + g_{i-1,0}))^2\} \\ &+ \sum_{i=1}^{n-1}\{(g_{i,N})^2 + (6g_{i,N} - 2(g_{i+1,n} + g_{i-1,N}))^2\}. \end{aligned} \tag{19}$$

Since

$$(8g_{N,j} - 2(g_{n,j+1} + g_{N,j-1}))^2 \le 3[(8g_{N,j})^2 + (2g_{n,j+1})^2 + (2g_{N,j-1})^2],$$

we have

$$\begin{aligned}
&\sum_{i=1}^{n-1}\{(g_{n,j})^2 + (8g_{N,j} - 2(g_{n,j+1} + g_{N,j-1}))^2\} \\
&\quad \le 193\sum_{i=1}^{n-1}(g_{n,j})^2 + 12\sum_{i=1}^{n-1}\{(g_{n,j+1})^2 + (g_{N,j-1})^2\} \\
&\quad \le 217\sum_{i=1}^{n-1}(g_{n,j})^2 + 12\{(g_{n,0})^2 + (g_{N,N})^2\} \\
&\quad \le 217h^{-1}\{\sum_{i=1}^{n-1} h(g_{n,j})^2 + h/2((g_{n,0})^2 + (g_{N,N})^2)\} \\
&\quad \le Ch^{-1}\int_{AB} g^2 dl. \qquad (20)
\end{aligned}$$

Similarly, we have

$$\sum_{i=1}^{n-1}\{(g_{0,j})^2 + (8g_{0,j} - 2(g_{0,j+1} + g_{0,j-1}))^2\} \le Ch^{-1}\int_{CD} g^2 dl, \qquad (21)$$

$$\sum_{i=1}^{n-1}\{(g_{i,0})^2 + (8g_{i,0} - 2(g_{i+1,0} + g_{i-1,0}))^2\} \le Ch^{-1}\int_{AD} g^2 dl, \qquad (22)$$

$$\sum_{i=1}^{n-1}\{(g_{i,N})^2 + (6g_{i,N} - 2(g_{i+1,N} + g_{i-1,N}))^2\} \le Ch^{-1}\int_{BC} g^2 dl. \qquad (23)$$

Hence we obtain from (19)-(23)

$$||\mathbf{b_g}|| \le Ch^{-1/2}\sqrt{\int_\Gamma g^2 dl} = Ch^{-1/2}||g||_{0,\Gamma}. \qquad (24)$$

Combining (16)-(18) and (24), we have

$$\begin{aligned}
||\mathbf{b}|| &\le ||\mathbf{b_f}|| + ||\mathbf{b_g}|| + ||\mathbf{b_{g^*}}|| + ||\mathbf{b_{g^{**}}}|| \\
&\le C\{h^3||f||_{0,S} + h^{-1/2}||g||_{0,\Gamma} + h^{1/2}||g^*||_{0,\Gamma_D} + h^{3/2}||g^{**}||_{0,\Gamma_N}\}. \qquad (25)
\end{aligned}$$

This is the desired result (15) and the proof of Lemma 2.1 is completed. □

Let us consider the corresponding eigenvalue problems of (4)-(6)

$$w_{xxxx} + 2w_{xxyy} + w_{yyyy} = \mu w, \text{ in } S, \qquad (26)$$

$$w = 0, \text{ on } \Gamma, \qquad (27)$$

$$\frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_D, \frac{\partial^2 w}{\partial n^2} = 0 \text{ on } \Gamma_N. \tag{28}$$

Also the linear algebraic eigenvalue problem of (9) is given by $\mathbf{Ay} = \lambda \mathbf{y}$. Denote the leading eigenpair of (26)-(28) by $(\mu_{\min}, w^h_{\min})$. Hence we have $\lambda^h_{\min} \simeq h^4 \mu_{\min}$ and $w^h_{\min} \simeq w_{\min}$. Since the total number of (i, j) is $(N-1)^2$, the orthogonal vector of $w^h_{\min}$ is given by

$$y_n = Ch\{\cdots, (w^h_{\min})_{i,j}, \cdots\}^T, \tag{29}$$

where $C > 0$ is a bounded constant independent of h. Below, let us derive the bounds of $\beta_n = \mathbf{y_n^T b} = \mathbf{y_n^T}(\mathbf{b_f} + \mathbf{b_g} + \mathbf{b_{g^*}} + \mathbf{b_{g^{**}}})$. We have the following lemma.

Lemma 2.2. *There exist the following approximations,*

$$\mathbf{y_n^T b_f} \simeq Ch^3 \int\int_s f w_{\min} ds, \tag{30}$$

$$\mathbf{y_n^T b_{g^*}} \simeq -Ch^3 \int_{\Gamma_D} g^* \frac{\partial^2 w_{\min}}{\partial n^2} dl, \tag{31}$$

$$\mathbf{y_n^T b_{g^{**}}} \simeq Ch^3 \int_{\Gamma_N} g^{**} \frac{\partial w_{\min}}{\partial n} dl, \tag{32}$$

$$\mathbf{y_n^T b_g} \simeq Ch^3 \int_{\Gamma} g(\frac{\partial^3 w_{\min}}{\partial n^3} + 2\frac{\partial^3 w_{\min}}{\partial n \partial s^2}) dl, \tag{33}$$

where C is the constant given in (29), and n and s are the exterior normal and tangent directions of $\Gamma = \partial S$, respectively.

Proof. We shall prove (30)-(33) one by one. First we have from (11)

$$\begin{aligned} \mathbf{y}_n^T b_f &= Ch \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} h^4 f_{ij} (w^h_{\min})_{i,j} = Ch^3 \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} h^2 f_{ij} (w^h_{\min})_{i,j} \\ &\simeq Ch^3 \int\int_s f w_{\min} ds, \end{aligned} \tag{34}$$

where h is small. Also we have from (12)

$$\mathbf{y_n^T b_{g^*}} = -2Ch^2 \sum_{i=1}^{n-1} \{g^*_{i,0}(w^h_{\min})_{i,1} + g^*_{N,i}(w^h_{\min})_{N-1,i} + g^*_{0,i}(w^h_{\min})_{1,i}\}. \tag{35}$$

Since $(w^h_{\min})_{i,0} = 0$ and $(w^h_{\min})_{i,1} \simeq (w^h_{\min})_{i,-1}$, we obtain

$$\frac{\partial^2 w_{\min}}{\partial n^2} \simeq \frac{(w^h_{\min})_{i,-1} - 2(w^h_{\min})_{i,0} + (w^h_{\min})_{i,1}}{h^2} = \frac{2(w^h_{\min})_{i,1}}{h^2}.$$

Then

$$-2Ch^2 \sum_{i=1}^{n-1} g^*_{i,0}(w^h_{\min})_{i,1} \simeq -Ch^4 \sum_{i=1}^{n-1} g^*_{i,0} \frac{\partial^2 w_{\min}}{\partial n^2} \simeq -Ch^3 \int_{AB} g^* \frac{\partial^2 w_{\min}}{\partial n^2} dl. \tag{36}$$

Hence we have from (35)

$$\mathbf{y_n^T b_{g^*}} \simeq -Ch^3 \int_{\Gamma_D} g^* \frac{\partial^2 w_{\min}}{\partial n^2} dl. \tag{37}$$

This is the second desired result (31). Next, since $(w^h_{\min})_{i,N} = 0$, we have from (13)

$$\begin{aligned}\mathbf{y_n^T b_{g^{**}}} &\simeq -Ch^3 \sum_{i=1}^{n-1} g^{**}_{i,N} (w^h_{\min})_{i,N-1} \simeq Ch^4 \sum_{i=1}^{n-1} g^{**}_{i,N} \frac{\partial w_{\min}}{\partial n}|_{i,N-1} \\ &\simeq Ch^3 \int_{\Gamma_N} g^{**} \frac{\partial w_{\min}}{\partial n} dl. \end{aligned} \tag{38}$$

This is the third desired result (32).

Below, let us prove (33), which is nontrivial. Let

$$\mathbf{b_g} = \mathbf{b_g}|_{AB} + \mathbf{b_g}|_{BC} + \mathbf{b_g}|_{CD} + \mathbf{b_g}|_{DA}, \tag{39}$$

where the components of $\mathbf{b_g}|_{AB}, \mathbf{b_g}|_{BC}, \mathbf{b_g}|_{CD}$ and $\mathbf{b_g}|_{DA}$ are given in (14). Then

$$\mathbf{y_n^T b_g} = \mathbf{y_n^T}(\mathbf{b_g}|_{AB} + \mathbf{b_g}|_{BC} + \mathbf{b_g}|_{CD} + \mathbf{b_g}|_{DA}). \tag{40}$$

For the first term on the right hand of the above equation, we have

$$\begin{aligned}\mathbf{y_n^T b_g}|_{AB} &= Ch \sum_{i=1}^{n-1} [8g_{N,j} - 2(g_{n,j+1} + g_{N,j-1})](w^h_{\min})_{N-1,j} \\ &\quad - Ch \sum_{i=1}^{n-1} g_{N,j} (w^h_{\min})_{N-2,j} \\ &= Ch \sum_{i=1}^{n-1} [4(w^h_{\min})_{N-1,j} - (w^h_{\min})_{N-2,j}] g_{N,j} \\ &\quad + Ch \sum_{i=1}^{n-1} [4g_{N,j} - 2(g_{n,j+1} + g_{N,j-1})](w^h_{\min})_{N-1,j}. \end{aligned} \tag{41}$$

Assume that $(w^h_{\min})_{N,j} = 0$ and $(w^h_{\min})_{N+1,j} \simeq (w^h_{\min})_{N-1,j}$ according to the boundary approximation, we obtain

$$\begin{aligned}(\frac{\partial^3 w_{\min}}{\partial n^3})_{N,j} &= h^{-3}\{(w^h_{\min})_{N+1,j} - 3(w^h_{\min})_{N,j} + 3(w^h_{\min})_{N-1,j} - (w^h_{\min})_{N-2,j}\} \\ &\simeq h^{-3}\{4(w^h_{\min})_{N-1,j} - (w^h_{\min})_{N-2,j}\}. \end{aligned} \tag{42}$$

Hence we have the approximation

$$4(w^h_{\min})_{N-1,j} - (w^h_{\min})_{N-2,j} \simeq h^3 (\frac{\partial^3 w_{\min}}{\partial n^3})_{N,j}, \tag{43}$$

which indicates

$$Ch\sum_{i=1}^{n-1}[4(w^h_{\min})_{N-1,j}-(w^h_{\min})_{N-2,j}]g_{N,j}$$
$$\simeq Ch^3\sum_{i=1}^{n-1}h(\frac{\partial^3 w_{\min}}{\partial n^3})_{N,j}g_{N,j}\simeq Ch^3\int_{AB}g\frac{\partial^3 w_{\min}}{\partial n^3}dl. \quad (44)$$

Next, we have

$$Ch\sum_{i=1}^{n-1}[4g_{N,j}-2(g_{n,j+1}+g_{N,j-1})](w^h_{\min})_{N-1,j}$$
$$=Ch\sum_{i=1}^{n-1}\{4(w^h_{\min})_{N-1,j}-2[(w^h_{\min})_{N-1,j+1}+(w^h_{\min})_{N-1,j-1}]\}g_{N,j}, \quad (45)$$

by noting $(w^h_{\min})_{N-1,0}=(w^h_{\min})_{N-1,N}=0$. Moreover, assume

$$\frac{\partial^2 w_{\min}}{\partial y^2}|_{N-1,j}\simeq -h^{-2}\{2(w^h_{\min})_{N-1,j}-[(w^h_{\min})_{N-1,j+1}+(w^h_{\min})_{N-1,j-1}]\}. \quad (46)$$

Since $\frac{\partial^2 w_{\min}}{\partial y^2}|_{AB}=0$ from $w_{\min}|_{AB}=0$, we have from (45),

$$Ch\sum_{i=1}^{n-1}[4g_{N,j}-2(g_{n,j+1}+g_{N,j-1})](w^h_{\min})_{N-1,j}$$
$$\simeq -2Ch\sum_{i=1}^{n-1}h^2(\frac{\partial^2 w_{\min}}{\partial n^2})_{N-1,j}g_{N,j}\simeq 2Ch^3\sum_{i=1}^{n-1}h(\frac{\partial^3 w_{\min}}{\partial y^2\partial n})_{N-1,j}g_{N,j}$$
$$\simeq 2Ch^3\int_{AB}g\frac{\partial^3 w_{\min}}{\partial y^2\partial n}dl\simeq 2Ch^3\int_{AB}g\frac{\partial^3 w_{\min}}{\partial s^2\partial n}dl. \quad (47)$$

Hence we obtain from (41), (44) and (47),

$$\mathbf{y_n^T b_g}|_{AB}\simeq Ch^3\int_{AB}g(\frac{\partial^3 w_{\min}}{\partial n^3}+2\frac{\partial^3 w_{\min}}{\partial n\partial s^2})dl. \quad (48)$$

Similarly, we have

$$\mathbf{y_n^T b_g}|_{DA}\simeq Ch^3\int_{DA}g(\frac{\partial^3 w_{\min}}{\partial n^3}+2\frac{\partial^3 w_{\min}}{\partial n\partial s^2})dl, \quad (49)$$

$$\mathbf{y_n^T b_g}|_{CD}\simeq Ch^3\int_{DC}g(\frac{\partial^3 w_{\min}}{\partial n^3}+2\frac{\partial^3 w_{\min}}{\partial n\partial s^2})dl. \quad (50)$$

Finally, let us consider $\mathbf{y_n^T b_g}|_{BC}$ from (14),

$$\begin{aligned}\mathbf{y_n^T b_g}|_{BC} =& Ch\sum_{i=1}^{n-1}[6g_{i,N} - 2(g_{i+1,N} + g_{i-1,N})](w^h_{\min})_{i,N-1} \\ & - Ch\sum_{i=1}^{n-1} g_{i,n}(w^h_{\min})_{N-2,j} \\ =& Ch\sum_{i=1}^{n-1}[2(w^h_{\min})_{i,N-1} - (w^h_{\min})_{i,N-2}]g_{i,N} \\ & + Ch\sum_{i=1}^{n-1}[4g_{i,N} - 2(g_{i+1,N} + g_{i-1,N})](w^h_{\min})_{i,N-1}. \end{aligned} \tag{51}$$

For the (i, j) near BC, since $(w^h_{\min})_{i,N} = 0$, we have $(w^h_{\min})_{i,N+1} \simeq -(w^h_{\min})_{i,N-1}$, and

$$\begin{aligned}&(\frac{\partial^3 w_{\min}}{\partial n^3})_{i,N} = (\frac{\partial^3 w_{\min}}{\partial y^3})_{i,N} \\ &\simeq h^{-3}\{(w^h_{\min})_{i,N+1} - 3(w^h_{\min})_{i,N} + 3(w^h_{\min})_{i,N-1} - (w^h_{\min})_{i,N-2}\} \\ &\simeq h^{-3}\{2(w^h_{\min})_{i,N-1} - (w^h_{\min})_{i,N-2}\}. \end{aligned} \tag{52}$$

Hence, we obtain

$$\begin{aligned}& Ch\sum_{i=1}^{n-1}[2(w^h_{\min})_{i,N-1} - (w^h_{\min})_{i,N-2}]g_{i,N} \\ &\simeq Ch^3\sum_{i=1}^{n-1} hg_{i,N}(\frac{\partial^3 w_{\min}}{\partial n^3})_{i,N} \simeq Ch^3\int_{BC} g\frac{\partial^3 w_{\min}}{\partial n^3}dl. \end{aligned} \tag{53}$$

Moreover, because of $\frac{\partial^2 w_{\min}}{\partial x^2}|_{CB} = 0$, from the second term on the most right-hand side of (51), we have

$$\begin{aligned}& Ch\sum_{i=1}^{n-1}[4g_{i,N} - 2(g_{i+1,N} + g_{i-1,N})](w^h_{\min})_{i,N-1} \\ &\simeq Ch\sum_{i=1}^{n-1}\{4(w^h_{\min})_{i,N-1} - 2[(w^h_{\min})_{i+1,N-1} + (w^h_{\min})_{i-1,N-1}]\}g_{i,N} \\ &\simeq -2Ch^3\sum_{i=1}^{n-1} g_{i,N}(\frac{\partial^2 w_{\min}}{\partial x^2})_{i,N-1} \simeq 2Ch^3\sum_{i=1}^{n-1} hg_{i,N}(\frac{\partial^3 w_{\min}}{\partial x^2\partial n})_{i,N} \\ &\simeq Ch^3\int_{BC} g\frac{\partial^3 w_{\min}}{\partial n\partial s^2}dl. \end{aligned} \tag{54}$$

Combining (51), (53) and (54), we have

$$\mathbf{y_n^T b_g}|_{CB} \simeq Ch^3\int_{BC} g(\frac{\partial^3 w_{\min}}{\partial n^3} + 2\frac{\partial^3 w_{\min}}{\partial n\partial s^2})dl. \tag{55}$$

Hence from (48)-(50) and (55), we obtain

$$\mathbf{y_n^T b_g} \simeq Ch^3 \int_\Gamma g(\frac{\partial^3 w_{\min}}{\partial n^3} + 2\frac{\partial^3 w_{\min}}{\partial n \partial s^2})dl. \tag{56}$$

This is the last desired result (33) and the proof of Lemma 2.2 is completed. □

2.2 Bounds of Cond_EE

We are ready to derive the bounds of Cond_EE. We have from (40) and Lemma 2.2,

$$\begin{aligned}\beta_n = y_n^T b \simeq & Ch^3\{\int\int_s f w_{\min} ds - \int_{\Gamma_D} g^* \frac{\partial^2 w_{\min}}{\partial n^2} dl \\ & + \int_{\Gamma_N} g^{**} \frac{\partial w_{\min}}{\partial n} dl + \int_\Gamma g(\frac{\partial^3 w_{\min}}{\partial n^3} + 2\frac{\partial^3 w_{\min}}{\partial n \partial s^2})dl\}.\end{aligned} \tag{57}$$

Denote $p(v) = \frac{\partial^3 v}{\partial n^3} + 2\frac{\partial^3 v}{\partial n \partial s^2}$ and assume **A1**:

$$\begin{aligned}T_1 = & \int_{\Gamma_N} g^{**} \frac{\partial w_{\min}}{\partial n} dl + \int_\Gamma g(\frac{\partial^3 w_{\min}}{\partial n^3} + 2\frac{\partial^3 w_{\min}}{\partial n \partial s^2})dl \\ & + \int\int_s f w_{\min} ds - \int_{\Gamma_D} g^* \frac{\partial^2 w_{\min}}{\partial n^2} dl \\ = & O(1).\end{aligned} \tag{58}$$

We obtain the simplest effective condition number from Lemma 2.1 and (58),

$$\text{Cond_EE} = \frac{||b||}{|\beta_n|} \leq C\{||f||_{0,S} + h^{-3.5}||g||_{0,\Gamma} + h^{-2.5}||g^*||_{0,\Gamma_D} + h^{-1.5}||g^{**}||_{0,\Gamma_N}\}. \tag{59}$$

We state this result as a theorem.

Theorem 2.1. *Let **A1** and all conditions in Lemma 2.1 hold. Then the bounds (59) of Cond_EE exist.*

From Theorem 2.1 we have the following corollary.

Corollary 2.1 *Let all conditions in Theorem 2.1 hold. For the homogeneous boundary conditions $g = g^* = g^{**} = 0$, the simplest effective condition number has the bounds*

$$Cond_EE = O(1). \tag{60}$$

From Theorem 2.1, the worst effects on Cond_EE are resulted from $u|_\Gamma = g$. Actually, we can use a transformation $v = u - \bar{u}$, where $\bar{u}|_\Gamma = g$, $\partial u/\partial n|_{\Gamma_D} = g^*$, and $\partial^2 u/\partial n^2|_{\Gamma_N} = g^{**}$, to give the new biharmonic equation with the homogeneous boundary conditions

$$\Delta^2 v = f - \Delta^2 \bar{u} \text{ in } S,$$

$$v = 0 \text{ on } \Gamma,\ \partial v/\partial n = 0 \text{ on } \Gamma_D,\ \partial^2 u/\partial n^2 = 0, \text{ on } \Gamma_N.$$

Hence we may achieve the small bound (60) of effective condition number which offer an excellent stability for numerical biharmonic equations.

2.3 Bounds of Cond_eff

Below, we will derive the bounds of effective condition number in (1),

$$\text{Cond_eff}=\frac{||\mathbf{b}||}{\lambda_n||\mathbf{x}||}. \tag{61}$$

First, we give a lemma.

Lemma 2.3. *For the stiffness matrix* $\mathbf{A}$ *in (9), there exists the bound*

$$\lambda_{\min}=\lambda_{\min}(\mathbf{A})\geq ch^4, \tag{62}$$

where c *is the constant independent of* h.

Proof. First, consider the simple support boundary condition on the entire boundary ∂S, and denote $\mathbf{A}^*$ the corresponding stiffness matrix. Since the eigenvectors are known as

$$w_{p,q}=2\{...,\sin(p\pi ih)\sin(q\pi jh),...\}/N,$$

where $1\leq p,q\leq N-1$, we obtain the eigenvalues

$$\lambda_{p,q}(\mathbf{A}^*)=16\{\sin^2(p\pi h/2)+\sin^2(q\pi h/2)\}^2,$$

to give the minimal eigenvalue

$$\lambda_{\min}(\mathbf{A}^*)=\lambda_{1,1}(\mathbf{A}^*)=64\sin^4(\pi h/2)\simeq 4\pi^4h^4. \tag{63}$$

Next, consider the mixed type of the clamped and simple support boundary conditions in (5)-(7). The key difference is that the first coefficient in the DE increases from 19 to 21. Hence, we conclude that only difference between matrices $\mathbf{A}$ and $\mathbf{A}^*$ lies in that the diagonal entries of $\mathbf{A}$ are larger than those of $\mathbf{A}^*$. Then we have

$$\mathbf{A}=\mathbf{A}^*+\mathbf{D},$$

where $\mathbf{D}$ is a diagonal matrix with $\mathbf{D}$=Diag$\{d_{1,1},...,d_{n,n}\}$ with $d_{i,i}\geq 0$, and $n=(N-1)^2$. Hence, we obtain

$$\begin{aligned}\lambda_{\min}(\mathbf{A})&=\min_{(x,x)\neq 0}\frac{(\mathbf{Ax},\mathbf{x})}{(\mathbf{x},\mathbf{x})}=\min_{(x,x)\neq 0}\frac{(\mathbf{A}^*\mathbf{x},\mathbf{x})+(\mathbf{Dx},\mathbf{x})}{(\mathbf{x},\mathbf{x})}\\&\geq\min_{(x,x)\neq 0}\frac{(\mathbf{A}^*\mathbf{x},\mathbf{x})}{(\mathbf{x},\mathbf{x})}=\lambda_{\min}(\mathbf{A}^*).\end{aligned} \tag{64}$$

From (63) and (64) we have

$$\lambda_{\min}(\mathbf{A})\geq\lambda_{\min}(\mathbf{A}^*)\geq ch^4. \tag{65}$$

This is the desired result (62), and completes the proof of Lemma 2.3. □

Since $f\in C(S)$, $u\in C(S)$ is bounded, to have

$$||x||=\sqrt{\sum_{i,j=1}u_{i,j}^2}\leq CN=O(h^{-1}). \tag{66}$$

Hence, we may assume that the majority of $u_{i,j} = O(1)$ and assumption **A2**:

$$||x|| \geq ch^{-1}, \tag{67}$$

where c is a constant independent of h. Based on (61), (67), **A2** and Lemmas 2.1, 2.3, we have the following theorem.

Theorem 2.2. *Let **A2** and all conditions in Lemma 2.1 hold. Then there exist the bounds,*

$$Cond_eff \leq C\{||f||_{0,S} + h^{-3.5}||g||_{0,\Gamma} + h^{-2.5}||g^*||_{0,\Gamma_D} + h^{-1.5}||g^{**}||_{0,\Gamma_N}\}, \tag{68}$$

where C is a constant independent of h.

Note that the bounds of effective condition number in Theorems 2.1 and 2.2 are exactly the same, but based on different assumptions, **A1** and **A2**. A linkage between **A1** and **A2** is provided in the next section.

3 A View on Assumption A1

We cite the Green formula in Li, et al. [14],

$$\begin{aligned}\int\int_s \Lambda_\sigma(u,v) = \int\int_s v\Delta^2 u - \int_{\partial S} m(u)v_n - \int_{\partial S} p(u)v \\ + 2(1-\sigma)([u_{xy}v]_3^4 - [u_{xy}v]_1^2),\end{aligned} \tag{69}$$

where $\Delta u = u_{xx} + u_{yy}$, $[v]_1^2 = v_2 - v_1$ and $1, 2, 3, 4$ are four corners of S. The notations are

$$\Lambda_\sigma(u,v) = \Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}), \tag{70}$$

$$m(u) = -u_{nn} - \sigma u_{ss}, p(u) = u_{nnn} + (2-\sigma)u_{ssn}, \tag{71}$$

where $0 \leq \sigma < 1$ and n and s are the normal and tangent directions along the boundary ∂S of S, respectively. The notations in (70) and(71) are $u_{xy} = \partial^2 u/\partial x \partial y$, $u_{xx} = \partial^2 u/\partial x^2, u_{nn} = \partial^2 u/\partial n^2$, $u_{ss} = \partial^2 u/\partial s^2$ etc.

For the clamped and simply support boundary condition, $v|_\Gamma = 0$, there are no corner effects term, because $2(1-\sigma)([u_{xy}v]_3^4 - [u_{xy}v]_1^2) = 0$. Choosing $\sigma = 0$, we have from (69)

$$\begin{aligned}\int\int_s \Lambda_\sigma(u,w) &= \int\int_s (-2u_{xy}v_{xy} + u_{xx}v_{xx} + u_{yy}v_{yy}) \\ &= \int\int_s w\Delta^2 u - \int_\Gamma m(u)w_n - \int_\Gamma p(u)w,\end{aligned} \tag{72}$$

where $m(u) = -u_{nn}, p(u) = u_{nnn} + 2u_{ssn}$. Let $w = w_{\min}$ satisfy

$$\Delta^2 w = \mu w, w|_\Gamma = 0, \partial w/\partial n|_{\Gamma_D} = 0, \partial^2 w/\partial n^2|_{\Gamma_D} = 0.$$

We obtain from (72)

$$\int\int_s \Lambda_1(u,w) = \int\int_s w\Delta^2 u + \int_\Gamma \frac{\partial^2 u}{\partial n^2}\frac{\partial w}{\partial n} - \int_\Gamma p(u)w$$
$$= \int\int_s fw + \int_{\Gamma_N} g^{**}\frac{\partial w}{\partial n}. \tag{73}$$

On the other hand, we have from (72),

$$\int\int_s \Lambda_0(u,w) = \mu\int\int_s uw + \int_{\Gamma_D} g^*\frac{\partial^2 w}{\partial n^2} - \int_\Gamma p(u)g. \tag{74}$$

Combining (73) and (74), we have

$$\int\int_s fw - \int_{\Gamma_D} g^*\frac{\partial^2 w}{\partial n^2} + \int_{\Gamma_N} g^{**}\frac{\partial w}{\partial n} + \int_\Gamma p(u)g = \mu\int\int_s uw. \tag{75}$$

Hence, we may restate assumption **A1** in (66) as **A1*** :

$$\int\int_s uw_{\min} = O(1), \tag{76}$$

by noting $\mu_{\min} = O(1)$. Hence Theorem 2.1 may be rewritten as follows.

Theorem 3.1. *Let **A1*** and all conditions in Lemma 2.1 hold. Then there exist the bounds (60) of Cond_EE.*

To close this section, let us explore a linkage between assumptions **A1** (i.e., **A1***) and **A2**. In fact, assumption **A1*** leads to **A2**. We confirm this conclusion by contradiction. Suppose that **A2** does not hold, then we have

$$||\mathbf{x}|| = 0(h^{-1}), \text{ as } h \to 0. \tag{77}$$

Since function $w_{\min}$ denotes the displacements of the vibrating plate corresponding to the minimal frequency (see Courant and Hilbert [9]), we conclude that $w_{\min} > 0$ in S and that the majority of $w_{\min}(x_i, y_j)$ are $O(1)$. Hence we obtain

$$\int\int_s uw_{\min} \simeq \sum_{i,j} w_{\min}(x_i,y_j)u(x_i,y_j) \simeq \sum_{i,j} w_{\min}(x_i,y_j)u_{i,j}$$
$$\le \sqrt{\sum_{i,j} w^2_{\min}(x_i,y_j)}\sqrt{\sum_{i,j} u^2_{i,j}} = O(N)0(h^{-1}) = 0(1). \tag{78}$$

This gives

$$\int\int_s uw_{\min} = 0(1), \tag{79}$$

which contradict assumption **A1***.

4 Numerical experiments

The uniform rectangles $\square_{ij}$ are chosen with $h = 1/N$, where N is the division number along AB (see Figure 1). We choose the biharmonic equation with three different boundary conditions, called Models I, II and III.

Model I. Choose the exact solution

$$u(x,y) = [x(1-x)\sin(\pi x)][y(1-y)^2\sin(\pi y)]$$

to satisfy

$$\Delta^2 u = f, \text{ in } s,$$

$$u|_\Gamma = 0, \partial u/\partial n|_{AB\cup AD\cup CD} = 0, \partial^2 u/\partial n^2|_{BC} = 0,$$

where $S = [0,1] \times [0,1]$ and $\Gamma = \partial S = AB \cup BC \cup CD \cup DA$.

Model II. Choose the exact solution

$$u(x,y) = [x(1-x)\sin(\pi x)][y(1-y)\sin(\pi y)]$$

to satisfy

$$\Delta^2 u = f, \text{ in } s,$$

$$u|_\Gamma = 0, \partial u/\partial n|_{AB\cup AD\cup CD} = 0, \partial^2 u/\partial n^2|_{BC} = 2\pi x(1-x)\sin(\pi x).$$

Model III. Choose the exact solution

$$u(x,y) = [x(1-x)\sin\frac{\pi x}{2}][y(1-y)^2\sin(\pi y)]$$

to satisfy

$$\Delta^2 u = f, \text{ in } s,$$

$$u|_\Gamma = 0, \partial u/\partial n|_{AB\cup CD} = 0, \partial u/\partial n|_{AB} = -y(1-y)^2\sin(\pi y), \partial^2 u/\partial n^2|_{BC} = 0.$$

The difference equations are established as in Section 2.1 and the numerical solutions are obtained from the Gaussian elimination. The numerical results are listed in Tables 1, 2, 3 for Models I, II and III, respectively. Interestingly, the computed values of Cond_E and Cond_EE are the same in three significant digits.

Let us examine the numerical asymptotes of Cond_E and Cond_EE as the N grows. First, for Model I with the homogeneous boundary conditions, $g = g^* = g^{**} = 0$ in (5) and (6), the following rates

$$\text{Cond_E}=\text{Cond_EE}=O(1), \tag{80}$$

are observed from Table 1, to verify perfectly Theorem 2.1 and Corollary 2.1.

Next, for Model II with the boundary conditions (5) and (6) with

$$g = g^* = 0, g^{**} = 2\pi x(1-x)\sin(\pi x). \tag{81}$$

we can see numerically from Table 2,

$$\frac{\text{Cond_EE}|_{N=64}}{\text{Cond_EE}|_{N=32}} = \frac{21.0641}{7.8290} = 2.6905 \simeq 2^{1.5} = 2.83, \tag{82}$$

Table 1: The condition numbers and effective condition numbers for Model I

N	8	16	32	64
$\lambda_{max}(A)$	59.38	62.79	63.69	63.92
$\lambda_{min}(A)$	0.1841	0.124(-1)	0.800(-3)	0.603(-4)
Cond	0.322(3)	0.506(4)	0.796(5)	0.106(7)
$\|\|b\|\|_2$	0.542(-1)	0.712(-2)	0.920(-3)	0.117(-3)
β_n	0.230(-1)	0.282(-2)	0.351(-3)	0.439(-4)
Cond_E	2.3565	2.5248	2.6211	2.6651
Cond_EE	2.3565	2.5248	2.6211	2.6651

Table 2: The condition numbers and effective condition numbers for Model II

N	8	16	32	64
$\lambda_{max}(A)$	59.38	62.79	63.69	63.92
$\lambda_{min}(A)$	0.1841	0.124(-1)	0.800(-3)	0.603(-4)
Cond	0.322(3)	0.506(4)	0.796(5)	0.106(7)
$\|\|b\|\|_2$	0.906(-1)	0.198(-1)	0.577(-2)	0.194(-2)
β_n	0.478(-1)	0.590(-2)	0.737(-3)	0.921(-4)
Cond_E	1.8954	3.3559	7.8290	21.0641
Cond_EE	1.8954	3.3559	7.8290	21.0641

to indicate the following rates

$$\text{Cond_E} = \text{Cond_EE}{=}O(h^{-1.5}), \tag{83}$$

which also verify the Theorem 2.1.

Lastly, for Model III with the following boundary conditions in (5) and (6)

$$g = 0, g^*|_{AB} = -y(1-y)^2 \sin(\pi y), g^*|_{AD\cup CD} = 0, g^{**} = 0,$$

we can see numerically from Table 3,

$$\frac{\text{Cond_EE}|_{N=64}}{\text{Cond_EE}|_{N=32}} = \frac{590.116}{104.727} = 5.6348 \simeq 2^{2.5} = 5.66, \tag{84}$$

to indicate the following rates,

$$\text{Cond_E} = \text{Cond_EE}{=}O(h^{-2.5}), \tag{85}$$

which also validate Theorem 2.1.

Table 3: The condition numbers and effective condition numbers for Model III

N	8	16	32	64
$\lambda_{max}(A)$	59.38	62.79	63.69	63.92
$\lambda_{min}(A)$	0.1841	0.124(-1)	0.800(-3)	0.603(-4)
Cond$_.$	0.322(3)	0.506(4)	0.796(5)	0.106(7)
$\|\|b\|\|_2$	0.767(-1)	0.417(-1)	0.288(-1)	0.203(-1)
β_n	0.172(-1)	0.219(-2)	0.275(-3)	0.344(-4)
Cond_E	4.459	19.041	104.727	590.116
Cond_EE	4.459	19.041	104.727	590.116

To close this paper, let us make a few concluding remarks.

1. The bounds of effective condition number are smaller or even much smaller than those of the traditional Cond$_.$. This paper explores new stability analysis, based on effective condition number.

2. The effective condition number is applied to the standard 13-point difference equation for biharmonic equations. Their bounds are derived, to show Cond_EE = Cond_E = $O(h^{-3.5})$, which are smaller than Cond$_. = O(h^{-4})$ when h is small. However, for special cases, the effective condition number may be much smaller. For instance, the Cond_EE = Cond_E = $O(1)$ is achieved for the homogeneous boundary conditions. This result is surprising, compared to Cond$_. = O(h^{-4})$. The small bounds of effective condition number are significant to stability of numerical biharmonic equations, because its instability is severe, compared with the stability of numerical Poisson's equation.

3. To reduce the bounds of effective condition number, we may employ a transformation $v = u - \bar{u}$, where the function $\bar{u}$ satisfies the non-homogeneous

boundary conditions. Then we obtain $\Delta^2 v = f - \Delta^2 \bar{u}$ with the homogeneous boundary conditions and the small effective condition number as $O(1)$ is achieved.

4. Numerical experiments are carried out for the mixed types of the clamped and simply support boundary conditions, in which the computed results coincide with the new stability analysis perfectly.

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STABILITY PROBLEMS OF CUBIC MAPPINGS WITH THE FIXED POINT ALTERNATIVE

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Abstract. We investigate the general solution of the cubic functional equation $3f(2x+y)+f(2x-y)+6f(y)=8f(x+y)+24f(x)$ on the real vector spaces and then we prove the generalized Hyers-Ulam-Rassias stability in quasi-β-normed spaces, the stability by using a subadditive function and then the stability by using the alternative fixed point method.

1. Introduction

The stability problem for functional equations is originated by S.M. Ulam [32] concerning the stability of group homomorphisms. The first result concerning the stability of functional equations was presented by Hyers [8] on the assumption that the spaces are Banach spaces. The famous Hyers stability result that appeared in [8] was generalized by T. Aoki [1] for the stability of the additive mapping involving a sum of powers of p-norms and by Th.M. Rassias [24] for the stability of the linear mapping by considering the Cauchy difference to be unbounded.

The result of Th.M. Rassias lead mathematicians working in stability of functional equations to establish what is known today as Hyers-Ulam-Rassias stability or Cauchy-Rassias stability as well as to introduce new definitions of stability concepts.

In [18] and [19], J.M. Rassias investigated a similar stability theorem for linear and nonlinear mappings with the unbounded Cauchy difference.

During the last three decades, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors [2], [5],[6],[7],[9],[25],[10],[26],[27], and [28].

The quadratic function $f(x)=cx^2$ ($c\in\mathbb{R}$) satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)\,. \tag{1.1}$$

This question is called the quadratic functional equation, and every solution of the equation (1.1) is called a quadratic function. In fact, a function $f:X\to Y$ is a solution of the equation (1.1) if and only if there exists a bilinear function $B:X\times X\to Y$ such that $f(x)=B(x,x)$ for all $x\in X$.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was first proved by Skof [31] for functions $f:X\to Y$, where X is a normed space and Y is a Banach space.

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The cubic function $f(x) = cx^3$ ($c \in \mathbb{R}$) satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)\,. \tag{1.2}$$

The equation (1.2) was solved by Jun and Kim [12]. Similar to a quadratic functional equation, actually, they proved that a function $f : X \to Y$ is a solution of the equation (1.2) if and only if there exists a function $F : X \times X \times X \to Y$ such that $f(x) = F(x, x, x)$ for all $x \in X$, and F is symmetric for each fixed one variable and is additive for fixed two variables; see [12]. We promise that by a cubic function we mean every solution of the equation (1.2) is called a cubic function. Also, Chu and Kang [15, Lemma 2.1] proved that the equation (1.2) is equivalent to the following equation:

$$f(x+2y) + f(x-2y) + f(2x) = 2f(x) + 4f(x+y) + 4f(x-y)\,. \tag{1.3}$$

Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We will consider the definition and some preliminary results of a quasi-β-norm on a linear space.

Definition 1.1. Let X be a linear space over a field $\mathbb{K}$. A *quasi-β-norm* $||\cdot||$ is a real-valued function on X satisfying the followings:

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.
(2) $||\lambda x|| = |\lambda|^\beta \cdot ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.
(3) There is a constant $K \geq 1$ such that $||x+y|| \leq K(||x|| + ||y||)$ for all $x, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi-β-normed space* if $||\cdot||$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $||\cdot||$. A *quasi-Banach space* is a complete quasi-β-normed space.

A quasi-β-norm $||\cdot||$ is called a (β, p)-norm ($0 < p \leq 1$) if

$$||x+y||^p \leq ||x||^p + ||y||^p\,,$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space; see [3], [29] and [23].

In this paper, we consider the following the cubic functional equation:

$$3f(2x+y) + f(2x-y) + 6f(y) = 8f(x+y) + 24f(x) \tag{1.4}$$

for all $x, y \in X$. We investigate the generalized Hyers-Ulam-Rassias stability problem in quasi-β-normed spaces and then the stability by using a subadditive function and then the stability by using the alternative fixed point method for the cubic function $f : X \to Y$ satisfying the equation (1.4).

2. Cubic functional equations

Lemma 2.1. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation (1.3) for all $x, y \in X$ if and only if f is cubic.*

Proof. See [15, Lemma 2.1]. □

Lemma 2.2. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the equation (1.4) if and only if f is cubic.*

Proof. Lemma 2.1 implies that it is enough to show that (1.4) and (1.3) are equivalent. Letting $x = y = 0$ in the equation (1.4), we have $f(0) = 0$. Also putting $y = 0$ in the equation (1.4), we get $4f(2x) = 32f(x)$, that is, $f(2x) = 8f(x)$, for all $x \in X$. Now letting $x = \frac{x-y}{2}$ in the equation (1.4),

$$3f(x)+f(x-2y)+6f(y)=f(x+y)+3f(x-y)\,, \tag{2.1}$$

for all $x\,, y\in X$. Also, letting $y=2y$ in the equation (1.4), we have

$$3f(2x+2y)+f(2x-2y)+48f(y)=8f(x+2y)+24f(x)\,,$$

or

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y)\,, \tag{2.2}$$

for all $x\,, y\in X$. Adding equations (2.1) and (2.2),

$$\begin{aligned}&f(x+2y)+3f(x)+3f(x)+f(x-2y)+6f(y)\\&=f(x+y)+3f(x-y)+3f(x+y)+f(x-y)+6f(y)\,,\end{aligned}$$

i.e.,

$$f(x+2y)+f(x-2y)+f(2x)=2f(x)+4f(x+y)+4f(x-y)\,,$$

for all $x\,, y\in X$. Hence (1.4) implies (1.3). Conversely, it is easy to show that $f(0)=0\,, f(-x)=-f(x)$, and $f(2x)=8f(x)$, for all $x\in X$. Letting $x=2x$ in the equation (1.3), we have

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)\,, \tag{2.3}$$

for all $x\,, y\in X$. Now switching x and y in the equation (1.3), we get

$$f(2x+y)-f(2x-y)+f(2y)=2f(y)+4f(x+y)-4f(x-y)\,, \tag{2.4}$$

for all $x\,, y\in X$. By (2.3) and (2.4),

$$3f(2x+y)+f(2x-y)+f(2y)=2f(y)+8f(x+y)+24f(x)\,,$$

for all $x\,, y\in X$. Since $f(2y)=8f(y)$, we have the desired result. □

3. Stabilities

Throughout this section, let X be a quasi-β-normed space and let Y be a quasi-β-Banach space with a quasi-β-norm $||\cdot||_Y$. Let K be the modulus of concavity of $||\cdot||_Y$. We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.4). After then we will study the stability by using a subadditive function and alternative fixed point method. For a given mapping $f: X\to Y$, let

$$Df(x,y)=\quad 3f(2x+y)+f(2x-y)-8f(x+y)-24f(x)+6f(y)\,,$$

$x,\, y\in X$.

Theorem 3.1. *Suppose that there exists a mapping $\phi: X^2\to\mathbb{R}^+:=[0,\infty)$ for which a mapping $f: X\to Y$ satisfies $f(0)=0$,*

$$||Df(x,y)||_Y\le\phi(x,y) \tag{3.1}$$

and the series $\sum_{j=0}^{\infty}\left(\frac{K}{2^{3\beta}}\right)^j\phi(2^jx,2^jy)$ *converges for all* $x,\,y\in X$. *Then there exists a unique cubic mapping* $C:X\to Y$ *which satisfies the equation (1.4) and the inequality*

$$||f(x)-C(x)||_Y\leq\frac{K}{32^\beta}\sum_{j=0}^{\infty}\Big(\frac{K}{2^{3\beta}}\Big)^j\phi(2^jx,0)\,, \tag{3.2}$$

for all $x\in X$.

Proof. By letting $y=0$ in the inequality (3.1), since $f(0)=0$, we have

$$||Df(x,0)||_Y=||4f(2x)-32f(x)||_Y\leq\phi(x,0)\,,$$

for all $x\in X$. Then

$$||f(x)-\frac{1}{2^3}f(2x)||_Y\leq\frac{1}{32^\beta}\phi(x,0)\,, \tag{3.3}$$

for all $x\in X$. Now, putting $x=2x$ and dividing by $2^{3\beta}$ in the inequality (3.3), we get

$$\frac{1}{2^{3\beta}}||f(2x)-\frac{1}{2^3}f(2^2x)||_Y\leq\frac{1}{32^\beta}\Big(\frac{1}{2^{3\beta}}\Big)\phi(2x,0)\,, \tag{3.4}$$

for all $x\in X$. Combining two equations (3.3) and (3.4), we have

$$\begin{aligned}||f(x)-\Big(\frac{1}{2^3}\Big)^2f(2^2x)||_Y &\leq K(\|\,f(x)-\frac{1}{2^3}f(2x)\,\|_Y+\|\,\frac{1}{2^3}f(2x)-(\frac{1}{2^3})^2f(2^2x)\,\|_Y)\\ &\leq \frac{K}{32^\beta}(\phi(x\,,0)+\frac{1}{2^{3\beta}}\phi(2x,,0))\,,\end{aligned}$$

for all $x\in X$. Inductively, continue this way,

$$||f(x)-(\frac{1}{2^3})^nf(2^n\,x)||_Y\leq\frac{K}{32^\beta}\sum_{j=0}^{n-1}(\frac{K}{2^{3\beta}})^j\phi(2^jx,0)\,,$$

for all $x\in X$, and all $n\in N$. Let $n\,,m\in N$ with $n<m$. Replacing x by $2^n\,x$ and multiplying $\left(\frac{1}{2^{2\beta}}\right)^n$ in the inequality (3.3),

$$(\frac{1}{2^{3\beta}})^n\,||f(2^n\,x)-\frac{1}{2^3}f(2^{n+1}x)||_Y\leq(\frac{1}{2^{3\beta}})^n\phi(2^n\,x,0)\,,$$

for all $x\in X$. Also, inductively

$$||(\frac{1}{2^3})^n\,f(2^n\,x)-(\frac{1}{2^3})^m\,f(2^m\,x)||_Y\leq\frac{K}{32^\beta}\sum_{j=n}^{m-1}(\frac{K}{2^{3\beta}})^j\phi(2^j\,x\,,0)\,, \tag{3.5}$$

for all $x\in X$. As $m\to\infty$, the right-hand side in the inequality (3.5) close to 0. Hence $\{\Big(\frac{1}{2^3}\Big)^nf(2^nx)\}$ is a Cauchy sequence in the quasi-β-Banach space Y. Thus we can define a mapping $C:X\to Y$ by

$$C(x)=\lim_{n\to\infty}\Big(\frac{1}{2^3}\Big)^nf(2^nx)\,,$$

for all $x\in X$. Hence we have

$$||f(x)-C(x)||_Y\leq\frac{K}{32^\beta}\sum_{j=0}^{\infty}\Big(\frac{K}{2^{3\beta}}\Big)^j\phi(2^jx,0)\,,$$

for all $x\in X$. Now, we will show that the mapping $C:X\to Y$ satisfies the equation (1.4) and then the mapping is unique.

$$\begin{aligned}
&\Big(\frac{1}{2^{3\beta}}\Big)^n ||Df(2^n x, 2^n y)||_Y \\
&= \Big(\frac{1}{2^{3\beta}}\Big)^n ||3f(2^n(2x+y)) + f(2^n(2x-y)) - 8f(2^n(x+y)) - 24f(2^n x) + 6f(2^n y)||_Y \\
&\le \Big(\frac{K}{2^{3\beta}}\Big)^n ||3f(2^n(2x+y)) + f(2^n(2x-y)) - 8f(2^n(x+y)) - 24f(2^n x) + 6f(2^n y)||_Y \\
&\le \Big(\frac{K}{2^{3\beta}}\Big)^n \phi(2^n x, 2^n y)\,,
\end{aligned}$$

for all $x, y \in X$. By taking $n \to \infty$, we know that the mapping $C : X \to Y$ is cubic.

Now, assume that there exists a cubic mapping $T : X \to Y$ satisfying the inequalities (3.1) and (3.2). Then

$$\begin{aligned}
||T(x) - C(x)||_Y &= \Big(\frac{1}{2^{3\beta}}\Big)^n ||T(2^n x) - C(2^n x)||_Y \\
&\le \Big(\frac{1}{2^{3\beta}}\Big)^n K\Big(||T(2^n x) - f(2^n x)||_Y + ||f(2^n x) - C(2^n x)||_Y\Big) \\
&\le \frac{2K^2}{32^\beta} \sum_{j=0}^{\infty} \Big(\frac{K}{2^{3\beta}}\Big)^{n+j} \phi(2^{n+j} x, 0)\,,
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of $C : X \to Y$. □

Theorem 3.2. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||Df(x, y)||_Y \le \phi(x, y) \tag{3.6}$$

and the series $\sum_{j=1}^{\infty} \Big(2^{3\beta} K\Big)^j \phi(2^{-j} x, 2^{-j} y)$ converges for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ which satisfies the equation (1.4) and the inequality

$$||f(x) - C(x)||_Y \le \frac{1}{4^\beta} \sum_{j=1}^{\infty} \Big(2^{3\beta} K\Big)^j \phi(2^{-j} x, 0)\,, \tag{3.7}$$

for all $x \in X$.

Proof. Letting $y = 0$ in the inequality (3.6),

$$||Df(x, 0)||_Y = ||4f(2x) - 32f(x)||_Y \le \phi(x, 0)\,,$$

for all $x \in X$. Now, replacing x by $\frac{x}{2}$, we have $||4f(x) - 32f(\frac{x}{2})||_Y \le \phi(\frac{x}{2}, 0)$, that is, $||f(x) - 2^3 f(\frac{x}{2})||_Y \le \frac{1}{4^\beta}\phi(\frac{x}{2}, 0)$, for all $x \in X$. Let $n, m \in N$ with $n < m$. Similar to the inequality (3.5) of the proof Theorem 3.1, we get

$$||2^{3n} f(\frac{x}{2^n}) - 2^{3m} f(\frac{x}{2^m})||_Y \le \frac{1}{4^\beta} \sum_{j=n}^{m-1} (2^{3\beta} K)^j \phi(2^{-j} x, 0)\,,$$

for all $x \in X$. The remain parts of the proof follow from the proof of Theorem 3.1. □

Now we will recall a subadditive function and then investigate the stability under the condition that the space Y is a (β, p)-Banach space. The basic definitions of subadditive functions follow from the reference [23].

A function $\phi : A \to B$ having a domain A and a codomain $(B, \leq)$ that are both closed under addition is called

(1) *a subadditive function* if $\phi(x+y) \leq \phi(x) + \phi(y)$,
(2) *a contractively subadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x+y) \leq L(\phi(x) + \phi(y))$,
(3) *a expansively superadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x+y) \geq \frac{1}{L}(\phi(x) + \phi(y))$,

for all $x, y \in A$.

Theorem 3.3. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||Df(x,y)||_Y \leq \phi(x,y) \tag{3.8}$$

for all x, $y \in X$ and the map ϕ is contractively subadditive with a constant L such that $2^{1-3\beta}L < 1$. Then there exists a unique cubic mapping $C : X \to Y$ which satisfies the equation (1.4) and the inequality

$$||f(x) - C(x)||_Y \leq \frac{\phi(x,0)}{4^\beta \sqrt[p]{2^{3\beta p} - (2L)^p}}, \tag{3.9}$$

for all $x \in X$.

Proof. By the inequalities (3.3) and (3.5) of the proof of Theorem 3.1, we have

$$\begin{aligned}
&||\frac{1}{2^{3n}} f(2^n x) - \frac{1}{2^{3m}} f(2^m x)||_Y^p \\
&\leq \sum_{j=n}^{m-1} \Big(\frac{1}{2^{3\beta}}\Big)^{jp} ||f(2^j x) - \frac{1}{2^3} f(2^{j+1}x)||_Y^p \\
&\leq \frac{1}{32^{\beta p}} \sum_{j=n}^{m-1} \Big(\frac{1}{2^{3\beta}}\Big)^{jp} \phi(2^j x, 0)^p \\
&\leq \frac{1}{32^{\beta p}} \sum_{j=n}^{m-1} \Big(\frac{1}{2^{3\beta}}\Big)^{jp} (2L)^{jp} \phi(x,0)^p \\
&= \frac{\phi(x,0)^p}{32^{\beta p}} \sum_{j=n}^{m-1} \Big(2^{1-3\beta} L\Big)^{jp},
\end{aligned}$$

that is,

$$||\Big(\frac{1}{2^3}\Big)^n f(2^n x) - \Big(\frac{1}{2^3}\Big)^m f(2^m x)||_Y^p \leq \frac{\phi(x,0)^p}{32^{\beta p}} \sum_{j=n}^{m-1} \Big(2^{1-3\beta} L\Big)^{jp}, \tag{3.10}$$

for all $x \in X$, and for all n and m with $n < m$. Hence $\{\frac{1}{2^{3n}} f(2^n x)\}$ is a Cauchy sequence in the space Y. Thus we may define a mapping $C : X \to Y$ by

$$C(x) = \lim_{n \to \infty} \frac{1}{2^{3n}} f(2^n x),$$

for all $x \in X$. Now, we will show that the map $C : X \to Y$ is a cubic mapping. Indeed,

$$\begin{aligned} ||DC(x,y)|| &= \lim_{n\to\infty} \frac{||Df(2^n x, 2^n y)||_Y^p}{2^{3\beta pn}} \\ &\le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)^p}{2^{3\beta pn}} \\ &\le \lim_{n\to\infty} \phi(x,y)^p (2^{1-3\beta}L)^{pn} = 0\,, \end{aligned}$$

for all $x \in X$. Hence the mapping $C : X \to Y$ is a cubic mapping. Note that the inequality (3.10) implies the inequality (3.9) by letting $n = 0$ and taking $m \to \infty$. Assume that there exists another mapping $T : X \to Y$ satisfying (1.4) and (3.9). Then

$$\begin{aligned} ||T(x) - \Big(\frac{1}{2^3}\Big)^n f(2^n x)||_Y^p &= \Big(\frac{1}{2^{3\beta}}\Big)^{pn} ||T(2^n x) - f(2^n x)||_Y^p \\ &\le \Big(\frac{1}{2^{3\beta}}\Big)^{pn} \frac{\phi(2^n x, 0)^p}{4^{\beta p}(2^{3\beta p} - (2L)^p)} \\ &\le \Big(2^{1-3\beta}L\Big)^{pn} \frac{\phi(x,0)^p}{4^{\beta p}(2^{3\beta p} - (2L)^p)}\,, \end{aligned}$$

that is,

$$||T(x) - \Big(\frac{1}{2^3}\Big)^n f(2^n x)||_Y \le \Big(2^{1-3\beta}L\Big)^n \frac{\phi(x,0)}{4^{\beta p}\sqrt[p]{(2^{3\beta p} - (2L)^p)}}\,,$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of $C : X \to Y$. □

Theorem 3.4. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||Df(x,y)||_Y \le \phi(x,y) \tag{3.11}$$

for all $x, y \in X$ and the map ϕ is expansively superadditive with a constant L such that $2^{3\beta-1}L < 1$. Then there exists a unique cubic mapping $C : X \to Y$ which satisfies the equation (1.4) and the inequality

$$||f(x) - C(x)||_Y \le \frac{\phi(x,0)}{4^\beta L\sqrt[p]{2^p - (2^{3\beta}L)^p}}\,, \tag{3.12}$$

for all $x \in X$.

Proof. By letting $y = 0$ in the equation (3.11), we have

$$||4f(2x) - 32^3 f(x)||_Y \le \phi(x,0)\,,$$

and then replacing x by $\frac{x}{2}$,

$$||f(x) - 2^3 f(\frac{x}{2})||_Y \le \frac{1}{4^\beta}\phi(\frac{x}{2},0)\,, \tag{3.13}$$

for all $x \in X$. For all n and m with $n < m$, inductively we have

$$||2^{3n} f(\frac{x}{2^n}) - 2^{3m} f(\frac{x}{2^m})||_Y^p \le \frac{\phi(x,0)^p}{4^{\beta p}(2L)^p} \sum_{j=n}^{m-1} \Big(2^{3\beta-1}L\Big)^{jp}\,, \tag{3.14}$$

for all $x \in X$. The remains follow from the proof of Theorem 3.3. □

Now, we will investigate the stability of the given cubic functional equation (1.4) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point.

Theorem 3.5 (The alternative of fixed point [22], [30])**.** *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* L *. Then for each given* $x \in \Omega$ *, either*

$$d(T^n x,\, T^{n+1}x) = \infty \text{ for all } n \geq 0\,,$$

or there exists a natural number n_0 *such that*

(1) $d(T^n x,\, T^{n+1}x) < \infty$ *for all* $n \geq n_0$ *;*
(2) *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T *;*
(3) y^* *is the unique fixed point of* T *in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\}\,;$$

(4) $d(y, y^*) \leq \frac{1}{1-L}\, d(y, Ty)$ *for all* $y \in \triangle$ *.*

Theorem 3.6. *Let* $f : X \to Y$ *be a function with* $f(0) = 0$ *for which there exists a function* $\phi : X^2 \to [0, \infty)$ *such that there exists a constant* L *,* $0 < L < 1$ *, satisfying the inequalities*

$$\| Df(x, y) \|_Y \leq \phi(x, y) \tag{3.15}$$

$$\phi(2x, 2y) \leq 8^{\beta} L\phi(x, y)\,,$$

for all $x, y \in X$ *. Then there exists a unique cubic function* $C : X \to Y$ *defined by* $\lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}} = C(x)$ *such that*

$$\| f(x) - C(x) \|_Y \leq \frac{1}{32^{\beta}(1 - L)}\, \phi(x, 0)\,, \tag{3.16}$$

for all $x \in X$ *.*

Proof. Consider the set

$$\Omega = \{g | g : X \to Y\,, g(0) = 0\}$$

and introduce the generalized metric on Ω *,*

$$d(g,\, h) = inf\{K \in (0, \infty) \mid \| g(x) - h(x) \|_Y \leq K\phi(x, 0)\,, x \in X\}\,.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{8}\, g(2x)\,, \quad g \in \Omega$$

for all $x \in X$ *.* Note that for all $g, h \in \Omega$ *,* let $K \in (0,\, \infty)$ be an arbitrary constant with $d(g, h) \leq K$ *.* Then

$$\begin{aligned}
& \| g(x) - h(x) \|_Y \leq K\phi(x, 0)\,, \text{ for all } x \in X\,, \\
\Rightarrow \quad & \| \frac{1}{8}g(2x) - \frac{1}{8}h(2x) \|_Y \leq \frac{1}{8^{\beta}} K\phi(2x, 0)\,, \text{ for all } x \in X\,, \\
\Rightarrow \quad & \| \frac{1}{8}g(2x) - \frac{1}{8}h(2x) \|_Y \leq LK\phi(x, 0)\,, \text{ for all } x \in X\,, \\
\Rightarrow \quad & d(Tg,\, Th) \leq LK\,.
\end{aligned}$$

Hence we have that

$$d(Tg,\, Th) \leq L\, d(g,\, h)\,,$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. By setting $y = 0$ in the inequality (3.15) and dividing both sides by 32^β, then we have

$$\| f(x) - \frac{1}{8} f(2x) \|_Y \leq \frac{1}{32^\beta} \phi(x, 0),$$

for all $x \in X$, that is, $d(Tf, f) \leq \frac{1}{32^\beta} < \infty$. We can apply the fixed point alternative and since $\lim_{r\to\infty} d(T^r f, C) = 0$, there exists a fixed point C of T in Ω such that

$$C(x) = \lim_{n\to\infty} \frac{f(2x)}{2^{3n}}, \tag{3.17}$$

for all $x \in X$. Letting $x = 2^n x$ and $y = 2^n y$ in the equation (3.15) and dividing by $2^{3n\beta}$,

$$\begin{aligned} \| DC(x, y) \|_Y &= \lim_{n\to\infty} \frac{\| Df(2^n x, 2^n y) \|_Y}{2^{3n\beta}} \\ &\leq \lim_{n\to\infty} \frac{1}{2^{3n\beta}} \phi(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} L^n \phi(x, y) = 0, \end{aligned}$$

for all $x, y \in X$; that is it satisfies the equation (1.4). By Lemma 2.2, the C is cubic. Also, the fixed point alternative guarantees that such a C is the unique function. Again using the fixed point alternative, we have

$$d(f, C) \leq \frac{1}{1 - L} d(Tf, f).$$

Hence we may conclude that

$$d(f, C) \leq \frac{1}{1 - L} d(Tf, f) \leq \frac{1}{32^\beta (1 - L)},$$

which implies the equation (3.16). □

Theorem 3.7. *Let $f : X \to Y$ be a function with $f(0) = 0$ for which there exists a function $\phi : X^2 \to [0, \infty)$ such that there exists a constant L, $0 < L < 1$, satisfying the inequalities*

$$\begin{aligned} \| Df(x, y) \|_Y &\leq \phi(x, y) \\ \phi(x, y) &\leq \frac{L}{8^\beta} \phi(2x, 2y), \end{aligned} \tag{3.18}$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \to Y$ defined by $\lim_{n\to\infty} 2^{3n} f(\frac{1}{2^n} x) = C(x)$ such that

$$\| f(x) - C(x) \|_Y \leq \frac{L}{32^\beta (1 - L)} \phi(x, 0), \tag{3.19}$$

for all $x \in X$.

Proof. We will use the same notation for Ω and d as in the proof of Theorem 3.6 and then we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = 8\, g(\frac{x}{2}), \quad g \in \Omega$$

for all $x \in X$. Then

$$\begin{aligned} ||Tg(x) - Th(x)||_Y &= 8^\beta ||g(\frac{x}{2}) - h(\frac{x}{2})||_Y \\ &\leq 8^\beta K \phi(\frac{x}{2}, 0) \leq LK \phi(x, 0), \end{aligned}$$

for all $x \in X$, that is, $d(Tg, Th) \leq LK$. Hence $d(Tg, Th) \leq Ld(g, h)$, for any $g, h \in \Omega$. Thus T is a strictly self-mapping of Ω with the Lipschitz constant L. By letting $x = \frac{x}{2}$ and $y = 0$ in the inequality (3.18) and dividing both sides by 4^β, then

$$||f(x) - 8f(\frac{x}{2})||_Y \leq \frac{1}{4^\beta}\phi(\frac{x}{2}, 0) \leq \frac{L}{32^\beta}\phi(x, 0)\,,$$

for all $x \in X$. Hence $d(Tf, f) \leq \frac{L}{32^\beta} < \infty$. The remains follows from the proof of Theorem 3.6. □

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A New Representation For The Fuzzy Systems In Terms Of Some Additive And Multiplicative Subsystem Inferences

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Abstract

A new representation for fuzzy systems in terms of additive and multiplicative subsystem inferences of single variable is proposed. This representation enables an approximate functional characterization of the inferred output. The form of the approximating function is dictated by the choice of polynomial, sinusoidal, or other designs of subsystem inferences. The paper is organized as follows. The first section gives a brief review on product sum fuzzy inference and introduces the concepts of additive and multiplicative decomposable systems. The second section presents the proposed subsystem inference representation. The third section focuses on subsystem inference of single variable.The sections 4 to 6 discuss the cases of polynomial, sinusoidal, orthonormal and other designs of subsystem inferences, and their applications. The section 7 presents some conclusions. The paper one finishes with its references.

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Keywords: subsystem inference representation, fuzzy systems, membership function, approximating function, universal approximators.

1 Introduction

Consider a fuzzy system of n input variables $a, b, \ldots, y, z$ with input membership functions $A_i,\ i = \overline{1, m_a},\ B_j,\ j = \overline{1, m_b}, \ldots, Y_h,\ h = \overline{1, m_y},\ Z_r,\ r = \overline{1, m_z}$ and the $m_a \cdot m_b \cdot \ldots \cdot m_y \cdot m_z$ fuzzy rules:

$$\Re_N :\ A_i,\ B_j, \ldots, Y_h,\ Z_r \rightarrow\ \delta(u_{i,j,\ldots,h,r}), \tag{1}$$

where $N = [(i-1) \cdot m_b \ldots \cdot m_z + \ldots + (h-1) \cdot m_z + r]$ and $u_{i,j,\ldots,h,r}$ is the corresponding rule consequent.

Given certain values for the variables $a, b, \ldots, y, z$, the output $\mathbf{u}$ is inferred by

$$\mathbf{u}=\frac{\sum_{i=1}^{m_a}\sum_{j=1}^{m_b}\ldots\sum_{h=1}^{m_y}\sum_{r=1}^{m_z}u_{i,j,\ldots,h,r}\cdot\mu_{A_i}(a)\cdot\mu_{B_j}(b)\cdot\ldots\mu_{Y_h}(y)\cdot\mu_{Z_r}(z)}{\sum_{i=1}^{m_a}\sum_{j=1}^{m_b}\ldots\sum_{h=1}^{m_y}\sum_{r=1}^{m_z}\mu_{A_i}(a)\cdot\mu_{B_j}(b)\cdot\ldots\mu_{Y_h}(y)\cdot\mu_{Z_r}(z)},\tag{2}$$

where $\mu_{A_i}(a),\ \mu_{B_j}(b),\ldots,\mu_{Y_h}(y),\ \mu_{Z_r}(z)$ are respectively, the membership degrees of $a,b,\ldots,y,z$ with respect to membership functions $A_i,\ B_j,\ldots,Y_h,\ Z_r$.

The inferred output $\mathbf{u}$ of the fuzzy system from (1) is based on the $m_a\times m_b\times\ldots\times m_y\times m_z$ hyper-dimensional matrix $U_{m_a,\ m_b,\ldots,m_y,\ m_z}$, which contains the rule consequents $u_{i,j,\ldots,h,r}$ as elements, namely we can write (2) as

$$\mathbf{u}\equiv\mathcal{I}_{a,b,\ldots,y,z}(U_{m_a,\ m_b,\ldots,m_y,\ m_z}).$$

If there exist $m_a+m_b+\ldots+m_y+m_z$ parameters $\alpha_i,\ \beta_j,\ldots,\eta_h,\zeta_r$, for $i=\overline{1,m_a},\ j=\overline{1,m_b},\ldots,h=\overline{1,m_y},\ r=\overline{1,m_z}$, such that

$$u_{i,j,\ldots,h,r}=\alpha_i+\beta_j+\ldots+\eta_h+\zeta_r\tag{3}$$

(2) becomes

$$\mathbf{u}=\frac{\sum_{i=1}^{m_a}\alpha_i\cdot\mu_{A_i}(a)}{\sum_{i=1}^{m_a}\mu_{A_i}(a)}+\frac{\sum_{j=1}^{m_b}\beta_j\cdot\mu_{B_j}(b)}{\sum_{j=1}^{m_b}\mu_{B_j}(b)}+\ldots+\frac{\sum_{r=1}^{m_z}\zeta_r\cdot\mu_{Z_r}(z)}{\sum_{r=1}^{m_z}\mu_{Z_r}(z)}.$$

This leads to $\mathbf{u}=\mathbf{u}^A+\mathbf{u}^B+\ldots+\mathbf{u}^Y+\mathbf{u}^Z$, where $\mathbf{u}^A,\mathbf{u}^B,\ldots,\mathbf{u}^Y,\mathbf{u}^Z$ are the inferred outputs of subsystems $A,B\ldots,Y,Z$.

The fuzzy rules of the subsystems are given by $m_a+m_b+\ldots+m_y+m_z$ parameters $\alpha_i,\ \beta_j,\ldots,\eta_h,\zeta_r$, for $i=\overline{1,m_a},\ j=\overline{1,m_b},\ldots,h=\overline{1,m_y},\ r=\overline{1,m_z}$, such that

$$\begin{aligned}&Subsystem\ A:\ A_i\to\delta(\alpha_i),\ for\ i=\overline{1,m_a}\\&Subsystem\ B:\ B_j\to\delta(\beta_j),\ for\ j=\overline{1,m_b}\\&\qquad\vdots\\&Subsystem\ Y:\ Y_h\to\delta(\eta_h),\ for\ h=\overline{1,m_y}\\&Subsystem\ Z:\ Z_r\to\delta(\zeta_r),\ for\ r=\overline{1,m_z}.\end{aligned}\tag{4}$$

Equation (3) is the necessary and sufficient condition for fuzzy systems (1) to be decomposable into n additive subsystems, each operating on the single variable of $a,b,\ldots,y,z$.

On the other hand, if there exist $m_a+m_b+\ldots+m_y+m_z$ variables $\alpha_i,\ \beta_j,\ldots,\eta_h,\zeta_r$, for $i=\overline{1,m_a},\ j=\overline{1,m_b},\ldots,h=\overline{1,m_y},\ r=\overline{1,m_z}$, such that

$$u_{i,j,\ldots,h,r}=\alpha_i\cdot\beta_j\cdot\ldots\cdot\eta_h\cdot\zeta_r,\tag{5}$$

$\mathbf{u}$ can be expressed as

$$\mathbf{u} = \left(\frac{\sum_{i=1}^{m_a} \alpha_i \cdot \mu_{A_i}(a)}{\sum_{i=1}^{m_a} \mu_{A_i}(a)} \right) \cdot \left(\frac{\sum_{j=1}^{m_b} \beta_j \cdot \mu_{B_j}(b)}{\sum_{j=1}^{m_b} \mu_{B_j}(b)} \right) \cdot \ldots \cdot \left(\frac{\sum_{r=1}^{m_z} \zeta_r \cdot \mu_{Z_r}(z)}{\sum_{r=1}^{m_z} \mu_{Z_r}(z)} \right).$$

This results in $\mathbf{u} = \mathbf{u}^A \cdot \mathbf{u}^B \cdot \ldots \cdot \mathbf{u}^Y \cdot \mathbf{u}^Z$, where $\mathbf{u}^A, \mathbf{u}^B, \ldots, \mathbf{u}^Y, \mathbf{u}^Z$ are the inferred outputs of subsystems $A, B \ldots, Y, Z$ given in (4).

Equation (5) is the necessary and sufficient condition for fuzzy systems (1) to be decomposable into n multiplicative subsystems, each operating on the single variable of $a, b, \ldots, y, z$.

2 Subsystem Inference Representation

Consider a fuzzy system with two input variables a and b with rule consequents embedded in the $m_a \times m_b$ matrix

$$U_{m_a,m_b} = \begin{pmatrix} u_{1,1} & u_{1,2} & \ldots & u_{1,m_b} \\ u_{2,1} & u_{2,2} & \ldots & u_{2,m_b} \\ \ldots & \ldots & \ldots & \ldots \\ u_{m_a,1} & u_{m_a,2} & \ldots & u_{m_a,m_b} \end{pmatrix}.$$

The inferred output is

$$\mathcal{I}_{a,b}(U_{m_a,m_b}) = \frac{\mathcal{M}_A^T \cdot U_{m_a,m_b} \cdot \mathcal{M}_B}{\left(\sum_{i=1}^{m_a} \mu_{A_i}(a) \right) \cdot \left(\sum_{j=1}^{m_b} \mu_{B_j}(b) \right)}, \tag{6}$$

where the column vector $\mathcal{M}_A$ and respectively $\mathcal{M}_B$ contain the membership degrees of a and b:

$$\mathcal{M}_A = \begin{pmatrix} \mu_{A_1}(a) \\ \mu_{A_2}(a) \\ \vdots \\ \mu_{A_{m_a}}(a) \end{pmatrix}, \quad \mathcal{M}_B = \begin{pmatrix} \mu_{B_1}(b) \\ \mu_{B_2}(b) \\ \vdots \\ \mu_{B_{m_b}}(b) \end{pmatrix}. \tag{7}$$

Let a set of linear independent m_a by one column vectors $f_A^{(q)}$, $q = \overline{1, m_a}$, be selected for variable a and, similarly, a linear independent set of m_b by one column vectors $f_B^{(l)}$, $l = \overline{1, m_b}$, be selected for variable b. One can associate with each vector $f_A^{(q)} = [f_A^{(q)}(1)\ f_A^{(q)}(2) \ldots f_A^{(q)}(m_a)]^T$ a subsystem $A^{(q)}$ with the following fuzzy rules: *Subsystem* $A^{(q)}$: *Rule* i : $A_i \to \delta(f_A^{(q)}(i))$, *for* $i = \overline{1, m_a}$.

The inferred output of subsystem $A^{(q)}$ becomes

$$\mathcal{I}_a(f_A^{(q)}) = \frac{\left(\sum_{i=1}^{m_a} f_A^{(q)}(i) \cdot \mu_{A_i}(a) \right)}{\sum_{i=1}^{m_a} \mu_{A_i}(a)} = \frac{\mathcal{M}_A^T \cdot f_A^{(q)}}{\sum_{i=1}^{m_a} \mu_{A_i}(a)}.$$

Similarly, for each column vector $f_B^{(l)} = [f_B^{(l)}(1)\ f_B^{(l)}(2) \dots f_B^{(l)}(m_b)]^T$, one constructs a subsystem $B^{(l)}$ with the fuzzy rules:

$$Subsystem\ B^{(l)}:\ Rule\ j:\ B_j \to \delta(f_B^{(l)}(j)),\ for\ j = \overline{1, m_b}$$

and inferred output

$$\mathcal{I}_b(f_B^{(l)}) = \frac{\left(\sum_{j=1}^{m_b} f_B^{(l)}(j) \cdot \mu_{B_j}(b)\right)}{\sum_{j=1}^{m_b} \mu_{B_j}(b)} = \frac{(f_B^{(l)})^T \cdot \mathcal{M}_B}{\sum_{j=1}^{m_b} \mu_{B_j}(b)}.$$

The selection of vectors $f_A^{(q)}$ and $f_B^{(l)}$ should depend on the kind of approximation function one desires for the problem at hand, be it polynomial, sinusoidal, or other designs.

The inferred output $\mathcal{I}_{a,b}(U_{m_a,m_b})$ is now expressible in terms of the inferred outputs $\mathcal{I}_a(f_A^{(q)})$ and $\mathcal{I}_b(f_B^{(l)})$. Using the fact that the $m_a \times m_a$ and $m_b \times m_b$ matrices

$$F_A = (f_A^{(1)},\ f_A^{(2)}, \dots, f_A^{(m_a)})$$

and

$$F_B = (f_B^{(1)},\ f_B^{(2)}, \dots, f_B^{(m_b)})$$

are invertible, (6) becomes

$$\mathcal{I}_{a,b}(U_{m_a,m_b}) = \left(\frac{\mathcal{M}_A^T \cdot F_A}{\sum_{i=1}^{m_a} \mu_{A_i}(a)}\right) \cdot \nu_{m_a,m_b} \cdot \left(\frac{(F_B)^T \cdot \mathcal{M}_B}{\sum_{j=1}^{m_b} \mu_{B_j}(b)}\right), \tag{8}$$

where the $m_a \times m_b$ matrix ν_{m_a,m_b} is defined by

$$\nu_{m_a,m_b} \equiv F_A^{-1} \cdot U_{m_a,m_b} \cdot (F_B^T)^{-1}. \tag{9}$$

With

$$\left(\frac{\mathcal{M}_A^T \cdot F_A}{\sum_{i=1}^{m_a} \mu_{A_i}(a)}\right) = \left[\frac{\mathcal{M}_A^T \cdot f_A^{(1)}}{\sum_{i=1}^{m_a} \mu_{A_i}(a)},\ \frac{\mathcal{M}_A^T \cdot f_A^{(2)}}{\sum_{i=1}^{m_a} \mu_{A_i}(a)}, \dots, \frac{\mathcal{M}_A^T \cdot f_A^{(m_a)}}{\sum_{i=1}^{m_a} \mu_{A_i}(a)}\right] =$$

$$= \left(\mathcal{I}_a(f_A^{(1)}),\ \mathcal{I}_a(f_A^{(2)}), \dots, \mathcal{I}_a(f_A^{(m_a)})\right) \tag{10}$$

and similarly,

$$\left(\frac{(F_B)^T \cdot \mathcal{M}_B}{\sum_{j=1}^{m_b} \mu_{B_j}(b)}\right) = \begin{pmatrix} \mathcal{I}_b(f_B^{(1)}) \\ \mathcal{I}_b(f_B^{(2)}) \\ \vdots \\ \mathcal{I}_b(f_B^{(m_b)}) \end{pmatrix} \tag{11}$$

(8) becomes

$$\mathcal{I}_{a,b}(U_{m_a,m_b}) = \left(\mathcal{I}_a(f_A^{(1)}),\ \mathcal{I}_a(f_A^{(2)}),\ldots,\mathcal{I}_a(f_A^{(m_a)})\right)\cdot\nu_{m_a,m_b}\cdot\begin{pmatrix}\mathcal{I}_b(f_B^{(1)})\\ \mathcal{I}_b(f_B^{(2)})\\ \vdots\\ \mathcal{I}_b(f_B^{(m_b)})\end{pmatrix} =$$

$$= \sum_{q=1}^{m_a}\sum_{l=1}^{m_b}\nu_{q,l}\cdot\mathcal{I}_a(f_A^{(q)})\cdot\mathcal{I}_b(f_B^{(l)}), \tag{12}$$

where $\nu_{q,l}$ are elements of the matrix ν_{m_a,m_b}.

To recover U_{m_a,m_b} from ν_{m_a,m_b}, one has

$$U_{m_a,m_b} = F_A\cdot\nu_{m_a,m_b}\cdot(F_B^T). \tag{13}$$

Equation (12) expresses the original inference in terms of multiplicative subsystem inferences of single variable a and b. A fuzzy system of two input variables is hence generally expressible as additive sum of $m_a \times m_b$ systems each of which is multiplicative decomposable into two subsystems of single variable. The previous result can be extended to a fuzzy system with a general number of n input variables. The inferred output is expressible in the following form:

$$\mathcal{I}_{a,b,c,d,\ldots,x,y,z}(U_{m_a,m_b,m_c,m_d,\ldots,m_x,m_y,m_z}) =$$

$$= \sum_{i=1}^{m_a}\sum_{j=1}^{m_b}\sum_{k=1}^{m_c}\sum_{s=1}^{m_d}\cdots\sum_{t=1}^{m_x}\sum_{h=1}^{m_y}\sum_{r=1}^{m_z}\nu_{i,j,k,s,\ldots,t,h,r}\cdot$$

$$\cdot\mathcal{I}_a(f_A^{(i)})\cdot\mathcal{I}_b(f_B^{(j)})\cdot\mathcal{I}_c(f_C^{(k)})\cdot\mathcal{I}_d(f_D^{(s)})\cdot\ldots\cdot\mathcal{I}_x(f_X^{(t)})\cdot\mathcal{I}_y(f_Y^{(h)})\cdot\mathcal{I}_z(f_Z^{(r)}). \tag{14}$$

Equation (14) expresses a general fuzzy system of n variables into $m_a \times m_b \times \ldots \times m_y \times m_z$ systems each of which is multiplicative decomposable into n subsystems of single variable.

3 Subsystem Inference of Single Variable

This section is devoted to gaining a better understanding of single variable subsystem inference. Let the subsystem in consideration be $A^{(q)}$ with $m_a = 3$ input membership function, namely

$$Subsystem\ A^{(q)}:\ A_i \to \delta(f_A^{(q)}(i)),\ for\ i = \overline{1, m_a}.$$

The inferred output is

$$\mathcal{I}_a(f_A^{(q)}) = \frac{\sum_{i=1}^{3} f_A^{(q)}(i)\cdot\mu_{A_i}(a)}{\sum_{i=1}^{3}\mu_{A_i}(a)}. \tag{15}$$

If the input membership function $A_i,\ i = \overline{1,3}$ are triangular with at most two overlapping and sum of membership degrees equals one (see Fig.1) then (15) becomes

$$\mathcal{I}_a(f_A^{(q)}) = \sum_{i=1}^{3} f_A^{(q)}(i) \cdot \mu_{A_i}(a). \tag{16}$$

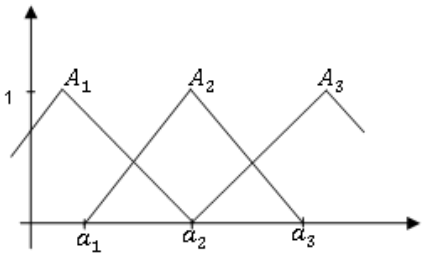


Fig. 1. Triangular input membership functions.

Since the membership degrees are given as

$$\mu_{A_1}(a) = \begin{cases} 1 - \frac{a-a_1}{a_2-a_1},\ a_1 \le a \le a_2, \\ \\ 0,\ a_2 < a \le a_3, \end{cases}$$

$$\mu_{A_2}(a) = \begin{cases} \frac{a-a_1}{a_2-a_1},\ a_1 \le a \le a_2, \\ \\ 1 - \frac{a-a_2}{a_3-a_2},\ a_2 < a \le a_3, \end{cases}$$

$$\mu_{A_3}(a) = \begin{cases} 0,\ a_1 \le a \le a_2, \\ \\ \frac{a-a_2}{a_3-a_2},\ a_2 < a \le a_3, \end{cases}$$

we shall have

$$\mathcal{I}_a(f_A^{(q)}) = \begin{cases} f_A^{(q)}(1) + (f_A^{(q)}(2) - f_A^{(q)}(1)) \cdot \frac{a-a_1}{a_2-a_1},\ for\ a_1 \le a \le a_2, \\ f_A^{(q)}(2) + (f_A^{(q)}(3) - f_A^{(q)}(2)) \cdot \frac{a-a_2}{a_3-a_2},\ for\ a_2 < a \le a_3. \end{cases} \tag{17}$$

Fig. 2 shows $\mathcal{I}_a(f_A^{(q)})$ as a function of a. As a increases from a_1 to a_2, $\mathcal{I}_a(f_A^{(q)})$ follows a straight line from $f_A^{(q)}(1)$ to $f_A^{(q)}(2)$, and then from $f_A^{(q)}(2)$ to $f_A^{(q)}(3)$ as a increases from a_2 to a_3.

Extending to the case of a general m_a, $\mathcal{I}_a(f_A^{(q)})$ would follow a continuous, piecewise straight line joining the points $f_A^{(q)}(i)$ at $a = a_i,\ i = \overline{1, m_a}$.

Fig 3 depicts the case where the input membership function are not triangular but still at most two of them overlap each other with sum of membership degrees equals one. Let A_1 deviates from being triangular by an amount of $Q_{1,2}$ within $a_1 \le a \le a_2$.

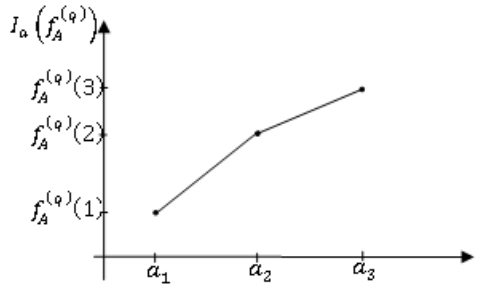


Fig. 2. Subsystem inference with triangular input membership functions.

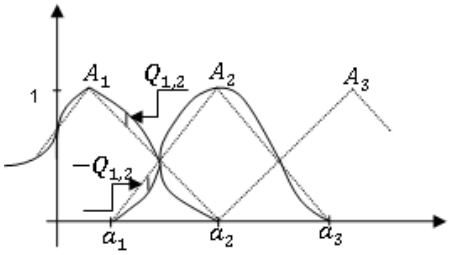


Fig. 3. Non triangular input membership functions.

For simplicity, $Q_{1,2}$ is assumed here to be zero at $a = a_1$ and $a = a_2$. In the case when $a_1 \leq a \leq a_2$, the membership degrees become

$$\mu_{A_1}(a) = 1 - \frac{a - a_1}{a_2 - a_1} + Q_{1,2}, \quad \mu_{A_2}(a) = \frac{a - a_1}{a_2 - a_1} - Q_{1,2}$$

and

$$\mathcal{I}_a(f_A^{(q)}) = f_A^{(q)}(1) + (f_A^{(q)}(2) - f_A^{(q)}(1)) \cdot \frac{a - a_1}{a_2 - a_1} - (f_A^{(q)}(2) - f_A^{(q)}(1)) \cdot Q_{1,2}. \quad (18)$$

The first two terms of (18) correspond to $\mathcal{I}_a(f_A^{(q)})$ when A_1 and A_2 are triangular. The third term gives the results when A_1 and A_2 deviate from being triangular which is equal to $-(f_A^{(q)}(2) - f_A^{(q)}(1)) \cdot Q_{1,2}$. The resulting $\mathcal{I}_a(f_A^{(q)})$ is depicted in Fig. 4 for $a_1 \leq a \leq a_2$. A similar derivation follows for $a_2 < a \leq a_3$.

Subsystem inferences for general profiles of input membership functions will be obtain similarly.

Fig 5 shows the case where the input membership function are both trapezoidal and triangular but still at most two of them overlap each other with sum of membership degrees equals one.

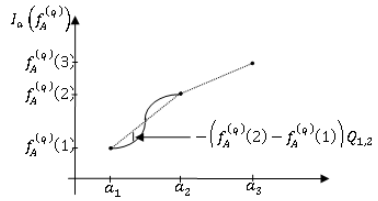


Fig. 4. Subsystem inference with non triangular input membership functions.

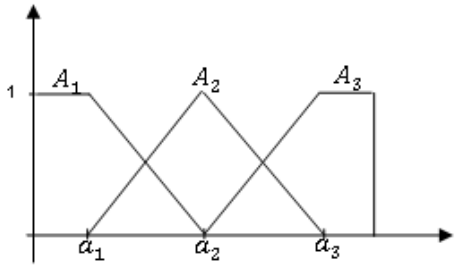


Fig. 5. The membership functions for small, medium and big.

$$\mu_{A_1}(a) = \begin{cases} 1 - \frac{a-a_1}{a_2-a_1}, \ a_1 \leq a \leq a_2, \\ \\ 0, \ a_2 < a \leq a_3, \end{cases}$$

$$\mu_{A_2}(a) = \begin{cases} \frac{a-a_1}{a_2-a_1}, \ a_1 \leq a \leq a_2, \\ \\ 1 - \frac{a-a_2}{a_3-a_2}, \ a_2 < a \leq a_3, \end{cases}$$

$$\mu_{A_3}(a) = \begin{cases} 0, \ a_1 \leq a \leq a_2, \\ \\ \frac{a-a_2}{a_3-a_2}, \ a_2 < a \leq a_3, \end{cases}$$

4 Polynomial Subsystem Inferences

A special choice of subsystem inferences is now introduced for the triangular input membership functions depicted in Fig. 1.

Let the set of linear independent vectors $f_A^{(q)}, \ q = \overline{1, m_a}$, be selected as follows:

$$f_A^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad f_A^{(2)} = \begin{pmatrix} 0 \\ (a_2 - a_1) \\ (a_3 - a_1) \\ \vdots \\ (a_{m_a-1} - a_1) \\ (a_{m_a} - a_1) \end{pmatrix}, \ldots,$$

$$\ldots, \quad f_A^{(q)} = \begin{pmatrix} 0 \\ (a_2 - a_1)^{q-1} \\ (a_3 - a_1)^{q-1} \\ \vdots \\ (a_{m_a-1} - a_1)^{q-1} \\ (a_{m_a} - a_1)^{q-1} \end{pmatrix}, \ldots, \quad f_A^{(m_a)} = \begin{pmatrix} 0 \\ (a_2 - a_1)^{m_a-1} \\ (a_3 - a_1)^{m_a-1} \\ \vdots \\ (a_{m_a-1} - a_1)^{m_a-1} \\ (a_{m_a} - a_1)^{m_a-1} \end{pmatrix} \quad (19)$$

The resulting subsystem inference $\mathcal{I}_a(f_A^{(q)})$ depicting in Fig. 6 shows a continuous piecewise straight line joining the points $(a_i,\ (a_i - a_1)^{q-1})$.

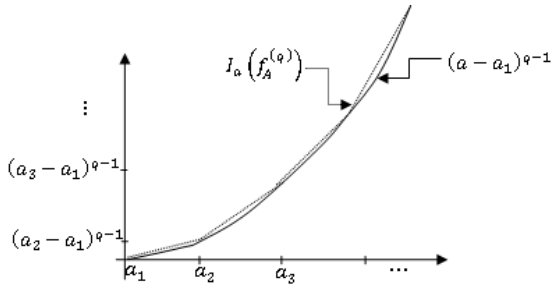


Fig. 6. Polynomial subsystem inference.

The inference $\mathcal{I}_a(f_A^{(q)})$ constitutes a piecewise linear approximation to the polynomial term $(a - a_1)^{q-1}$.

The vectors $f_A^{(q)}$ in (19) are termed *polynomial* subsystem vectors and the corresponding set of $\mathcal{I}_a(f_A^{(q)})$ *polynomial* subsystem inferences.

With column vectors $f_B^{(j)},\ j = \overline{1, m_b}$, selected similarly for variable b

$$f_B^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, f_B^{(2)} = \begin{pmatrix} 0 \\ (b_2 - b_1) \\ (b_3 - b_1) \\ \vdots \\ (b_{m_b-1} - b_1) \\ (b_{m_b} - b_1) \end{pmatrix}, \ldots, f_B^{(m_b)} = \begin{pmatrix} 0 \\ (b_2 - b_1)^{m_b-1} \\ (b_3 - b_1)^{m_b-1} \\ \vdots \\ (b_{m_b-1} - b_1)^{m_b-1} \\ (b_{m_b} - b_1)^{m_b-1} \end{pmatrix} \quad (20)$$

and substituting $\mathcal{I}_a(f_A^{(1)}) = 1$ and $\mathcal{I}_b(f_B^{(1)}) = 1$ in (12), the inferred output becomes

$$\mathcal{I}_{a,b}(U_{m_a,m_b}) = \nu_{1,1} + \sum_{l=2}^{m_b} \nu_{1,l} \cdot \mathcal{I}_b(f_B^{(l)}) + \sum_{q=2}^{m_a} \nu_{q,1} \cdot \mathcal{I}_a(f_A^{(q)}) +$$

$$+ \sum_{q=2}^{m_a} \sum_{l=2}^{m_b} \nu_{q,l} \cdot \mathcal{I}_a(f_A^{(q)}) \cdot \mathcal{I}_b(f_B^{(l)}). \tag{21}$$

This shows that $\mathcal{I}_{a,b}(U_{m_a,m_b})$ is an approximation to the polynomial function

$$p(a,b) = \nu_{1,1} + \sum_{l=2}^{m_b} \nu_{1,l} \cdot (b - b_1)^{l-1} + \sum_{q=2}^{m_a} \nu_{q,1} \cdot (a - a_1)^{q-1} +$$

$$+ \sum_{q=2}^{m_a} \sum_{l=2}^{m_b} \nu_{q,l} \cdot (a - a_1)^{q-1} \cdot (b - b_1)^{l-1}. \tag{22}$$

The function $p(a,b)$ contains polynomial terms up to orders of $m_a - 1$ and $m_b - 1$ in a and b. For the present case where input membership functions of Fig. 1 are used, $\mathcal{I}_{a,b}(U_{m_a,m_b})$ yields exact values of $p(a,b)$ at the $m_a \times m_b$ rectangular grid points (a_q, b_l), $q = \overline{1, m_a}$, $l = \overline{1, m_b}$ and approximates in between points via linear interpolation over the corner values.

Conversely, the function $p(a,b)$ can be viewed as an approximate output of the fuzzy system. Subsystem inference representation produces an approximate functional characterization of the inferred output.

The approximation error $E(a,b)$ between $\mathcal{I}_{a,b}(U_{m_a,m_b})$ and $p(a,b)$ is given by

$$E(a,b) = \mathcal{I}_{a,b}(U_{m_a,m_b}) - p(a,b) = \sum_{l=2}^{m_b} \nu_{1,l} \cdot E^{(1,l)} + \sum_{q=2}^{m_a} \nu_{q,1} \cdot E^{(q,1)} +$$

$$+ \sum_{q=2}^{m_a} \sum_{l=2}^{m_b} \nu_{q,l} \cdot E^{(q,l)}, \tag{23}$$

where $E^{(q,l)}$ is the error of the (q,l) term

$$E^{(q,l)} = \mathcal{I}_a(f_A^{(q)}) \cdot \mathcal{I}_b(f_B^{(l)}) - (a - a_1)^{q-1} \cdot (b - b_1)^{l-1}. \tag{24}$$

The results can be extended to a system with n fuzzy variables $a, b, c, \ldots, y, z$ having $m_a, m_b, m_c, \ldots, m_y, m_z$ input membership functions. The approximating polynomial function would be of order up to $m_a - 1$, $m_b - 1$, $m_c - 1, \ldots$, and $m_z - 1$ in $a, b, c, \ldots$ and z.

Example Consider a two variable fuzzy system with $m_a = m_b = 3$. Let the input membership functions A_i and B_j be located at $a_1 = -0.5$, $a_2 = 0$, $a_3 = 0.5$, and $b_1 = -2$, $b_2 = 0$, $b_3 = 2$. The rule consequents $u_{i,j}$ are embedded in the matrix

$$U_{3,3} = \begin{pmatrix} -11 & -5 & -1 \\ -5 & 0 & 5 \\ 2 & 6 & 10 \end{pmatrix}.$$

Let the vectors $f_A^{(i)}$ and $f_B^{(j)}$ be selected according to (19)

$$f_A^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad f_A^{(2)} = \begin{pmatrix} -0.5 \\ 0 \\ 0.5 \end{pmatrix}, \quad f_A^{(3)} = \begin{pmatrix} 0.25 \\ 0 \\ 0.25 \end{pmatrix},$$

$$f_B^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad f_B^{(2)} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}, \quad f_B^{(3)} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}.$$

With $F_A = (f_A^{(1)},\ f_A^{(2)}, f_A^{(3)})$ and $F_B = (f_B^{(1)},\ f_B^{(2)}, f_B^{(3)})$, using (9) we shall have

$$\nu_{3,3} \equiv F_A^{-1} \cdot U_{3,3} \cdot \left(F_B^T\right)^{-1} = \begin{pmatrix} 0 & 2.5 & 0 \\ 11 & -0.5 & 0.25 \\ 2 & -1 & -0.5 \end{pmatrix}.$$

This yields a subsystem inference representation for the system (see (21)):

$$\mathcal{I}_{a,b}(U_{3,3}) = 2.5 \cdot \mathcal{I}_b(f_B^{(2)}) + 11 \cdot \mathcal{I}_a(f_A^{(2)}) + 2 \cdot \mathcal{I}_a(f_A^{(3)}) - 0.5 \cdot \mathcal{I}_a(f_A^{(2)}) \cdot \mathcal{I}_b(f_B^{(2)}) +$$

$$+0.25 \cdot \mathcal{I}_a(f_A^{(2)}) \cdot \mathcal{I}_b(f_B^{(3)}) - \mathcal{I}_a(f_A^{(3)}) \cdot \mathcal{I}_b(f_B^{(2)}) - 0.5 \cdot \mathcal{I}_a(f_A^{(3)}) \cdot \mathcal{I}_b(f_B^{(3)}). \quad (25)$$

Equation (25) indicates that $\mathcal{I}_{a,b}(U_{3,3})$ is an approximation of the polynomial function

$$p(a,b) = 2.5 \cdot b + 11 \cdot a + 2 \cdot a^2 - 0.5 \cdot a \cdot b + 0.25 \cdot a \cdot b^2 - a^2 \cdot b - 0.5 \cdot a^2 \cdot b^2.$$

The error of approximation from (23) is

$$E(a,b) = 2.5 \cdot E^{(1,2)} + 11 \cdot E^{(2,1)} + 2 \cdot E^{(3,1)} - 0.5 \cdot E^{(2,2)} + 0.25 \cdot E^{(2,3)} -$$

$$-E^{(3,2)} - 0.5 \cdot E^{(3,3)},$$

where, using (24)

$$E^{(q,l)} = \mathcal{I}_a(f_A^{(q)}) \cdot \mathcal{I}_b(f_B^{(l)}) - a^{q-1} \cdot b^{l-1},\ (\forall)\ q, l = \overline{1,3}.$$

Of the seven terms in (25), the first three are subsystem inferences of a or b only. Together, they comprise an additive decomposable system.

The remaining terms of (25) are cross products of subsystem inferences of a and b, and each constituting a multiplicative decomposable system.

Kosko [9], Wang [16] and later Alci [1], Kim [8] have shown that fuzzy systems are universal approximators to continuous functions on compact domain.

The same conclusion can be reached here with subsystem inference representation.

A fuzzy system is expressible in terms of polynomial inferences, as in (21). As m_a and m_b tend to large values, the polynomial inferences $\mathcal{I}_a(f_A^{(i)})$ and $\mathcal{I}_b(f_B^{(j)})$ converge uniformly to the polynomial terms $a^{(i-1)}$ and $b^{(j-1)}$.

Since polynomials are universal approximators (see [12]), the same can be concluded regarding fuzzy system.

5 Other Designs of Subsystem Inferences

The vectors $f_A^{(q)},\ q=\overline{1,m_a}$ can also be selected to emulate other functions. In the example of sinusoidal subsystem inferences

$$f_A^{(1)}=\begin{pmatrix}1\\1\\1\\\vdots\\1\\1\end{pmatrix}, f_A^{(2)}=\begin{pmatrix}0\\\cos(a_1-\tilde{a})\\\cos(a_2-\tilde{a})\\\vdots\\\cos(a_{m_a-1}-\tilde{a})\\\cos(a_{m_a}-\tilde{a})\end{pmatrix}, \ldots, f_A^{(3)}=\begin{pmatrix}0\\\sin(a_1-\tilde{a})\\\sin(a_2-\tilde{a})\\\vdots\\\sin(a_{m_a-1}-\tilde{a})\\\sin(a_{m_a}-\tilde{a})\end{pmatrix},$$

$$\ldots, f_A^{(2r)}=\begin{pmatrix}\cos(r\cdot(a_1-\tilde{a}))\\\cos(r\cdot(a_2-\tilde{a}))\\\cos(r\cdot(a_3-\tilde{a}))\\\vdots\\\cos(r\cdot(a_{m_a-1}-\tilde{a}))\\\cos(r\cdot(a_{m_a}-\tilde{a}))\end{pmatrix}, f_A^{(2r+1)}=\begin{pmatrix}\sin(r\cdot(a_1-\tilde{a}))\\\sin(r\cdot(a_2-\tilde{a}))\\\sin(r\cdot(a_3-\tilde{a}))\\\vdots\\\sin(r\cdot(a_{m_a-1}-\tilde{a}))\\\sin(r\cdot(a_{m_a}-\tilde{a}))\end{pmatrix}, \quad (26)$$

and depending on whether m_a is even or odd

$$f_{A\ (m_a\ even)}^{(m_a)}=\begin{pmatrix}\cos(\frac{m_a}{2}\cdot(a_1-\tilde{a}))\\\cos(\frac{m_a}{2}\cdot(a_2-\tilde{a}))\\\cos(\frac{m_a}{2}\cdot(a_3-\tilde{a}))\\\vdots\\\cos(\frac{m_a}{2}\cdot(a_{m_a-1}-\tilde{a}))\\\cos(\frac{m_a}{2}\cdot(a_{m_a}-\tilde{a}))\end{pmatrix},$$

$$f_{A\ (m_a\ odd)}^{(m_a)}=\begin{pmatrix}\sin(\frac{m_a-1}{2}\cdot(a_1-\tilde{a}))\\\sin(\frac{m_a-1}{2}\cdot(a_2-\tilde{a}))\\\sin(\frac{m_a-1}{2}\cdot(a_3-\tilde{a}))\\\vdots\\\sin(\frac{m_a-1}{2}\cdot(a_{m_a-1}-\tilde{a}))\\\sin(\frac{m_a-1}{2}\cdot(a_{m_a}-\tilde{a}))\end{pmatrix}. \quad (27)$$

The vectors (26) and (27) give rise to subsystem inferences $\mathcal{I}_a(f_A^{(q)})$ which approximate the sinusoidal terms of $\sin(r\cdot(a-\tilde{a}))$ and $\cos(r\cdot(a-\tilde{a}))$. Same as before for polynomial inferences, should the input membership functions of Fig.1 be adopted, the fuzzy inferred output constitutes a piecewise linear approximation of the sinusoidal function; $\mathcal{I}_{a,b}(U_{m_a,m_b})$ will be an approximation to:

1) the sinusoidal function (if m_a is odd):

$$p(a,b)=\nu_{1,1}+\sum_{l=2}^{m_b}\nu_{1,l}\sin\left(\frac{l-1}{2}(b-\tilde{b})\right)+\sum_{q=2}^{m_a}\nu_{q,1}\cdot\sin\left(\frac{q-1}{2}(a-\tilde{a})\right)$$
$$+\sum_{q=2}^{m_a}\sum_{l=2}^{m_b}\nu_{q,l}\cdot\sin\left(\frac{q-1}{2}\cdot(a-\tilde{a})\right)\cdot\sin\left(\frac{l-1}{2}\cdot(b-\tilde{b})\right) \tag{28}$$

2) the cosinusoidal function (if m_a is even):

$$p(a,b)=\nu_{1,1}+\sum_{l=2}^{m_b}\nu_{1,l}\cdot\cos\left(\frac{l}{2}\cdot(b-\tilde{b})\right)+\sum_{q=2}^{m_a}\nu_{q,1}\cdot\cos\left(\frac{q}{2}\cdot(a-\tilde{a})\right)$$
$$+\sum_{q=2}^{m_a}\sum_{l=2}^{m_b}\nu_{q,l}\cdot\cos\left(\frac{q}{2}\cdot(a-\tilde{a})\right)\cdot\cos\left(\frac{l}{2}\cdot(b-\tilde{b})\right). \tag{29}$$

With appropriate designs, sinusoidal inferences can be further manipulated into an orthonormal set. Define $f_A^{(q)}),\ q=\overline{1,m_a}$ as

$$f_A^{(1)}=\frac{1}{\sqrt{m_a}}\begin{pmatrix}1\\1\\1\\\vdots\\1\\1\end{pmatrix},\dots,$$

$$f_A^{(2r)}=\sqrt{\frac{2}{m_a}}\begin{pmatrix}\cos(\frac{2\pi}{m_a}\cdot\ r\cdot(1-\varepsilon))\\\cos(\frac{2\pi}{m_a}\cdot\ r\cdot(2-\varepsilon))\\\cos(\frac{2\pi}{m_a}\cdot\ r\cdot(3-\varepsilon))\\\vdots\\\cos(\frac{2\pi}{m_a}\cdot\ r\cdot(m_a-1-\varepsilon))\\\cos(\frac{2\pi}{m_a}\cdot\ r\cdot(m_a-\varepsilon))\end{pmatrix},$$

$$f_A^{(2r+1)} = \sqrt{\frac{2}{m_a}} \begin{pmatrix} \sin(\frac{2\pi}{m_a} \cdot \ r \cdot (1-\varepsilon)) \\ \sin(\frac{2\pi}{m_a} \cdot \ r \cdot (2-\varepsilon)) \\ \sin(\frac{2\pi}{m_a} \cdot \ r \cdot (3-\varepsilon)) \\ \vdots \\ \sin(\frac{2\pi}{m_a} \cdot \ r \cdot (m_a-1-\varepsilon)) \\ \sin(\frac{2\pi}{m_a} \cdot \ r \cdot (m_a-\varepsilon)) \end{pmatrix}, \tag{30}$$

and and depending on whether m_a is even or odd

$$\begin{matrix} f_A^{(m_a)} \\ (m_a \ even) \end{matrix} = \sqrt{\frac{2}{m_a}} \begin{pmatrix} \cos(\frac{2\pi}{m_a} \cdot \frac{m_a}{2} \cdot (1-\varepsilon)) \\ \cos(\frac{2\pi}{m_a} \cdot \frac{m_a}{2} \cdot (2-\varepsilon)) \\ \cos(\frac{2\pi}{m_a} \cdot \frac{m_a}{2} \cdot (3-\varepsilon)) \\ \vdots \\ \cos(\frac{2\pi}{m_a} \cdot \frac{m_a}{2} \cdot (m_a-1-\varepsilon))) \\ \cos(\frac{2\pi}{m_a} \cdot \frac{m_a}{2} \cdot (m_a-\varepsilon)) \end{pmatrix},$$

$$\begin{matrix} f_A^{(m_a)} \\ (m_a \ odd) \end{matrix} = \sqrt{\frac{2}{m_a}} \begin{pmatrix} \sin(\frac{2\pi}{m_a} \cdot \frac{m_a-1}{2} \cdot (1-\varepsilon)) \\ \sin(\frac{2\pi}{m_a} \cdot \frac{m_a-1}{2} \cdot (2-\varepsilon)) \\ \sin(\frac{2\pi}{m_a} \cdot \frac{m_a-1}{2} \cdot (3-\varepsilon)) \\ \vdots \\ \sin(\frac{2\pi}{m_a-1} \cdot \frac{m_a}{2} \cdot (m_a-1-\varepsilon))) \\ \sin(\frac{2\pi}{m_a} \cdot \frac{m_a-1}{2} \cdot (m_a-\varepsilon)) \end{pmatrix}, \tag{31}$$

where ε is an integer parameter, between 1 and m_a.

The vectors in (30) and (31) satisfy the orthonormal condition

$$(f_A^{(q_1)})^T \cdot f_A^{(q_2)} = \delta_{q_1,q_2}. \tag{32}$$

Equation (32) can be readily observed when q_1 and q_2 are both numbers, i.e, $q_1 = 2r_1 + 1$ and $q_2 = 2r_2 + 1$ for $r_1, r_2 \geq 1$. In this case

$$(f_A^{(q_1)})^T \cdot f_A^{(q_2)} =$$

$$= \frac{1}{m_a} \sum_{i=1}^{m_a} \left[\cos\left(\frac{2\pi \cdot (i-\varepsilon)}{m_a} \cdot (r_1 - r_2) \right) - \cos\left(\frac{2\pi \cdot (i-\varepsilon)}{m_a} \cdot (r_1 + r_2) \right) \right]$$

and orthonormality is derived using the formulae from Sireţchi [14]:

$$\sum_{i=0}^{n-1}\cos(\theta+i\cdot h)=\frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}\cos\left(\theta+\frac{n-1}{2}h\right),$$

$$\sum_{i=0}^{n-1}\sin(\theta+i\cdot h)=\frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}\sin\left(\theta+\frac{n-1}{2}h\right)$$

from which we have for integer $p<K$:

$$\sum_{k=1}^{K}\sin\left(\frac{2\pi\cdot k}{K}\cdot p\right)=0,$$

$$\sum_{k=1}^{K}\cos\left(\frac{2\pi\cdot k}{K}\cdot p\right)=\begin{cases}0\ for\ p\neq 0,\\ \\ K\ for\ p=0.\end{cases}$$

A similar approach can be used to show orthonormality for other cases of q_1 and q_2.

In the example of exponential subsystem inferences,

$$f_A^{(1)}=\begin{pmatrix}1\\1\\1\\\vdots\\1\\1\end{pmatrix},\quad f_A^{(2)}=\begin{pmatrix}1\\e^{(a_2-a_1)}\\e^{(a_3-a_1)}\\\vdots\\e^{(a_{m_a-1}-a_1)}\\e^{(a_{m_a}-a_1)}\end{pmatrix},\ldots,$$

$$\ldots,\quad f_A^{(q)}=\begin{pmatrix}1\\e^{(q-1)\cdot(a_2-a_1)}\\e^{(q-1)\cdot(a_3-a_1)}\\\vdots\\e^{(q-1)\cdot(a_{m_a-1}-a_1)}\\e^{(q-1)\cdot(a_{m_a}-a_1)}\end{pmatrix},\ldots,\quad f_A^{(m_a)}=\begin{pmatrix}1\\e^{(m_a-1)\cdot(a_2-a_1)}\\e^{(m_a-1)\cdot(a_3-a_1)}\\\vdots\\e^{(m_a-1)\cdot(a_{m_a-1}-a_1)}\\e^{(m_a-1)\cdot(a_{m_a}-a_1)}\end{pmatrix}$$

the inference $\mathcal{I}_a(f_A^{(q)})$ constitutes a piecewise linear approximation to the exponential term $e^{(q-1)\cdot(a-a_1)}$.

6 Applications

Application 1 We shall use the polynomial and exponential inferences together for the fuzzy approximation of the differential equation solution

$$\begin{cases} y''' = xe^{-x}, \\ y(0) = 0, \\ y'(0) = 2, \\ y''(0) = 2. \end{cases} \tag{33}$$

We shall consider the fuzzy systems with two fuzzy variables: a is for approximating the polynomial term of x, and variables b is for realizing the exponential term e^{-x}.

Trapezoidal input membership functions of Fig. 8 and Fig. 9 are assumed for all variables.

Let the domains of interest for a and b be $[-0.7, 0.3]$ and $[0.2, 0.8]$.

Trapezoidal input membership functions for the fuzzy variables: a

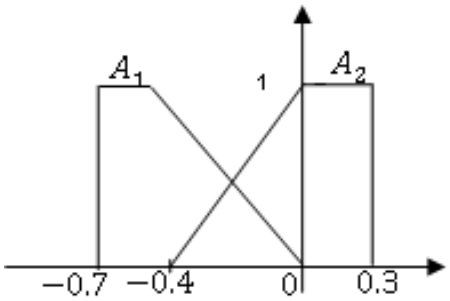


Fig. 7. Trapezoidal input membership functions for the fuzzy variables: a.

$$\mu_{A_1}(a) = \begin{cases} 1, \ x \in [-0.7, -0.4], \\ 1 - \frac{x+0.4}{0.4}, \ x \in (-0.4, 0], \end{cases}$$

$$\mu_{A_2}(a) = \begin{cases} \frac{x+0.4}{0.4}, \ x \in [-0.4, 0], \\ 1, \ x \in (0, 0.3], \end{cases}$$

For the variables a and b, one sets:

$$m_a = m_b = 2,$$

$$a_1 = -0.4, \ a_2 = 0, \ b_1 = 0.3, \ b_2 = 0.7,$$

$$f_A^{(1)} = f_B^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$f_A^{(2)} = \begin{pmatrix} 0 \\ a_2 - a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.4 \end{pmatrix}, \ f_B^{(2)} = \begin{pmatrix} 1 \\ e^{b_1 - b_2} \end{pmatrix} = \begin{pmatrix} 1 \\ e^{-0.4} \end{pmatrix}.$$

The inferred output of subsystem $A^{(2)}$ becomes

$$\mathcal{I}_a(f_A^{(2)}) = f_A^{(2)}(1) \cdot \mu_{A_1}(a) + f_A^{(2)}(2) \cdot \mu_{A_2}(a),$$

namely

$$\mathcal{I}_a(f_A^{(2)}) = \begin{cases} 0.4 \cdot \frac{x+0.4}{0.4}, \ x \in [-0.4, 0], \\ 0.4, \ x \in (0, 0.3]; \end{cases}$$

Trapezoidal input membership functions for the fuzzy variables: b

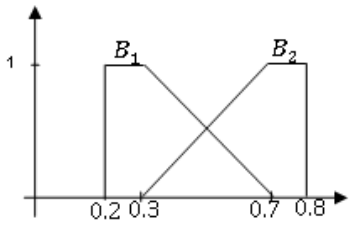


Fig. 8. Trapezoidal input membership functions for the fuzzy variable b.

$$\mu_{B_1}(b) = \begin{cases} 1, \ b \in [0.2, 0.3), \\ 1 - \frac{b-0.3}{0.4}, \ b \in [0.3, 0.7] \end{cases}$$

$$\mu_{B_2}(b) = \begin{cases} \frac{b-0.3}{0.4}, b \in [0.3, 0.7], \\ 1, \ b \in (0.7, 0.8], \end{cases}$$

The inferred output of subsystem $B^{(2)}$ becomes

$$\mathcal{I}_b(f_B^{(2)}) = f_B^{(2)}(1) \cdot \mu_{B_1}(b) + f_B^{(2)}(2) \cdot \mu_{B_2}(b),$$

namely

$$\mathcal{I}_b(f_B^{(2)}) = \begin{cases} 1, \ b \in [0.2, 0.3) \\ 1 + (1 - e^{-0.4}) \cdot \frac{b-0.3}{0.4}, \ b \in [0.3, 0.7], \\ e^{-0.4}, \ b \in (0.7, 0.8]; \end{cases}$$

therefore

$$\mathcal{I}_b(f_B^{(2)}) = \begin{cases} e^{-0.4},\ x \in [0.2231, 0.3567); \\ 1 + (1 - e^{-0.4}) \cdot \frac{b-0.3}{0.4},\ x \in [0.3567, 1.204], \\ 1,\ x \in (1.204, 1.6092] \end{cases}$$

The values $\nu_{q,l}$ are obtained from (12) and (33): $\nu_{1,1} = 0,\ \nu_{1,2} = 0,\ \nu_{2,1} = 0,\ \nu_{2,2} = 1$.

Using (12) we have

$$\mathcal{I}_{a,b}(U_{2,2}) = \sum_{q=1}^{2}\sum_{l=1}^{2} \nu_{q,l} \cdot \mathcal{I}_a(f_A^{(q)}) \cdot \mathcal{I}_b(f_B^{(l)}) = \nu_{2,2} \cdot \mathcal{I}_a(f_A^{(2)}) \cdot \mathcal{I}_b(f_B^{(2)}).$$

Hence, a fuzzy system to achieve (33) is

$$y''' = \mathcal{I}_{a,b}(U_{2,2}) = \begin{cases} 0.4e^{-0.4},\ x \in [0.2231, 0.3]; \\ 0,\ x \in [-0.4, 0] \cup (0.3, 1.6092] \end{cases} \tag{34}$$

where $y(0) = 0,\ y'(0) = 2,\ y''(0) = 2$ and $U_{2,2}$ is given by (13).

Equation (34) will solve analytically using Matlab 7.0 using the function *dsolve*; let $y_1(x)$ the analytical solution of the equation (33), $y_2(x)$ and $y_3(x)$ the analytical solutions of the equation (34) for $x \in [-0.4, 0] \cup (0.3, 1.6092]$ and respectively for $x \in [0.2231, 0.3]$.

We shall obtain:

$$|y_1(-0.1) - y_2(-0.1)| = 4.3376e - 006,$$

$$|y_1(0.4) - y_2(0.4)| = 9.1184e - 004,$$

$$|y_1(0.7) - y_2(0.7)| = 0.0076,$$

$$|y_1(1.25) - y_2(1.25)| = 0.0636,$$

$$|y_1(0.25) - y_3(0.25)| = 5.5080e - 004,$$

$$|y_1(0.3) - y_3(0.3)| = 9.0670e - 004.$$

Application 2 We can use the polynomial and sinusoidal inferences together for the fuzzy approximation of the function

$$v(x, y) = \frac{1}{\cos 2x}(-2y - 2\sin x - 0.5\sin 2x + 2x\cos x). \tag{35}$$

We shall consider generating the function v by two fuzzy systems: one for the numerator of v and one for its denominator. The one for the numerator is assigned three fuzzy variables, $a,\ b$ and c. Variable a is for approximating the sine

and cosine terms of x, and variables b and c are for realizing the linear terms x and y. Triangular input membership functions of Fig. 1 are assumed for all variables.

Let the domains of interest for x and y be $[-(L_x/2), L_x/2]$ and $[-(L_y/2), L_y/2]$. For variable a let be m_a membership functions evenly spaced at locations

$$a_i = -(L_x/2) + (i-1) \cdot \frac{L_x/2 - (-L_x/2)}{m_a - 1},\ (\forall)\ i = \overline{1, m_a}.$$

Sinusoidal subsystem inferences are adopted for a by choosing the vectors $f_A^{(q)}$ according to (26):

$$f_A^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix},\quad f_A^{(2)} = \begin{pmatrix} \cos(a_1) \\ \cos(a_2) \\ \cos(a_3) \\ \vdots \\ \cos(a_{m_a-1}) \\ \cos(a_{m_a}) \end{pmatrix},\ f_A^{(3)} = \begin{pmatrix} \sin(a_1) \\ \sin(a_1) \\ \sin(a_2) \\ \vdots \\ \sin(a_{m_a-1}) \\ \sin(a_{m_a}) \end{pmatrix},$$

$$f_A^{(4)} = \begin{pmatrix} \cos(2a_1) \\ \cos(2a_2) \\ \cos(2a_3) \\ \vdots \\ \cos(2a_{m_a-1}) \\ \cos(2a_{m_a}) \end{pmatrix},\ f_A^{(5)} = \begin{pmatrix} \sin(2a_1) \\ \sin(2a_2) \\ \sin(2a_3) \\ \vdots \\ \sin(2a_{m_a-1}) \\ \sin(2a_{m_a}) \end{pmatrix}.$$

For variables b and c, one sets:

$$m_b = m_c = 2,$$

$$b_1 = -(L_x/2),\ b_2 = L_x/2,\ c_1 = -(L_y/2),\ c_2 = L_y/2,$$

$$f_B^{(1)} = f_C^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ f_B^{(2)} = \begin{pmatrix} -(L_x/2) \\ L_x/2 \end{pmatrix},\ f_C^{(2)} = \begin{pmatrix} -(L_y/2) \\ L_y/2 \end{pmatrix}.$$

The values $\nu_{q,l,r}$ are obtained from (14) and (35): $\nu_{1,1,2} = -2,\ \nu_{2,2,1} = 2,\ \nu_{3,1,1} = -2,\ \nu_{5,1,1} = -0.5$ and zero otherwise.

Using (13), the (i,j,k) element of matrix $U_{m_a,2,2}$ which yields the rule consequent $u_{i,j,k}$ is given by:

$$U_{m_a,2,2}(i,j,k) = \sum_{q=1}^{m_a} \sum_{l=1}^{2} \sum_{r=1}^{2} \nu_{q,l,r} \cdot f_A^{(q)}(i) \cdot f_B^{(l)}(j) \cdot f_C^{(r)}(k) =$$

$$-2 \cdot f_A^{(1)}(i) \cdot f_B^{(1)}(j) \cdot f_C^{(2)}(k) + 2 \cdot f_A^{(2)}(i) \cdot f_B^{(2)}(j) \cdot f_C^{(1)}(k) -$$
$$-2 \cdot f_A^{(3)}(i) \cdot f_B^{(1)}(j) \cdot f_C^{(1)}(k) - 0.5 \cdot f_A^{(5)}(i) \cdot f_B^{(1)}(j) \cdot f_C^{(1)}(k).$$

The fuzzy system for the denominator utilizes a single fuzzy variable of d. The same input membership functions as for a is adopted, i.e., $m_d = m_a$ and $d_i = a_i$. To effectuate a subsystem inference with approximates the term $\cos(2x)$, $f_D^{(q)}$ is selected as

$$f_D^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad f_D^{(2)} = \begin{pmatrix} \cos(d_1) \\ \cos(d_2) \\ \cos(d_3) \\ \vdots \\ \cos(d_{m_d-1}) \\ \cos(d_{m_d}) \end{pmatrix},$$

$$f_D^{(3)} = \begin{pmatrix} \sin(d_1) \\ \sin(d_1) \\ \sin(d_2) \\ \vdots \\ \sin(d_{m_d-1}) \\ \sin(d_{m_d}) \end{pmatrix}, \quad f_D^{(4)} = \begin{pmatrix} \cos(2d_1) \\ \cos(2d_2) \\ \cos(2d_3) \\ \vdots \\ \cos(2d_{m_d-1}) \\ \cos(2d_{m_d}) \end{pmatrix}.$$

Note. Since the selection of vectors $f_A^{(q1)}, f_B^{(q2)}, f_C^{(q3)}, f_D^{(q4)}$ should depend on the kind of approximation function one desires we can choose

$$f_A^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad f_A^{(2)} = \begin{pmatrix} \sin(a_1) \\ \sin(a_2) \\ \sin(a_3) \\ \vdots \\ \sin(a_{m_a-1}) \\ \sin(a_{m_a}) \end{pmatrix},$$

$$f_A^{(3)} = \begin{pmatrix} \cos(a_1) \\ \cos(a_1) \\ \cos(a_2) \\ \vdots \\ \cos(a_{m_a-1}) \\ \cos(a_{m_a}) \end{pmatrix}, \quad f_A^{(4)} = \begin{pmatrix} \sin(2a_1) \\ \sin(2a_2) \\ \sin(2a_3) \\ \vdots \\ \sin(2a_{m_a-1}) \\ \sin(2a_{m_a}) \end{pmatrix}$$

in order to approximate the numerator of v and

$$f_D^{(1)} = \begin{pmatrix} \cos(2d_1) \\ \cos(2d_2) \\ \cos(2d_3) \\ \vdots \\ \cos(2d_{m_d-1}) \\ \cos(2d_{m_d}) \end{pmatrix}$$

to approximate its denominator.

7 Conclusions

A new representation for fuzzy systems in terms of additive and multiplicative subsystem inferences of single variable is proposed. This representation enables an approximate functional characterization of the inferred output.

The form of the approximating function is dictated by the choice of polynomial, sinusoidal, or other designs of subsystem inferences.

With polynomial subsystem inferences, the approximating function is a sum of polynomial terms of orders depending on the numbers of input membership functions. The constant, linear, and nonlinear parts of the fuzzy inference can hence be identified.

With sinusoidal inferences, the approximating function is comprised of sine, cosine, and cross product terms. With proper scaling, the sinusoidal inferences produce a set of orthonormal inferences.

The present work also includes two applications about constructing a fuzzy approximator for a function expressible in terms of sums and products of functions of single variable. In this case, subsystem inferences which emulate the various single variable functions are adopted.

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An Estimate for the Discrete Derivative Green's Function for the 4D Variable Coefficient Elliptic Equation*

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Abstract

In this paper, definitions of the regularized derivative Green's function $\partial_{Z,\ell}G_Z^*$ and the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h$ in four dimensions are first given. With complicated arguments, the $W^{1,1}$-seminorm estimate for $\partial_{Z,\ell}G_Z^*$ is then derived. Finally, applying the triangle inequality and the error estimate of the finite element approximation results in the $W^{1,1}$-seminorm estimate with order $O(|\ln h|^{\frac{5}{4}})$ for $\partial_{Z,\ell}G_Z^h$.

Keywords: regularized derivative Green's function; discrete derivative Green's function; $W^{1,1}$-seminorm estimate

1. Introduction and preliminaries

It is well known that estimates for the Green's function play very important roles in the study of superconvergence (especially, pointwise superconvergence) of the finite element method (FEM) (see [1–8]). Recently, for the 3D Poisson equation, we studied the estimates for the Green's function and its applications to superconvergence of the FEM (see [5, 9]). However, for the higher-dimensional variable coefficient elliptic equation, it is difficult to derive estimates for the Green's function. The present paper, for the 4D variable coefficient elliptic equation, gives an estimate with order $O(|\ln h|^{\frac{5}{4}})$ for the discrete derivative Green's function.

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Here we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

We consider the following variable coefficient elliptic problem:

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{4} \partial_j(a_{ij}\partial_i u) + a_0 u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^4$ is a bounded polytopic domain and $\partial_i u = \partial_{x_i} u(x_1, x_2, x_3, x_4), i = 1, 2, 3, 4$ stand for usual partial derivatives. We assume a_{ij}, a_0, and f are sufficiently smooth given functions, and $a_{ij} = a_{ji}$. Let $\{\mathcal{T}^h\}$ be a regular family of uniform partitions of $\bar{\Omega}$ (see [10]). Denote the space of continuous piecewise m-degree ($m \geq 1$) polynomials with respect to this kind of partitions by S^h and let $S_0^h = S^h \cap H_0^1(\Omega)$. Discretizing the above elliptic equation using S_0^h as approximating space means finding $u_h \in S_0^h$ such that $a(u_h\,,\, v) = (f\,,\, v)$ for all $v \in S_0^h$, where

$$a(u_h\,,\, v) \equiv \int_\Omega (\sum_{i,j=1}^{4} a_{ij}\partial_i u_h \partial_j v + a_0 u_h v)\, dx_1 dx_2 dx_3 dx_4,$$

and

$$(f\,,\, v) \equiv \int_\Omega f v\, dx_1 dx_2 dx_3 dx_4.$$

This gives the Galerkin orthogonality relation

$$a(u - u_h\,,\, v) = 0 \ \ \forall\, v \in S_0^h. \tag{1.2}$$

For every $Z \in \Omega$, we define the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h$ and the L^2-projection $P_h u \in S_0^h$ such that (see [11])

$$(v,\, \partial_{Z,\ell}\delta_Z^h) = \partial_\ell v(Z) \ \ \forall\, v \in S_0^h, \tag{1.3}$$

$$(u - P_h u,\, v) = 0 \ \ \forall\, v \in S_0^h, \tag{1.4}$$

where $\ell \in \mathcal{R}^4$ and $|\ell| = 1$. $\partial_\ell v(Z)$ stands for the onesided directional derivative.

Let $\partial_{Z,\ell} G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the elliptic problem $\mathcal{L}G_Z^* = \partial_{Z,\ell}\delta_Z^h$. We may call $\partial_{Z,\ell} G_Z^*$ the regularized derivative Green's function, and denote the finite element approximation to $\partial_{Z,\ell} G_Z^*$ by $\partial_{Z,\ell} G_Z^h$ (the so-called discrete derivative Green's function). Thus,

$$a(\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h, v) = 0 \ \ \forall\, v \in S_0^h. \tag{1.5}$$

The main result of this work is the following optimal estimate:

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,\,1} \leq C|\ln h|^{\frac{5}{4}}.$$

2. An estimate for the discrete derivative Green's function $\partial_{Z,\ell} G_Z^h$

To derive the estimate for the discrete derivative Green's function, we introduce the weight function defined by

$$\phi \equiv \phi(X) = \left(|X - \bar{X}|^2 + \theta^2\right)^{-2} \ \forall\, X \in \bar{\Omega}, \tag{2.1}$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [4, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} \left|D^\beta v\right|^2, |\nabla^n v|_{\phi^\alpha,\Omega} = \left(\int_\Omega \phi^\alpha \, |\nabla^n v|^2 \, dX\right)^{\frac{1}{2}}, \|v\|^2_{m,\phi^\alpha,\Omega} = \sum_{n=0}^{m} |\nabla^n v|^2_{\phi^\alpha,\Omega},$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, $|\beta| = \beta_1 + \beta_2 + \beta_3 + \beta_4$, and $\beta_i \geq 0$, $i = 1, 2, 3, 4$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha,\Omega} = \left(\int_\Omega \phi^\alpha |v|^2 \, dX\right)^{\frac{1}{2}}.$$

We assume that there exists an q_0 $(1 < q_0 \leq \infty)$ such that

$$\mathcal{L} : \ W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \longrightarrow L^q(\Omega) \ (1 < q < q_0)$$

is a homeomorphism (see [1]). It means that for all $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, we have the so-called a priori estimate:

$$\|v\|_{2,q,\Omega} \leq C(q) \|\mathcal{L} v\|_{0,q,\Omega}, \tag{2.2}$$

where $C(q)$ denotes a positive constant only depending on q. Next, we give some lemmas used in the proofs of our main results.

Similar to the arguments in [1] and [11], we may obtain the following Lemma 2.1 and 2.2.

Lemma 2.1. *For ϕ the weight function defined by (2.1), we have the following estimates*:

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n) \phi^{\alpha + \frac{n}{4}}, \ \alpha \in \mathcal{R}, \ n = 1, 2. \tag{2.3}$$

Lemma 2.2. *For ϕ the weight function defined by (2.1), we have the following estimates*:

$$\int_\Omega \phi^\alpha \, dX \leq C(\alpha - 1)^{-1} \theta^{-4(\alpha-1)} \ \forall\, \alpha > 1, \tag{2.4}$$

$$\int_\Omega \phi \, dX \leq C(k) |\ln \theta|, \ \theta \leq k < 1. \tag{2.5}$$

In addition, with the similar arguments in [5], we may derive

Lemma 2.3. *For $\partial_{Z,\ell}\delta_Z^h$ the discrete derivative δ function defined by (1.3), we have the following estimates*:

$$|\partial_{Z,\ell}\delta_Z^h(X)| \le Ch^{-5}\mathrm{e}^{-Ch^{-1}|X-Z|}, \tag{2.6}$$

$$\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\frac{3}{2}}} \le C, \tag{2.7}$$

$$\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-1}} \le Ch^{-1}, \tag{2.8}$$

where $X, Z \in \bar{\Omega}$, and C is independent of h, X, and Z.

Lemma 2.4. *For $P_h w$ the L^2-projection of $w \in W^{1,q}(\Omega) \cap C(\bar{\Omega})$, and $1 \le q \le \infty$, we have the following stability estimate*:

$$\|P_h w\|_{1,q} \le C\|w\|_{1,q}. \tag{2.9}$$

Lemma 2.5. *Let $w \in W^{1,4}(\Omega)$, then we have*

$$\|w\|_{0,q} \le Cq^{\frac{3}{4}}\|w\|_{1,4}, \tag{2.10}$$

where $q >> 1$.

Remark 1. This lemma can be found in [2].

Lemma 2.6. *For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function, we have weighted-norm estimate*

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{2}}} \le C\left|\ln h\right|^{\frac{3}{4}}. \tag{2.11}$$

Proof. For simplicity, we write $g = \partial_{Z,\ell}G_Z^*$. Consider the problem

$$\begin{cases} \mathcal{L}w = \phi^{-\frac{1}{2}}g, & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Thus, from an inverse estimate, the stability estimate (2.9), a priori estimate (2.2), (2.10), the Sobolev Embedding Theorem [12], and the Poincaré inequality,

$$\begin{aligned} \|g\|^2_{\phi^{-\frac{1}{2}}} &= (\phi^{-\frac{1}{2}}g\,,\,g) = a(w\,,\,g) = (\partial_{Z,\ell}\delta_Z^h\,,\,w) = \partial_\ell P_h w(Z) \\ &\le |P_h w|_{1,\infty} \le Ch^{-\frac{4}{q}}\,|P_h w|_{1,q} \le Ch^{-\frac{4}{q}}|w|_{1,q} \\ &\le Ch^{-\frac{4}{q}}q^{\frac{3}{4}}\|w\|_{2,4} \le Ch^{-\frac{4}{q}}q^{\frac{3}{4}}\|\phi^{-\frac{1}{2}}g\|_{0,4} \\ &\le Ch^{-\frac{4}{q}}q^{\frac{3}{4}}\left\|\phi^{-\frac{1}{2}}g\right\|_1 \le Ch^{-\frac{4}{q}}q^{\frac{3}{4}}\left|\phi^{-\frac{1}{2}}g\right|_1. \end{aligned} \tag{2.13}$$

Taking $q = |\ln h|$ in (2.13) yields

$$\|g\|^2_{\phi^{-\frac{1}{2}}} \le C\left|\ln h\right|^{\frac{3}{4}}\left|\phi^{-\frac{1}{2}}g\right|_1 \le C\varepsilon\left|\phi^{-\frac{1}{2}}g\right|_1^2 + C\varepsilon^{-1}\left|\ln h\right|^{\frac{3}{2}}. \tag{2.14}$$

In addition,

$$\left|\phi^{-\frac{1}{2}}g\right|_1^2 \le a(\phi^{-\frac{1}{2}}g\,,\,\phi^{-\frac{1}{2}}g) \le C\left|a(g\,,\,\phi^{-1}g)\right| + C\,\|g\|^2_{\phi^{-\frac{1}{2}}}\,. \tag{2.15}$$

Nevertheless,

$$\left|a(g\,,\,\phi^{-1}g)\right| = \left|(\partial_{Z,\ell}\delta_Z^h\,,\,\phi^{-1}g)\right| \le \frac{1}{2}\left\|\partial_{Z,\ell}\delta_Z^h\right\|^2_{\phi^{-\frac{3}{2}}} + \frac{1}{2}\,\|g\|^2_{\phi^{-\frac{1}{2}}}\,. \tag{2.16}$$

Combining (2.14)–(2.16) with a suitably small $\varepsilon > 0$ gives

$$\|g\|^2_{\phi^{-\frac{1}{2}}} \le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|^2_{\phi^{-\frac{3}{2}}} + C\,|\ln h|^{\frac{3}{2}}\,. \tag{2.17}$$

The result (2.11) follows from (2.7) and (2.17). □

Lemma 2.7. *For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function, we have the following $W^{1,1}$-seminorm estimate*:

$$|\partial_{Z,\ell}G_Z^*|_{1,1} \le C\,|\ln h|^{\frac{5}{4}}\,. \tag{2.18}$$

Proof. Obviously,

$$|\partial_{Z,\ell}G_Z^*|^2_{1,1} \le \int_\Omega \phi\,dX\cdot\|\nabla(\partial_{Z,\ell}G_Z^*)\|^2_{\phi^{-1}}\,. \tag{2.19}$$

Furthermore,

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|^2_{\phi^{-1}} \le a(\partial_{Z,\ell}G_Z^*\,,\,\phi^{-1}\partial_{Z,\ell}G_Z^*) + C\,\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{-\frac{1}{2}}}\,. \tag{2.20}$$

Combined with (2.16) yields

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|^2_{\phi^{-1}} \le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|^2_{\phi^{-\frac{3}{2}}} + C\,\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{-\frac{1}{2}}}\,.$$

From (2.7) and (2.11),

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|^2_{\phi^{-1}} \le C\,|\ln h|^{\frac{3}{2}}\,. \tag{2.21}$$

Applying (2.5), (2.19), and (2.21) immediately results in (2.18). □

Lemma 2.8. *For $\Pi_m u \in S_0^h$ the interpolant of u, there hold the following interpolation error estimates:*

$$|u-\Pi_m u|_{n,\,p,\,e} \le Ch^{k+1-n}|u|_{k+1,\,p,\,e}\ \forall\, e\in\mathcal{T}^h, \tag{2.22}$$

$$|\nabla^n(u-\Pi_m u)|_{\phi^\alpha,\,e} \le Ch^{k+1-n}\left|\nabla^{k+1}u\right|_{\phi^\alpha,\,e}\ \forall\, e\in\mathcal{T}^h, \tag{2.23}$$

$$|\nabla^s(\phi^\alpha v-\Pi_m(\phi^\alpha v))|_{\phi^{-\beta}} \le C^*\frac{h}{\theta}\left(|v|_{\phi^{2\alpha+\frac{1}{2}-\beta}} + |\nabla v|_{\phi^{2\alpha-\beta}}\right)\ \forall\, v\in S_0^h, \tag{2.24}$$

where $\alpha, \beta \in R$, $C^* = C^*(\alpha, \beta)$, $n = 0, 1, 2$, $s = 0, 1$, $\theta \geq 4|\alpha|h$, $1 \leq p \leq \infty$, $1 \leq k \leq m$, *and* k *is an integer,* C *is independent of* α, h *and* u.

Remark 2. The proof of Lemma 2.8 is almost the same as that of Lemma 4 in [1] (see p.110 in [1]).

Lemma 2.9. *For* $\partial_{Z,\ell} G_Z^*$ *and* $\partial_{Z,\ell} G_Z^h$, *the regularized derivative Green's function and the discrete derivative Green's function, respectively, we have the following estimate*:

$$\left|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right|_{1,\,1} \leq C|\ln h|^{\frac{1}{2}}. \tag{2.25}$$

Proof. For simplicity, we write $g = \partial_{Z,\ell} G_Z^*$, and $g_h = \partial_{Z,\ell} G_Z^h$. Obviously,

$$|g - g_h|_{1,\,1}^2 \leq \int_\Omega \phi \, dX \cdot |g - g_h|_{1,\,\phi^{-1}}^2 . \tag{2.26}$$

Moreover, similar to the proof of (2.43) in [5], we have

$$|g - g_h|_{1,\,\phi^{-1}}^2 \leq Ch^2 \left|\nabla^2 g\right|_{\phi^{-1}}^2 + \hat{C} \left\|g - g_h\right\|_{\phi^{-\frac{1}{2}}}^2 . \tag{2.27}$$

Further,

$$\begin{aligned} \left\|g - g_h\right\|_{\phi^{-\frac{1}{2}}}^2 &= (\phi^{-\frac{1}{2}}(g - g_h),\, g - g_h) = a(w,\, g - g_h) = a(w - \Pi_m w,\, g - g_h) \\ &\leq \varepsilon \, |g - g_h|_{1,\,\phi^{-1}}^2 + C(\varepsilon) \, |w - \Pi_m w|_{1,\,\phi}^2 \\ &\quad + \eta \left\|g - g_h\right\|_{\phi^{-\frac{1}{2}}}^2 + C(\eta) \left\|w - \Pi_m w\right\|_\phi^2 . \end{aligned} \tag{2.28}$$

where $\mathcal{L}w = \phi^{-\frac{1}{2}}(g - g_h)$. Taking $0 < \eta < 1$ and applying the weighted interpolation error estimate (2.23) yield

$$\left\|g - g_h\right\|_{\phi^{-\frac{1}{2}}}^2 \leq \varepsilon \, |g - g_h|_{1,\,\phi^{-1}}^2 + C(\varepsilon) h^2 \left|\nabla^2 w\right|_\phi^2 . \tag{2.29}$$

From a priori estimate (2.2), (2.4), the Sobolev Embedding Theorem [12], and the Poincaré inequality,

$$\begin{aligned} \left|\nabla^2 w\right|_\phi^2 &\leq \left(\int_\Omega \phi^2 dX\right)^{\frac{1}{2}} \left|\nabla^2 w\right|_{0,\,4}^2 \leq C\theta^{-2} \left\|\phi^{-\frac{1}{2}}(g - g_h)\right\|_{0,\,4}^2 \\ &\leq C\theta^{-2} \left\|\phi^{-\frac{1}{2}}(g - g_h)\right\|_{1,\,2}^2 \leq C\theta^{-2} \left|\phi^{-\frac{1}{2}}(g - g_h)\right|_{1,\,2}^2 \\ &\leq C\theta^{-2} a(\phi^{-\frac{1}{2}}(g - g_h),\, \phi^{-\frac{1}{2}}(g - g_h)) \\ &\leq C\theta^{-2} \left[\left|a(g - g_h,\, \phi^{-1}(g - g_h))\right| + \left\|g - g_h\right\|_{\phi^{-\frac{1}{2}}}^2\right] \\ &\leq C\theta^{-2} \left(|g - g_h|_{1,\,\phi^{-1}}^2 + \left\|g - g_h\right\|_{\phi^{-\frac{1}{2}}}^2\right). \end{aligned}$$

Thus,

$$\|g-g_h\|^2_{\phi^{-\frac{1}{2}}} \le \varepsilon\,|g-g_h|^2_{1,\,\phi^{-1}} + C(\varepsilon)\gamma^{-2}\left(|g-g_h|^2_{1,\,\phi^{-1}} + \|g-g_h\|^2_{\phi^{-\frac{1}{2}}}\right).$$

Taking an $\varepsilon \in (0,1)$ such that $0 < 4\varepsilon\hat{C} < 1$, and Choosing $\gamma \in [4,+\infty)$ such that $0 < C(\varepsilon)\gamma^{-2} < \min(\varepsilon, \frac{1}{2})$, we then have

$$\|g-g_h\|^2_{\phi^{-\frac{1}{2}}} \le 4\varepsilon\,|g-g_h|^2_{1,\,\phi^{-1}}\,. \tag{2.30}$$

Combining (2.27) and (2.30) yields

$$|g-g_h|^2_{1,\,\phi^{-1}} \le Ch^2\left|\nabla^2 g\right|^2_{\phi^{-1}}. \tag{2.31}$$

It is easy to see

$$\begin{aligned}
\left|\nabla^2 g\right|^2_{\phi^{-1}} &= \int_\Omega \phi^{-1}\left|\nabla^2 g\right|^2 dX = \int_\Omega \left(\phi^{-\frac{1}{2}}\left|\nabla^2 g\right|\right)^2 dX \\
&\le C\left(\int_\Omega \left|\nabla^2(\phi^{-\frac{1}{2}}g)\right|^2 dX + \int_\Omega \left|g\nabla^2\phi^{-\frac{1}{2}}\right|^2 dX + \int_\Omega \left|\nabla\phi^{-\frac{1}{2}}\cdot\nabla g\right|^2 dX\right) \\
&\le C\left(\left\|\nabla^2(\phi^{-\frac{1}{2}}g)\right\|^2_0 + \|g\|^2_0 + |g|^2_{1,\,\phi^{-\frac{1}{2}}}\right) \\
&\le C\left(\left\|\mathcal{L}(\phi^{-\frac{1}{2}}g)\right\|^2_0 + \|g\|^2_0 + |g|^2_{1,\,\phi^{-\frac{1}{2}}}\right) \\
&\le C\left(\|\mathcal{L}g\|^2_{\phi^{-1}} + \|g\|^2_0 + |g|^2_{1,\,\phi^{-\frac{1}{2}}}\right) \\
&\le C\left(\left\|\partial_{Z,\ell}\delta^h_Z\right\|^2_{\phi^{-1}} + \|g\|^2_0 + a(g,\,\phi^{-\frac{1}{2}}g)\right) \\
&= C\left(\left\|\partial_{Z,\ell}\delta^h_Z\right\|^2_{\phi^{-1}} + \|g\|^2_0 + (\partial_{Z,\ell}\delta^h_Z,\,\phi^{-\frac{1}{2}}g)\right) \\
&\le C\left\|\partial_{Z,\ell}\delta^h_Z\right\|^2_{\phi^{-1}} + C\,\|g\|^2_0\,,
\end{aligned}$$

namely,

$$\left|\nabla^2 g\right|^2_{\phi^{-1}} \le C\left\|\partial_{Z,\ell}\delta^h_Z\right\|^2_{\phi^{-1}} + C\,\|g\|^2_0\,. \tag{2.32}$$

We will now estimate $\|g\|_0$. Obviously,

$$\begin{aligned}
\|g\|^2_0 &= (g,\,g) = a(v,\,g) = (\partial_{Z,\ell}\delta^h_Z,\,v) = \partial_\ell P_h v(Z) \\
&\le |P_h v|_{1,\infty} \le Ch^{-1}|P_h v|_{1,4} \le Ch^{-1}|v|_{1,4} \\
&\le Ch^{-1}\|v\|_2 \le Ch^{-1}\|g\|_0,
\end{aligned}$$

where $\mathcal{L}v = g$. Thus,

$$\|g\|_0 \le Ch^{-1}. \tag{2.33}$$

Combining (2.8), (2.31)–(2.33), we have

$$|g - g_h|^2_{1,\,\phi^{-1}} \le C. \tag{2.34}$$

The result (2.25) follows from (2.5), (2.26), and (2.34). □

Combining (2.18) and (2.25) immediately gives the following result.

Theorem 2.1. *For $\partial_{Z,\ell} G_Z^h$ the discrete derivative Green's function, we have the following estimate*:

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,\,1} \le C|\ln h|^{\frac{5}{4}}. \tag{2.35}$$

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Superconvergence Analysis of Cubic Block Finite Elements of Intermediate Families *

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Abstract

For the Poisson equation in three dimensions, using the properties of the interpolation operator of projection type, we derive weak estimates for cubic block finite elements of intermediate families (CBFEIF). In addition, estimates for the discrete Green's functions are given. Finally, we prove that derivatives and function values of the finite element approximation and the corresponding interpolant of projection type are superclose in the pointwise sense of the L^∞-norm.

Mathematics subject classification: 65N30.
Key words: cubic block finite elements of intermediate families; interpolant of projection type; superconvergence; weak estimate; discrete Green's function

I. INTRODUCTION AND PRELIMINARIES

To the best of our knowledge, there have been some studies concerned with the superconvergence of the three-dimensional (3D) finite element (FE) method [1–22]. Specifically, many superconvergence results were obtained for the 3D tensor-product FE and the serendipity FE, as well as for the tetrahedral FE. As for the serendipity FE, early in 1977, Zlámal [23] proved a superconvergence of the gradient at Gaussian points for serendipity rectangular elements (with incomplete cubic polynomials) applied to a variable coefficient elliptic equation. This result was later generalized to the 3D space (see [22]). Zhang and Lin [21] also discussed the serendipity element. Recently, the authors have reported some superconvergence results for several 3D FE, which are used to approximate the sufficiently smooth solution of the elliptic equation (see [12–17]). Among of them, the quadratic serendipity block FE considered in [22] was discussed and the pointwise superconvergence of the derivatives of the FE approximation was derived (see [16]). In this article, we will address the problem of the pointwise superconvergence of derivatives and function values of the CBFEIF.

In this article, we use letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

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The model problem considered in this article is

$$-\Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{1.1}$$

Here $\Omega \subset R^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. We assume the given function f is sufficiently smooth and Ω is partitioned into a regular rectangulation $\mathcal{T}^h$ with mesh size $h \in (0, 1)$ such that $\bar{\Omega} = \cup_{e\in\mathcal{T}^h} \bar{e}$.

We now give the standard variational formulation of the above problem. Let

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dxdydz, \quad (f, v) = \int_\Omega fv \, dxdydz.$$

Then the weak formulation of (1.1) reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u, v) = (f, v) \text{ for all } v \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

We briefly describe the organization of this article. In Section 2, we discuss weak estimates for the CBFEIF, namely, Theorem 2.1, which is crucial for the superconvergence analysis. Two superconvergence results of the FE approximation are presented in Section 3.

II. WEAK ESTIMATES FOR THE CBFEIF

In this section, using the properties of the interpolation operator of projection type, we derive weak estimates for the CBFEIF.

We write an element

$$e = (x_e - h_e, x_e + h_e) \times (y_e - k_e, y_e + k_e) \times (z_e - d_e, z_e + d_e) \equiv I_1 \times I_2 \times I_3. \tag{2.1}$$

Let $\{l_j(x)\}_{j=0}^\infty$, $\{\tilde{l}_j(y)\}_{j=0}^\infty$, and $\{\bar{l}_j(z)\}_{j=0}^\infty$ be the normalized orthogonal Legendre polynomial systems on $L^2(I_1)$, $L^2(I_2)$, and $L^2(I_3)$, respectively. It is easy to check that $\{l_i(x)\tilde{l}_j(y)\bar{l}_k(z)\}_{i,j,k=0}^\infty$ is the normalized orthogonal polynomial system on $L^2(e)$. Set

$$\omega_0(x) = \tilde{\omega}_0(y) = \bar{\omega}_0(z) = 1, \ \omega_{j+1}(x) = \int_{x_e-h_e}^x l_j(\xi) \, d\xi,$$

$$\tilde{\omega}_{j+1}(y) = \int_{y_e-k_e}^y \tilde{l}_j(\xi) \, d\xi, \ \bar{\omega}_{j+1}(z) = \int_{z_e-d_e}^z \bar{l}_j(\xi) \, d\xi, \ j \geq 0,$$

which are called Lobatto functions. Suppose $u \in H^3(e)$. Then we have the following expansion (see [14]):

$$u(x, y, z) = \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z), \ (x, y, z) \in e, \tag{2.2}$$

where

$$\beta_{000} = u(x_e - h_e, y_e - k_e, z_e - d_e),$$

$$\beta_{i00} = \int_{I_1} \partial_x u(x, y_e - k_e, z_e - d_e) l_{i-1}(x) \, dx,$$

$$\beta_{0j0} = \int_{I_2} \partial_y u(x_e - h_e, y, z_e - d_e)\tilde{l}_{j-1}(y)\, dy,$$

$$\beta_{00k} = \int_{I_3} \partial_z u(x_e - h_e, y_e - k_e, z)\bar{l}_{k-1}(z)\, dz,$$

$$\beta_{ij0} = \int_{I_1\times I_2} \partial_x\partial_y u(x, y, z_e - d_e) l_{i-1}(x)\tilde{l}_{j-1}(y)\, dxdy,$$

$$\beta_{0jk} = \int_{I_2\times I_3} \partial_y\partial_z u(x_e - h_e, y, z)\tilde{l}_{j-1}(y)\bar{l}_{k-1}(z)\, dydz,$$

$$\beta_{i0k} = \int_{I_1\times I_3} \partial_x\partial_z u(x, y_e - k_e, z) l_{i-1}(x)\bar{l}_{k-1}(z)\, dxdz,$$

$$\beta_{ijk} = \int_e \partial_x\partial_y\partial_z u\, l_{i-1}(x)\tilde{l}_{j-1}(y)\bar{l}_{k-1}(z)\, dxdydz,\ i, j, k \geq 1.$$

We denote by T_{38}^3 the 3D cubic polynomial space of the intermediate family with 38 degrees of freedom, namely,

$$q(x, y, z) = \sum_{(i,j,k)\in I_{38}^3} a_{ijk} x^i y^j z^k,\ q \in T_{38}^3, \tag{2.3}$$

where

$$\begin{aligned} I_{38}^3 \quad = \quad & \{(i, j, k) | 0 \leq i, j, k \leq 3,\, i + j + k \leq 4\} \\ & \bigcup \{(3, 1, 1), (1, 3, 1), (1, 1, 3), (2, 2, 1), (2, 1, 2), (1, 2, 2)\}. \end{aligned} \tag{2.4}$$

Define the interpolation operator of projection type by Π_e: $H^3(e) \to T_{38}^3(e)$ such that

$$\Pi_e u(x, y, z) = \sum_{(i,j,k)\in I_{38}^3} \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z). \tag{2.5}$$

Also Π_e possesses the following properties:

$$|u - \Pi_e u|_{m,\,p,\,e} \leq Ch^{4-m}|u|_{4,\,p,\,e},\ 1 \leq p \leq \infty,\ 0 \leq m \leq 4, \tag{2.6}$$

$$\Pi_e u(x_e \pm h_e, y_e \pm k_e, z_e \pm d_e) = u(x_e \pm h_e, y_e \pm k_e, z_e \pm d_e), \tag{2.7}$$

$$\int_{l_i} (u - \Pi_e u) v\, dl = 0,\ \forall\, v \in P_1(l_i),\ i = 1, \cdots, 12, \tag{2.8}$$

$$\int_{\sigma_i} \Pi_e u\, d\sigma = \int_{\sigma_i} u\, d\sigma,\ i = 1, \cdots, 6, \tag{2.9}$$

where l_i, $i = 1, \ldots, 12$ are the edges of e, and σ_i, $i = 1, \cdots, 6$ the faces of e.

We now introduce the CBFEIF space defined by

$$S_0^h(\Omega) = \left\{ v \in H_0^1(\Omega) : v|_e \in T_{38}^3(e)\ \forall\, e \in \mathcal{T}^h \right\}. \tag{2.10}$$

Then the CBFEIF method to approximate the solution of (1.2) is given by

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h\,,\, v) = (f\,,\, v) \text{ for all } v \in S_0^h(\Omega). \end{cases}$$

Obviously, there holds the following Galerkin orthogonality:

$$a(u - u_h,\, v) = 0\ \ \forall\, v \in S_0^h(\Omega). \tag{2.11}$$

From (2.7)–(2.9),

$$\Pi : H^3(\Omega) \cap H_0^1(\Omega) \to S_0^h(\Omega),$$

where $(\Pi u)|_e = \Pi_e u$.

Theorem 2.1. *Suppose $\{\mathcal{T}^h\}$ is a regular family of rectangular partitions of Ω, and $u \in W^{5,\infty}(\Omega) \cap H_0^1(\Omega)$. Then for all $v \in S_0^h(\Omega)$, the interpolation operator Π satisfies the following weak estimates*:

$$|a(u - \Pi u,\, v)| \le Ch^4 \|u\|_{5,\infty,\Omega} |v|_{1,1,\Omega}, \tag{2.12}$$

$$|a(u - \Pi u,\, v)| \le Ch^5 \|u\|_{5,\infty,\Omega} |v|^h_{2,1,\Omega}, \tag{2.13}$$

where $|v|^h_{2,1,\Omega} = \sum_e |v|_{2,1,e}$.

Proof. For simplicity, we write $\lambda_{ijk} = \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z)$. By the orthogonality of the Lobatto functions and the Legendre polynomials, we have

$$\int_e \nabla(u - \Pi u) \cdot \nabla v \, dxdydz = \int_e \nabla r \cdot \nabla v \, dxdydz, \, \forall\, v \in S_0^h(\Omega),$$

where $r = \lambda_{023} + \lambda_{014} + r^*$. The r^* contains finite high-order terms as well as terms similar to λ_{023} and λ_{014}. So it suffices to bound the two terms λ_{023} and λ_{014}.

As for the integral

$$J_{023} \equiv \int_e \nabla\lambda_{023} \cdot \nabla v \, dxdydz. \tag{2.14}$$

Obviously,

$$J^x_{023} \equiv \int_e \partial_x\lambda_{023}\partial_x v \, dxdydz = 0. \tag{2.15}$$

In addition,

$$J^y_{023} \equiv \int_e \partial_y\lambda_{023}\partial_y v \, dxdydz = \beta_{023} \int_e \tilde{l}_1(y)\bar{\omega}_3(z)\partial_y v \, dxdydz, \tag{2.16}$$

then

$$|J^y_{023}| \le |\beta_{023}| \int_e |\tilde{l}_1(y)\bar{\omega}_3(z)||\partial_y v| \, dxdydz \le Ch^4 \|u\|_{5,\infty,\Omega} |v|_{1,1,e}. \tag{2.17}$$

Integration by parts with respect to z in (2.16) yields

$$|J^y_{023}| \le Ch^5 \|u\|_{5,\infty,\Omega} |v|_{2,1,e}. \tag{2.18}$$

For the integral

$$J^z_{023} \equiv \int_e \partial_z\lambda_{023}\partial_z v \, dxdydz = \beta_{023} \int_e \tilde{\omega}_2(y)\bar{l}_2(z)\partial_z v \, dxdydz, \tag{2.19}$$

we have

$$|J^z_{023}| \le |\beta_{023}| \int_e |\tilde{\omega}_2(y)\bar{l}_2(z)||\partial_z v| \, dxdydz \le Ch^4 \|u\|_{5,\infty,\Omega} |v|_{1,1,e}. \tag{2.20}$$

Similarly, integration by parts with respect to z in (2.19) yields

$$|J^z_{023}| \le Ch^5 \|u\|_{5,\infty,\Omega} |v|_{2,1,e}. \tag{2.21}$$

Combining (2.14), (2.15), (2.17), (2.18), (2.20), and (2.21) gives

$$|J_{023}| \le Ch^4 \|u\|_{5,\infty,\Omega} |v|_{1,1,e}, \tag{2.22}$$

and

$$|J_{023}| \le Ch^5 \|u\|_{5,\infty,\Omega} |v|_{2,1,e}. \tag{2.23}$$

Next we bound the term

$$J_{014} \equiv \int_e \nabla \lambda_{014} \cdot \nabla v \, dxdydz.$$

Obviously,

$$J_{014}^x \equiv \int_e \partial_x \lambda_{014} \partial_x v \, dxdydz = 0, \tag{2.24}$$

and

$$J_{014}^z \equiv \int_e \partial_z \lambda_{014} \partial_z v \, dxdydz = \beta_{014} \int_e \tilde{\omega}_1(y) \bar{l}_3(z) \partial_z v \, dxdydz = 0. \tag{2.25}$$

Thus we have

$$J_{014} = J_{014}^y \equiv \int_e \partial_y \lambda_{014} \partial_y v \, dxdydz = \beta_{014} \int_e \tilde{l}_0(y) \bar{\omega}_4(z) \partial_y v \, dxdydz. \tag{2.26}$$

It is easy to prove

$$|J_{014}| \le Ch^4 \|u\|_{5,\infty,\Omega} |v|_{1,1,e}, \tag{2.27}$$

and

$$|J_{014}| \le Ch^5 \|u\|_{5,\infty,\Omega} |v|_{2,1,e}. \tag{2.28}$$

From (2.22), (2.23), (2.27), and (2.28),

$$\left| \int_e \nabla r \cdot \nabla v \, dxdydz \right| \le Ch^4 \|u\|_{5,\infty,\Omega} |v|_{1,1,e}, \tag{2.29}$$

and

$$\left| \int_e \nabla r \cdot \nabla v \, dxdydz \right| \le Ch^5 \|u\|_{5,\infty,\Omega} |v|_{2,1,e}. \tag{2.30}$$

Hence summing over all elements yields

$$|a(u - \Pi u, v)| \le \sum_e \left| \int_e \nabla r \cdot \nabla v \, dxdydz \right| \le Ch^4 \|u\|_{5,\infty,\Omega} |v|_{1,1,\Omega},$$

and

$$|a(u - \Pi u, v)| \le \sum_e \left| \int_e \nabla r \cdot \nabla v \, dxdydz \right| \le Ch^5 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h.$$

This completes the proof of Theorem 2.1. □

III. SUPERCONVERGENCE OF THE FINITE ELEMENT APPROXIMATION

To derive superconvergence of the finite element approximation, we need to introduce discrete Green's function $G_Z^h \in S_0^h(\Omega)$ and discrete derivative Green's function $\partial_{Z,\ell} G_Z^h \in S_0^h(\Omega)$, defined by

$$a(v, G_Z^h) = v(Z), \; a(v, \partial_{Z,\ell} G_Z^h) = \partial_\ell v(Z) \; \forall \, v \in S_0^h(\Omega). \tag{3.1}$$

Here, for every $Z \in \Omega$ and any direction $\ell \in R^3$, $|\ell| = 1$, $\partial_{Z,\ell} G_Z^h$ and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell} G_Z^h = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z}^h - G_Z^h}{|\Delta Z|},\ \partial_\ell v(Z) = \lim_{|\Delta Z| \to 0} \frac{v(Z+\Delta Z) - v(Z)}{|\Delta Z|},\ \Delta Z = |\Delta Z| \ell.$$

Remark 1. Since $\Delta Z = |\Delta Z| \ell$, that is, ΔZ is of the same direction as ℓ. Thus, provided that the direction ℓ is given, the above limits exist. Hence, no matter what direction is given, the above definition has good meaning.

Lemma 3.1. *For G_Z^h and $\partial_{Z,\ell} G_Z^h$, the discrete Green's function and the discrete derivative Green's function, defined by* (3.1)*, we have the following estimates*:

$$\left| G_Z^h \right|_{2,\,1,\,\Omega}^h \le C \left| \ln h \right|^{\frac{2}{3}}. \tag{3.2}$$

$$\left| \partial_{Z,\ell} G_Z^h \right|_{1,\,1,\,\Omega} \le C \left| \ln h \right|^{\frac{4}{3}}. \tag{3.3}$$

Remark 2. The proof of Lemma 3.1 may be found in [17] and [18].

Finally, we give the following superconvergence estimates between u_h and Πu.

Theorem 3.1. *Suppose $\{\mathcal{T}^h\}$ is a regular family of rectangular partitions of Ω, and $u \in W^{5,\infty}(\Omega) \cap H_0^1(\Omega)$. For u_h and Πu, the CBFEIF approximation and the corresponding interpolant of projection type to u, respectively. Then we have the following superconvergence estimates*:

$$|u_h - \Pi u|_{1,\,\infty,\,\Omega} \le C h^4 \left| \ln h \right|^{\frac{4}{3}} \|u\|_{5,\,\infty,\,\Omega}. \tag{3.4}$$

$$|u_h - \Pi u|_{0,\,\infty,\,\Omega} \le C h^5 \left| \ln h \right|^{\frac{2}{3}} \|u\|_{5,\,\infty,\,\Omega}. \tag{3.5}$$

Proof. For every $Z \in \Omega$, from (2.11) and (3.1),

$$\partial_\ell \left(u_h - \Pi u \right)(Z) = a(u_h - \Pi u,\, \partial_{Z,\ell} G_Z^h) = a(u - \Pi u,\, \partial_{Z,\ell} G_Z^h).$$

Combined with (2.12) and (3.3), we get the result (3.4). In addition,

$$(u_h - \Pi u)(Z) = a(u_h - \Pi u\,,\, G_Z^h) = a(u - \Pi u\,,\, G_Z^h).$$

Combined with (2.13) and (3.2), we get the result (3.5).

This completes the proof of Theorem 3.1. □

Comments

• Although the degrees of freedom of the CBFEIF are much less than those of the tensor-product cubic block finite element (with 64 degrees of freedom), the CBFEIF is of the same accuracy as the tensor-product element on the regular meshes. Therefore, in this sense, the superconvergent results of the CBFEIF are more significant.

• For the other CBFEIF with 50 degrees of freedom, the finite element space is

$$S_0^h(\Omega) = \left\{ v \in H_0^1(\Omega) : v|_e \in T_{50}^3(e)\ \forall\, e \in \mathcal{T}^h \right\}$$

where $T_{50}^3 = \text{span}\{x^i y^j z^k : (i,j,k) \in I_{50}^3\}$, and

$$\begin{aligned} I_{50}^3 &= \{(i,j,k)|0 \le i,j,k \le 3,\, i+j+k \le 5\} \\ &\quad \bigcup\{(3,2,1),(3,1,2),(2,3,1),(2,1,3),(1,3,2),(1,2,3)\}. \end{aligned}$$

We also construct the interpolation operator of projection type Π_e: $H^3(e) \to T_{50}^3(e)$ defined by

$$\Pi_e u(x,y,z) = \sum_{(i,j,k)\in I_{50}^3} \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z), \quad (x,y,z) \in e.$$

Further, we may get the global interpolation operator of projection type Π: $H^3(\Omega)\cap H_0^1(\Omega) \to S_0^h(\Omega)$, where $(\Pi u)|_e = \Pi_e u$.

Similar to the proof of Theorem 2.1, we may derive the following weak estimates:

(1) on regular meshes,

$$|a(u-\Pi u,\, v)| \le Ch^4 \|u\|_{5,\,\infty,\,\Omega}|v|_{1,\,1,\,\Omega},$$

$$|a(u-\Pi u,\, v)| \le Ch^5 \|u\|_{5,\,\infty,\,\Omega}|v|_{2,\,1,\,\Omega}^h.$$

(2) on uniform meshes,

$$|a(u-\Pi u,\, v)| \le Ch^5 \|u\|_{6,\,\infty,\,\Omega}|v|_{1,\,1,\,\Omega},$$

$$|a(u-\Pi u,\, v)| \le Ch^6 \|u\|_{6,\,\infty,\,\Omega}|v|_{2,\,1,\,\Omega}^h.$$

From the above statements, no matter what meshes (regular or uniform) are given, the CBFEIF with 50 degrees of freedom is of the same accuracy as the tensor-product cubic block finite element.

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Positive solutions of an nth order multi-point boundary value problem

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Abstract
By using the Avery-Henderson fixed point theorem and the five functionals fixed point theorem, this paper investigates the multiplicity of positive solutions of an nth order multi-point boundary value problem. In addition, we also give some examples to demostrate our results.

MSC: 34B15
Keywords: Positive solutions, fixed point theorem in cones.

1 Introduction

In this paper, we consider the nth order multi-point boundary value problem (BVP)

$$\begin{cases} y^{(n)}(t)+Q(t,y,y',...,y^{(n-2)})=P(t,y,y',...,y^{(n-1)}), & t\in[a,b], \\ y^{(i)}(a)=0, & 0\le i\le n-3, \\ \sum_{i=1}^{m}\alpha_i y^{(n-2)}(\xi_i)=y^{(n-2)}(b) \end{cases} \tag{1.1}$$

where $\lambda>0$ is a parameter, $n\ge 3$, $m\ge 1$ are integers, $a<\xi_1<\xi_2<...<\xi_m<b$, $\alpha_i\in(0,\infty)$ for $1\le i\le m$, $\sum_{i=1}^{m}\alpha_i<1$ and $D=b-a-\sum_{i=1}^{m}\alpha_i(\xi_i-a)>0$.

The existence and multiplicity of positive solutions for second order multi-point boundary value problems have received a great deal of attention [6, 7, 9 − 16]. There are fewer results in the literature on nth order multi-point boundary value problems [3 − 5, 8].

We cite some appropriate references here [4, 5].
Graef and Yang [5] considered higher order multi-point boundary value problem

$$\begin{cases} u^{(n)}(t)+\lambda g(t)f(u(t))=0, & 0<t<1, \\ u(0)=u'(0)=...=u^{(n-3)}(0)=u^{(n-2)}(0) \\ \quad=\sum_{i=1}^{m}a_i u^{(n-2)}(\xi_i)-u^{(n-2)}(1)=0. \end{cases} \tag{1.2}$$

The authors established some existence and nonexistence results for positive solutions of the (1.2) by using Krasnosel'skii fixed point theorem.

Graef, Henderson, Wong and Yang [4] were interested in the following nth order three-point boundary value problem

$$\begin{cases} u^{(n)}(t)=F(t,u(t)), & t\in[0,1], \\ u^{(i)}(0)=0, & 0\le i\le n-3, \\ u^{(n-2)}(p)=0, \quad u^{(n-1)}(1)=0. \end{cases} \tag{1.3}$$

They studied the existence of at least three positive solutions of the (1.3). For this purpose, they used the Leggett-Williams fixed point theorem and the five functionals fixed point theorem.

In this paper, motivated by above research efforts on multi-point boundary value problems, criteria for the existence of at least two or three positive solutions of the BVP (1.1) are established by using the Avery-Henderson fixed point theorem and the five functionals fixed point theorem.

Moreover, the examples are also included to illustrate our results. Thus, our results are new for differential equations.

2 Preliminary lemmas

We give the following conditions and lemmas which will be used later.

We assume that there exist continuous functions $f : [0,\infty) \to (0,\infty)$, and $p,\ p_1,\ q,\ q_1 : [a,b] \to \mathbb{R}$ such that

$(H1)$ The maps $u \in \mathbb{R}^n \to P(t,u) \in \mathbb{R}$ and $u \in \mathbb{R}^{n-1} \to Q(t,u) \in \mathbb{R}$ are continuous for all $t \in [a,b]$;

$(H2)$ For $y \in [0,\infty)$,

$$q(t) \leq \frac{Q(t,y,y_1,...y_{n-2})}{f(y)} \leq q_1(t),$$

$$p(t) \leq \frac{P(t,y,y_1,...y_{n-1})}{f(y)} \leq p_1(t);$$

$(H3)$ $q(t) - p_1(t)$ is nonnegative for $t \in [a,b]$.

Let $G_2(t,s)$ be the Green's function for the boundary value problem

$$\begin{cases} y''(t) = 0, & t \in [a,b], \\ y(a) = 0, & \\ \sum_{i=1}^{m} \alpha_i y(\xi_i) = y(b). & \end{cases}$$

Then,

$$G_2(t,s) = \begin{cases} \frac{(b-t)(s-a) - \sum_{j=i}^{m} \alpha_j(\xi_j - t)(s-a) + \sum_{j=1}^{i-1} \alpha_j(\xi_j - a)(t-s)}{b-a-\sum_{i=1}^{m} \alpha_i(\xi_i - a)}, \\ a \leq t \leq b, \quad \xi_{i-1} \leq s \leq \min\{\xi_i, t\}, \quad i = \overline{1, m+1}, \\ \frac{(t-a)[b-s-\sum_{j=i}^{m} \alpha_j(\xi_j - s)]}{b-a-\sum_{i=1}^{m} \alpha_i(\xi_i - a)}, \\ a \leq t \leq b, \quad \max\{\xi_{i-1}, t\} \leq s \leq \xi_i, \quad i = \overline{1, m+1}. \end{cases}$$

Lemma 2.1 *There exist a number $k \in (0,1)$ and a continuous function $\psi : [a,b] \to \mathbb{R}^+$ such that*

$$G_2(t,s) \leq \phi(s) \quad t \in [a,b],\ s \in [a,b]$$

and

$$G_2(t,s) \geq k\phi(s) \quad \textit{for} \quad t \in [\xi_1, b],\ s \in [a,b],$$

where

$$\phi(s) = \tfrac{(b-s)(s-a)}{D},$$

$$k = \min_{2 \leq i \leq m} \{ \tfrac{1}{b} \textstyle\sum_{j=i}^{m} \alpha_j(b - \xi_j),\ \tfrac{\xi_1 - a}{b-a}[1 - \sum_{j=i}^{m} \alpha_j] \}. \tag{2.1}$$

Proof. Now, we will show that we may take $\phi(s) = \frac{(b-s)(s-a)}{D}$.

Upper bounds:

Case 1. Consider $a \leq s \leq \xi_1,\ s \leq t$. Then

$$G_2(t,s) = \frac{b-t-\sum_{j=1}^{m} \alpha_j(\xi_j - t)}{D}(s-a)$$

$$= \frac{b-\sum_{j=1}^{m} \alpha_j \xi_j + t(\sum_{j=1}^{m} \alpha_j - 1)}{D}(s-a).$$

Since $\sum_{j=1}^{m} \alpha_j < 1$, the maximum occurs when $t = s$ and then

$$G_2(t,s) \leq \frac{b-s+\sum_{j=1}^{m} \alpha_j(s-\xi_j)}{D}(s-a)$$

$$\leq \frac{(b-s)(s-a)}{D},$$

since $\sum_{j=1}^{m} \alpha_j(s-\xi_j) \leq 0$ for $s \leq \xi_1$ and $\xi_j \in (a,b)$ with $a < \xi_1 < \xi_2 < ... < \xi_{m-2} < b$.

Case 2. For $\xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m+1,\ \xi_{i-1} \leq s \leq \xi_i, \quad 2 \leq i \leq r,\ s \leq t$, we have

$$G_2(t,s) = \frac{(b-t)(s-a) - \sum_{j=i}^{m} \alpha_j(\xi_j - t)(s-a) + \sum_{j=1}^{i-1} \alpha_j(\xi_j - a)(t-s)}{D}$$

$$= \frac{(b-t)(s-a) - \sum_{j=i}^{m} \alpha_j(\xi_j - s)(s-a) + \sum_{j=1}^{m} \alpha_j(t-s)(s-a) + \sum_{j=1}^{i-1} \alpha_j(\xi_j - s)(t-s)}{D}$$

$$\leq \frac{b-t+\sum_{j=1}^{m} \alpha_j(t-s)}{D}(s-a)$$

$$\leq \frac{b-s\sum_{j=1}^{m} \alpha_j + t(\sum_{j=1}^{m} \alpha_j - 1)}{D}(s-a)$$

since $\sum_{j=i}^{m} \alpha_j(s-\xi_j) \leq 0$ and $\sum_{j=1}^{i-1} \alpha_j(\xi_j - s) \leq 0$ for $\xi_{i-1} \leq s \leq \xi_i,\ 2 \leq i \leq m+1$.

Since $\sum_{j=1}^{m} \alpha_j < 1$, the maximum occurs when $t = s$ so

$$G_2(t,s) \leq \frac{(b-s)(s-a)}{D}.$$

Case 3. For $\xi_{r-1} \leq t \leq \xi_r,\ 1 \leq r \leq m,\ \xi_{i-1} \leq s \leq \xi_i, \quad r \leq i \leq m,\ t \leq s$, we obtain

$$G_2(t,s) = \frac{(t-a)[b-s-\sum_{j=i}^{m} a_j(\xi_j - s)]}{D}$$

$$\leq \frac{(b-s)(t-a)}{D}$$

$$\leq \frac{(b-s)(s-a)}{D}$$

since $\sum_{j=i}^{m} \alpha_j(\xi_j - s) \geq 0$ for $\xi_{i-1} \leq s \leq \xi_i,\ 2 \leq i \leq m$.

Case 4. For $\xi_m \leq s \leq b, \quad t \leq s$, we clearly have

$$G_2(t,s) \leq \frac{(b-s)(s-a)}{D}.$$

Lower bounds: We shall show that we may take an arbitrary interval $[\xi_1, b] \subset (a,b]$.

We are looking for $\min\{G_2(t,s) : t \in [\xi_1, b]\}$ as a function of s of the same form as the upper bound.

Case 1. Consider $a \leq s \leq \xi_1,\ s \leq t$, we get

$$G_2(t,s) = \frac{b-t-\sum_{j=1}^{m}\alpha_j(\xi_j-t)}{D}(s-a)$$

$$= \frac{b-\sum_{j=1}^{m}\alpha_j\xi_j+t(\sum_{j=1}^{m}\alpha_j-1)}{D}(s-a).$$

Since $\sum_{j=1}^{m}\alpha_j < 1$, the minimum occurs when $t=b$ and then

$$G_2(t,s) \geq \frac{b-\sum_{j=1}^{m}\alpha_j\xi_j+b(\sum_{j=1}^{m}\alpha_j-1)}{D}(s-a)$$

$$> \frac{(b-s)(s-a)}{D}\frac{1}{b}\sum_{j=1}^{m}\alpha_j(b-\xi_j).$$

Case 2. For $\xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m+1,\ \xi_{i-1} \leq s \leq \xi_i,\quad 2 \leq i \leq r,\ s \leq t$, we have

$$G_2(t,s) = \frac{(b-t)(s-a)-\sum_{j=i}^{m}\alpha_j(\xi_j-s)(s-a)+\sum_{j=1}^{m}\alpha_j(t-s)(s-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-s)(t-s)}{D}$$

$$= \frac{t[(\sum_{j=1}^{m}\alpha_j-1)(s-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-s)]+[b-s\sum_{j=1}^{m}\alpha_j-\sum_{j=i}^{m}\alpha_j(\xi_j-s)](s-a)-s\sum_{j=1}^{i-1}\alpha_j(\xi_j-s)}{D}.$$

Since $(\sum_{j=1}^{m}\alpha_j-1)(s-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-s) < 0$, the minimum occurs when $t=b$, then

$$G_2(t,s) \geq \frac{-\sum_{j=i}^{m}\alpha_j(\xi_j-b)(s-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-a)(b-s)}{D}$$

$$\geq \frac{1}{D}\sum_{j=i}^{m}\alpha_j(b-\xi_j)(s-a)$$

$$> \frac{(b-s)(s-a)}{D}\frac{1}{b}\sum_{j=i}^{m}\alpha_j(b-\xi_j).$$

Case 3. For $\xi_{r-1} \leq t \leq \xi_r,\ 1 \leq r \leq m,\ \xi_{i-1} \leq s \leq \xi_i,\quad r \leq i \leq m,\ t \leq s$, we obtain

$$G_2(t,s) = \frac{(t-a)[b-s-\sum_{j=i}^{m}\alpha_j(\xi_j-s)]}{D}$$

$$= \frac{(t-a)[(b-s)(1-\sum_{j=i}^{m}\alpha_j)-\sum_{j=i}^{m}\alpha_j(\xi_j-b)]}{D}$$

$$\geq \frac{(t-a)(b-s)}{D}[1-\sum_{j=i}^{m}\alpha_j]$$

$$\geq \frac{(\xi_1-a)(b-s)}{D}[1-\sum_{j=i}^{m}\alpha_j]$$

$$\geq \frac{(s-a)(b-s)}{D}\frac{\xi_1-a}{b-a}[1-\sum_{j=i}^{m}\alpha_j].$$

Case 4. For $\xi_m \leq s \leq b,\quad t \leq s$, we have

$$G_2(t,s) = \frac{(t-a)(b-s)}{D}$$

$$\geq \frac{(\xi_1-a)(b-s)}{D}$$

$$\geq \frac{(s-a)(b-s)}{D}\frac{\xi_1-a}{b-a}.$$

Thus we can take

$$k = \min_{2\leq i\leq m}\{\tfrac{1}{b}\sum_{j=i}^{m}\alpha_j(b-\xi_j),\ \tfrac{\xi_1-a}{b-a}[1-\sum_{j=i}^{m}\alpha_j]\}.$$

Lemma 2.2 immediately comes from [5, Lemma 2.1].

Lemma 2.2 *If y satisfies the boundary conditions*

$$\begin{cases} y^{(i)}(a)=0, \quad 0\leq i\leq n-2, \\ \sum_{i=1}^{m}\alpha_i y^{(n-2)}(\xi_i)=y^{(n-2)}(b) \end{cases}$$

and

$$y^{(n)}(t)\leq 0, \quad t\in[a,b],$$

then

$$y^{(n-2)}(t)\geq 0.$$

Let $\mathbb{B}$ be the Banach space defined by

$$\mathbb{B}=\{y : y^{(n)} \text{ is continuous on } [a,b],\ y^{(i)}(a)=0 \quad 0\leq i\leq n-3\}, \tag{2.2}$$

with the norm $\|y\|=\max_{t\in[a,b]}|y^{(n-2)}(t)|$ and let

$$\mathcal{P}=\{y\in\mathbb{B} : y^{(n-2)}(t)\geq 0,\ \ \min_{t\in[\xi_1,b]}y^{(n-2)}(t)\geq k\|y\|\ \} \tag{2.3}$$

where k is as in (2.1).

Solving the BVP (1.1) is equivalent to finding fixed points of the operator $A:\mathbb{B}\to\mathbb{B}$ defined by

$$Ay(t)=\int_a^b G_n(t,s)[Q(s,y,y',...y^{(n-2)})-P(s,y,y',...y^{(n-1)})]ds, \quad t\in[a,b]. \tag{2.4}$$

It can be verified that

$$G_2(t,s)=G_n^{(n-2)}(t,s). \tag{2.5}$$

From (2.5), it follows that

$$(Ay)^{(n-2)}(t)=\int_a^b G_2(t,s)[Q(s,y,y',...y^{(n-2)})-P(s,y,y',...y^{(n-1)})]ds. \tag{2.6}$$

Solving the BVP (1.1) in $\mathbb{B}$ is equivalent to finding fixed points of the operator $A^{(n-2)}$ defined by (2.6).

Lemma 2.3 *The operator A is completely continuous such that $A(\mathcal{P})\subset\mathcal{P}$.*

Proof. From the continuity of $G_2(t,s)$ and $f(t)$ it follows that the operator A defined by (2.4) is completely continuous in $\mathbb{B}$. By Lemma 2.1, Lemma 2.2, and definition of $\mathcal{P}$, we get $A(\mathcal{P})\subset\mathcal{P}$.

Lemma 2.4 follows from [4, Lemma 3.3].

Lemma 2.4 *(a) Let $y\in B$. Then,*

$$|y^{(i)}(t)|\leq\tfrac{(t-a)^{n-2-i}}{(n-2-i)!}\|y\|, \quad t\in[a,b], \quad 0\leq i\leq n-3.$$

In particular,

$$|y(t)| \leq \frac{(b-a)^{n-2}}{(n-2)!}\|y\| \quad t \in [a,b]. \tag{2.7}$$

(*b*) *Let* $y \in P$. *Then*

$$y^{(i)}(t) \geq 0, \quad t \in [a,b], \quad 0 \leq i \leq n-3,$$

and

$$y^{(i)}(t) \geq \frac{(t-\xi_1)^{n-2-i}}{(n-2-i)!}k\|y\|, \quad t \in [\xi_1,b], \quad 0 \leq i \leq n-3.$$

In particular,

$$y(t) \geq \frac{(\xi_2-\xi_1)^{n-2}}{(n-2)!}k\|y\| \quad t \in [\xi_2,b]. \tag{2.8}$$

3 Existence of two positive solutions

In this section, using Theorem 3.1, the Avery-Henderson fixed-point theorem, we prove the existence of at least two positive solutions of the BVP (1.1).

Theorem 3.1 [2] *Let* $\mathcal{P}$ *be a cone in a real Banach space* S. *If* η *and* ψ *are increasing, nonnegative continuous functionals on* $\mathcal{P}$, *let* θ *be a nonnegative continuous functional on* $\mathcal{P}$ *with* $\theta(0) = 0$ *such that, for some positive constants* r *and* M,

$$\psi(u) \leq \theta(u) \leq \eta(u) \text{ and } \|u\| \leq M\psi(u)$$

for all $u \in \overline{\mathcal{P}(\psi,r)}$. *Suppose that there exist positive numbers* $p < q < r$ *such that*

$$\theta(\lambda u) \leq \lambda\theta(u), \quad \text{for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta,q).$$

If $A : \overline{\mathcal{P}(\psi,r)} \to \mathcal{P}$ *is a completely continuous operator satisfying*

(*i*) $\psi(Au) > r$ *for all* $u \in \partial\mathcal{P}(\psi,r)$,
(*ii*) $\theta(Au) < q$ *for all* $u \in \partial\mathcal{P}(\theta,q)$,
(*iii*) $\mathcal{P}(\eta,p) \neq \emptyset$ *and* $\eta(Au) > p$ *for all* $u \in \partial\mathcal{P}(\eta,p)$,

then A *has at least two fixed points* u_1 *and* u_2 *such that*

$$p < \eta(u_1) \text{ with } \theta(u_1) < q \quad \text{and} \quad q < \theta(u_2) \text{ with } \psi(u_2) < r.$$

Again define the Banach space $\mathbb{B}$ by (2.2), the cone $\mathcal{P} \subset \mathbb{B}$ by (2.3) and the operator $A : \mathcal{P} \to \mathcal{P}$ by (2.4).

Let the nonnegative, increasing, continuous functionals ψ, θ, and η be defined on the cone $\mathcal{P}$ by

$$\psi(y) := \min_{t\in[\xi_1,b]} y^{(n-2)}(t), \ \theta(y) := \max_{t\in[\xi_1,b]} y^{(n-2)}(t), \ \eta(y) := \max_{t\in[a,b]} y^{(n-2)}(t) \tag{3.1}$$

and let $\mathcal{P}(\psi,r) := \{y \in \mathcal{P} : \psi(y) < r\}$.

Finally, define constants

$$R := \int_a^b \phi(s)(q_1(s) - p(s))ds, \tag{3.2}$$

$$N := \int_{\xi_2}^b \phi(s)(q(s) - p_1(s))ds. \tag{3.3}$$

Theorem 3.2 *Assume* $(H1)-(H3)$ *hold. Suppose there exist positive numbers* $0<p<q<r$ *such that the function f satisfies the following conditions:*

$$(i)\ f(y)>\tfrac{p}{Nk}\ \text{for } t\in[\xi_2,b]\ \text{and } y\in[\tfrac{kp(\xi_2-\xi_1)^{n-2}}{(n-2)!},\tfrac{p(b-a)^{n-2}}{(n-2)!}],$$
$$(ii)\ f(y)<\tfrac{q}{R}\ \text{for } t\in[a,b]\ \text{and } y\in[0,\tfrac{q(b-a)^{n-2}}{k(n-2)!}],$$
$$(iii)\ f(y)>\tfrac{r}{Nk}\ \text{for } t\in[\xi_2,b]\ \text{and } y\in[\tfrac{r(\xi_2-\xi_1)^{n-2}}{(n-2)!},\tfrac{r(b-a)^{n-2}}{k(n-2)!}]$$

where k, m, M are as defined in (2.1), (3.2) *and* (3.3) *respectively. Then the BVP (1.1) has at least two positive solutions* y_1 *and* y_2 *such that*

$$p<\max_{t\in[a,b]}y_1(t)\quad with\quad \max_{t\in[\xi_1,b]}y_1(t)<q,$$

$$q<\max_{t\in[\xi_1,b]}y_2(t)\quad with\quad \min_{t\in[\xi_1,b]}y_2(t)<r.$$

Proof. From (3.1), for each $y\in\mathcal{P}$ we have

$$\psi(y)\le\theta(y)\le\eta(y), \tag{3.4}$$

$$\|y\|\le\tfrac{1}{k}\min_{t\in[\xi_1,b]}y^{(n-2)}(t)=\tfrac{1}{k}\psi(y)\le\tfrac{1}{k}\theta(y)\le\tfrac{1}{k}\eta(y). \tag{3.5}$$

For any $y\in\mathcal{P}$, (3.4) and (3.5) imply

$$\psi(y)\le\theta(y)\le\eta(y),\quad \|y\|\le\frac{1}{k}\psi(y).$$

For all $y\in\mathcal{P},\ \lambda\in[0,1]$ we have

$$\theta(\lambda y)=\max_{t\in[\xi_1,b]}(\lambda y)^{(n-2)}(t)=\lambda\max_{t\in[\xi_1,b]}y^{(n-2)}(t)=\lambda\theta(y).$$

It is clear that $\theta(0)=0$.

We now show that the remaining conditions of Theorem 3.1 are satisfied.

Firstly, we shall verify that the condition (iii) of Theorem 3.1 is satisfied. Since $0\in\mathcal{P}$ and $p>0$, $\mathcal{P}(\eta,p)\neq\emptyset$. Since $y\in\partial\mathcal{P}(\eta,p),\ kp\le y^{(n-2)}(t)\le\|y\|=p$ for $t\in[\xi_2,b]$. In view of (2.7) and (2.8), it follows that $y(t)\in[\frac{kp(\xi_2-\xi_1)^{n-2}}{(n-2)!},\frac{p(b-a)^{n-2}}{(n-2)!}]$, for $t\in[\xi_2,b]$. Therefore,

$$\begin{aligned}\eta(Ay)&=\max_{t\in[a,b]}(Ay)^{(n-2)}(t)\\&\ge(Ay)^{(n-2)}(b)\\&=\int_a^b G_2(b,s)[Q(s,y,y',...y^{(n-2)})-P(s,y,y',...y^{(n-1)})]ds\\&\ge\int_a^b G_2(b,s)(q(s)-p_1(s))f(y(s))ds\\&>\tfrac{p}{N}\int_{\xi_2}^b\phi(s)(q(s)-p_1(s))ds\ge p\end{aligned}$$

using (3.3) and hypothesis (i).

Now we shall show that the condition (ii) of Theorem 3.1 is satisfied. Since $y\in\partial\mathcal{P}(\theta,q)$, from (3.5) we have that $0\le y^{(n-2)}(t)\le\|y\|\le q/k$ for $t\in[a,b]$. By (2.7), we have $y(t)\in[0,\frac{q(b-a)^{n-2}}{k(n-2)!}]$, for $t\in[a,b]$. Thus,

$$\theta(Ay)=\max_{t\in[\xi_1,b]}(Ay)^{(n-2)}(t)$$

$$= \max_{t\in[\xi_1,b]} \int_a^b G_2(t,s)[Q(s,y,y',...y^{(n-2)}) - P(s,y,y',...y^{(n-1)})]ds$$

$$\leq \max_{t\in[\xi_1,b]} \int_a^b G_2(t,s)(q_1(s)-p(s))f(y(s))ds$$

$$< \frac{q}{R}\int_a^b \phi(s)(q_1(s)-p(s))ds = q$$

by hypothesis (ii) and (3.2).

Finally, using hypothesis (iii) and (3.3), we shall show that the condition (i) of Theorem 3.1 is satisfied. Since $y \in \partial\mathcal{P}(\psi,r)$, from (3.5) we have that $\min_{t\in[\xi_1,b]} y^{(n-2)}(t) = r$ and $r \leq y^{(n-2)}(t) \leq \|y\| \leq r/k$. This implies $y(t) \in [\frac{r(\xi_2-\xi_1)^{n-2}}{(n-2)!}, \frac{r(b-a)^{n-2}}{k(n-2)!}]$, for $t \in [\xi_2, b]$. By the concavity of $(Ay)^{(n-2)}$,

$$\psi(Ay) = \min_{t\in[\xi_1,b]} (Ay)^{(n-2)}(t) = \min\{(Ay)^{(n-2)}(\xi_1), (Ay)^{(n-2)}(b)\}.$$

Now, we suppose $\psi(Ay) = (Ay)^{(n-2)}(\xi_1)$. Thus,

$$\psi(Ay) = (Ay)^{(n-2)}(\xi_1) = \int_a^b G_2(\xi_1,s)[Q(s,y,y',...y^{(n-2)}) - P(s,y,y',...y^{(n-1)})]ds$$

$$\geq \int_{\xi_2}^b k\phi(s)(q(s)-p_1(s))f(y(s))ds$$

$$> \frac{r}{N}\int_{\xi_2}^b \phi(s)(q(s)-p_1(s))ds \geq r$$

Now, we suppose $\psi(Ay) = (Ay)^{(n-2)}(b)$. Then

$$\psi(Ay) = (Ay)^{(n-2)}(b) \geq \int_a^b G_2(b,s)(q(s)-p_1(s))f(y(s))ds$$

$$> \frac{r}{N}\int_{\xi_2}^b \phi(s)(q(s)-p_1(s))ds \geq r.$$

This completes the proof. □

4 Existence of three positive solutions

In this section, using Theorem 4.1, the five functionals fixed-point theorem, we prove the existence of at least three positive solutions of the BVP (1.1).

Let $\gamma,\ \beta, \theta$ be nonnegative continuous concave functionals on $\mathcal{P}$. For nonnegative real numbers l, p, q, r, and h, define the convex sets

$$\mathcal{P}(\gamma,h) = \{y \in \mathcal{P} : \gamma(y) \leq h\}$$

$$\mathcal{P}(\gamma,\alpha,p,h) = \{y \in \mathcal{P} : p \leq \alpha(y),\ \gamma(y) \leq h\}$$

$$\mathcal{Q}(\gamma,\beta,r,h) = \{y \in \mathcal{P} : \beta(y) \leq r, \gamma(y) \leq h\}$$

$$\mathcal{P}(\gamma,\theta,\alpha,p,q,h) = \{y \in \mathcal{P} : p \leq \gamma(y), \theta(y) \leq q, \gamma(y) \leq h\},$$

$$\mathcal{Q}(\gamma,\beta,\psi,l,r,h) = \{y \in \mathcal{P} : l \leq \psi(y), \beta(y) \leq r, \gamma(y) \leq h\}.$$

Theorem 4.1 [1] *Suppose there exist $h > 0$ and $m > 0$ such that $\alpha(y) \leq \beta(y)$ and $\|y\| \leq m\gamma(y)$ for all $y \in \overline{\mathcal{P}(\gamma,h)}$. Let $\overline{\mathcal{P}(\gamma,h)} \to \overline{\mathcal{P}(\gamma,h)}$ be a completely continuous operator. If there exist nonnegative real numbers p, q, r and l with $0 < r < p$ such that*

$$(i)\ \{y \in \mathcal{P}(\gamma,\theta,\alpha,p,q,h) : \alpha(y) > p\} \neq \emptyset \text{ and } \alpha(Ay) > p \text{ for } y \in \mathcal{P}(\gamma,\theta,\alpha,p,q,h);$$

(ii) $\{y \in \mathcal{Q}(\gamma,\beta,\psi,l,r,h) : \beta(y) < r\} \neq \emptyset$ *and* $\beta(Ay) < r$ *for* $y \in \mathcal{Q}(\gamma,\beta,\psi,l,r,h)$;

(iii) $\alpha(Ay) > p$ *for* $y \in \mathcal{P}(\gamma,\alpha,p,h)$ *with* $\theta(Ay) > q$;

(iv) $\beta(Ay) < r$ *for* $y \in \mathcal{Q}(\gamma,\beta,r,h)$ *with* $\theta(Ay) < l$.

Then A *has at least three positive solutions* $y_1,\ y_2$ *and* y_3 *in* $\overline{\mathcal{P}(\gamma,h)}$ *satisfying*

$$\beta(y_1) < r,\ p < \alpha(y_2),\ r < \beta(y_3) \text{ with } \alpha(y_3) < p.$$

Again define the Banach space $\mathbb{B}$ by (2.2), the cone $\mathcal{P} \subset \mathbb{B}$ by (2.3) and the operator $A : \mathcal{P} \to \mathcal{P}$ by (2.4).

Let $l = 0, m = 1$ and define the nonnegative, continuous, concave functionals α, ψ and the nonnegative, continuous, convex functionals γ, β, θ on $\mathcal{P}$ by

$$\alpha(y) = \min_{t\in[\xi_1,b]} y(t),\ \psi(y) \equiv 0,\ \gamma(y) = \beta(y) = \theta(y) = \|y\|.$$

In addition to this

$$\alpha(y) \leq \beta(y) \text{ and } \|y\| \leq m\gamma(y) \text{ for } y \in \mathcal{P}.$$

Theorem 4.2 *Assume* $(H1)-(H3)$ *hold. Suppose there exist positive numbers* $0 < r < p < h$ $(h > RN^{-1}k^{-1}p)$ *such that*

$(D1)$ $f(y) \leq \frac{r}{R}$ *for* $t \in [a,b]$ *and* $y \in [0, \frac{r(b-a)^{n-2}}{(n-2)!}]$,

$(D2)$ $f(y) \leq \frac{\epsilon_0}{Nk}p$ *for* $t \in [a,b]$ *and* $y \in [\frac{r(b-a)^{n-2}}{(n-2)!}, \frac{h(b-a)^{n-2}}{(n-2)!}]$

$(D3)$ $f(y) > \frac{1}{Nk}p$ *for* $t \in [\xi_2,b]$ *and* $y \in [\frac{p(\xi_2-\xi_1)^{n-2}}{(n-2)!}, \frac{h(b-a)^{n-2}}{(n-2)!}]$,

where $\epsilon_0 \geq 1$ *with* $\epsilon_0 RN^{-1}k^{-1}p < h$, k, R, N *are as defined in* (2.1), (3.2) *and* (3.3) *respectively. Then the boundary value problem* (1.1) *has at least three positive solutions* $y_1,\ y_2,$ *and* y_3 *satisfying*

$$\beta(y_1) < r,\quad \alpha(y_2) > p,\quad r < \beta(y_3) \text{ with } \alpha(y_3) < p.$$

Proof. The conditions of the five functionals fixed point theorem (Theorem 4.1) will be shown to be satisfied. From Lemma 2.3 $A : \mathcal{P} \to \mathcal{P}$. For $y \in \overline{\mathcal{P}(\gamma,h)}$, by using (2.7), we get $y(t) \in [0, \frac{h(b-a)^{n-2}}{(n-2)!}]$, for $t \in [a,b]$. Since $R \geq Nk$, $r < p$ and $\epsilon_0 \geq 1$, we have $\frac{r}{R} < \frac{\epsilon_0}{Nk}p$. Then we obtain

$$\begin{aligned}(Ay)^{(n-2)}(t) &\leq \textstyle\int_a^b G_2(t,s)(q_1(s)-p(s))f(y(s))ds\\ &\leq \textstyle\int_a^b \phi(s)(q_1(s)-p(s))f(y(s))ds\\ &\leq \tfrac{\epsilon_0}{Nk}p\textstyle\int_a^b \phi(s)(q_1(s)-p(s))ds\\ &= \epsilon_0 RN^{-1}k^{-1}p < h\end{aligned}$$

using hypothesis $(D2)$. Therefore $A : \overline{\mathcal{P}(\gamma,h)} \to \overline{\mathcal{P}(\gamma,h)}$.

In the following, we now prove that the conditions of Theorem 4.1 satisfy with $q = h$.

It is obvious that

$$\{y \in \mathcal{P}(\gamma, \theta, \alpha, p, q, h) : \alpha(y) > p\} = \{y \in \mathcal{P} : \min_{t\in[\xi_1,b]} y^{(n-2)}(t) > p, \quad \|y\| \leq h\} \neq \emptyset.$$

Let $y \in \mathcal{P}(\gamma, \theta, \alpha, p, q, h)$, then $p \leq \min_{t\in[\xi_1,b]} y^{(n-2)}(t)$ and $\|y\| \leq h$. This implies $y(t) \in [\frac{p(\xi_2-\xi_1)^{n-2}}{(n-2)!}, \frac{h(b-a)^{n-2}}{(n-2)!}]$ for $t \in [\xi_2, b]$.

By the concavity of $(Ay)^{(n-2)}$, there are two cases:

either $\alpha(Ay) = (Ay)^{(n-2)}(\xi_1)$, or $\alpha(Ay) = (Ay)^{(n-2)}(b)$.

First, suppose $\alpha(Ay) = (Ay)^{(n-2)}(\xi_1)$.

$$\alpha(Ay) = (Ay)^{(n-2)}(\xi_1) = \int_a^b G_2(\xi_1, s)[Q(s, y, y', ...y^{(n-2)}) - P(s, y, y', ...y^{(n-1)})]ds$$

$$\geq k\int_a^b \phi(s)(q(s) - p_1(s))f(y(s))ds > k\frac{p}{Nk}\int_{\xi_2}^b \phi(s)(q(s) - p_1(s))ds = p.$$

Now, we suppose $\alpha(Ay) = (Ay)^{(n-2)}(b)$. Then,

$$\alpha(Ay) = (Ay)^{(n-2)}(b) = \int_a^b G_2(b, s)[Q(s, y, y', ...y^{(n-2)}) - P(s, y, y', ...y^{(n-1)})]ds$$

$$\geq k\int_a^b \phi(s)(q(s) - p_1(s))f(y(s))ds > k\frac{p}{Nk}\int_{\xi_2}^b \phi(s)(q(s) - p_1(s))ds = p.$$

Hence, using (3.3) and hypothesis $(D3)$, condition (i) of Theorem 4.1 holds.

It is clear that

$$\{y \in \mathcal{P}(\gamma, \beta, \psi, l, r, h) : \beta(y) < r\} = \{y \in \mathcal{P} : \|y\| < r\} \neq \emptyset.$$

By hypothesis $(D1)$ and (3.2), we get for $y \in \mathcal{P}(\gamma, \beta, \psi, l, r, h)$,

$$(Ay)^{(n-2)}(t) \leq \int_a^b G_2(t, s)(q_1(s) - p(s))f(y(s))ds$$

$$\leq \int_a^b \phi(s)(q_1(s) - p(s))f(y(s))ds$$

$$\leq \frac{r}{R}\int_a^b \phi(s)(q_1(s) - p(s))ds = r.$$

Thus, condition (ii) of Theorem 4.1 holds.

Since

$$\mathcal{P}(\gamma, \alpha, p, h) = \{y \in \mathcal{P} : \min_{t\in[\xi_1,b]} y^{(n-2)}(t) \geq p, \quad \|y\| \leq h\},$$

we get $\alpha(Ay) > p$ for $y \in \mathcal{P}(\gamma, \alpha, p, h)$. Therefore, condition (iii) of Theorem 4.1 is satisfied.

Finally, as far as (iv) is concerned, we omit the condition (iv) since $\psi(Ay) < l = 0$ is impossible. Since all conditions of Theorem 4.1 are verified, the BVP (1.1) has at least three positive solutions such that

$$\|y_1\| < r, \quad \min_{t\in[\xi_2,b]} y_2(t) > p, \quad r < \|y_3\|, \ with \ \min_{t\in[\xi_2,b]} y_3(t) < p.$$

5 Examples

Example 5.1. Let us introduce an example to illustrate the applicability of Theorem 4.2. Consider the BVP:

$$\begin{cases} y^{(iv)}(t) + (g(t,y,y',y'')+4)\frac{22000e^y}{e^y+2640e^{18}} = g(t,y,y',y'',y''')\frac{22000e^y}{e^y+2640e^{18}}, \quad t\in[0,5], \\ y(0) = y'(0) = 0, \\ \frac{1}{3}y'(1) + \frac{1}{7}y'(2) = y'(5). \end{cases} \tag{5.1}$$

Then $a = 0,\ b = 5,\ \alpha_1 = \frac{1}{3},\ \alpha_2 = \frac{1}{7},\ \xi_1 = 1,\ \xi_2 = 2,\ P(t,y,y',y'',y''') = g(t,y,y',y'')\frac{22000e^y}{e^y+2640e^{18}}$ and $Q(t,y,y',y'') = (g(t,y,y',y'')+2)\frac{22000e^y}{e^y+2640e^{18}}$, where the map $u\in\mathbb{R}^3 \to g(t,u)\in\mathbb{R}$ is continuous for all $t\in[0,5]$.

Taking $f(y) = \frac{22000e^y}{e^y+2640e^{18}}$, we get

$$\frac{Q(t,y,y',y'')}{f(y)} = g(t,y,y',y'')+4 \quad \text{and} \quad \frac{P(t,y,y',y'',y''')}{f(y)} = g(t,y,y',y'').$$

Hence, we may choose

$$q(t) = g(t,y,y',y'')+1, \quad q_1(t) = g(t,y,y',y'')+4,$$

and

$$p(t) = p_1(t) = g(t,y,y',y'').$$

Clearly f is continuous and increasing on $[0,\infty)$.

By (2.1), (3.2) and (3.3), we get $k = \frac{3}{35},\ R = \frac{875}{46},\ N = \frac{567}{184}$. If we take $p = e^{-17},\ q = e^{-12},$ and $r = 5800$ then

$$0 < p < q < r.$$

It is clear that $(i),\ (ii),\ (iii)$ are satisfied. Thus by Theorem 3.2, the BVP (5.1) has at least two positive solutions y_1, y_2 with

$$e^{-17} < \eta(y_1) \ \text{ with } \ \theta(y_1) < e^{-12} \ \text{ and } \ e^{-12} < \theta(y_2) \ \text{ with } \ \psi(y_2) < 5800.$$

Example 5.2. Let us introduce an example to illustrate the applicability of Theorem 3.2. Consider the BVP:

$$\begin{cases} y'''(t) + (g(t,y,y')+2)\frac{5000y^2}{y^2+1} = g(t,y,y')\frac{5000y^2}{y^2+1}, \quad t\in[1,10], \\ y(1) = y'(1) = 0, \\ \frac{1}{2}y'(2) + \frac{1}{5}y'(5) + \frac{1}{10}y'(7) = y'(10). \end{cases} \tag{5.2}$$

Then $a = 1,\ b = 10,\ \alpha_1 = \frac{1}{2},\ \alpha_2 = \frac{1}{5},\ \alpha_3 = \frac{1}{10},\ \xi_1 = 2,\ \xi_2 = 5, \xi_3 = 7,\ P(t,y,y',y'') = g(t,y,y')\frac{5000y^2}{y^2+1}$ and $Q(t,y,y') = (g(t,y,y')+2)\frac{5000y^2}{y^2+1}$, where the map $u\in\mathbb{R}^2 \to g(t,u)\in\mathbb{R}$ is continuous for all $t\in[1,10]$.

Taking $f(y) = \frac{5000y^2}{y^2+1}$, we get

$$\frac{Q(t,y,y')}{f(y)} = g(t,y,y') + 2 \quad \text{and} \quad \frac{P(t,y,y',y'')}{f(y)} = g(t,y,y').$$

Hence, we may choose

$$q(t) = g(t,y,y') + 1, \quad q_1(t) = g(t,y,y') + 2,$$

and

$$p(t) = p_1(t) = g(t,y,y').$$

Clearly f is continuous and increasing on $[0,\infty)$.

By (2.1), (3.2) and (3.3), we get $k = \frac{3}{100},\ R = \frac{1215}{71},\ N = \frac{2125}{213}$. If we take $p = 1000,\ h = 975000,\ r = 10^{-7}$ and $\epsilon_0 = 17$ then

$$0 < r < p < h, \quad h > RN^{-1}k^{-1}p \ \text{ and } \ \epsilon_0 RN^{-1}k^{-1}p < h.$$

It is clear that $(D1)$, $(D2)$, $(D3)$ are satisfied. Thus by Theorem 3.2, the BVP (5.2) has at least two positive solutions y_1, y_2 with

$$\|y_1\| < 10^{-7}, \ \ \min_{t\in[2,10]} y_2^{(n-2)}(t) > 10^3, \ \ 10^{-7} < \|y_3\| \ \text{ with } \ \min_{t\in[2,10]} y_3^{(n-2)} < 10^3.$$

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WEAK ϕ-CONTRACTION ON PARTIAL METRIC SPACES

ERDAL KARAPINAR

Abstract. In this manuscript, the notion of weak ϕ-contraction is considered on partial metric space. It is shown that a self mapping T on a complete partial metric space X has a fixed point if they satisfied weak ϕ-contraction.

1. Introduction and Preliminaries

In 1992, Matthews [1, 2] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which $d(x,x)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (See e.g. [3, 4, 5, 6])

A partial metric space (See e.g.[1, 2]) is a pair $(X, p : X \times X \to \mathbb{R}^+)$ (where $\mathbb{R}^+$ denotes the set of all non negative real numbers) such that

(PM1) $p(x,y) = p(y,x)$ (symmetry)
(PM2) If $p(x,x) = p(x,y) = p(y,y)$ then $x = y$ (equality)
(PM3) $p(x,x) \leq p(x,y)$ (small self-distances)
(PM4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$ (triangularity)

for all $x, y, z \in X$. We use the abbreviation PMS for the partial metric space (X,p).

Notice that for a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

is a (usual) metric on X. Notice also that each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1. *(See e.g.*[1, 2, 6] *)*

(*i*) *A sequence* $\{x_n\}$ *in a PMS* (X,p) *converges to* $x \in X$ *if and only if* $p(x,x) = \lim_{n\to\infty} p(x,x_n)$,
(*ii*) *A sequence* $\{x_n\}$ *in a PMS* (X,p) *is called a Cauchy if and only if* $\lim_{n,m\to\infty} p(x_n,x_m)$ *exists (and finite),*
(*iii*) *A PMS* (X,p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges, with respect to* τ_p*, to a point* $x \in X$ *such that* $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(*iv*) *A mapping* $f : X \to X$ *is said to be continuous at* $x_0 \in X$*, if for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

The following lemma will be useful in the proof of main theorem.

Lemma 2. *(See e.g.*[1, 2, 6] *)*

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(A) *A sequence* $\{x_n\}$ *is Cauchy in a PMS* (X,p) *if and only if* $\{x_n\}$ *is Cauchy in a metric space* (X,d_p),
(B) *A PMS* (X,p) *is complete if and only if a metric space* (X,d_p) *is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m) \tag{1.2}$$

Boyd and Wong [7] introduced the notion of Φ-contraction: a self mapping T on a metric space X is called Φ-contraction if there exists an upper semi-continuous function $\Phi:[0,\infty)\to[0,\infty)$ such that

$$d(Tx,Ty)\leq \Phi(d(x,y)) \quad \text{for all } \ x,y\in X.$$

Alber and Guerre-Delabriere [8], generalized the notion of Φ-contraction by defining the notion of weak ϕ-contraction for Hilbert spaces: A self mapping T on a metric space X is called weak ϕ-contraction if $\phi:[0,\infty)\to[0,\infty)$ is a strictly increasing map with $\phi(0)=0$ and

$$d(Tx,Ty)\leq d(x,y)-\phi(d(x,y)), \text{ for all } x,y\in X.$$

They also proved the existence of fixed points in Hilbert spaces. If we replace Hilbert spaces with an arbitrary Banach spaces [8] still we have fixed points(See e.g. [9]). We should notice that for a lower semi-continuous mapping ϕ, the function $\Phi(u)=u-\phi(u)$ coincides with Boyd and Wong types.

In fixed point theory, Φ-contraction and weak ϕ-contraction have been studied by many authors, (See e.g.,[10, 11, 12, 13, 14], also [15]). In this manuscript, by using weak ϕ-contraction on a complete partial metric space, we obtain a unique fixed point.

2. Main Results

We start with the definition of weak ϕ-contraction.

Definition 3. *Let* (X,p) *be a PMS. An operator* $T:Y\to Y$ *is called a weak* ϕ*-contraction if there exists a continuous, non-decreasing function* $\phi:[0,\infty)\to[0,\infty)$ *with* $\phi(t)>0$ *for* $t\in(0,\infty)$ *and* $\phi(0)=0$*, such that*

$$d(Tx,Ty)\leq d(x,y)-\phi(d(x,y)) \tag{2.1}$$

for any $x,\ y\in X$.

Theorem 4. *Let* (X,p) *be a complete partial metric space. Suppose that* $\phi:[0,\infty)\to[0,\infty)$ *is a continuous, sublinear, non-decreasing function* $\phi:[0,\infty)\to[0,\infty)$ *with* $\phi(t)>0$ *for* $t\in(0,\infty)$ *and* $\phi(0)=0$*. Suppose also that* $T:X\to X$ *is a weak* ϕ*-contraction. Then,* T *has a unique fixed point.*

Proof. Let $x_0\in X$ and set $x_{n+1}=Tx_n$. Notice that, if $x_n=x_{n+1}$ for any $n\geq 0$, then obviously T has a fixed point. Thus, suppose $x_n\neq x_{n+1}$ for all $n\geq 0$. Due to (2.1), we have

$$p(x_{n+1},x_{n+2})=p(Tx_n,Tx_{n+1})\leq p(x_n,x_{n+1})-\phi(p(x_n,x_{n+1})).$$

Define $t_n=p(x_n,x_{n+1})$. Then one can obtain

$$t_{n+1}\leq t_n-\phi(t_n)\leq t_n \tag{2.2}$$

which implies that $\{t_n\}$ is a non-negative non-increasing sequence. Hence, $\{t_n\}$ converges to L where $L\geq 0$. So there are two cases: $L>0$ or $L=0$. Assume that

$L > 0$. Regarding that ϕ is non-decreasing, we get $0 < \phi(L) \leq \phi(t_n)$. Due to (2.2), we have $t_{n+1} \leq t_n - \phi(t_n) \leq t_n - \phi(L)$ and so

$$t_{n+2} \leq t_{n+1} - \phi(t_{n+1}) \leq t_n - \phi(t_n) - \phi(t_{n+1}) \leq t_n - 2\phi(L).$$

Inductively we obtain $t_{n+k} \leq t_n - k\phi(L)$ which is a contradiction for large enough $k \in \mathbb{N}$. Hence we have $L = 0$. Thus, we have $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$.

Due to (PM3) we have

$$p(x_n, x_n) \leq p(x_{n+1}, x_n) \text{ and } \ p(x_{n+1}, x_{n+1}) \leq p(x_{n+1}, x_n)$$

and hence

$$\lim_{n\to\infty} p(x_n, x_n) = 0 \text{ and } \lim_{n\to\infty} p(x_{n+1}, x_{n+1}) = 0 \tag{2.3}$$

By definition,

$$d_p(x_{n+1}, x_n) = 2p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1}) - p(x_n, x_n) \tag{2.4}$$

and thus regarding (2.3) and (2.4) one can obtain that $\lim_{n\to\infty} d_p(x_{n+1}, x_n) = 0$.

Take $\varepsilon > 0$. Choose $N_0 \in \mathbb{N}$ in a way that $d_p(x_{N_0}, x_{N_0+1}) \leq \min\{\frac{\varepsilon}{2}, \phi(\frac{\varepsilon}{2})\}$. We assert that T is a self mapping on the closed ball $B_{d_p}(x_{N_0}, \varepsilon)$. To proof our assertion, take $x \in B_{d_p}(x_{N_0}, \varepsilon)$.

Case 1. $d_p(x_{N_0}, x) \leq \frac{\varepsilon}{2}$, then due to (1.1),(2.1) and triangle inequality, we have

$$\begin{aligned} d_p(Tx, x_{N_0}) &\leq d_p(Tx, Tx_{N_0}) + d_p(Tx_{N_0}, x_{N_0}) \\ &\leq 2p(Tx, Tx_{N_0}) - p(Tx_{N_0}, Tx_{N_0}) - p(Tx, Tx) + d_p(Tx_{N_0}, x_{N_0}) \\ &= 2[p(x, x_{N_0}) - \phi(p(x, x_{N_0}))] - [p(x_{N_0}, x_{N_0}) - \phi(p(x_{N_0}, x_{N_0}))] \\ &-[p(x, x) - \phi(p(x, x))] + d_p(x_{N_0+1}, x_{N_0}) \\ &\leq 2p(x, x_{N_0}) - p(x_{N_0}, x_{N_0}) - p(x, x) \\ &-[2\phi(p(x, x_{N_0})) - \phi(p(x_{N_0}, x_{N_0})) - \phi(p(x, x))] + d_p(x_{N_0+1}, x_{N_0}) \\ &= d_p(x, x_{N_0}) - [2\phi(p(x, x_{N_0})) - \phi(p(x_{N_0}, x_{N_0})) - \phi(p(x, x))] \\ &+d_p(x_{N_0+1}, x_{N_0}) \end{aligned} \tag{2.5}$$

Due to (PM3), we have

$$p(x, x) \leq p(x, x_{N_0}) \ \text{ and } \ p(x_{N_0}, x_{N_0}) \leq p(x, x_{N_0})$$

by the properties of ϕ

$$\phi(p(x, x)) \leq \phi(p(x, x_{N_0})) \ \text{ and } \ \phi(p(x_{N_0}, x_{N_0})) \leq \phi(p(x, x_{N_0}))$$

which implies that

$$0 \leq 2\phi(p(x, x_{N_0})) - \phi(p(x, x)) - \phi(p(x, x_{N_0})) \tag{2.6}$$

Combining (2.5) and (2.6) we get

$$\begin{aligned} d_p(Tx, x_{N_0}) &\leq d_p(x, x_{N_0}) + d_p(x_{N_0+1}, x_{N_0}) \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{2.7}$$

Now consider the other case: $d_p(x, x_{N_0}) > \frac{\varepsilon}{2}$. Thus, $\frac{\varepsilon}{2} < d_p(x, x_{N_0}) \leq \varepsilon$ which implies that $\phi(\frac{\varepsilon}{2}) \leq \phi(d_p(x, x_{N_0}))$. Regarding (1.1), we have

$$d_p(x, x_{N_0}) = 2p(x, x_{N_0}) - p(x_{N_0}, x_{N_0}) - p(x, x)$$

and by properties of ϕ,

$$\begin{aligned}\phi(\tfrac{\varepsilon}{2}) &\leq \phi(d_p(x,x_{N_0})) = \phi(2p(x,x_{N_0}) - p(x_{N_0},x_{N_0}) - p(x,x)) \\ &\leq 2\phi(p(x,x_{N_0})) - \phi(p(x_{N_0},x_{N_0})) - \phi(p(x,x))\end{aligned} \tag{2.8}$$

Thus, due to (2.1), (2.8) and triangle inequality, we have

$$\begin{aligned}d_p(Tx,x_{N_0}) &\leq d_p(Tx,Tx_{N_0}) + d_p(Tx_{N_0},x_{N_0}) \\ &= 2p(Tx,Tx_{N_0}) - p(Tx_{N_0},Tx_{N_0}) - p(Tx,Tx) + d_p(Tx_{N_0},x_{N_0}) \\ &\leq 2[p(x,x_{N_0}) - \phi(p(x,x_{N_0}))] - [p(x_{N_0},x_{N_0}) - \phi(p(x_{N_0},x_{N_0}))] \\ &-[p(x,x) - \phi(p(x,x))] + d_p(Tx_{N_0},x_{N_0}) \\ &\leq 2p(x,x_{N_0}) - p(x_{N_0},x_{N_0} - p(x,x) + d_p(Tx_{N_0},x_{N_0}) \\ &-[2\phi(p(x,x_{N_0})) - \phi(p(x_{N_0},x_{N_0})) - \phi(p(x,x))] \\ &\leq d_p(x,x_{N_0}) + d_p(x_{N_0+1},x_{N_0}) \\ &-[2\phi(p(x,x_{N_0})) - \phi(p(x_{N_0},x_{N_0})) - \phi(p(x,x))] \\ &\leq \varepsilon + \phi(\tfrac{\varepsilon}{2}) - \phi(\tfrac{\varepsilon}{2}) \\ &\leq \varepsilon.\end{aligned} \tag{2.9}$$

Thus, in any case, $Tx \in B_{d_p}(x_{N_0},\varepsilon)$. In other words, T is a self mapping on the closed ball $B_{d_p}(x_{N_0},\varepsilon)$ and thus $x_n \in B_{d_p}(x_{N_0},\varepsilon)$ for each $n > N_0$. Therefore, the sequence $\{x_n\}$ is Cauchy in X. Since (X,p) is complete, then by Lemma 1.2 (X,d_p) is complete either. Taking into account that $B_{d_p}(x_{N_0},\varepsilon)$ is closed, the sequence $\{x_n\}$ is convergent in X, say $z \in X$, that is $\lim_{n\to\infty} d_p(x_n,z) = 0$. By Lemma 1.2,

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m) \tag{2.10}$$

Claim that $p(z,z) = 0$. Consider

$$p(x_{n+1},x_{n+1}) \leq p(x_n,x_n) - \phi(p(x_n,x_n)). \tag{2.11}$$

Letting $n \to \infty$ and taking continuity of ϕ into account, we obtain $p(z,z) \leq p(z,z) - \phi(p(z,z))$ which is possible only if $p(z,z) = 0$.

We assert that $Tz = z$. Due to (PM4), we have

$$\begin{aligned}p(Tz,z) &\leq p(Tz,Tx_n) + p(Tx_n,z) - p(Tx_n,Tx_n) \\ &\leq p(z,x_n) - \phi(p(z,x_n)) + p(x_{n+1},z) - [p(x_n,x_n) - \phi(p(x_n,x_n))]\end{aligned} \tag{2.12}$$

Letting $n \to \infty$ and regarding the continuity of ϕ, the expression (2.12) yields that $p(Tz,z) \leq 0$. Hence $Tz = z$.

Now we show that z is unique fixed point of T. Assume the contrary, that is, there exists $w \in X$ such that $z \neq w$ and $p(w,Tw) = 0$.

$$p(z,w) = p(Tz,Tw) \leq p(z,w) - \phi(p(z,w))$$

which is a contradiction. Thus z is a unique fixed point of T. □

Example 5. *Suppose $X = [0,1]$ and $q(x,y) = |x-y|$ and also $p(x,y) = \max\{x,y\}$. Then (X,q) and (X,p) are PMS. Suppose $T : X \to X$ such that $Tx = \frac{x}{2}$ for all $x \in X$ and $\phi(t) : [0,\infty) \to [0,\infty)$ such that $\phi(t) = \sqrt{t}$. Clearly, all conditions*

of Theorem 4 are satisfied by (X, q) and (X, p). In any case T has a unique fixed point, indeed $x = 0$.

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Non-Archimedean stability of an AQQ functional equation

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Abstract. Recently, Mohammadi et al. [24] proved the Hyers-Ulam stability of the following additive-quadratic-quartic functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 2f(x+y)+2f(-x-y)+2f(x-y) \\ &+ 2f(y-x)-4f(-x)-2f(x)+f(2y) \\ &+ f(-2y)-4f(y)-4f(-y) \end{aligned}$$

in random normed spaces.

In this paper, we prove the Hyers-Ulam stability of the above functional equation in non-Archimedean normed spaces.

1. Introduction and Preliminaries

The stability problem of functional equations is originated from a question of Ulam [34] concerning the stability of group homomorphisms. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [30] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. *let f be an approximately additive mapping from a normed vector space E into a Banach space E', i.e., f satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^r+\|y\|^r)$$

for all $x, y \in E$, where ϵ and r are constants with $\epsilon > 0$ and $0 \leq r < 1$. Then the mapping $L : E \to E'$ defined by $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies

$$\|f(x+y)-L(x)\| \leq \frac{2\epsilon}{2-2^r}\|x\|^r$$

for all $x \in E$.

The paper of Th.M. Rassias [30] has provided a lot of influence in the development of what we call *Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

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is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [33] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [8] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [3]–[5], [7, 11, 12], [16]–[19], [22, 26, 28, 29, 31, 32]).

In 1897, Hensel [14] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [9, 20, 21, 27].

In [23], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.1}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.1), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Definition 1.1. *By a* non-Archimedean field *we mean a field* $\mathbb{K}$ *equipped with a function(valuation)* $|\cdot| : \mathbb{K} \to [0,\infty)$ *such that for all* $r, s \in \mathbb{K}$*, the following conditions hold:*
(i) $|r| = 0$ *if and only if* $r = 0$
(ii) $|rs| = |r||s|$
(iii) $|r + s| \leq max\{|r|, |s|\}$.

Remark 1.1. *Clearly* $|1| = |-1| = 1$ *and* $|n| \leq 1$ *for all* $n \in \mathbb{N}$.

Definition 1.2. *Let* X *be a vector space over a scalar field* $\mathbb{K}$ *with a non-Archimedean non-trivial valuation* $|\cdot|$*. A function* $||\cdot|| : X \to \mathbb{R}$ *is a* non-Archimedean norm *(valuation) if it satisfies the following conditions:*
(i) $||x|| = 0$ *if and only if* $x = 0$
(ii) $||rx|| = |r|||x||$ $(r \in \mathbb{K}, x \in X)$
(iii) The strong triangle inequality(ultrametric); namely

$$||x + y|| \leq max\{||x||, ||y||\}, \qquad x, y \in X.$$

Then $(X, ||\cdot||)$ *is called a non-Archimedean space.*

Definition 1.3. *A sequence* $\{x_n\}$ *is* Cauchy *if and only if* $\{x_{n+1} - x_n\}$ *converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.*

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: "for $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$".

Example 1.1. *Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-*adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$ where $|a_k| \leq p-1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact filed.*

Arriola and Beyer [2] investigated the Hyers-Ulam stability of approximate additive functions $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous function for which there exists a fixed ϵ:

$$|f(x+y) - f(x) - f(y)| \leq \epsilon$$

for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive function $T : \mathbb{Q}_p \to \mathbb{R}$ such that

$$|f(x) - T(x)| \leq \epsilon$$

for all $x \in \mathbb{Q}_p$. Moslehian and Th.M. Rassias [25] proved the Hyers-Ulam stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean normed spaces.

In this paper, we prove the Hyers-Ulam stability of the following additive-quadratic-quartic functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 2f(x+y)+2f(-x-y)+2f(x-y) \\ &+ 2f(y-x)-4f(-x)-2f(x)+f(2y) \\ &+ f(-2y)-4f(y)-4f(-y), \end{aligned} \tag{1.2}$$

in non-Archimedean normed spaces.

2. Non-Archimedean stability of the functional equation (1.2): odd case

One can easily show that an odd mapping $f : X \to Y$ satisfies

$$\begin{aligned} f(x+2y)+f(x-2y) &= 2f(x+y)+2f(-x-y)+2f(x-y) \\ &+ 2f(y-x)-4f(-x)-2f(x)+f(2y) \\ &+ f(-2y)-4f(y)-4f(-y) \end{aligned}$$

if and only if the odd mapping mapping $f : X \to Y$ is an additive mapping, that is,

$$f(x+2y)+f(x-2y) = 2f(x).$$

One can easily show that an even mapping $f : X \to Y$ satisfies

$$\begin{aligned} f(x+2y)+f(x-2y) &= 2f(x+y)+2f(-x-y)+2f(x-y) \\ &+ 2f(y-x)-4f(-x)-2f(x)+f(2y) \\ &+ f(-2y)-4f(y)-4f(-y) \end{aligned}$$

if and only if the even mapping mapping $f : X \to Y$ is a quadratic-quartic mapping, that is,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y).$$

It was shown in [10] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that

$$f(x)=\frac{1}{12}g(x)-\frac{1}{12}h(x).$$

Remark 2.1. *In the rest of this section, let*

$$\begin{aligned}\chi_f(x,y) &= f(x+2y)+f(x-2y)-2f(x+y)\\ &- 2f(-x-y)-2f(x-y)-2f(y-x)+4f(-x)+2f(x)\\ &- f(2y)-f(-2y)+4f(y)+4f(-y)\end{aligned}$$

for all $x, y \in G$. Also, G is an additive semigroup and X is a complete non-Archimedean space.

Theorem 2.1. *Let $\zeta : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty}|3|^{-n}\zeta(3^n x, 3^n y)=0, \qquad x, y \in G. \tag{2.1}$$

Let for each $x \in G$ the limit

$$\Delta(x)=\lim_{n\to\infty} max\Big\{|3|^{-k}\zeta(3^k x, 3^k x) : 0 \le k < n\Big\}$$

exists for all $x \in G$. Suppose that $f : G \to X$ is an odd mapping satisfying the inequality

$$\Big\|\chi_f(x,y)\Big\| \le \zeta(x,y) \tag{2.2}$$

for all $x, y \in G$. Then the limit

$$\alpha(x) := \lim_{n\to\infty} 3^{-n} f(3^n x)$$

exist for all $x \in G$ and $\alpha(x) : G \to X$ is an additive mapping satisfying

$$||f(x)-\alpha(x)|| \le |3|^{-1}\Delta(x), \qquad x \in G. \tag{2.3}$$

Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} max\Big\{|3|^{-j}\zeta(3^j x, 3^j x) : k \le j < n+k\Big\}=0, \tag{2.4}$$

then $\alpha(x)$ is the unique mapping satisfying (2.3).

Proof. Putting $x = y$ in (2.2), we have

$$\Big\|3^{-1}f(3x)-f(x)\Big\| \le |3|^{-1}\zeta(x,x). \tag{2.5}$$

Replacing x by $3^n x$ in (2.5), we get

$$\Big\|3^{-(n+1)}f(3^{n+1}x)-3^{-n}f(3^n x)\Big\| \le |3|^{-(n+1)}\zeta(3^n x, 3^n x). \tag{2.6}$$

It follows from (2.1) and (2.6) that the sequence $\Big\{3^{-n}f(3^n x)\Big\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, $\Big\{3^{-n}f(3^n x)\Big\}_{n=1}^{\infty}$ is convergent. Set

$$\alpha(x) := \lim_{n\to\infty} 3^{-n} f(3^n x).$$

Using induction we see that

$$\Big\|3^{-n}f(3^nx)-f(x)\Big\| \quad \leq \quad |3|^{-1}max\Big\{|3|^{-k}\zeta(3^kx,3^kx):0\leq k<n\Big\}. \tag{2.7}$$

Indeed, (2.7) holds for $n=1$ by (2.5). Now, if (2.7) holds for n, then by (2.6), we obtain

$$\begin{aligned}
&\Big\|3^{-(n+1)}f(3^{n+1}x)-f(x)\Big\| \\
&\quad=\Big\|3^{-(n+1)}f(3^{n+1}x)\pm 3^{-n}f(3^nx)-f(x)\Big\| \\
&\quad\leq max\Big\{\Big\|3^{-(n+1)}f(3^{n+1}x)-3^{-n}f(3^nx)\Big\|\ ,\ \Big\|3^{-n}f(3^nx)-f(x)\Big\|\Big\} \\
&\quad\leq |3|^{-1}max\Big\{|3|^{-n}\zeta(3^nx,3^nx), max\Big\{|3|^{-k}\zeta(3^kx,3^kx):0\leq k<n\Big\}\Big\} \\
&\quad=|3|^{-1}max\Big\{|3|^{-k}\zeta(3^kx,3^kx):0\leq k<n+1\Big\}.
\end{aligned} \tag{2.8}$$

So for all $n\in\mathbb{N}$ and all $x\in G$, (2.7) holds. By taking n to approach infinity in (2.8), one obtains (2.3).

If $\beta(x)$ is another mapping satisfies (2.5), then for all $x\in G$, we get

$$\begin{aligned}
||\alpha(x)-\beta(x)|| &= \lim_{k\to\infty}|3|^{-k}||\alpha(3^kx)-\beta(3^kx)|| \\
&\leq \lim_{k\to\infty}|3|^{-k}max\Big\{||\alpha(3^kx)-f(3^kx)||\ ,\ ||f(3^kx)-\beta(3^kx)||\Big\} \\
&\leq |3|^{-1}\lim_{k\to\infty}\lim_{n\to\infty}max\Big\{|3|^{-j}\zeta(3^jx,3^jx):k\leq j<n+k\Big\}=0.
\end{aligned}$$

Therefore for all $x\in G$, we obtain $\alpha(x)=\beta(x)$. □

Corollary 2.1. *Let $\xi:[0,\infty)\to[0,\infty)$ be a function satisfying*

$$\xi(|3|t)\leq\xi(|3|)\xi(t)\ (t\geq 0),\qquad \xi(|3|)<|3|.$$

Let $\kappa>0$ and $f:G\to X$ an odd mapping satisfying the inequality

$$\Big\|\chi_f(x,y)\Big\|\leq\kappa\Big(\xi(|x|)+\xi(|y|)\Big)$$

for all $x,y\in G$. Then the limit

$$\alpha(x):=\lim_{n\to\infty}3^{-n}f(3^nx)$$

exist for all $x\in G$ and $\alpha(x):G\to X$ is a unique additive mapping satisfying

$$||f(x)-\alpha(x)||\leq\frac{2\kappa\xi(|x|)}{|3|},\qquad x\in G.$$

Proof. Using induction one can show that for all $n\in\mathbb{N}$,

$$\xi(|3|^nt)\leq(\xi(|3|))^n\xi(|t|))\leq|3|^n\xi(|t|).$$

Define $\zeta:G^2\to[0,\infty)$ by $\zeta(x,y):=\kappa\Big(\xi(|x|)+\xi(|y|)\Big)$. Since $|3|^{-1}\xi(|3|)<1$, we have

$$\lim_{n\to\infty}|3|^{-n}\zeta(3^nx,3^ny)\leq\lim_{n\to\infty}\Big(|3|^{-1}\xi(|3|)\Big)^n\zeta(x,y)=0,\qquad x,y\in G.$$

Also for all $x\in G$

$$\Delta(x)=\lim_{n\to\infty}max\Big\{|3|^{-k}\zeta(3^kx,3^kx):0\leq k<n\Big\}=\zeta(x,x)=2\kappa\xi(|x|)$$

exist and for all $x \in G$ we have

$$\lim_{k\to\infty}\lim_{n\to\infty} max\Big\{|3|^{-j}\zeta(3^j x, 3^j x) : k \le j < n+k\Big\} = 0.$$

By Theorem 2.1, we complete the proof. □

Corollary 2.2. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|3|t) \le \xi(|3|)\xi(t)\ (t \ge 0), \qquad \xi(|3|) < |3|.$$

Let $\kappa > 0$ and $f : G \to X$ an odd mapping satisfying the inequality

$$\Big\|\chi_f(x,y)\Big\| \le \kappa\Big(\xi(|x|)\cdot\xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\alpha(x) := \lim_{n\to\infty} 3^{-n} f(3^n x)$$

exist for all $x \in G$ and $\alpha(x) : G \to X$ is a unique additive mapping satisfying

$$||f(x) - \alpha(x)|| \le \frac{\kappa\xi^2(|x|)}{|3|}, \qquad x \in G.$$

Theorem 2.2. *Let $\zeta : G^2 \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty} |3|^n \zeta\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) = 0 \tag{2.9}$$

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\Delta(x) = \lim_{n\to\infty} max\Big\{|3|^k \zeta\Big(\frac{x}{3^k}, \frac{x}{3^k}\Big) : 0 \le k < n\Big\}$$

exists. Suppose that $f : G \to X$ is an odd mapping satisfying the inequality

$$\Big\|\chi_f(x,y)\Big\| \le \zeta(x,y) \tag{2.10}$$

for all $x, y \in G$. Then the limit

$$\alpha(x) := \lim_{n\to\infty} 3^n f\Big(\frac{x}{3^n}\Big)$$

exist for all $x \in G$ and $\alpha : G \to X$ is an additive mapping satisfying

$$||f(x) - \alpha(x)|| \le |3|^{-1}\Delta(x), \qquad x \in G. \tag{2.11}$$

Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} max\Big\{|3|^j \zeta\Big(\frac{x}{3^j}, \frac{x}{3^j}\Big) : k \le j < n+k\Big\} = 0,$$

then $\alpha(x)$ is the unique mapping satisfying (2.11).

Proof. Letting $y = x$ in (2.10), we obtain

$$\Big\|3^{-1}f(3x) - f(x)\Big\| \le |3|^{-1}\zeta(x,x). \tag{2.12}$$

Replacing x by $\frac{x}{3^n}$ in (2.12), we get

$$\Big\|3^{n-1}f\Big(\frac{x}{3^{n-1}}\Big) - 3^n f\Big(\frac{x}{3^n}\Big)\Big\| \le |3|^{n-1}\zeta\Big(\frac{x}{3^n}, \frac{x}{3^n}\Big). \tag{2.13}$$

It follows from (2.9) and (2.13) that the sequence $\left\{3^n f\left(\frac{x}{3^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, $\left\{3^n f\left(\frac{x}{3^n}\right)\right\}_{n=1}^{\infty}$ is convergent. It follows from (2.13) that

$$\begin{aligned}\left\|3^n f\left(\frac{x}{3^n}\right)-3^p f\left(\frac{x}{3^p}\right)\right\| &= \left\|\sum_{k=p+1}^{n} 3^{k-1} f\left(\frac{x}{3^{k-1}}\right)-3^k f\left(\frac{x}{3^k}\right)\right\| \\ &\leq max\left\{\left\|3^{k-1} f\left(\frac{x}{3^{k-1}}\right)-3^k f\left(\frac{x}{3^k}\right)\right\| :\ p+1\leq k<n\right\} \\ &\leq |3|^{-1} max\left\{|3|^k \zeta\left(\frac{x}{3^k},\frac{x}{3^k}\right) :\ p+1\leq k<n\right\}\end{aligned}$$

for all $x\in G$ and all non-negative integers n,p with $n-1>p\geq 0$. Letting $p=0$ and passing the limit $n\to\infty$ in the last inequality, we obtain (2.11).

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.3. *Let $\xi:[0,\infty)\to[0,\infty)$ be a function satisfying*

$$\xi(|3|^{-1}t)\leq \xi(|3|^{-1})\xi(t)\ (t\geq 0),\qquad \xi(|3|^{-1})<|3|^{-1}.$$

Let $\kappa>0$ and $f:G\to X$ an odd mapping satisfying the inequality

$$\left\|\chi_f(x,y)\right\|\leq \kappa\Big(\xi(|x|)+\xi(|y|)\Big)$$

for all $x,y\in G$. Then the limit

$$\alpha(x):=\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$$

exist for all $x\in G$ and $\alpha:G\to X$ is a unique additive mapping satisfying

$$||f(x)-\alpha(x)||\leq \frac{2\kappa\xi(|x|)}{|3|},\qquad x\in G.$$

Corollary 2.4. *Let $\xi:[0,\infty)\to[0,\infty)$ be a function satisfying*

$$\xi(|3|^{-1}t)\leq \xi(|3|^{-1})\xi(t)\ (t\geq 0),\qquad \xi(|3|^{-1})<|3|^{-1}.$$

Let $\kappa>0$ and $f:G\to X$ an odd mapping satisfying the inequality

$$\left\|\chi_f(x,y)\right\|\leq \kappa\Big(\xi(|x|)\cdot\xi(|y|)\Big)$$

for all $x,y\in G$. Then the limit

$$\alpha(x):=\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$$

exist for all $x\in G$ and $\alpha:G\to X$ is a unique additive mapping satisfying

$$||f(x)-\alpha(x)||\leq \frac{\kappa\xi^2(|x|)}{|3|},\qquad x\in G.$$

However, the following example shows that the same result of Theorem 1.1 is not true in non-Archimedean normed spaces.

Example 2.1. *Let $p>2$ and let $f:\mathbb{Q}_p\to\mathbb{Q}_p$ be defined by $f(x)=2$. Then for $\epsilon=1$,*

$$|f(x+y)-f(x)-f(y)|=1\leq\epsilon$$

for all $x, y \in \mathbb{Q}_p$. However, the sequences $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ and $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ are not Cauchy. In fact, by using the fact that $|2| = 1$, we have

$$\left|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\right| = |2^{-n} \cdot 2 - 2^{-(n+1)} \cdot 2| = |2^{-n}| = 1$$

and

$$\left|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right| = |2^n.2 - 2^{(n+1)}.2| = |2^{n+1}| = 1$$

for all $x, y \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence these sequences are not convergent in $\mathbb{Q}_p$.

3. Non-Archimedean stability of the functional equation (1.2): even case

In this section, we prove the Hyers-Ulam stability of the functional equation $\chi(x, y) = 0$ in non-Archimedean spaces: an even case.

Theorem 3.1. *Let $\zeta : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |16|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0 \tag{3.1}$$

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\Delta^1(x) = \lim_{n\to\infty} max\Big\{|16|^k \zeta\Big(\frac{x}{2^{k-1}}, \frac{x}{2^k}\Big), |4| \cdot |16|^k \zeta\Big(\frac{x}{2^k}, \frac{x}{2^k}\Big) : 0 \le k < n\Big\}$$

exists. Suppose that $f : G \to X$ is an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \le \zeta(x, y) \tag{3.2}$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 16^n \Big(f\Big(\frac{x}{2^{n-1}}\Big) - 4f\Big(\frac{x}{2^n}\Big)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a quartic mapping satisfying

$$\Big\|f(2x) - 4f(x) - \beta(x)\Big\| \le |16|^{-1} \Delta^1(x) \tag{3.3}$$

for all $x \in G$. Moreover, if

$$\lim_{k\to\infty} \lim_{n\to\infty} max\Big\{|16|^j \zeta\Big(\frac{x}{2^{j-1}}, \frac{x}{2^j}\Big), |4| \cdot |16|^j \zeta\Big(\frac{x}{2^j}, \frac{x}{2^j}\Big) : k \le j < n + k\Big\} = 0,$$

then $\beta(x)$ is the unique mapping satisfying (3.3).

Proof. Letting $x = y$ in (3.2), we get

$$\Big\|f(3y) - 6f(2y) + 15f(y)\Big\| \le \zeta(y, y) \tag{3.4}$$

for all $y \in G$. Replacing x by $2y$ in (3.2), we obtain

$$\Big\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\Big\| \le \zeta(2y, y) \tag{3.5}$$

for all $y \in G$. By (3.4) and (3.5), we have

$$\begin{aligned}\Big\|f(4x)-20f(2x)+64f(x)\Big\| &= \Big\|f(4x)-4f(3x)+4f(2x)+4f(x) \\ &+ 4(f(3x)-6f(2x)+15f(x))\Big\| \\ &\leq max\Big\{\|f(4x)-4f(3x)+4f(2x)+4f(x)\|, \\ &\quad \|4(f(3x)-6f(2x)+15f(x))\|\Big\} \\ &\leq max\Big\{\zeta(2x,x), |4|\zeta(x,x)\Big\}\end{aligned} \tag{3.6}$$

for all $x \in G$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\Big\|g(2x)-16g(x)\Big\| \leq max\Big\{\zeta(2x,x), |4|\zeta(x,x)\Big\}. \tag{3.7}$$

Replacing x by $\frac{x}{2}$ in (3.7), we obtain

$$\Big\|g(x)-16g\Big(\frac{x}{2}\Big)\Big\| \leq max\Big\{\zeta\Big(x,\frac{x}{2}\Big), |4|\zeta\Big(\frac{x}{2},\frac{x}{2}\Big)\Big\}. \tag{3.8}$$

Replacing x by $\frac{x}{2^{n-1}}$ in (3.8), we get

$$\begin{aligned}&\Big\|16^{n-1}g\Big(\frac{x}{2^{n-1}}\Big)-16^n g\Big(\frac{x}{2^n}\Big)\Big\| \\ &\quad \leq |16|^{-1}max\Big\{|16|^n\zeta\Big(\frac{x}{2^{n-1}},\frac{x}{2^n}\Big), |4|\cdot|16|^n\zeta\Big(\frac{x}{2^n},\frac{x}{2^n}\Big)\Big\}.\end{aligned} \tag{3.9}$$

It follows from (3.1) and (3.9) that the sequence $\Big\{16^n g\Big(\frac{x}{2^n}\Big)\Big\}_{n=1}^{\infty}$ is convergent. Proceeding as in the proof of Theorem 2.1, we obtain that there exists a mapping $\beta(x)$ such that

$$\beta(x) = \lim_{n\to\infty} 16^n g\Big(\frac{x}{2^n}\Big) = \lim_{n\to\infty} 16^n\Big(f\Big(\frac{x}{2^{n-1}}\Big) - 4f\Big(\frac{x}{2^n}\Big)\Big).$$

On the other hand,

$$\begin{aligned}\beta(2x)-16\beta(x) &= \lim_{n\to\infty}\Big[16^n g\Big(\frac{x}{2^{n-1}}\Big) - 16^{n+1}g\Big(\frac{x}{2^n}\Big)\Big] \\ &= \lim_{n\to\infty} 16\Big[16^{n-1}g\Big(\frac{x}{2^{n-1}}\Big) - 16^n g\Big(\frac{x}{2^n}\Big)\Big] \\ &= 0.\end{aligned}$$

So the mapping $\beta(x) : G \to X$ is quartic.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.1. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|2|^{-1}t) \leq \xi(|2|^{-1})\xi(t)\ (t \geq 0), \qquad \xi(|2|^{-1}) < |2|^{-4}.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x,y)\Big\| \leq \kappa\Big(\xi(|x|) + \xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 16^n\Big(f\Big(\frac{x}{2^{n-1}}\Big) - 4f\Big(\frac{x}{2^n}\Big)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a unique quartic mapping satisfying

$$\Big\|f(2x) - 4f(x) - \beta(x)\Big\| \leq \frac{2\kappa\ \xi(|x|)}{|16|}, \qquad x \in G.$$

Proof. Defining $\zeta : G^2 \to [0, \infty)$ by $\zeta(x, y) := \kappa(\xi(|\frac{x}{2}|) + \xi(|y|))$, we obtain the result. □

Corollary 3.2. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|^{-1}t) \leq \xi(|2|^{-1})\xi(t)\ (t \geq 0), \qquad \xi(|2|^{-1}) < |2|^{-4}.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \kappa\Big(\xi(|x|) \cdot \xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 16^n \bigg(f\Big(\frac{x}{2^{n-1}}\Big) - 4f\Big(\frac{x}{2^n}\Big)\bigg)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a unique quartic mapping satisfying

$$\Big\|f(2x) - 4f(x) - \beta(x)\Big\| \leq \frac{\kappa\xi^2(|x|)}{|16|}, \qquad x \in G.$$

Theorem 3.2. *Let $\zeta : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |16|^{-n}\zeta(2^n x, 2^n y) = 0$$

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\Delta^2(x) = \lim_{n\to\infty} max\Big\{|16|^{-k}\zeta(2^{k+1}x, 2^k x), |4| \cdot |16|^{-k}\zeta(2^k x, 2^k x) : 0 \leq k < n\Big\}$$

exists. Suppose that $f : G \to X$ is an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \zeta(x, y)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 16^{-n}\Big(f(2^{n+1}x) - 4f(2^n x)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a quartic mapping satisfying

$$\Big\|f(2x) - 4f(x) - \beta(x)\Big\| \leq |16|^{-1}\Delta^2(x), \qquad x \in G. \tag{3.10}$$

Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} max\Big\{|16|^{-j}\zeta(2^{j+1}x, 2^j x), |4| \cdot |16|^{-j}\zeta(2^j x, 2^j x) : k \leq j < n + k\Big\} = 0,$$

then $\beta(x)$ is the unique mapping satisfying (3.10).

Corollary 3.3. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t)\ (t \geq 0), \qquad \xi(|2|) < |2|^4.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \kappa\Big(\xi(|x|) + \xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 16^{-n}\Big(f(2^{n+1}x) - 4f(2^n x)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a unique quartic mapping satisfying

$$\Big\|f(2x) - 4f(x) - \beta(x)\Big\| \leq \frac{2\kappa\xi(|x|)}{|16|}, \qquad x \in G.$$

Corollary 3.4. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \ (t \geq 0), \qquad \xi(|2|) < |2|^4.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x,y)\Big\| \leq \kappa\Big(\xi(|x|) \cdot \xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 16^{-n}\Big(f(2^{n+1}x) - 4f(2^n x)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a unique quartic mapping satisfying

$$\Big\|f(2x) - 4f(x) - \beta(x)\Big\| \leq \frac{\kappa\xi^2(|x|)}{|16|}, \qquad x \in G.$$

Theorem 3.3. *Let $\zeta : G^2 \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty} |4|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0, \qquad x, y \in G.$$

Let for each $x \in G$ the limit

$$\Delta^3(x) = \lim_{n\to\infty} max\Big\{|4|^k\zeta\Big(\frac{x}{2^{k-1}}, \frac{x}{2^k}\Big), |4|^{k+1}\zeta\Big(\frac{x}{2^k}, \frac{x}{2^k}\Big) : 0 \leq k < n\Big\}$$

exists. Suppose that $f : G \to X$ is an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x,y)\Big\| \leq \zeta(x,y)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 4^n\left(f\Big(\frac{x}{2^{n-1}}\Big) - 16f\Big(\frac{x}{2^n}\Big)\right)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a quadratic mapping satisfying

$$\Big\|f(2x) - 16f(x) - \beta(x)\Big\| \leq |4|^{-1}\Delta^3(x), \qquad x \in G. \tag{3.11}$$

Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} max\Big\{|4|^j\zeta\Big(\frac{x}{2^{j-1}}, \frac{x}{2^j}\Big), |4|^{j+1}\zeta\Big(\frac{x}{2^j}, \frac{x}{2^j}\Big) : k \leq j < n+k\Big\} = 0,$$

then $\beta(x)$ is the unique mapping satisfying (3.11).

Proof. Letting $g(x) := f(2x) - 16f(x)$ for all $x \in X$ in (3.6), we get

$$\Big\|g(x) - 4g\Big(\frac{x}{2}\Big)\Big\| \leq max\Big\{\zeta\Big(x, \frac{x}{2}\Big), |4|\zeta\Big(\frac{x}{2}, \frac{x}{2}\Big)\Big\}.$$

Proceeding as in the proof of Theorems 2.1 and 3.1, we obtain that the sequence $\left\{4^n g\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is convergent to a mapping $\beta(x)$, i.e.,

$$\beta(x) = \lim_{n\to\infty} 4^n g\Big(\frac{x}{2^n}\Big) = \lim_{n\to\infty} 4^n \bigg(f\Big(\frac{x}{2^{n-1}}\Big) - 16 f\Big(\frac{x}{2^n}\Big)\bigg).$$

On the other hand,

$$\begin{aligned}\beta(2x) - 4\beta(x) &= \lim_{n\to\infty}\Big[4^n g\Big(\frac{x}{2^{n-1}}\Big) - 4^{n+1} g\Big(\frac{x}{2^n}\Big)\Big] \\ &= \lim_{n\to\infty} 4\Big[4^{n-1} g\Big(\frac{x}{2^{n-1}}\Big) - 4^n g\Big(\frac{x}{2^n}\Big)\Big] \\ &= 0.\end{aligned}$$

So the mapping $\beta(x) : G \to X$ is quadratic.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.5. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|2|^{-1}t) \le \xi(|2|^{-1})\xi(t)\ (t \ge 0), \qquad \xi(|2|^{-1}) < |2|^{-2}.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x,y)\Big\| \le \kappa\Big(\xi(|x|) + \xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 4^n \bigg(f\Big(\frac{x}{2^{n-1}}\Big) - 16 f\Big(\frac{x}{2^n}\Big)\bigg)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a unique quadratic mapping satisfying

$$\Big\|f(2x) - 16f(x) - \beta(x)\Big\| \le \frac{2\kappa\ \xi(|x|)}{|4|}, \qquad x \in G.$$

Corollary 3.6. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|2|^{-1}t) \le \xi(|2|^{-1})\xi(t)\ (t \ge 0), \qquad \xi(|2|^{-1}) < |2|^{-2}.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x,y)\Big\| \le \kappa\Big(\xi(|x|) \cdot \xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 4^n \bigg(f\Big(\frac{x}{2^{n-1}}\Big) - 16 f\Big(\frac{x}{2^n}\Big)\bigg)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a quadratic mapping satisfying

$$\Big\|f(2x) - 16f(x) - \beta(x)\Big\| \le \frac{\kappa\xi^2(|x|)}{|4|}, \qquad x \in G.$$

Theorem 3.4. *Let $\zeta : G^2 \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty} |4|^{-n}\zeta(2^n x, 2^n y) = 0$$

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\Delta^4(x) = \lim_{n\to\infty} max\Big\{|4|^{-k}\zeta(2^{k+1}x, 2^k x), |4|^{-k+1}\zeta(2^k x, 2^k x) : 0 \le k < n\Big\}$$

exists. Suppose that $f : G \to X$ is an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x,y)\Big\| \leq \zeta(x,y),$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 4^{-n}\Big(f(2^{n+1}x) - 16f(2^n x)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a quadratic mapping satisfying

$$\Big\|f(2x) - 16f(x) - \beta(x)\Big\| \leq |4|^{-1}\Delta^4(x), \qquad x \in G. \tag{3.12}$$

Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} max\Big\{|4|^{-j}\zeta(2^{j+1}x, 2^j x), |4|^{-j+1}\zeta(2^j x, 2^j x) : k \leq j < n+k\Big\} = 0,$$

then $\beta(x)$ is the unique mapping satisfying (3.12).

Corollary 3.7. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \ (t \geq 0), \qquad \xi(|2|) < |2|^2.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\eta_f(x,y)\Big\| \leq \kappa\Big(\xi(|x|) + \xi(|y|)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 4^{-n}\Big(f(2^{n+1}x) - 16f(2^n x)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a unique quadratic mapping satisfying

$$\Big\|f(2x) - 16f(x) - \beta(x)\Big\| \leq \frac{2\kappa\xi(|x|)}{|4|}, \qquad x \in G.$$

Corollary 3.8. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \ (t \geq 0), \qquad \xi(|2|) < |2|^2.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x,y)\Big\| \leq \kappa\Big(\xi(||x||) \cdot \xi(||y||)\Big)$$

for all $x, y \in G$. Then the limit

$$\beta(x) := \lim_{n\to\infty} 4^{-n}\Big(f(2^{n+1}x) - 16f(2^n x)\Big)$$

exist for all $x \in G$ and $\beta(x) : G \to X$ is a unique quadratic mapping satisfying

$$\Big\|f(2x) - 16f(x) - \beta(x)\Big\| \leq \frac{\kappa\xi^2(|x|)}{|4|}, \qquad x \in G.$$

4. Non-Archimedean stability of the functional equation (1.2)

Theorem 4.1. *Let $\zeta : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |4|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0$$

for all $x, y \in G$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \zeta(x, y)$$

for all $x, y \in G$. Then there exist a quartic mapping $\alpha : G \to X$ and a quadratic mapping $\beta : G \to X$ such that

$$\Big\|f(x) - \frac{1}{12}\alpha(x) - \frac{1}{12}\beta(x)\Big\| \leq max\Big\{\frac{1}{|192|}\Delta^1(x), \frac{1}{|48|}\Delta^3(x)\Big\}$$

for all $x \in G$.

Proof. Since $\lim_{n\to\infty} |4|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0$, $\lim_{n\to\infty} |16|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0$. The result follows from Theorems 3.1 and 3.3. □

Corollary 4.1. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|^{-1}t) \leq \xi(|2|^{-1})\xi(t) \ (t \geq 0), \qquad |2|^2\xi(|2|^{-1}) < 1.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \kappa\Big(\xi(|x|) + \xi(|y|)\Big)$$

for all $x, y \in G$. Then there exist a quartic mapping $\alpha : G \to X$ and a quadratic mapping $\beta : G \to X$ such that

$$\Big\|f(x) - \frac{1}{12}\alpha(x) - \frac{1}{12}\beta(x)\Big\| \leq \frac{2\kappa\ \xi(|x|)}{|192|}$$

for all $x \in G$.

Corollary 4.2. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|^{-1}t) \leq \xi(|2|^{-1})\xi(t) \ (t \geq 0), \qquad |2|^2\xi(|2|^{-1}) < 1.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \kappa\Big(\xi(|x|) \cdot \xi(|y|)\Big)$$

for all $x, y \in G$. Then there exist a quartic mapping $\alpha : G \to X$ and a quadratic mapping $\beta : G \to X$ such that

$$\Big\|f(x) - \frac{1}{12}\alpha(x) - \frac{1}{12}\beta(x)\Big\| \leq \frac{\kappa\ \xi^2(|x|)}{|192|}$$

for all $x \in G$.

Theorem 4.2. *Let $\zeta : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |16|^{-n}\zeta(2^n x, 2^n y) = 0$$

for all $x, y \in G$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \zeta(x, y)$$

for all $x, y \in G$. Then there exist a quartic mapping $\alpha : G \to X$ and a quadratic mapping $\beta : G \to X$ such that

$$\Big\|f(x) - \frac{1}{12}\alpha(x) - \frac{1}{12}\beta(x)\Big\| \leq max\Big\{\frac{1}{|192|}\Delta^2(x), \frac{1}{|48|}\Delta^4(x)\Big\}$$

for all $x \in G$.

Proof. Since $\lim_{n\to\infty} |16|^{-n}\zeta(2^n x, 2^n y) = 0$, $\lim_{n\to\infty} |4|^{-n}\zeta(2^n x, 2^n y) = 0$. The result follows from Theorems 3.2 and 3.4. □

Corollary 4.3. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \ (t \geq 0), \qquad \xi(|2|) < |2|^4.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \kappa\Big(\xi(|x|) + \xi(|y|)\Big)$$

for all $x, y \in G$. Then there exist a quartic mapping $\alpha : G \to X$ and a quadratic mapping $\beta : G \to X$ such that

$$\Big\|f(x) - \frac{1}{12}\alpha(x) - \frac{1}{12}\beta(x)\Big\| \leq \frac{2\kappa\xi(|x|)}{|192|}$$

for all $x \in G$.

Corollary 4.4. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \ (t \geq 0), \qquad \xi(|2|) < |2|^4.$$

Let $\kappa > 0$ and $f : G \to X$ an even mapping satisfying $f(0) = 0$ and the inequality

$$\Big\|\chi_f(x, y)\Big\| \leq \kappa\Big(\xi(|x|) \cdot \xi(|y|)\Big)$$

for all $x, y \in G$. Then there exist a quartic mapping $\alpha : G \to X$ and a quadratic mapping $\beta : G \to X$ such that

$$\Big\|f(x) - \frac{1}{12}\alpha(x) - \frac{1}{12}\beta(x)\Big\| \leq \frac{\kappa\xi^2(|x|)}{|192|}$$

for all $x \in G$.

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Generalized fixed point theorem for three mappings on Three quasi-metric spaces

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Abstract. A new fixed point theorem in three quasi-metric spaces is proved. This result generalizes and extends several of well-known fixed point theorems for metric spaces to quasi-metric spaces. Several corollaries determined by the form of implicit relations are obtained.

Keywords: Cauchy sequence, complete quasi-metric space, fixed point, implicit relation.

Mathematics Subject Classification: 47H10, 54H25

1. Introduction and Preliminaries

Fisher [3], Popa [11] and Nešić′ [7] proved some fixed point theorems on two metric spaces. Nung [8], Jain et al [5] and Kikina [6] proved similar results for three metric spaces. Using the implicit relation, other authors unified and generalized some of the well-known theorems. So Telci [12] and later Aliouche and Fisher [1] realized the generalization for two mappings on two metric spaces. Our aim is to unify, generalize and extend all the above theorems in quasi-metric spaces that are more general than the metric ones (see below the definition of a quasi-metric space).

In this paper, using a new class of implicit relations, we prove a theorem as a corollary of which are taken propositions that extend and generalize theorems: Fisher [3], Nung [8], Jain et al [5], Popa [11], Nešić′ [7], Kikina [6], Telci [12], Aliouche and Fisher [1] etc, from metric spaces to quasi-metric spaces.

Now we give some standard definitions and notation.

Quasi-metric spaces concept is treated differently by many authors. In this paper our concept is in line with this treated on [9], [10], [13], [14], [15], [2], [4] etc:

Definition1.1 Let X be a set. A nonnegative symmetric function d defined on $X \times X$ is called a quasi-distance on X if and only if there exists a constant $k \geq 1$ such that for all x, y and $z \in X$ the following conditions hold:

(1) $d(x,y) = 0 \Leftrightarrow x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq k[d(x,z) + d(z,y)]$.

Inequality (3) is often called quasi-triangular *inequality* and k is often called the

quasi-triangular coefficient of d .

It is well known that, exactly as for a distance, the balls $B(a,r)=\{x\in X : d(x,a)<r\}$ form a base for a complete system of neighborhoods of $a\in X$ and $\tau=\{Q\subset X : \forall a\in Q, \exists r>0, B(a,r)\subset Q\}$ is a topology on *X*.

The space *X* endowed with the topology τ , is called a quasi-metric space.

Metric spaces are a special case of quasi-metric space (for k=1), but contrary to a distance, the "open" balls $B(a,r)$ are not always open sets in this topology and consequently a quasi-distance d is not always a continuous function of its variables with respect to the topology. For this reason, in some articles on fixed point theory, authors require the continuity of the used quasi-distance, as complementary conditions (see among others [2], [4]). The quasi-metric space constitutes a more general setting than the metric ones in many applications (see [9]).

The following example illustrates the existence of the quasi-distance for an arbitrary coefficient $k\geq 1$.

Example 1.2 *Let* $X=R\times R$ *and* $x=(x_1,x_2)\in X, y=(y_1,y_2)\in X$. *The function* $d: X\times X\to R^+$ *such that*

$$d(x,y)=\begin{cases} k|x_1-y_1|+|x_2-y_2|, \text{ for } |x_1-y_1|\leq|x_2-y_2| \\ |x_1-y_1|+k|x_2-y_2|, \text{ for } |x_1-y_1|>|x_2-y_2| \end{cases}$$

is a quasi-distance.

Let we verify the satisfying of the three conditions of the definition 1.1.

(1) $d(x,y)=0\Leftrightarrow\begin{cases} x_1-y_1=0 \\ x_2-y_2=0 \end{cases}\Leftrightarrow\begin{cases} x_1=y_1 \\ x_2=y_2 \end{cases}\Leftrightarrow x=(x_1,x_2)=y=(y_1,y_2)$

(2) $d(x,y)=d(y,x), \forall x,y\in X$, since $|x_1-y_1|=|y_1-x_1|$ and $|x_2-y_2|=|y_2-x_2|$.

(3) $d(x,y)\leq k\left[|x_1-y_1|+|x_2-y_2|\right]=k\left[|x_1-z_1+z_1-y_1|+|x_2-z_2+z_2-y_2|\right]\leq$

$\leq k\left[|x_1-z_1|+|z_1-y_1|+|x_2-z_2|+|z_2-y_2|\right]=$

$=k\left[|x_1-z_1|+|x_2-z_2|\right]+k\left[|z_1-y_1|+|z_2-y_2|\right]\leq$

$\leq k\left[d(x,z)+d(z,y)\right], \forall x,y,z\in X.$

We emphases the fact that this quasi-distance d is not distance. For example: if $k=2$ $x=(\alpha,0), y=(0,\alpha)$, $z=(0,0)$, where $\alpha>0$, we have:

$$d(x,y)=3\alpha>d(x,z)+d(z,y)=2\alpha$$

Definition 1.3 A sequence $\{x_n\}$ in a quasi-metric space (X,d) is called Cauchy sequence if $\lim_{n\to\infty} d(x_n,x_{n+p})=0$

Definition 1.4 Let (X,d) be a quasi-metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x\in X$ (denoted by $\lim_{n\to\infty} x_n=x$) if $\lim_{n\to\infty} d(x_n,x)=0$.

Definition 1.5 A quasi-metric space (X,d) is called complete, if every Cauchy sequence is convergent.

2. Implicit relation

Before stating the main theorem we define a new class of implicit functions, whose role will be crucial in the main result of this paper.

Definition 2.1 Let $R_+ = [0, +\infty)$. The set of all functions with 5 variables $\varphi : R_+^5 \to R$ satisfying the properties:

(a). φ is upper semi-continuous in each coordinate variable

(b). If $\varphi(u,0,u,v,w) \le 0$ or $\varphi(u,u,0,v,w) \le 0$ for all $u,v,w \ge 0$, then there exists a real constant $0 \le c < 1$ such that $u \le c \max\{v,w\}$

will be noted by Φ_5 and every such function will be called a Φ_5-function with constant c.

Some examples of Φ_k-function are as follows:

Example 2.2 Let $F : R_+ \to R_+$ a continuous function with $F(0) = 0$.

The function $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1^p - c\max\{t_2^p, t_3^p, t_4^p, t_5^p\} - F(\min\{t_2^p, t_3^p, t_4^p, t_5^p\})$, where $0 \le c < 1$ and $p \ge 0$, is Φ_k-function with constant $c_1 = \sqrt[p]{c} < 1$

Proof: (a) is clear since φ is continuous.

Suppose that $u,v,w \ge 0$ and then

$\varphi(u,0,u,v,w) = u^p - c\max\{0, u^p, v^p, w^p\} - F(\min\{0, u^p, v^p, w^p\})$

$= u^p - c\max\{u^p, v^p, w^p\} - F(0) = u^p - c\max\{u^p, v^p, w^p\} \le 0$

We have $u \le \max\{v,w\}$ since in contrary, (if $u > \max\{v,w\}$),

then $u^p \le c\max\{u^p, v^p, w^p\} = cu^p < u^p$, a contradiction.

Therefore, $u^p \le c\max\{v^p, w^p\}$ *and so* $u \le c_1 \max\{v,w\}$, where $c_1 = \sqrt[p]{c} < 1$. Similarly, if $\varphi(u,u,0,v,w) \le 0$, then $u \le c_1 \max\{v,w\}$. The proof of (b) is completed.

Example 2.3 The function $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1 - (a_2 t_2^p + a_3 t_3^p + a_4 t_4^p + a_5 t_5^p)^{1/p}$, where $p > 0$ and $0 \le a_i, \sum_{i=2}^{5} a_i < 1, \quad i = 2,3,4,5$, is Φ_5-function with constant $c = (a_2 + a_3 + a_4 + a_5)^{1/p}$

Proof: (a) is clear since φ is continuous.

Suppose that $u,v,w \ge 0$ and then

$\varphi(u,0,u,v,w) = u - (a_2 0 + a_3 u^p + a_4 v^p + a_5 w^p)^{1/p} \le 0$

If $u \ge \max\{v,w\}$, then

$u \le (a_2 0 + a_3 u^p + a_4 v^p + a_5 w^p)^{1/p} \le (a_2 u^p + a_3 u^p + a_4 u^p + a_5 u^p)^{1/p} =$

$= [(a_2 + a_3 + a_4 + a_5) u^p]^{1/p} = (a_2 + a_3 + a_4 + a_5)^{1/p} u \le cu < u$,

a contradiction, where $c = (a_2 + a_3 + a_4 + a_5)^{1/p} < 1$. Therefore,

$u \le (a_2 0 + a_3 u^p + a_4 v^p + a_5 w^p)^{1/p} \le [(a_2 + a_3 + a_4 + a_5)\max\{v^p, w^p\}]^{1/p}$

$= c\max\{v,w\}$. Similarly, if $\varphi(u,u,0,v,w) \le 0$, then $u \le c\max\{v,w\}$.

The proof of (b) is completed.

Definition 2.4 The set of all continuous functions with 4 variables $f : R_+^4 \to R$ satisfying the properties:

(a'). f is non decreasing in respect with each variable.

(b'). $f(t,t,t,t) \le t,\ t \in R_+$

will be noted $\mathbb{F}_4$ and every such function will be called a $\mathbb{F}_4$-function

Denote $I_4 = \{1,2,3,4\}$. Some examples of $\mathbb{F}_4$-function are as follows:

1. $f(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$

2. $f(t_1,t_2,t_3,t_4) = [\max\{t_i t_j : i, j \in I_4\}]^{1/2}$

3. $f(t_1,t_2,t_3,t_4) = [\max\{t_1t_2, t_2t_3, t_3t_4\}]^{1/2}$

4. $f(t_1,t_2,t_3,t_4) = [\max\{t_1^p, t_2^p, t_3^p, t_4^p\}]^{1/p},\ p > 0$

5. $f(t_1,t_2,t_3,t_4) = (a_1t_1^p + a_2t_2^p + a_3t_3^p + a_4t_4^p)^{1/p}$, where $p > 0$ and $0 \le a_i,\ a_1 + a_2 + a_3 + a_4 \le 1$

The proof is done for the example 5:

(a') It is obvious that the function f is non decreasing in respect with each variable

(b') We have: $f(t,t,t,t) = (a_1t^p + a_2t^p + a_3t + a_4t^p)^{1/p} = [(a_1 + a_2 + a_3 + a_4)t^p]^{1/p} =$

$= (a_1 + a_2 + a_3 + a_4)^{1/p} t \le t$. The proof of (b') is completed.

Definition 2.5 The set of all continuous functions with 4 variables $g : R_+^4 \to R$ satisfying the property:

$$t_1t_2t_3t_4 = 0 \Rightarrow g(t_1,t_2,t_3,t_4) = 0,$$

(The function g takes the value zero at the points for which at least one of the coordinates is zero), will be noted G_4 and every such function will be called a G_4-function.

Some examples of G_4-function are as follows:

6. $g(t_1,t_2,t_3,t_4) = \min\{t_1,t_2,t_3,t_4\}$

7. $g(t_1,t_2,t_3,t_4) = \min\{t_1^p, t_2^p, t_3^p, t_4^p\}$

Let $\mathcal{F}$ be the set of continuous mappings $F : R_+ \to R_+$ such that $F(0) = 0$ (for example $F(t) = t^\lambda, \lambda > 0$).

The following relationship between $\mathbb{F}_4$-functions and Φ_5-functions holds:

Lemma 2.6 If $f \in \mathbb{F}_4$, $g \in G_4$, $F \in \mathcal{F}$ and $0 \le c < 1$, then the function $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1 - cf(t_2,t_3,t_4,t_5) - F(g(t_2,t_3,t_4,t_5))$ is Φ_5-function with constant c.

Proof. **(a')** is clear since φ is continuous

Suppose that $u,v,w \ge 0$ and then

$\varphi(u,0,u,v,w) = u - cf(0,u,v,w) - F(g(0,u,v,w)) =$

$= u - cf(0,u,v,w) - F(0) = u - cf(0,u,v,w) \le 0$ (*)

We have $u \le \max\{v,w\}$ since in contrary, (if $u > \max\{v,w\}$), by using the properties of f we get: $f(0,u,v,w) \le f(u,u,u,u) \le u$ and by (*) it follows $u \le cu < u$, a contradiction.

Therefore, after replacing the coordinates of the point $(0,u,v,w)$ by $\max\{v,w\}$ and using the properties of f we get $u \le c\max\{v,w\}$. Similarly, if $\varphi(u,u,0,v,w) \le 0$, then $u \le c\max\{v,w\}$. The proof of **(b)** is completed.

The above lemma gives us the possibility to construct other functions of type Φ_5-function:

Example 2.7 $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1 - c[\max\{t_2t_3, t_3t_4, t_4t_5\}]^{1/2} - F(\min\{t_2,t_3,t_4,t_5\})$, where $0 \le c < 1$.

Example 2.8 $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1 - (a_2t_2^p + a_3t_3^p + a_4t_4^p + a_5t_5^p)^{1/p} - \min\{t_2^p, t_3^p, t_4^p, t_5^p\}$, where $p > 0$ and $0 \le a_i, \sum_{i=2}^{5} a_i < 1$, etc.

3. Main results

We prove now the main theorem.

Theorem 3.1 *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete quasi-metric spaces with* coefficients k_1,k_2,k_3 *and continuous quasi-distances d,ρ,σ respectively. Suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, such that at least one of them is a continuous mapping. Let $\varphi_i \in \Phi_5$ with* coefficients c_i *for $i = 1,2,3$. If $c_i \in [0, \frac{1}{k}), k = \max\{k_1,k_2,k_3\}$ and the following inequalities hold:*

$$\varphi_1(d(RSy,RSTx), d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx)) \le 0 \qquad (1)$$

$$\varphi_2(\rho(TRz,TRSy), \rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy)) \le 0 \qquad (2)$$

$$\varphi_3(\sigma(STx,STRz), \sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz)) \le 0 \qquad (3)$$

for all $x \in X, y \in Y$ and $z \in Z$, then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$ and STR has a unique fixed point $\gamma \in Z$. Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

Proof. Let $x_0 \in X$ be an arbitrary point. We define the sequences $(x_n),(y_n)$ and (z_n) in X,Y and Z respectively as follows:

$$x_n = (RST)^n x_0, y_n = Tx_{n-1}, z_n = Sy_n, n = 1,2,...$$

We prove that (x_n), (y_n) *and* (z_n) ere Cauchy sequences.

Denote

$$d_n = d(x_n, x_{n+1}), \rho_n = \rho(y_n, y_{n+1}), \sigma_n = \sigma(z_n, z_{n+1}), n = 1,2,...$$

We will assume that $x_n \ne x_{n+1}$, $y_n \ne y_{n+1}$ *and* $z_n \ne z_{n+1}$ for all *n*, otherwise if $x_n = x_{n+1}$ for some *n*, then $y_{n+1} = y_{n+2}, z_{n+1} = z_{n+2}$ we can take $\alpha = x_{n+1}, \beta = y_{n+1}, \gamma = z_{n+1}$.

By the inequality (2), for $y = y_n$ and $z = z_{n-1}$ we get:

$$\varphi_2(\rho(TRz_{n-1},TRSy_n),\rho(y_n,TRz_{n-1}),\rho(y_n,TRSy_n),\sigma(z_{n-1},Sy_n),d(Rz_{n-1},RSy_n))=$$
$$=\varphi_2(\rho(y_n,y_{n+1}),\rho(y_n,y_n),\rho(y_n,y_{n+1}),\sigma(z_{n-1},z_n),d(x_{n-1},x_n))=$$
$$=\varphi_2(\rho_n,0,\rho_n,\sigma_{n-1},d_{n-1})\le 0$$

And from **(b),** we have

$$\rho_n \le c_2 \max\{\sigma_{n-1},d_{n-1}\} \qquad (4)$$

Where c_2 is a real constant satisfying (b) for φ_2

By (3) for $z=z_n$ and $x=x_{n-1}$ we get:

$$\varphi_3(\sigma(STx_{n-1},STRz_n),\sigma(z_n,STx_{n-1}),\sigma(z_n,STRz_n),d(x_{n-1},Rz_n),\rho(Tx_{n-1},TRz_n))=$$
$$=\varphi_3(\sigma(z_n,z_{n+1}),\sigma(z_n,z_n),\sigma(z_n,z_{n+1}),d(x_{n-1},x_n),\rho(y_n,y_{n+1}))=$$
$$=\varphi_3(\sigma_n,0,\sigma_n,d_{n-1},\rho_n)\le 0$$

And from **(b),** we have $\sigma_n \le c_3 \max\{d_{n-1},\rho_n\}$.

By this inequality and by (4) it follows

$$\sigma_n \le c_3 \max\{d_{n-1},\sigma_{n-1}\} \qquad (5)$$

In similar way, by (1) for $x=x_n$ and $y=y_n$ we get:

$$\varphi_1(d(x_n,x_{n+1}),d(x_n,x_n),d(x_n,x_{n+1}),\rho(y_n,y_{n+1}),\sigma(z_n,z_{n+1}))=$$
$$=\varphi_1(d_n,0,d_n,\rho_n,\sigma_n)\le 0$$

And from **(b),** we have $d_n \le c_1 \max\{\rho_n,\sigma_n\}$.

By this inequality and by (4), (5) it follows:

$$d_n \le c_1 \max\{d_{n-1},\sigma_{n-1}\} \qquad (6)$$

By the inequalities (4), (5) and (6), using the mathematical induction, we get:

$$d(x_n,x_{n+1}) \le c^{n-1}\max\{d(x_1,x_2),\sigma(z_1,z_2)\}=c^{n-1}l$$
$$\rho(y_n,y_{n+1}) \le c^{n-1}\max\{d(x_1,x_2),\sigma(z_1,z_2)\}=c^{n-1}l$$
$$\sigma(z_n,z_{n+1}) \le c^{n-1}\max\{d(x_1,x_2),\sigma(z_1,z_2)\}=c^{n-1}l$$

where $c=\max\{c_1,c_2,c_3\}$ and $l=\max\{d(x_1,x_2),\sigma(z_1,z_2)\}$

Let's prove that the sequences $(x_n),(y_n)$ and (z_n) are Cauchy sequences.

$$d(x_n,x_{n+p}) \le k_1[d(x_n,x_{n+1})+d(x_{n+1},x_{n+p})] \le$$
$$\le k_1 d(x_n,x_{n+1})+k_1^2[d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+p}))] \le$$
$$\le k_1 d(x_n,x_{n+1})+k_1^2 d(x_{n+1},x_{n+2})+k_1^3 d(x_{n+2},x_{n+3})+\ldots+$$
$$+k_1^{p-1}[d(x_{n+p-2},x_{n+p-1})+d(x_{n+p-1},x_{n+p})] \le$$
$$\le k_1 c^{n-1}l+k_1^2 c^n l+\ldots+k_1^{p-1}c^{n+p-3}l+k_1^{p-1}c^{n+p-2}l \le$$
$$\le k_1 c^{n-1}l[1+(k_1c)+(k_1c)^2+\ldots+(k_1c)^{p-1}]=$$
$$=k_1 c^{n-1}l\frac{1-(k_1c)^p}{1-k_1c}<k_1c^{n-1}\frac{l}{1-k_1c}$$

Because $k_1c<1$. Letting n tend to infinity, we have:

$\lim_{n\to\infty} d_1(x_n, x_{n+p}) = 0$. So, (x_n) is a Cauchy sequence for $c < \frac{1}{k_1}$

In the same way we show that the sequences (y_n) and (z_n) are Cauchy sequences respectively for $c < \frac{1}{k_2}$ and $c < \frac{1}{k_3}$. For $c = \max\{c_1, c_2, c_3\} < \frac{1}{k}$ where $k = \max\{k_1, k_2, k_3\}$ all of three sequences are Cauchy sequences. Since the quasi-metric spaces $(X, d), (Y, \rho)$ and (Z, σ) are complete quasi-metric spaces we have:

$$\lim_{n\to\infty} x_n = \alpha \in X, \lim_{n\to\infty} y_n = \beta \in Y, \lim_{n\to\infty} z_n = \gamma \in Z .$$

Nov supposes that T is a continuous mapping, we have:

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n \Rightarrow T\alpha = \beta. \qquad (7)$$

To prove that α is a fixed point of ST. Using the inequality (3), for $x = \alpha$ and $z = z_n$ we obtain:

$$\varphi_3(\sigma(ST\alpha, STRz_n), \sigma(z_n, ST\alpha), \sigma(z_n, STRz_n), d(\alpha, Rz_n), \rho(T\alpha, TRz_n)) =$$
$$= \varphi_3(\sigma(ST\alpha, z_{n+1}), \sigma(z_n, ST\alpha), \sigma(z_n, z_{n+1}), d(\alpha, x_n), \rho(T\alpha, y_{n+1})) \le 0$$

Letting n tend to infinity and using **(a),** the continuity of quasi-distance d, ρ, σ and (7**)**, we have: $\varphi_3(\sigma(ST\alpha, \gamma), \sigma(\gamma, ST\alpha), 0, 0, 0) \le 0$

From **(b),** we have $\sigma(ST\alpha, \gamma) = 0$ and so

$$ST\alpha = \gamma \qquad (8)$$

Using the inequality (1), for $y = T\alpha$ and $x = x_n$ we obtain:

$\varphi_1(d(RST\alpha, RSTx_n), d(x_n, RST\alpha), d(x_n, RSTx_n), \rho(T\alpha, Tx_n), \sigma(ST\alpha, STx_n)) \le 0$

Letting n tend to infinity and using **(a),** the continuity of quasi-distance d, ρ, σ, (8) and (7), we have: $\varphi_1(d(RST\alpha, \alpha), d(\alpha, RST\alpha), 0, 0, 0) \le 0$.It follows from **(b)** that $d(RST\alpha, \alpha) = 0$ and so

$$RST\alpha = \alpha \qquad (9)$$

By (9), (8) and (7) we obtain:

$$T\alpha = \beta$$
$$ST\alpha = S\beta = \gamma$$
$$RST\alpha = RS\beta = R\gamma = \alpha$$

And

$$TRS\beta = T(RST\alpha) = T\alpha = \beta$$
$$STR\gamma = S(TRS\beta) = S\beta = \gamma$$

Thus, we proved that the points α, β *and* γ are fixed points of RST, TRS and STR respectively. In the same conclusion we would arrive if one of the mappings R or T would be continuous. Let we prove now the uniqueness of the fixed points α, β and γ. Assume that there is α' a fixed point of RST different from α.

By (1) for $x = \alpha'$ and $y = T\alpha$ we get:

$$\varphi_1(d(RST\alpha, RST\alpha'), d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) =$$
$$= \varphi_1(d(\alpha, \alpha'), d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha') \le 0$$

And from **(b),** we have

$$d(\alpha,\alpha') \le c\max\{\rho(T\alpha,T\alpha'),\sigma(ST\alpha,ST\alpha')\} \qquad (10)$$

By (2) for $z = ST\alpha$ and $y = T\alpha'$ we have:

$$\varphi_2\begin{pmatrix}\rho(TRST\alpha,TRST\alpha'),\rho(T\alpha',TRST\alpha),\rho(T\alpha',TRST\alpha'),\\ \sigma(ST\alpha,ST\alpha'),d(RST\alpha',RST\alpha)\end{pmatrix} =$$

$$= \varphi_2\left(\rho(T\alpha,T\alpha'),\rho(T\alpha',T\alpha),0,\sigma(ST\alpha,ST\alpha'),d(\alpha',\alpha)\right) \le 0$$

and from **(b),** we have:

$$\rho(T\alpha,T\alpha') \le c\max\{\sigma(ST\alpha,ST\alpha'),d(\alpha',\alpha)\}$$

By this inequality and using (10), we get:

$$\rho(T\alpha,T\alpha') \le c\sigma(ST\alpha,ST\alpha') \qquad (11)$$

By the inequality (3), for $x = RST\alpha, z = ST\alpha'$, in similar way, we obtain:

$$\sigma(ST\alpha,ST\alpha') \le c\sigma(ST\alpha,ST\alpha')$$

and so

$$\sigma(ST\alpha,ST\alpha') = 0 \qquad (12)$$

Considering backwards from (12), (11) and (10) we have $d(\alpha,\alpha') = 0$.
Thus, we have again $\alpha = \alpha'$.In the same way, it is proved the uniqueness of β and γ. This completes the proof of theorem

4. Corollaries

Corollary 4.1 *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete quasi-metric spaces, for* coefficients k_1,k_2,k_3 *and continuous quasi-distances* d,ρ,σ *respectively. Suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X, *such that at least one of them is a continuous mapping. Let* $f_i \in \mathbb{F}_4, g_i \in G_4, F_i \in \mathcal{F}$ and $c_i \in [0,\frac{1}{k}), k = \max\{k_1,k_2,k_3\}$. *If the following inequalities hold:*

$$\begin{aligned}
d(RSy,RSTx) &\le c_1 f_1(d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx)) + \\
&+ F_1(g_1(d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx))\\
\rho(TRz,TRSy) &\le c_2 f_2(\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy)) + \\
&+ F_2(g_2(\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy))\\
\sigma(STx,STRz) &\le c_3 f_3(\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz)) + \\
&+ F_3(g_3(\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz))
\end{aligned}$$

for all $x \in X, y \in Y$ and $z \in Z$, then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$ and STR has a unique fixed point $\gamma \in Z$. Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

Proof. The proof follows by theorem 3.1 in the case $\varphi_i(t_1,t_2,t_3,t_4,t_5) = t_1 - c_i f_i(t_2,t_3,t_4,t_5) - F_i(g_i(t_2,t_3,t_4,t_5)$. The function φ_i is Φ_5-function (lemma 2.6)

The next corollary extends and generalizes the Theorem Nešić [10] from two mappings of two metric space to three mappings of tree quasi-metric spaces:

Corollary 4.2 *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete quasi-metric spaces, for* coefficients k_1,k_2,k_3 *and continuous quasi-distances* d,ρ,σ *respectively. Suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X, *such that at least one of them is a continuous mapping. Let* $F\in\mathcal{F}$. *If there exists* $c\in[0,\frac{1}{k^p}), k=\max\{k_1,k_2,k_3\}$ *and* $p\geq 0$ *such that the following inequalities hold*

$$(1')\quad d^p(RSy,RSTx)\leq c\max(d^p(x,RSy),d^p(x,RSTx),\rho^p(y,Tx),\sigma^p(Sy,STx))+$$
$$+F(\min(d^p(x,RSy),d^p(x,RSTx),\rho^p(y,Tx),\sigma^p(Sy,STx)).$$
$$(2')\quad \rho^p(TRz,TRSy)\leq c\max(\rho^p(y,TRz),\rho^p(y,TRSy),\sigma^p(z,Sy),d^p(Rz,RSy))+$$
$$+F(\min(\rho^p(y,TRz),\rho^p(y,TRSy),\sigma^p(z,Sy),d^p(Rz,RSy)).$$
$$(3')\quad \sigma^p(STx,STRz)\leq c\max(\sigma^p(z,STx),\sigma^p(z,STRz),d^p(x,Rz),\rho^p(Tx,TRz))+$$
$$+F(\min(\sigma^p(z,STx),\sigma^p(z,STRz),d^p(x,Rz),\rho^p(Tx,TRz))$$

for all $x\in X, y\in Y$ and $z\in Z$, then RST has a unique fixed point $\alpha\in X$, TRS has a unique fixed point $\beta\in Y$ and STR has a unique fixed point $\gamma\in Z$. Further, $T\alpha=\beta, S\beta=\gamma$ and $R\gamma=\alpha$.

Proof. The proof follows by Theorem 3.1 in the case $\varphi_1=\varphi_2=\varphi_3=\varphi\in\Phi_5$ such that $\varphi(t_1,t_2,t_3,t_4,t_5)=t_1^p-c\max\{t_2^p,t_3^p,t_4^p,t_5^p\}-F(\min\{t_2^p,t_3^p,t_4^p,t_5^p\})$**.** The function φ is Φ_k-function with constant $c_1=\sqrt[p]{c}<1$ (example 2.2).

Corollary 4.3 From the corollary 4.2 in case $k_1=k_2=k_3=1$, we obtain the proposition which extends the Theorem Nešić [7] from two in three metric spaces.

Corollary 4.4 *Theorem Nung [8] is taken by Corollary 4.3 for* $p=1$, $k_1=k_2=k_3=1$ *and* $F=0$.

Corollary 4.5 Theorem Jain et al (Theorem **3,** [5]) *is taken* by Theorem 3.1 *in case* $k_1=k_2=k_3=1$ and $\varphi_1=\varphi_2=\varphi_3=\varphi\in\mathbb{F}_5$ such that $\varphi(t_1,t_2,t_3,t_4,t_5)=t_1^2-c\max\{t_2t_4,t_4t_3,t_3t_5,t_5t_2\}$. The function φ is Φ_5- *function*.

Corollary 4.6 Theorem Jain et al (Theorem **4,** [5]) is taken by theorem 3.1 in the case $k_1=k_2=k_3=1$ and $\varphi_1=\varphi_2=\varphi_3=\varphi\in\mathbb{F}_5$ such that

$$\varphi(t_1,t_2,t_3,t_4,t_5)=t_1\max\{t_2,t_3\}-ct_5\max\{t_3,t_5\}.$$

The function φ is Φ_5- *function* which it can be proved easily.

Corollary 3.6 T*heorem Kikina (Theorem 2.1, [6]) is taken by* theorem 3.1 *in the case* $k_1=k_2=k_3=1$ and $\varphi_1=\varphi_2=\varphi_3=\varphi$ *such that*

$$\varphi(t_1,t_2,t_3,t_4,t_5)=t_1^p-c\max\{t_2^p,t_3^p,t_4^p\}-F(\min\{t_2^p,t_3^p,t_4^p\}).$$

Corollary 3.7 (Quasi-metric version of Theorem Nešić [7]) *Let* $(X,d),(Y,\rho)$ *be complete quasi- metric spaces with coefficients* k_1,k_2 *respectively and continuous quasi-distances* d,ρ. *Suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* X *and* $F\in\mathcal{F}$ *If there exists* $c\in[0,\frac{1}{k^p})$, *where* $k=\max\{k_1,k_2\}$ *and* $p\in N$ *such that the*

following inequalities hold:

$$(1'') \qquad d^p(Sy,STx) \le c\max\{d^p(x,Sy), d^p(x,STx), \rho^p(y,Tx)\} + F(\min\{d^p(x,Sy), d^p(x,STx), \rho^p(y,Tx)\})$$

$$(2'') \qquad \rho^p(Tx,TSy) \le c\max\{\rho^p(y,Tx), \rho^p(y,TSy), d^p(x,Sy)\} + F(\min\{\rho^p(y,Tx), \rho^p(y,TSy), d^p(x,Sy)\})$$

for all $x \in X, y \in Y$, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta, S\beta = \alpha$.

Proof. The proof follows by Corollary 4.2 in the case $Z = X, \sigma = d$ and the mapping R as the identity mapping in X. Then the inequality $(1')$ takes the form $(1'')$, the inequality $(2')$ takes the form $(2'')$ and the inequality (3) is always satisfied since his left side is $\sigma^p(STx, STx) = 0$. Thus, the satisfying of the conditions $(1')$, $(2')$ and $(3')$ is reduced in satisfying of the conditions $(1'')$, $(2'')$.

The mappings T and S may be not continuous, while from the mappings T, S and R for which we applied Corollary 4.2, the identity mapping R is continuous. This completes the proof. In the special case $k_1 = k_2 = 1$ we get Theorem Nešić [7].

We emphasize the fact that in the Theorem Nešić the mappings F_1 and F_2 can be replaced by $F(t) = \max\{F_1(t), F_2(t)\}$ and c_1, c_2 can be replaced by $c = \max\{c_1, c_2\}$.

Corollary 3.8 Theorem Fisher (Theorem 1, [3]) is taken by Corollary 3.7 for $k_1 = k_2 = 1$, $F = 0$ and $p = 1$. We also emphasize here that the constants c_1, c_2 can be replaced by $c = \max\{c_1, c_2\}$.

Remark. As corollaries of these results we can obtain other propositions determined by the form of implicit relations.

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Random Fixed Point Theorems for Multivalued Subsequentially Limit-Contractive Maps Satisfying Inwardness Conditions*

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Abstract

Let (Ω, Σ) be a measurable space, with $\sum$ a sigma-algebra of subset of Ω, and let C be a non-empty bounded closed convex and separable subset of a Banach space X, satisfying property (D) , KC(X) the family of all compact convex subsets of X. We prove that a continuous 1-χ contractive mutivalued subsequentially limit-contractive map from C into $KC(X)$ satisfying an inwardness condition has a random fixed point. Our work also extends and improves stochastic version of the results of Shahzad and Lone [*Fixed point of multimaps which are not necessarily nonexpansive*, Fixed Point Theory Applications, 2 (2005), 169-176] and connected with Khan and Domlo, [*Random fixed points of multivalued inward random operators*, Journal of Applied Mathematics and Stochastic Analysis, Volume 2006 (2006), Article ID 19428, 8 pages], Plubtieng and Kumam [*Random fixed point theorems for multivalued nonexpansive non-selfrandom operators*, Journal of Applied Mathematics and Stochastic Analysis, 2006 (2006), Article ID 43796, 9 pages].

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1 Introduction

The theory of nonlinear analysis has come out as one of the momentous mathematical disciplines during the last 50 years. The study of random fixed point theorems was initiated by the Prague school of probability in the 1950s. Random operator theory is needed for the study of various classes of random equations (see [13] and references therein). Random fixed point theory has received much attention for the last two decades because of its importance in probabilistic functional analysis; the reader is referred to Itoh [13], Shahzad and Latif [38], Tan and Yaun [41], Yuan and Yu [47]. Generalizations of the random fixed point theorems for continuous selfmaps to the case of non-selfmaps have been considered by many authors (see e.g. Benavides et al. [9] Lorenzo [35], [35] Xu [45],[44]). The fixed point theory of multivalued nonexpansive mappings is much more complicated than the corresponding theory of single-valued nonexpansive mappings. It is natural to expect that the theory of nonself multivalued noncontinuous functions would be even harder. In 2002, Lorenzo Ramírez [35] was proved the existence of a random fixed point theorems for a random nonexpansive operator in the framework of a Banach spaces with a characteristic of noncompact convexity associated to the Kuratowski measure of noncompactness $\varepsilon_\alpha(X)$ is less than 1. On the other hand, Domínguez Benavides and Lorenzo Ramírez [7] was proved a fixed point theorem for a multivalued nonexpansive and 1-χ-contractive mapping in the framework of a Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ is less than 1. In 2004, Domínguez Benavides and Lorenzo Ramírez [8] was proved a fixed point theorem for a multivalued nonexpansive non-self mapping and 1-χ-contractive mapping in the framework of a Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\alpha(X)$ is less than 1.

In 2005, Shahzad and Lone [39] have replaced nonexpansive maps with SL maps in the results of Benavides and Ramírez [8]. In 2006, Dompongsa et. al [5] prove that if X is a reflexive Banach space satisfying the property (D) and E be a weakly compact convex subset of X, then every nonexpansive mapping $T : C \to KC(C)$ has a fixed point. Recently many authors have studied the problem of random fixed point theorems; see, for instance, [16-32, 34-35, 37, 40, 42-44, 46] and the references therein.

The purpose of the present paper is to prove some random fixed point theorems for multivalued nonexpansive non-self random operators. We will prove that the existence of a random fixed point theorem for a multivalued non-self random operator in the framework of a Banach spaces with property (D), and satisfying an inwardness condition. Our results can also be seen as are improves and extension of Theorem 3.2 in [19], [28], [39] and many authors.

2 Preliminaries and notations

We begin with establishing some preliminaries. By (Ω, Σ) we denote a measurable spaces with Σ is a sigma-algebra of subset of Ω. Let (X, d) be a metric spaces. We denote by $CL(X)(resp.CB(X), KC(X))$ the family of all nonempty closed (resp. closed bounded, compact convex) subset of X, and by H

the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A,B) = \max \quad \sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)$$

for $A, B \in CB(X)$, where $d(x,E) = \inf\{d(x,y)|y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X. Recall now that a multivalued mapping $T : C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^1(V) := \{x \in C : Tx \cap V \neq \emptyset\}$is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf.[2] and [3] for details). There is another defferent another defferent kind of continuity for multivalued operator: is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see Deimling [3])that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

A multivalued operator $T : \Omega \to 2^X$ is call $(\Sigma)-$ measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be be a *measurable selector* of a measurable multivalued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times C \to 2^X$ is call a random operator if, for each fixed $x \in C$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in C : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : C \to 2^X, F(\omega)$ may be nonempty. A measurable operator $x : \Omega \to C$ is said to be a *random fixed point of a operator* $T : \Omega \times C \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times C \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T(\omega, \cdot) : C \to 2^X$ is continuous.

If C is a closed convex subset of a Banach spaces X, then a multivalued mapping $T : C \to CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in C,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in C,$$

A random operator $T : \Omega \times C \to 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \to C$ is nonexpansive. For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1. (Wagner cf.[42]) *Let (X, d) be a complete separable metric spaces and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2. (Itoh 1977, cf.[13]) *Suppose $\{T_n\}$ is a sequence of measurable multivalued operator from Ω to $CB(X)$ and $T : \Omega \to CB(X)$ is an operator. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

Lemma 2.3. (Tan and Yuan cf.[41]) *Let X be a separable metric spaces and Y a metric spaces. If $f : \Omega \times X \to Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.*

As an easy application of Proposition 3 of Itoh[13] we have the following result.

Lemma 2.4. *Let C be s closed separable subset of a Banach space $X, T : \Omega \times C \to C$ a random continuous operator and $F : \Omega \to 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \to 2^C$ given by*

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measurable and so is the operator $cl\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5. (Domínguez Benavidel, Lopez Acedo and Xu cf.[9]) *Suppose C is a weakly closed nonempty separable subset of a Banach space $X, F : \Omega \to 2^X$ a measurable with weakly compact values, $f : \Omega \times C \to \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map. $R : \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

Then a multivalued mapping $T : C \to 2^X$ is called *γ-condensing (resp., 1-γ-contractive)* where $\gamma = \alpha(\cdot)$ or $\chi(\cdot)$ if, for each bounded subset B of C with $\gamma(B) > 0$, there holds the inequality

$$\gamma(T(B)) < \gamma(B) \quad (\text{resp.}\gamma(T(B)) \leq \gamma(B)).$$

Here $T(B) = \bigcup_{x \in B} Tx$. The random operator $T : \Omega \times C \to 2^X$ is said to be *1-γ-contractive* if, for each $\omega \in \Omega$ the map $T : (\omega, \cdot) \to 2^X$ is 1-γ-contractive.

In order to study the fixed point theory for non-self mappings, we must introduce some terminology for boundary condition. The inward set of C at $x \in C$ defined by

$I_C(x) := \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}$.

Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C does. A multivalued mapping $T : C \to 2^X\{\emptyset\}$ is said to be inward on C if

$Tx \subset I_C(x) \ \forall x \in C$.

Let $\bar{I}_C(x) := x + \{\lambda(z - x) : z \in C, \lambda \geq 1\}$. Note that for a convex C, we have $\bar{I}_C(x) = \overline{I_C(x)}$, and T is said to be weakly inward on C if

$Tx \subset \bar{I}_C(x) \ \forall x \in C$.

Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$r(C, \{x_n\}) = \inf \left\{ \limsup_n \|x_n - x\| : x \in C \right\},$$

$$A(C, \{x_n\}) = \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}.$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf \{\sup\{\|x - y\| : y \in D\} : x \in C\}.$$

Obviously, the convexity of C implies that $A(C, \{x_n\})$ is convex. Notice that $A(C, \{x_n\})$ is a nonempty weakly compact set if C is weakly compact, or C is a closed convex subset of a reflexive Banach spaces X.

Let S be a set $G \subset S$ and D be a directed set, we shall say that a net x_α in S eventually in G if there exist $\alpha_0 \in D$ such that $x_\alpha \in G$ for all $\alpha \geq \alpha_0$.

Definition 2.6. *A net $\{x_\alpha\}$ in a set S is call an ultranet if for each sunset $C \subset S$, either $\{x_\alpha\}$ is eventually in G or $\{x_\alpha\}$ is eventually in $S \setminus G$.*

The following facts concerning ultranets can be found in [15]:

(a) Every net in a set has an ultranet.

(b) If $f : S_1 \to S_2$ is a map and if $\{x_\alpha\}$ is an ultranet in S_1, then $\{f(x_\alpha)\}$ is a ultranet in S_2.

(c) If S is compact and $\{x_\alpha\}$ is a ultranet in S, then $\lim_\alpha x_\alpha$ exists.

Obviously, the convexity of C implies that $A(C, \{x_\alpha\})$ is convex. Notice that $A(C, \{x_\alpha\})$ is a nonempty weakly compact set if C is weakly compact, or C is a closed convex subset of a reflexive Banach spaces X.

Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\}$ is called *regular* with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; while $\{x_n\}$ is called *asymptotically uniform* with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.7. (Goebel[12] and Lim[34]). *Let $\{x_n\}$ and C be as above. Then we have*

(i) *There always exists a subsequence of $\{x_n\}$ which is regular with respect to C;*

(ii) *if C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to C.*

A sequence $\{x_n\}$ is called *asymptotically T-regular* if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Definition 2.8. (Shahzad and Lone [39]) . *Let C be a nonempty weakly compact convex subset of a Banach space X and $T : C \to KC(X)$. The map T is called subsequentially limit-contractive (SL) if for every asymptotically T-regular sequence $\{x_n\}$ in C, we have*

$$\limsup_{n\to\infty} H(Tx_n, Tx) \leq \limsup_{n\to\infty} \|x_n - x\|, \quad \text{for all} \quad x \in A(C, \{x_n\}). \tag{2.1}$$

Every nonexpansive map is an SL map but the converse does not hold (see Example 3.2 in [39]).

The property (D) introduced in ([5]) is defined as follows:

Definition 2.9. (Dhompongsa et. al [5]) *A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform relative to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular asymptotically uniform relative to E we have*

$$r(E, \{y_n\}) \leq \lambda r(E, \{x_n\}). \tag{2.2}$$

Definition 2.10. (Dhompongsa, Kaewcharoen, and Kaewkhao [4]) *A Banach space X is said to satisfy the Dominguez-Lorenzo condition if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset E of X, and for every bounded sequence $\{x_n\}$ in E, which is regular relative to E,*

$$r_E(A(E,\{x_n\})) \leq \lambda r(E,\{x_n\}). \tag{2.3}$$

Remark 2.11. *(i) It is clear from the definition that property (D) is weaker than the Domínguez-Lorenzo condition (see [4]) but stronger than weak normal structure (see [5, Theorem 3.3]).*

(ii) From [4, Theorem 2.7], it can be seen that every Banach spaces X with $\varepsilon_\beta(X) < 1$ satisfies the Dominguez-Lorenzo condition.

Moreover, we also need the following Lemma

The following result are now basic in the fixed point theorem for multivalued mappings.

Lemma 2.12. (Deimling [3]). *Let X be a Banach space and $\emptyset \neq D \subset X$ be closed bounded convex. Let $F : D \to 2^X$ be upper semicontinuous $\gamma-$condensing with closed convex values, where $\gamma(\cdot) = \alpha(\cdot) or \chi(\cdot)$. If $Fx \cap \overline{I_D(x)} \neq \emptyset$ for all $x \in C$, then F has a fixed point. (Here $I_D(x)$ is call the inward set at x defined by $I_D := \{x + \lambda(y - x) : \lambda \geq 0, y \in D\}$)*

3 The Main Results

In order to prove our first result, we need the following Lemma which is prove along the proof of Kirk-Massa theorem as it appear in [46].

Lemma 3.1. *Let C be a nonempty closed bounded convex subset of a Banach space X. $T : C \to KC(X)$ is an SL map such that $T(C)$ is a bounded set and which satisfies $Tx \subset I_C(x)$, $\forall x \in C. If \{x_n\}$ is asymptotically T-regular in C, then There always exists an ultrasubnet $\{x_{n_\alpha}\}$ of the positive integer n such that $Tx \cap I_A \neq \emptyset, \forall x \in A := A(C,\{x_{n_\alpha}\})$*

Proof. By Lemma 2.7, we may assume that $\{x_n\}$ is asymptotically uniform. Let $A := A(C,\{x_n\})$ and let $r := r(C,\{x_n\})$. Let $\{n_\alpha\}$ be an ultranet of the positive integer $\{n\}$. By the compactness of $T(\omega, x_{n_\alpha}(\omega))$ implies that for each n_α fixed $\omega \in \Omega$, we can take $y_{n_\alpha}(\omega) \in T(\omega, x_{n_\alpha}(\omega))$ such that

$$\|x_{n_\alpha}(\omega) - y_{n_\alpha}(\omega)\| = d(x_{n_\alpha}(\omega), T(\omega, x_{n_\alpha}(\omega))).$$

Since $T(\omega, x(\omega))$ is compact, for each $x(\omega) \in A(\omega)$, we can find $z_{n_\alpha}(\omega) \in T(\omega, x(\omega))$ such that

$$\begin{aligned} \|y_{n_\alpha}(\omega) - z_{n_\alpha}(\omega)\| &= d(y_{n_\alpha}(\omega), T(\omega, x(\omega))) \\ &\leq H(T(\omega, x_{n_\alpha}(\omega)), T(\omega, x(\omega))) \\ &\leq \|x_{n_\alpha}(\omega) - x(\omega)\|. \end{aligned}$$

Let $z(\omega) = \lim_\alpha z_{n_\alpha}(\omega) \in T(\omega, x(\omega))$. It should remain to prove $z(\omega) \in I_{A(\omega)}(x)$. Since T is an SL map and $x_{n_\alpha}(\omega)$ is asymptotically T-regular, it follows that

$$\limsup_\alpha H(T(\omega, x_{n_\alpha}(\omega)) - T(\omega, x(\omega))) \quad \leq \quad \limsup_\alpha \|x_{n_\alpha}(\omega) - x(\omega)\|$$

For all $x \in A(\omega)$ and $\omega \in \Omega$. Now

$$\begin{aligned} \|y_{n_\alpha}(\omega)) - z_{n_\alpha}(\omega)\| &= d(y_{n_\alpha}(\omega) - T(\omega, x(\omega))) \\ &\leq H(T(\omega, x_{n_\alpha}(\omega)) - T(\omega, x(\omega))), \end{aligned}$$

and so

$$\begin{aligned} \lim_\alpha \|x_{n_\alpha}(\omega) - z(\omega)\| &= \lim_\alpha \|y_{n_\alpha}(\omega) - z_{n_\alpha}(\omega)\| \\ &\leq \limsup_\alpha \|x_{n_\alpha}(\omega) - x(\omega)\| \\ &= r(\omega). \end{aligned}$$

where $r(\omega) = r(C, \{x_{n_\alpha}(\omega)\})$. Since $z(\omega) \in T(\omega, x(\omega)) \subseteq I_C(x)$, for each fixed $\omega \in \Omega$ there exist $\lambda \geq 0$ and $v(\omega) \in E$ such that

$$z(\omega) = x(\omega) + \lambda(v(\omega) - x(\omega)).$$

If $\lambda \leq 1$, then by the convexity of E, $z(\omega) \in E$ and hence $z(\omega) \in A(\omega) \subseteq I_{A(\omega)}(x)$ and we are done. So assume that $\lambda > 1$. Then we can write

$$v(\omega) = \mu z(\omega) + (1 - \mu)x(\omega) \text{ with } \mu = \frac{1}{\lambda} \in (0, 1).$$

It follows that

$$\begin{aligned} f(\omega, v(\omega)) &= \limsup_\alpha \|x_{n_\alpha}(\omega) - v(\omega)\| \\ &\leq \mu \limsup_\alpha \|x_{n_\alpha}(\omega) - z(\omega)\| + (1 - \mu) \limsup_\alpha \|x_{n_\alpha}(\omega) - x(\omega)\| \\ &\leq r(\omega). \end{aligned}$$

Therefore $v(\omega) \in A(\omega)$ and thus $z(\omega) = x(\omega) + \lambda(v(\omega) - x(\omega))$ belong to $I_{A(\omega)}(x)$. That is

$$T(\omega, x(\omega)) \cap I_{A(\omega)}(x) \neq \emptyset \ \ \forall x(\omega) \in A(\omega).$$

□

Next Theorem is stochastic version of Theorem 3.6 [5] extend for non-self mapping. With this observation, we are able to prove our main result. The proof below is inspired by same ideas in the proof of Theorem 3.2 in [19].

Theorem 3.2. *Let C be a nonempty closed bounded convex separable subset of a reflexive Banach spaces X satisfying property (D). If $T : \Omega \times C \to KC(X)$ is a continuous $1 - \chi-$ contractive SL map which satisfies the inwardness condition, then T has a random fixed point.*

Proof. Fix $x_0 \in C$, and consider the measurable function $x_0(\omega) \equiv x_0$. For each $n \geq 1$, define $T_n(\omega, \cdot) : C \to KC(X)$ by

$$T_n(\omega, x) = \frac{1}{n}x_0(\omega) + (\frac{n-1}{n})T(\omega, x), \ \ \forall x \in C.$$

Then $T_n(\omega, \cdot)$ is a multivalued contraction and $T_n(\omega, x) \subset I_C(x)$, $\forall x \in C$. Hence each T_n has a fixed point $z_n(\omega) \in C$. It is easily seen that

$$d(z_n(\omega), T(\omega, z_n(\omega))) \leq \frac{1}{n} diam C \to 0 \text{ as } n \to \infty.$$

Thus the set

$$F_n(\omega) = \{x \in C : d(x, T(\omega, x)) \leq \frac{1}{n} diamC\}.$$

is nonempty closed and convex. By Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ and $d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} diamC \to 0$ as $n \to \infty$. Defined a function $f : \Omega \times C \to \mathbb{R}^+ := [0, \infty)$ by

$$f(\omega, x) = \limsup_n \|x_n(\omega) - x\|, \ \ x \in C.$$

It is easily see that $f(\cdot, x)$ is measurable and $f(\omega, \cdot)$ is continuous and convex, therefore it is a weakly lower semicontinuous function. Since X is reflexive, C is a weakly compact. Hence, by Lemma 2.5, the marginal functions

$$r(\omega) := \inf_{x \in C} f(\omega, x)$$

and

$$A(\omega) := \{x \in C : f(\omega, x) = r(\omega)\}$$

are measurable. It is clearly that $A(\omega)$ is a weakly compact convex subset of C. For any $\omega \in \Omega$, by Lemma 2.7, we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect C. Denote $A(\omega) = A(C, \{x_{n_\alpha}(\omega)\})$, and $r(\omega) = r(C, \{x_{n_\alpha}(\omega)\})$. Moreover, since X satisfies property (D) with a corresponding $\lambda \in (0, 1)$, we have

$$r_C(A(\omega)) \leq \lambda r(C, \{x_{n_\alpha}(\omega)\}). \tag{3.1}$$

For each $\omega \in \Omega$ and $n \geq 1$, we define the multivalued contraction $T_n(\omega, \cdot) : A(\omega) \to KC(X)$ defined by

$$T_n(\omega, x) = \frac{1}{n} x_0(\omega) + (\frac{n-1}{n}) T(\omega, x),$$

for each $(\omega, x) \in \Omega \times C$. By Lemma 3.1, we have $T(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega)$. Since $I_{A(\omega)}(x)$ is convex, it follow that $T_n(\omega, \cdot)$ satisfies the boundary condition i.e.,

$$T_n(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega). \tag{3.2}$$

Since $T_n(\omega, \cdot)$ is $1 - \chi-$contractive mapping, it follows by [3, pp.382] that $T_n(\omega, \cdot)$ is χ- condensing. Hence by Lemma 2.12, $T_n(\omega, \cdot)$ has a fixed point $z_n(\omega) \in A(\omega), i.e. F(\omega) \cap A(\omega) \neq \emptyset$. Also it is easily seen that

$$dist(z_n(\omega), T(\omega, z_n(\omega))) \leq \frac{1}{n} diamC \to 0 \text{ as } n \to \infty.$$

Thus, $F_n^1(\omega) := \{x \in A(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n} diamC\} \neq \emptyset$ for each $n \geq 1$ is closed and measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1, and from definition of it we have $x_n^1(\omega) \in A(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \to 0$ as $n \to \infty$. Consider the function $f_2 : \Omega \times E \to \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_\alpha \|x_{n_\alpha}^1(\omega) - x\|, \ \ \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontunuous function. Then the marginal function

$$r_2(\omega) := \inf_{x \in A(\omega)} f_2(\omega, x)$$

and

$$A^1(\omega) := \{x \in A(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $A^1(\omega) = A(A(\omega), \{x^1_{n_\alpha}(\omega)\})$, it follows that $A^1(\omega)$ is a weakly compact and convex. Also $r_2(\omega) = r(A(\omega), \{x^1_{n_\alpha}(\omega)\})$. We proceed as before to obtain that

$$T(\omega, x(\omega)) \cap I_{A^1}(x(\omega)) \neq \emptyset \ \ \forall x(\omega) \in A^1 = A(A(\omega), \{x^1_{n_\alpha}(\omega)\}),$$

and by (3.1) we obtain that

$$r_E(A^1) \leq \lambda r(A(\omega), \{x^1_{n_\alpha}(\omega)\}) \leq \lambda r_C(A(\omega)). \tag{3.3}$$

By induction, for each $m \geq 1$, we take a sequence $\{x^m_n(\omega)\}_n \subseteq A^{m-1}$ such that $\lim_n d(x^m_n(\omega), T(\omega, x^m_n(\omega))) = 0$ for each fixed $\omega \in \Omega$. By means of the ultranet $\{x^m_{n_\alpha}(\omega)\}_\alpha$ we construct the set $A^m := A(C, \{x^m_{n_\alpha}(\omega)\})$ such that

$$r_C(A^m) \leq \lambda^m r_C(A(\omega)). \tag{3.4}$$

Choose $x_m(\omega)$ as a measurable selector of A^m. We shall prove that $\{x_m(\omega)\}_m$ is a Cauchy sequence. For each $m \geq 1$, we have

$$\begin{aligned} \|x_{m-1}(\omega) - x_m(\omega)\| &\leq \|x_{m-1}(\omega) - x^m_n(\omega)\| + \|x^m_n(\omega) - x_m(\omega)\| \\ &\leq dim A_{m-1}(\omega) + \|x^m_n(\omega) - x_m(\omega)\|. \end{aligned}$$

Since $dim A^m \leq 2r_C(A^m)$, taking upper limit as $n \to +\infty$, we have

$$\begin{aligned} \|x_{m-1}(\omega) - x_m(\omega)\| &\leq dim A^{m-1} + \limsup_n \|x^m_n(\omega) - x_m(\omega)\| \\ &= dim A^{m-1} + r(C, \{x^m_n(\omega)\}) \\ &\leq dim A^{m-1} + r_C(A^{m-1}\}) \\ &\leq 2r_C(A^{m-1}) + r_C(A^{m-1}) \\ &= 3\lambda^{m-1} r_C(A(\omega)). \end{aligned}$$

Since $\lambda < 1$, hence $\{x_m(\omega)\}_{m\geq 1}$ is a Cauchy sequence we conclude that there exists $x(\omega) \in C$ such that $x_m(\omega)$ converges to $x(\omega)$. Finally, we will show that $x(\omega)$ is a random fixed point of T. Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(x_m(\omega), T(\omega, x_m(\omega)) &\leq \|x_m(\omega) - x^m_n(\omega)\| + d(x^m_n(\omega), T(\omega, x^m_n(\omega))) \\ &+ H(T(\omega, x^m_n(\omega)), T(\omega, x_m(\omega))). \end{aligned}$$

Since T is an SL map, and $\{x^m_n(\omega)\}$ asymptotically T-regular for $m \geq 1$, hence we have

$$\limsup_{n\to\infty} H(T(\omega, x^m_n(\omega)), T(\omega, x_m(\omega))) \leq \limsup_{n\to\infty} \|x^m_n(\omega) - x_m(\omega)\|.$$

Taking the upper limit as $n \to +\infty$,

$$\begin{aligned} d(x_m(\omega), T(\omega, x_m(\omega)) &\leq 2 \limsup_n \|x_m(\omega) - x^m_n(\omega)\| \\ &\leq 2\lambda^{m+1} r_C(A(\omega)) \end{aligned}$$

Now taking the limit $m \to +\infty$ on both sides we get $d(x_m(\omega), T(\omega, x_m(\omega))) = 0$, and the continuity of $T(\omega, \cdot)$ implies that $d(x(\omega), T(\omega, x(\omega))) = 0$, that is, $x(\omega) \in T(\omega, x(\omega))$. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed bounded convex separable subset of a reflexive Banach spaces X satisfying the property (D) and $T : \Omega \times C \to KC(X)$ be a multivalued nonexpansive random operator and $1 - \chi-$ contractive mappings such that satisfies the inwardness condition, then T has a random fixed point.*

Proof. This follows immediately from Theorem 3.2. □

Corollary 3.4. ([19, Theorem 3.2]) *Let C be a nonempty closed bounded convex separable subset of a reflexive Banach spaces X satisfying the Domínguez-Lorenzo condition and $T : \Omega \times C \to KC(X)$ be a multivalued nonexpansive random operator and $1 - \chi-$ contractive mappings such that for each $\omega \in \Omega,\ T(\omega, C)$ is a bounded set, and which satisfies the inwardness condition, i.e., for each $\omega \in \Omega,\ T(\omega, x) \subset I_C(x),\ \forall x \in C$, then T has a random fixed point.*

Proof. Since every nonexpansive map is an SL map. This follows immediately from Theorem 3.2 and by Remark 2.11 the Domínguez-Lorenzo condition implies property (D). □

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Some properties of twisted Bernstein polynomials associated with twisted ζ-Euler numbers and polynomials

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Abstract

In this paper, we consider twisted ζ-Euler numbers and polynomials, and discuss p-adic ζ-integral representation of twisted Bernstein polynomials. By using these properties, we derive some identities on the twisted Bernstein polynomials.

1 Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = ^{-1}$.

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ has a limit $f'(a)$ as $(x,y) \to (a,a)$. Let $T_p = \cup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n}$ be the locally constant space, where $C_{p^n} = \{\zeta \in \mathbb{C}_p | \zeta^{p^n} = 1\}$ is the cyclic group of order of p^n (cf. [4,6-7,16]).

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We assume $\zeta \in T_p$ with $|1-\zeta|_p < 1$ and use the notation

$$[x]_\zeta = \frac{1-\zeta^x}{1-\zeta} \quad \text{and} \quad [x]_{-\zeta} = \frac{1-(-\zeta)^x}{1+\zeta}$$

for all $x \in \mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic ζ-integral on $\mathbb{Z}_p$ is defined by

$$I_{-\zeta}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-\zeta}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-\zeta}} \sum_{x=0}^{p^N-1} f(x)(-\zeta)^x. \quad \text{(see [10-18])}. \tag{1}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. From (1), we can derive

$$\zeta^n I_{-\zeta}(f_n) = (-1)^n I_{-\zeta}(f) + [2]_\zeta \sum_{l=0}^{n-1} (-1)^{n-1-l} \zeta^l f(l), \quad \text{(see [18])}. \tag{2}$$

As well known definition, the ζ-Euler numbers $\mathcal{E}_{n,\zeta}$ and polynomials $\mathcal{E}_{n,\zeta}(x)$ are given by the generating functions as follows:

$$\frac{[2]_\zeta}{\zeta e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\zeta} \frac{t^n}{n!}, \quad \text{and} \tag{3}$$

$$\frac{[2]_\zeta}{\zeta e^t + 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\zeta}(x) \frac{t^n}{n!}. \tag{4}$$

By (2), we can obtain the following equation

$$\zeta \int_{\mathbb{Z}_p} f(x+1) d\mu_{-\zeta}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-\zeta}(x) = (1+\zeta) f(0). \tag{5}$$

Thus, if we take $f(x) = e^{tx}\zeta^x$, then

$$\int_{\mathbb{Z}_p} e^{tx} \zeta^x d\mu_{-\zeta}(x) = \frac{1+\zeta}{\zeta e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\zeta} \frac{t^n}{n!}, \quad \text{and} \tag{6}$$

$$\int_{\mathbb{Z}_p} e^{t(x+y)} \zeta^y d\mu_{-\zeta}(y) = \frac{1+\zeta}{\zeta e^t + 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\zeta}(x) \frac{t^n}{n!}. \tag{7}$$

By (6) and (7), we have

$$\mathcal{E}_{n,\zeta} = \int_{\mathbb{Z}_p} x^n \zeta^x d\mu_{-\zeta}(x), \quad \text{and} \tag{8}$$

$$\mathcal{E}_{n,\zeta}(x) = \int_{\mathbb{Z}_p} (x+y)^n \zeta^y d\mu_{-\zeta}(y). \tag{9}$$

Let $C[0,1]$ denote the set of continuous functions on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced positive operator in the field of real numbers $\mathbb{R}$:

$$\begin{aligned} \mathbb{B}(f:x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{n,k}(x) \end{aligned} \tag{10}$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$ (see[1-3,5,8,9]). Here, $\mathbb{B}(f:x)$ is called the Bernstein operator of order n for f.

In this paper, we will define twisted Bernstein polynomials, we investigate the p-adic ζ-integral representation of twisted Bernstein polynomials and derive some identities on twisted Bernstein polynomials.

2 Twisted Bernstein polynomials

In this section, we assume $\zeta \in T_p$ with $|1-\zeta|_p < 1$. The twisted Bernstein operator $\mathbb{B}_\zeta(f:x)$ is defined by

$$\begin{aligned} \mathbb{B}_\zeta(f:x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \zeta^x x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{n,k,\zeta}(x) \end{aligned} \tag{11}$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$. Here, $B_{n,k,\zeta}(x)$ is called the twisted Bernstein polynomials.

From (7), we see that

$$\frac{\zeta+1}{\zeta e^t+1} e^{(1-x)t} = \sum_{k=0}^{\infty} \mathcal{E}_{n,\zeta}(1-x)\frac{t^n}{n!}. \tag{12}$$

Thus we get

$$\frac{\zeta+1}{\zeta e^t+1} e^{(1-x)t} = \frac{1+\zeta^{-1}}{1+\zeta^{-1}e^{-t}} e^{(-t)x} = \sum_{k=0}^{\infty} \mathcal{E}_{n,\zeta^{-1}}(x)(-1)^n \frac{t^n}{n!}, \tag{13}$$

By comparing the coefficients on the both sides of (12) and (13), we can derive the following theorem.

Theorem 1. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $n \in \mathbb{Z}_+$ and $x \in \mathbb{Z}_p$, we have*

$$\mathcal{E}_{n,\zeta}(1-x) = (-1)^n \mathcal{E}_{n,\zeta^{-1}}(x). \tag{14}$$

By (8) and (9), we have

$$\begin{aligned} \mathcal{E}_{n,\zeta}(x) &= \sum_{l=0}^{n} \binom{n}{l} x^l \int_{\mathbb{Z}_p} \zeta^y y^{n-l} d\mu_{-\zeta}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} x^l \mathcal{E}_{n-l,\zeta} \\ &= (x + \zeta \mathcal{E}_\zeta)^n. \end{aligned} \tag{15}$$

Thus, we obtain the following recurrence formula.

Theorem 2. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $n \in \mathbb{N}$, we have*

$$\mathcal{E}_{0,\zeta} = 1, \quad \zeta(\mathcal{E}_\zeta + 1)^n + \mathcal{E}_{n,\zeta} = \begin{cases} [2]_\zeta & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{16}$$

with usual convention about replacing $\mathcal{E}_\zeta^n$ by $\mathcal{E}_{n,\zeta}$.

By Theorem 1, we easily get

$$\begin{aligned} \int_{\mathbb{Z}_p} \zeta^x (1-x)^n d\mu_{-\zeta}(x) &= (-1)^n \int_{\mathbb{Z}_p} \zeta^x (x-1)^n d\mu_{-\zeta}(x) \\ &= (-1)^n \mathcal{E}_{n,\zeta}(-1) = \mathcal{E}_{n,\zeta^{-1}}(2). \end{aligned} \tag{17}$$

By (16) and (17), we obtain the following theorem.

Theorem 3. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $n \in \mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p} \zeta^x (1-x)^n d\mu_{-\zeta}(x) = 1 + \zeta + \zeta^2 \int_{\mathbb{Z}_p} \zeta^x x^n d\mu_{-\zeta^{-1}}(x). \tag{18}$$

By using the p-adic ζ-integral on $\mathbb{Z}_p$ for one twisted Bernstein polynomial in (18), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n,\zeta} d\mu_{-\zeta}(x) \\ &= \int_{\mathbb{Z}_p} \binom{n}{k} \zeta^x x^k (1-x)^{n-k} d\mu_{-\zeta}(x) \end{aligned}$$

$$= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} \int_{\mathbb{Z}_P} \zeta^x x^{n-j} d\mu_{-\zeta}(x)$$
$$= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} \mathcal{E}_{n-j,\zeta}$$
$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} \mathcal{E}_{k+l,\zeta}. \tag{19}$$

Therefore, we obtain the following theorem.

Theorem 4. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $k, n \in \mathbb{Z}_+$ with $n > k$, we have*

$$\int_{\mathbb{Z}_p} \zeta^x (1-x)^n d\mu_{-\zeta}(x) = \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \mathcal{E}_{k+l,\zeta}. \tag{20}$$

It is known that $B_{k,n,\zeta}(x) = \zeta^{-1} B_{n-k,n,\zeta^{-1}}(1-x)$. Thus, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n,\zeta}(x) d\mu_{-\zeta}(x) \\
&= \int_{\mathbb{Z}_p} \zeta^{-1} B_{n-k,n,\zeta^{-1}}(1-x) d\mu_{-\zeta}(x) \\
&= \zeta^{-1} \int_{\mathbb{Z}_p} B_{n-k,n,\zeta^{-1}}(1-x) d\mu_{-\zeta}(x) \\
&= \zeta^{-1} \binom{n}{n-k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} (\zeta^{-1})^{1-x} (1-x)^{n-l} d\mu_{-\zeta}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} \zeta^x (1-x)^{n-l} d\mu_{-\zeta}(x).
\end{aligned} \tag{21}$$

By Theorem 3 and (21), we obtain the following theorem.

Theorem 5. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $k, n \in \mathbb{Z}_+$ with $n > k$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n,\zeta}(x) d\mu_{-\zeta}(x) \\
&= \begin{cases} 1 + \zeta + \zeta^2 \mathcal{E}_{n,\zeta^{-1}} & \text{if } k = 0, \\ \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} (1 + \zeta + \zeta^2 \mathcal{E}_{n-l,\zeta^{-1}}) & \text{if } k > 0. \end{cases}
\end{aligned} \tag{22}$$

By Theorem 4 and Theorem 5, we have the following corollary.

Corollary 6. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $k, n \in \mathbb{Z}_+$ with $n > k$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \mathcal{E}_{k+l,\zeta} \\
&= \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \begin{cases} 1+\zeta+\zeta^2 \mathcal{E}_{n,\zeta^{-1}} & \text{if } k=0, \\ \binom{n}{k} 1+\zeta+\zeta^2 \mathcal{E}_{n-l,\zeta^{-1}} & \text{if } k>0. \end{cases}
\end{aligned} \tag{23}$$

Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$ and $m, n, k \in \mathbb{Z}_+$ with $m+n > 2k$. By (18), p-adic ζ-integral for multiplication of two twisted Bernstein polynomials on $\mathbb{Z}_p$ can be given by the following relation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n,\zeta^{\frac{1}{2}}}(x) B_{k,m,\zeta^{\frac{1}{2}}}(x) d\mu_{-\zeta}(x) \\
&= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} \zeta^x x^{2k} (1-x)^{n+m-2k} d\mu_{-\zeta}(x) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} (-1)^{l+2k} \int_{\mathbb{Z}_p} \zeta^x (1-x)^{2k-l} (1-x)^{n+m-2k} d\mu_{-\zeta}(x) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} (-1)^{l+2k} \int_{\mathbb{Z}_p} \zeta^x (1-x)^{n+m-l} d\mu_{-\zeta}(x) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} (-1)^{l+2k} \binom{2k}{l} \left(1+\zeta+\zeta^2 \int_{\mathbb{Z}_p} \zeta^x x^{n+m-l} d\mu_{-\zeta^{-1}}(x) \right)
\end{aligned} \tag{24}$$

By (24), we obtain the following theorem.

Theorem 7. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $k, m, n \in \mathbb{Z}_+$ with $m+n > 2k$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n,\zeta^{\frac{1}{2}}}(x) B_{k,m,\zeta^{\frac{1}{2}}}(x) d\mu_{-\zeta}(x) \\
&= \begin{cases} 1+\zeta+\zeta^2 \mathcal{E}_{n+m,\zeta^{-1}} & \text{if } k=0, \\ \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} (1+\zeta+\zeta^2 \mathcal{E}_{n+m-l,\zeta^{-1}}) & \text{if } k>0, \end{cases}
\end{aligned} \tag{25}$$

By using the p-adic ζ-integral on $\mathbb{Z}_p$ for two twisted Bernstein polynomials in (18), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n,\zeta^{\frac{1}{2}}}(x) B_{k,m,\zeta^{\frac{1}{2}}}(x) d\mu_{-\zeta}(x) \\
&= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} \zeta^x x^{2k} (1-x)^{n+m-2k} d\mu_{-\zeta}(x)
\end{aligned}$$

$$= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\int_{\mathbb{Z}_p}\zeta^x x^{2k+l}d\mu_{-\zeta}(x)$$
$$= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\mathcal{E}_{2k+l,\zeta}. \tag{26}$$

By comparing the coefficients on the both sides of (25) and (26), we can derive the following corollary.

Corollary 8. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $m, n, k \in \mathbb{Z}_+$ with $m+n > 2k$, we have*

$$\sum_{l=0}^{m+n-2k}\binom{m+n-2k}{l}(-1)^l\mathcal{E}_{2k+l,\zeta}$$
$$= \begin{cases} 1+\zeta+\zeta^2\mathcal{E}_{m+n,\zeta^{-1}} & \text{if } k=0, \\ \sum_{l=0}^{2k}\binom{k}{l}(-1)^{l+2k}\left(1+\zeta+\zeta^2\mathcal{E}_{m+n-l,\zeta^{-1}}\right) & \text{if } k>0. \end{cases} \tag{27}$$

Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$ and $n_1, n_2, \cdots, n_s, k \in \mathbb{Z}_+$ with $n_1+\cdots+n_s > sk$, by induction, we get

$$\int_{\mathbb{Z}_p} B_{k,n_1,\zeta^{\frac{1}{s}}}(x)\cdots B_{k,n_s,\zeta^{\frac{1}{s}}}(x)d\mu_{-\zeta}(x)$$
$$= \left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\int_{\mathbb{Z}_p}\zeta^x x^{sk}(1-x)^{n_1+\cdots+n_s-sk}d\mu_{-\zeta}(x)$$
$$= \left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\sum_{l=0}^{sk}(-1)^{l+sk}\binom{sk}{l}\left(1+\zeta+\zeta^2\mathcal{E}_{n_1+\cdots+n_s-l,\zeta^{-1}}\right). \tag{28}$$

By (28), we obtain the following theorem.

Theorem 9. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $n_1, n_2, \cdots, n_s, k \in \mathbb{Z}_+$ with $n_1+\cdots+n_s > sk$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n_1,\zeta^{\frac{1}{s}}}(x)\cdots B_{k,n_s,\zeta^{\frac{1}{s}}}(x)d\mu_{-\zeta}(x)$$
$$= \begin{cases} 1+\zeta+\zeta^2\mathcal{E}_{n_1+\cdots+n_s,\zeta^{-1}} & \text{if } k=0, \\ \left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\sum_{l=0}^{sk}(-1)^{l+sk}\binom{sk}{l}\left(1+\zeta+\zeta^2\mathcal{E}_{n_1+\cdots+n_s-l,\zeta^{-1}}\right) & \text{if } k>0. \end{cases} \tag{29}$$

By using the p-adic ζ-integral on $\mathbb{Z}_p$ for twisted Bernstein polynomials $B_{k,n_1,\zeta^{\frac{1}{s}}}(x)\cdots B_{k,n_s,\zeta^{\frac{1}{s}}}(x)$ in (18), we get

$$\int_{\mathbb{Z}_p} B_{k,n_1,\zeta^{\frac{1}{s}}}(x)\cdots B_{k,n_s,\zeta^{\frac{1}{s}}}(x)d\mu_{-\zeta}(x)$$

$$= \left(\prod_{i=1}^{s} \binom{n_i}{k}\right) \int_{\mathbb{Z}_p} \zeta^x x^{sk} (1-x)^{n_1+\cdots+n_s-sk} d\mu_{-\zeta}(x)$$
$$= \left(\prod_{i=1}^{s} \binom{n_i}{k}\right) \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \mathcal{E}_{sk+l,\zeta}. \tag{30}$$

By comparing the coefficients on the both sides of (29) and (30), we can derive the following corollary.

Corollary 10. *Let $\zeta \in T_p$ with $|1-\zeta|_p < 1$. For $n_1, \cdots, n_s, k \in \mathbb{Z}_+$ with $n_1+\cdots+n_s > sk$, we have*

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \mathcal{E}_{sk+l,\zeta}$$
$$= \begin{cases} 1+\zeta+\zeta^2 \mathcal{E}_{n_1+\cdots+n_s,\zeta} & \text{if } k=0, \\ \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} (1+\zeta+\zeta^2 \mathcal{E}_{n_1+\cdots+n_s-l,\zeta}) & \text{if } k>0. \end{cases} \tag{31}$$

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A note on special strong differential subordinations using multiplier transformation

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Abstract

In the present paper we establish several strong differential subordinations regardind the operator $I(m,\lambda,l)$, given by $I(m,\lambda,l): \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$, $I(m,\lambda,l) f(z,\zeta) := z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta) z^j$, where $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$ and $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$ is the class of normalized analytic functions.

Keywords: strong differential subordination, univalent function, convex function, best dominant, differential operator.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} :\ |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} :\ |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Definition 1.1. *[3] Let $f(z,\zeta)$, $H(x,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.2. *[3] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

Lemma 1.3. *[5, p. 71] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta]$ and $p(z,\zeta) + \frac{1}{\gamma} z p'(z,\zeta) \prec\prec h(z,\zeta)$, then $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, where $g(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$ is convex and it is the best dominant.*

Lemma 1.4. *[4] Let $g(z,\zeta)$ be a convex function in U, for all $\zeta \in \overline{U}$, and let $h(z,\zeta) = g(z,\zeta) + n\alpha z g'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $\alpha > 0$ and n is a positive integer. If $p(z,\zeta) = g(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots$, $z \in U$, $\zeta \in \overline{U}$, is holomorphic in U, for all $\zeta \in \overline{U}$, and $p(z,\zeta) + \alpha z p'(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $p(z,\zeta) \prec\prec g(z,\zeta)$ and this result is sharp.*

Definition 1.5. *[1] For $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, the operator $I(m,\lambda,l) f(z,\zeta)$ is defined by the following infinite series $I(m,\lambda,l) f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta) z^j$, for $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.6. *It follows from the above definition that $(l+1) I(m+1,\lambda,l) f(z,\zeta) = [l+1-\lambda] I(m,\lambda,l) f(z,\zeta) + \lambda z (I(m,\lambda,l) f(z,\zeta))'$, for $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.7. *For $l = 0$, $\lambda \geq 0$, the operator $D^m_\lambda = I(m,\lambda,0)$ was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator $S^m = I(m,1,0)$ for $\lambda = 1$.*

2 Main results

Definition 2.1. *Let $\delta \in [0,1)$, $\lambda, l \geq 0$, $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. A function $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ is said to be in the class $S_m(\delta,\lambda,l)$ if it satisfies the inequality*

$$\operatorname{Re}(I(m,\lambda,l) f(z,\zeta))' > \delta, \quad \text{for} \quad z \in U,\ \zeta \in \overline{U}. \tag{1}$$

Theorem 2.2. *The set $S_m(\delta,\lambda,l)$ is convex.*

Proof. Let the functions $f_j(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_{jk}(\zeta) z^j$, for $k = 1,2$, $z \in U, \zeta \in \overline{U}$, be in the class $S_m(\delta,\lambda,l)$. It is sufficient to show that the function $h(z,\zeta) = \eta_1 f_1(z,\zeta) + \eta_2 f_2(z,\zeta)$ is in the class $S_m(\delta,\lambda,l)$, with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

Since $h(z,\zeta) = z + \sum_{j=n+1}^{\infty} (\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)) z^j$, for $z \in U$, $\zeta \in \overline{U}$, then

$$I(m,\lambda,l) h(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m (\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)) z^j, \quad \text{for} \quad z \in U,\ \zeta \in \overline{U}. \tag{2}$$

Differentiating (2) we obtain $(I(m,\lambda,l) h(z,\zeta))' = 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m (\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)) j z^{j-1}$, for $z \in U$, $\zeta \in \overline{U}$.

Hence

$$\begin{aligned} \operatorname{Re}(I(m,\lambda,l) h(z,\zeta))' &= 1 + \operatorname{Re}\left(\eta_1 \sum_{j=n+1}^{\infty} j \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_{j1}(\zeta) z^{j-1}\right) \\ &\quad + \operatorname{Re}\left(\eta_2 \sum_{j=n+1}^{\infty} j \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_{j2}(\zeta) z^{j-1}\right). \end{aligned} \tag{3}$$

Taking into account that $f_1, f_2 \in S_m(\delta,\lambda,l)$ we deduce

$$\operatorname{Re}\left(\eta_k \sum_{j=n+1}^{\infty} j \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_{jk}(\zeta) z^{j-1}\right) > \eta_k(\delta - 1), \quad \text{for} \quad k = 1,2. \tag{4}$$

Using (4) we get from (3) $\operatorname{Re}(I(m,\lambda,l) h(z,\zeta))' > 1 + \eta_1(\delta-1) + \eta_2(\delta-1)$, for $z \in U$, $\zeta \in \overline{U}$, that is $\operatorname{Re}(I(m,\lambda,l) h(z,\zeta))' > \delta$, for $z \in U$, $\zeta \in \overline{U}$, which is equivalent that $S_m(\delta,\lambda,l)$ is convex. ■

Theorem 2.3. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + \frac{1}{c+2} z g'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, $c > 0$. If $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, $f \in S_m(\delta,\lambda,l)$ and $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta)\, dt$, $z \in U$, $\zeta \in \overline{U}$, then*

$$(I(m,\lambda,l) f(z,\zeta))' \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{5}$$

implies

$$(I(m,\lambda,l) F(z,\zeta))' \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F(z,\zeta)=(c+2)\int_0^z t^c f(t,\zeta)\,dt. \tag{6}$$

Differentiating (6), with respect to z, we have $(c+1)F(z,\zeta)+zF'(z,\zeta)=(c+2)f(z,\zeta)$ and

$$(c+1)I(m,\lambda,l)F(z,\zeta)+z\left(I(m,\lambda,l)F(z,\zeta)\right)'=(c+2)I(m,\lambda,l)f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}. \tag{7}$$

Differentiating (7) we have

$$\left(I(m,\lambda,l)F(z,\zeta)\right)'+\frac{1}{c+2}z\left(I(m,\lambda,l)F(z,\zeta)\right)''=\left(I(m,\lambda,l)f(z,\zeta)\right)',\quad z\in U,\ \zeta\in\overline{U}. \tag{8}$$

Using (8), the differential subordination (5) becomes

$$\left(I(m,\lambda,l)F(z,\zeta)\right)'+\frac{1}{c+2}z\left(I(m,\lambda,l)F(z,\zeta)\right)''\prec\prec g(z,\zeta)+\frac{1}{c+2}zg'(z,\zeta). \tag{9}$$

Denote

$$p(z,\zeta)=\left(I(m,\lambda,l)F(z,\zeta)\right)',\quad z\in U,\ \zeta\in\overline{U}. \tag{10}$$

Replacing (10) in (9) we obtain $p(z,\zeta)+\frac{1}{c+2}zp'(z,\zeta)\prec\prec g(z,\zeta)+\frac{1}{c+2}zg'(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$.

Using Lemma 1.4 we have $p(z,\zeta)\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}$, i.e. $\left(I(m,\lambda,l)F(z,\zeta)\right)'\prec\prec g(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$, and this result is sharp. ■

Theorem 2.4. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$. If $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\left(I(m,\lambda,l)f(z,\zeta)\right)'\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U} \tag{11}$$

holds, then

$$\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}$$

and this result is sharp.

Proof. Consider $p(z,\zeta)=\frac{I(m,\lambda,l)f(z,\zeta)}{z}=\frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j}{z}=1+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+...,\quad z\in U,\ \zeta\in\overline{U}$.

Let $I(m,\lambda,l)f(z,\zeta)=zp(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$. Differentiating we obtain $\left(I(m,\lambda,l)f(z,\zeta)\right)'=p(z,\zeta)+zp'(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$.

Then (11) becomes $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$.

By using Lemma 1.4, we have $p(z,\zeta)\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}$, i.e. $\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}$. ■

Theorem 2.5. *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\left(I(m,\lambda,l)f(z,\zeta)\right)'\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U} \tag{12}$$

holds, then

$$\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$ is convex and it is the best dominant.

Proof. With notation $p(z,\zeta)=\frac{I(m,\lambda,l)f(z,\zeta)}{z}=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^{j-1}$ and $p(0,\zeta)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$, $p(z,\zeta)+zp'(z,\zeta)=\left(I(m,\lambda,l)f(z,\zeta)\right)'$.

We have $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$. Since $p(z,\zeta)\in\mathcal{H}^*[1,n,\zeta]$, using Lemma 1.3, for $\gamma=1$, we obtain $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$, i.e. $\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec h(z,\zeta)$, $z\in U,\ \zeta\in\overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ■

Theorem 2.6. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}_{n\zeta}^{*}$ and the strong differential subordination*

$$\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U} \tag{13}$$

holds, then

$$\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\prec\prec g(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. For $f\in\mathcal{A}_{n\zeta}^{*}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$ we have $I(m,\lambda,l)f(z,\zeta)=z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.

Consider $p(z,\zeta)=\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}=\frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}a_j(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}a_j(\zeta)z^j}=\frac{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}a_j(\zeta)z^{j-1}}{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}a_j(\zeta)z^{j-1}}$.

We have $p'(z,\zeta)=\frac{(I(m+1,\lambda,l)f(z,\zeta))'}{I(m,\lambda,l)f(z,\zeta)}-p(z,\zeta)\cdot\frac{(I(m,\lambda,l)f(z,\zeta))'}{I(m,\lambda,l)f(z,\zeta)}$. Then $p(z,\zeta)+zp'(z,\zeta)=\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'$.

Relation (13) becomes $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ and by using Lemma 1.4 we obtain $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. ■

Theorem 2.7. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}_{n\zeta}^{*}$ and the strong differential subordination*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U} \tag{14}$$

holds, then

$$[I(m,\lambda,l)f(z,\zeta)]'\prec\prec g(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

This result is sharp.

Proof. Let

$$p(z,\zeta)=(I(m,\lambda,l)f(z,\zeta))' \tag{15}$$

$$=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m ja_j(\zeta)z^{j-1}=1+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+....$$

We obtain $p(z,\zeta)+z\cdot p'(z,\zeta)=I(m,\lambda,l)f(z,\zeta)+z(I(m,\lambda,l)f(z,\zeta))'=$
$I(m,\lambda,l)f(z,\zeta)+\frac{(l+1)I(m+1,\lambda,l)f(z,\zeta)-(l+1-\lambda)I(m,\lambda,l)f(z,\zeta)}{\lambda}=$
$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)$.

Using the notation in (15), the strong differential subordination becomes $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$.

By using Lemma 1.4, we have $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $(I(m,\lambda,l)f(z,\zeta))'\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and this result is sharp. ■

Theorem 2.8. *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}_{n\zeta}^{*}$ and the strong differential subordination*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)\prec\prec h(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U} \tag{16}$$

holds, then

$$(I(m,\lambda,l)f(z,\zeta))'\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$ is convex and it is the best dominant.

Proof. For $f \in \mathcal{A}_{n\zeta}^{*}$, $f(z,\zeta) = z+\sum_{j=n+1}^{\infty} a_j(\zeta) z^j$ we have $I(m,\lambda,l) f(z,\zeta) = z+\sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = (I(m,\lambda,l) f(z,\zeta))' = 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta) z^{j-1} \in \mathcal{H}^*[1,n,\zeta]$.

We have $p(z,\zeta) + zp'(z,\zeta) = \frac{l+1}{\lambda} I(m+1,\lambda,l) f(z,\zeta) + \left(2-\frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Then $\frac{l+1}{\lambda} I(m+1,\lambda,l) f(z,\zeta) + \left(2-\frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$ becomes $p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.3, for $\gamma = 1$, we obtain $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $(I(m,\lambda,l) f(z,\zeta))' \prec\prec g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ■

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On special strong differential subordinations using Sălăgean and Ruscheweyh operators

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Abstract

In the present paper we establish several strong differential subordinations regardind the new operator L_α^n, given by $L_\alpha^n : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$, $L_\alpha^n f(z,\zeta) = (1-\alpha)R^n f(z,\zeta) + \alpha S^n f(z,\zeta)$, where $R^n f(z,\zeta)$ denote the Ruscheweyh derivative, $S^n f(z,\zeta)$ is the Sălăgean operator and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$ is the class of normalized analytic functions with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$.

Keywords: strong differential subordination, univalent function, convex function, best dominant, differential operator.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} :\ |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} :\ |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$, with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Definition 1.1 *[2] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| \leq 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.1 *[2] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

Lemma 1.1 *[4, p. 71] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta]$ and $p(z,\zeta) + \frac{1}{\gamma} z p'(z,\zeta) \prec\prec h(z,\zeta)$, then $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, where $g(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$ is convex and it is the best dominant.*

Lemma 1.2 *[3] Let $g(z,\zeta)$ be a convex function in U, for all $\zeta \in \overline{U}$, and let $h(z,\zeta) = g(z,\zeta) + n\alpha z g'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $\alpha > 0$ and n is a positive integer. If $p(z,\zeta) = g(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots$, $z \in U$, $\zeta \in \overline{U}$, is holomorphic in U, for all $\zeta \in \overline{U}$, and $p(z,\zeta) + \alpha z p'(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $p(z,\zeta) \prec\prec g(z,\zeta)$ and this result is sharp.*

Definition 1.2 *(Sălăgean [6]) For $f \in \mathcal{A}_\zeta^*$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned} S^0 f(z,\zeta) &= f(z,\zeta) \\ S^1 f(z,\zeta) &= z f'(z,\zeta) \\ &\dots \\ S^{n+1} f(z,\zeta) &= z\left(S^n f(z,\zeta)\right)', \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 1.2 *If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $S^n f(z,\zeta) = z + \sum_{j=2}^{\infty} j^n a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Definition 1.3 *(Ruscheweyh [5]) For $f \in \mathcal{A}_\zeta^*$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta) \\ R^1 f(z,\zeta) &= z f'(z,\zeta) \\ &\dots \\ (n+1) R^{n+1} f(z,\zeta) &= z\left(R^n f(z,\zeta)\right)' + n R^n f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 1.3 *If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $R^n f(z,\zeta) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

2 Main results

Definition 2.1 *[1] Let $\alpha \geq 0$, $n \in \mathbb{N}$. Denote by L_α^n the operator given by $L_\alpha^n : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$L_\alpha^n f(z,\zeta) = (1-\alpha) R^n f(z,\zeta) + \alpha S^n f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 2.1 *L_α^n is a linear operator and if $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $L_\alpha^n f(z,\zeta) = z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^n\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 2.2 *For $\alpha = 0$, $L_0^n f(z,\zeta) = R^n f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$ and for $\alpha = 1$, $L_1^n f(z,\zeta) = S^n f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$.*

For $n = 0$, $L_\alpha^0 f(z,\zeta) = (1-\alpha) R^0 f(z,\zeta) + \alpha S^0 f(z,\zeta) = f(z,\zeta) = R^0 f(z,\zeta) = S^0 f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$ and for $n = 1$, $L_\alpha^1 f(z,\zeta) = (1-\alpha) R^1 f(z,\zeta) + \alpha S^1 f(z,\zeta) = z f'(z,\zeta) = R^1 f(z,\zeta) = S^1 f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$.

Definition 2.2 *Let $\delta \in [0,1)$, $\alpha \geq 0$ and $n \in \mathbb{N}$. A function $f(z,\zeta) \in \mathcal{A}_\zeta^*$ is said to be in the class $S_n(\delta,\alpha,\zeta)$ if it satisfies the inequality*

$$\operatorname{Re}\left(L_\alpha^n f(z,\zeta)\right)' > \delta, \qquad z \in U,\ \zeta \in \overline{U}. \tag{2.1}$$

Theorem 2.1 *The set $S_n(\delta,\alpha,\zeta)$ is convex.*

Proof. Let the functions $f_j(z,\zeta) = z + \sum_{j=2}^{\infty} a_{jk}(\zeta) z^j$, for $k = 1,2$, $z \in U$, $\zeta \in \overline{U}$ be in the class $S_n(\delta,\alpha,\zeta)$. It is sufficient to show that the function $h(z,\zeta) = \eta_1 f_1(z,\zeta) + \eta_2 f_2(z,\zeta)$ is in the class $S_n(\delta,\alpha,\zeta)$ with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

Since $h(z,\zeta) = z + \sum_{j=2}^{\infty} \left(\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)\right) z^j$, $z \in U$, $\zeta \in \overline{U}$, then

$$L_\alpha^n h(z,\zeta) = z + \sum_{j=2}^{\infty} \left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] \left(\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)\right) z^j, \quad z \in U,\ \zeta \in \overline{U}. \tag{2.2}$$

Differentiating (2.2) we obtain
$\left(L_\alpha^n h(z,\zeta)\right)' = 1 + \sum_{j=2}^{\infty} \left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] \left(\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)\right) j z^{j-1}$, $z \in U$, $\zeta \in \overline{U}$.

Hence

$$\begin{aligned} \operatorname{Re}\left(L_\alpha^n h(z,\zeta)\right)' &= 1 + \operatorname{Re}\left(\eta_1 \sum_{j=2}^{\infty} j \left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] a_{j1}(\zeta) z^{j-1}\right) \\ &\quad + \operatorname{Re}\left(\eta_2 \sum_{j=2}^{\infty} j \left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] a_{j2}(\zeta) z^{j-1}\right). \end{aligned} \tag{2.3}$$

Taking into account that $f_1, f_2 \in S_n(\delta, \alpha, \zeta)$ we deduce

$$\operatorname{Re}\left(\eta_k \sum_{j=2}^{\infty} j\left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] a_{jk}(\zeta) z^{j-1}\right) > \eta_k(\delta - 1), \quad k = 1, 2. \tag{2.4}$$

Using (2.4) we get from (2.3) $\operatorname{Re}\left(L_a^n h(z,\zeta)\right)' > 1 + \eta_1(\delta-1) + \eta_2(\delta-1) = \delta$, $z \in U$, $\zeta \in \overline{U}$, which is equivalent that $S_n(\delta, \alpha, \zeta)$ is convex. ■

Theorem 2.2 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + \frac{1}{c+2} z g'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, $c > 0$. If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in S_n(\delta, \alpha, \zeta)$ and $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta)\, dt$, $z \in U$, $\zeta \in \overline{U}$, then*

$$\left(L_\alpha^n f(z,\zeta)\right)' \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U} \tag{2.5}$$

implies

$$\left(L_\alpha^n F(z,\zeta)\right)' \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1} F(z,\zeta) = (c+2) \int_0^z t^c f(t,\zeta)\, dt. \tag{2.6}$$

Differentiating (2.6), with respect to z, we have $(c+1) F(z,\zeta) + z F'(z,\zeta) = (c+2) f(z,\zeta)$ and

$$(c+1) L_\alpha^n F(z,\zeta) + z \left(L_\alpha^n F(z,\zeta)\right)' = (c+2) L_\alpha^n f(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}. \tag{2.7}$$

Differentiating (2.7) we have

$$\left(L_\alpha^n F(z,\zeta)\right)' + \frac{1}{c+2} z \left(L_\alpha^n F(z,\zeta)\right)'' = \left(L_\alpha^n f(z,\zeta)\right)', \quad z \in U, \ \zeta \in \overline{U}. \tag{2.8}$$

Using (2.8), the differential subordination (2.5) becomes

$$\left(L_\alpha^n F(z,\zeta)\right)' + \frac{1}{c+2} z \left(L_\alpha^n F(z,\zeta)\right)'' \prec\prec g(z,\zeta) + \frac{1}{c+2} z g'(z,\zeta). \tag{2.9}$$

Denote

$$p(z,\zeta) = \left(L_\alpha^n F(z,\zeta)\right)', \quad z \in U, \ \zeta \in \overline{U}. \tag{2.10}$$

Replacing (2.10) in (2.9) we obtain $p(z,\zeta) + \frac{1}{c+2} z p'(z,\zeta) \prec\prec g(z,\zeta) + \frac{1}{c+2} z g'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Using Lemma 1.2 we have $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, *i.e.* $\left(L_\alpha^n F(z,\zeta)\right)' \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and this result is sharp. ■

Theorem 2.3 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + z g'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and the strong differential subordination*

$$\left(L_\alpha^n f(z,\zeta)\right)' \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U} \tag{2.11}$$

holds, then

$$\frac{L_\alpha^n f(z,\zeta)}{z} \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}$$

and this result is sharp.

Proof. By using the properties of operator L_α^n, we have
$L_\alpha^n f(z,\zeta) = z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^n\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = \frac{L_\alpha^n f(z,\zeta)}{z} = \frac{z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^n\right) a_j(\zeta) z^j}{z} = 1 + p_1(\zeta) z + p_2(\zeta) z^2 + ...$, $z \in U$, $\zeta \in \overline{U}$.

Let $L_\alpha^n f(z,\zeta) = z p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Differentiating we obtain $\left(L_\alpha^n f(z,\zeta)\right)' = p(z,\zeta) + z p'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Then (2.11) becomes $p(z,\zeta) + z p'(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + z g'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.2, we have $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, *i.e.* $\frac{L_\alpha^n f(z,\zeta)}{z} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. ■

Theorem 2.4 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and the strong differential subordination*

$$(L_\alpha^n f(z,\zeta))' \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U} \tag{2.12}$$

holds, then

$$\frac{L_\alpha^n f(z,\zeta)}{z} \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

where $g(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt$ is convex and it is the best dominant.

Proof. With notation $p(z,\zeta)=\frac{L_\alpha^n f(z,\zeta)}{z}=1+\sum_{j=2}^{\infty}\left(\alpha j^n+(1-\alpha)C_{n+j-1}^n\right)a_j(\zeta)z^{j-1}$ and $p(0,\zeta)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$, $p(z,\zeta)+zp'(z,\zeta)=(L_\alpha^n f(z,\zeta))'$

We have $p(z,\zeta)+zp'(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Since $p(z,\zeta) \in \mathcal{H}^*[1,1,\zeta]$, using Lemma 1.1, for $n=1$ and $\gamma=1$, we obtain $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{L_\alpha^n f(z,\zeta)}{z} \prec\prec g(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ■

Theorem 2.5 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and the strong differential subordination*

$$\left(\frac{zL_\alpha^{n+1}f(z,\zeta)}{L_\alpha^n f(z,\zeta)}\right)' \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U} \tag{2.13}$$

holds, then

$$\frac{L_\alpha^{n+1}f(z,\zeta)}{L_\alpha^n f(z,\zeta)} \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

and this result is sharp.

Proof. For $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$ we have
$L_\alpha^n f(z,\zeta)=z+\sum_{j=2}^{\infty}\left(\alpha j^n+(1-\alpha)C_{n+j-1}^n\right)a_j(\zeta)z^j$, $z \in U$, $\zeta \in \overline{U}$.
Consider $p(z,\zeta)=\frac{L_\alpha^{n+1}f(z,\zeta)}{L_\alpha^n f(z,\zeta)}=\frac{z+\sum_{j=2}^{\infty}\left(\alpha j^{n+1}+(1-\alpha)C_{n+j}^{n+1}\right)a_j(\zeta)z^j}{z+\sum_{j=2}^{\infty}\left(\alpha j^n+(1-\alpha)C_{n+j-1}^n\right)a_j(\zeta)z^j}=\frac{1+\sum_{j=2}^{\infty}\left(\alpha j^{n+1}+(1-\alpha)C_{n+j}^{n+1}\right)a_j(\zeta)z^{j-1}}{1+\sum_{j=2}^{\infty}\left(\alpha j^n+(1-\alpha)C_{n+j-1}^n\right)a_j(\zeta)z^{j-1}}$.

We have $p'(z,\zeta)=\frac{\left(L_\alpha^{n+1}f(z,\zeta)\right)'}{L_\alpha^n f(z,\zeta)}-p(z,\zeta)\cdot\frac{(L_\alpha^n f(z,\zeta))'}{L_\alpha^n f(z,\zeta)}$. Then $p(z,\zeta)+zp'(z,\zeta)=\left(\frac{zL_\alpha^{n+1}f(z,\zeta)}{L_\alpha^n f(z,\zeta)}\right)'$.

Relation (2.13) becomes $p(z,\zeta)+zp'(z,\zeta) \prec\prec h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$ and by using Lemma 1.2 we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{L_\alpha^{n+1}f(z,\zeta)}{L_\alpha^n f(z,\zeta)} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. ■

Theorem 2.6 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and the strong differential subordination*

$$\left(L_\alpha^{n+1}f(z,\zeta)\right)'+\frac{(1-\alpha)\,nz(R^n f(z,\zeta))''}{n+1} \prec\prec h(z,\zeta), \qquad z \in U, \ \zeta \in \overline{U} \tag{2.14}$$

holds, then

$$[L_\alpha^n f(z,\zeta)]' \prec\prec g(z,\zeta), \qquad z \in U, \ \zeta \in \overline{U}.$$

This result is sharp.

Proof. By using the properties of operator L_α^n, we obtain

$$L_\alpha^{n+1}f(z,\zeta)=(1-\alpha)R^{n+1}f(z,\zeta)+\alpha S^{n+1}f(z,\zeta), \qquad z \in U, \ \zeta \in \overline{U}. \tag{2.15}$$

Then (2.14) becomes $\left((1-\alpha)R^{n+1}f(z,\zeta)+\alpha S^{n+1}f(z,\zeta)\right)'+\frac{(1-\alpha)nz(R^n f(z,\zeta))''}{n+1} \prec\prec h(z,\zeta)$, with $z \in U$, $\zeta \in \overline{U}$.

After a short calculation, we obtain
$(1-\alpha)(R^n f(z,\zeta))'+\alpha(S^n f(z,\zeta))'+z\left((1-\alpha)(R^n f(z,\zeta))''+\alpha(S^n f(z,\zeta))''\right) \prec\prec h(z,\zeta)$, $z \in U, \zeta \in \overline{U}$.
Let

$$p(z,\zeta)=(1-\alpha)(R^n f(z,\zeta))'+\alpha(S^n f(z,\zeta))'=(L_\alpha^n f(z,\zeta))' \tag{2.16}$$

$$= 1 + \sum_{j=2}^{\infty} \left(\alpha j^{n+1} + (1-\alpha) j C_{n+j-1}^{n}\right) a_j(\zeta) z^{j-1} = 1 + p_1(\zeta) z + p_2(\zeta) z^2 + ...$$

Using the notation in (2.16), the strong differential subordination becomes $p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta)$.

By using Lemma 1.2, we have $p(z,\zeta) \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad i.e. \quad (L_\alpha^n f(z,\zeta))' \prec\prec g(z,\zeta),$ $z \in U,\ \zeta \in \overline{U},$ and this result is sharp. ■

Theorem 2.7 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta) = 1$. If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and the strong differential subordination*

$$[L_\alpha^{n+1} f(z,\zeta)]' + \frac{(1-\alpha) nz \left(R^n f(z,\zeta)\right)''}{n+1} \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U} \tag{2.17}$$

holds, then

$$(L_\alpha^n f(z,\zeta))' \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)\,dt$ is convex and it is the best dominant.

Proof. For $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$ we have
$L_\alpha^n f(z,\zeta) = z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^{n}\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = (L_\alpha^n f(z,\zeta))' = 1 + \sum_{j=2}^{\infty} \left(\alpha j^{n+1} + (1-\alpha) j C_{n+j-1}^{n}\right) a_j(\zeta) z^{j-1} \in \mathcal{H}^*[1,1,\zeta]$.

We have $p(z,\zeta) + zp'(z,\zeta) = [L_\alpha^{n+1} f(z,\zeta)]' + \frac{(1-\alpha) nz (R^n f(z,\zeta))''}{n+1}$, $z \in U$, $\zeta \in \overline{U}$.

Then $[L_\alpha^{n+1} f(z,\zeta)]' + \frac{(1-\alpha) nz (R^n f(z,\zeta))''}{n+1} \prec\prec h(z,\zeta)$, $z \in U, \zeta \in \overline{U}$ becomes $p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.1, for $n = 1$ and $\gamma = 1$, we obtain $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $(L_\alpha^n f(z,\zeta))' \prec\prec g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)\,dt \prec\prec h(z,\zeta)$, $z \in U, \zeta \in \overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ■

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Coupled fixed point theorem for generalized fuzzy meir–keeler contraction in fuzzy metric spaces

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Abstract: In this paper, motivated by the definition of coupled fixed point [T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393], we introduce the definition of generalized fuzzy Meir–Keeler type functions and we prove a coupled fixed point theorem under generalized fuzzy Meir–Keeler contractive condition.

Keywords: Coupled fixed point, Fuzzy Meir-Keeler Contraction, Fuzzy metric spaces.

1. Introduction

Recently, Bhaskar and Lakshmikantham [1] introduced the notion of a coupled fixed point of a mapping of two variables. More precisely, let X be a non-empty set and $F : X \times X \to X$ be a given mapping. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping F if

$$F(x, y) = x, \quad F(y, x) = y.$$

They also showed the uniqueness of a coupled fixed point and applied their theorems to the problems of the existence and uniqueness of a solution for a periodic boundary value problem.

Later, in [3], Sedghi and Shobe investigated some more coupled fixed point theorems in Fuzzy metric spaces.

In [4], Meir and Keeler generalized the well known Banach fixed point theorem as follows.

Theorem 1.1. *(Meir–Keeler* [4]*). Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \Rightarrow d(T(x), T(y)) < \epsilon. \qquad (x, y \in X)$$

Then, T admits a unique fixed point $x_0 \in X$ and for all $x \in X$, the sequence $\{T^n(x)\}$ converges to x_0.

On the other hand, B Samet [2] introduced the generalization Meir-Keeler type functions on partially ordered metric space as follows:

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Definition 1.2. *([2]). Let (X,d) be a partially ordered metric space and $F : X \times X \to X$ be a given mapping. We say that F is a generalized Meir–keeler type function if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$x \geq u, y \leq v, \epsilon \leq \frac{1}{2}[d(x,u) + d(y,v)] < \epsilon + \delta(\epsilon) \Longrightarrow d(F(x,y), F(u,v)) < \epsilon.$$

B. Samet in [2] proved some fixed point results for this type functions.

Motivated by the results of Bhaskar and Lakshmikantham [1], Sedghi and Shobe [3] and Samet [2], in this paper, we introduce the definition of generalized fuzzy Meir–Keeler type functions and we prove a coupled fixed point theorem under generalized fuzzy Meir–Keeler contractive condition.

The theory of fuzzy sets was introduced by Zadeh in 1965 [5]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy metric spaces. In the sequel, we adopt the usual terminology, notations and conventions of the theory of fuzzy metric spaces as in [6], [7]–[9].

We note that a binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a continuous $t-$norm if $([0,1], *)$ is an abelian topological monoid; i.e.,
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.
Two typical examples of continuous $t - norms$ are $a *_1 b = min\{a, b\}, a *_2 b = ab$.

Definition 1.3. *([2]).The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t–norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, for each $x, y, z \in X$ and $t, s > 0$*
$(FM-1)$ $M(x,y,t) > 0$,
$(FM-2)$ $M(x,y,t) = 1$ *if and only if* $x = y$,
$(FM-3)$ $M(x,y,t) = M(y,x,t)$,
$(FM-4)$ $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
$(FM-5)$ $M(x,y,.) : (0,\infty) \longrightarrow [0,1]$ *is continuous.*

Definition 1.4. *([2]).Let $(X, M, *)$ be a fuzzy metric space.*
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called the Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.5. *([2]). Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, y, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Definition 1.6. *([6]). Let $(X, M, *)$ be a fuzzy metric space. M is said to satisfies the n-property on $X^2 \times (0, \infty)$ if*

$$\lim_{n\to\infty}[M(x, y, k^n t)]^{(n^p)} = 1,$$

where $x, y \in X$, $k > 1$ and $p > 0$.

Example 1.7. *Let (X, d) be an ordinary metric space, $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = e^{\frac{-d(x,y)}{t}}$ for every $x, y \in X$ and $t > 0$. Then we have*

$$\lim_{n\to\infty}[M(x, y, k^n t)]^{(n^p)} = \lim_{n\to\infty}(e^{\frac{-d(x,y)}{k^n t}})^{n^p} = \lim_{n\to\infty} e^{\frac{-d(x,y)n^p}{k^n t}} = 1.$$

Therefore M is satisfies the n–property on $X^2 \times (0, \infty)$.

Lemma 1.8. *Let $(X, M, *)$ be a fuzzy metric space, $a * b \geq ab$ for all $a, b \in [0, 1]$ and let M satisfies $n - property$. Suppose $\{x_n\}$ is a sequence in X such that for all $n \in \mathbb{N}$,*

$$M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_n, t)$$

for every $0 < k < 1$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Lemma 1.9. *For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

2. Main Results

We start by introducing a definition.

Definition 2.1. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \longrightarrow X$ be given mapping. We say that F is a generalized fuzzy Meir–keeler type function if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for every $x, y, u, v \in X$,*

$$1 - k(\epsilon + \delta(\epsilon)) < M(x, u, t) * M(y, v, t) \leq 1 - \epsilon \Longrightarrow M(F(x, y), F(u, v), kt) > 1 - \epsilon \quad (1)$$

for every $0 < k < 1$.

Proposition 2.2. *Let $(X, M, *)$ be a fuzzy metric space, and let $F : X \times X \longrightarrow X$ be a given mapping. If*

$$M(F(x, y), F(u, v), kt) \geq M(x, u, t) * M(y, v, t); \forall x, y, u, v \in X, \quad 0 < k < 1 \quad (2)$$

Then F is a generalized fuzzy Meir–keeler type function.

Proof. Suppose that (2) is satisfied. For all $\epsilon > 0$, give
$\delta(\epsilon) = (\frac{1-k}{k})\epsilon > 0$, then if

$$1 - k(\epsilon + \delta(\epsilon)) < M(x, u, t) * M(y, v, t) \leq 1 - \epsilon$$

it follows from (2) that

$$M(F(x, y), F(u, v), kt) \geq M(x, u, t) * M(y, v, t) > 1 - k(\epsilon + \delta(\epsilon))$$

i.e. $M(F(x, y), F(u, v), kt) > 1 - \epsilon$. □

The following lemma will be useful in the proof of the main result.

Lemma 2.3. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \longrightarrow X$ be a given mapping. If F is a generalized fuzzy Meir–keeler type function, then*

$$M(F(x,y), F(u,v), kt) > M(x,u,t) * M(y,v,t)$$

for all $x, y, u, v \in X$ whenever $x \neq u$ or $y \neq v$.

Proof. Let $x, y, u, v \in X$ such that $x \neq u$ or $y \neq v$. then,

$$M(x,u,t) * M(y,v,t) < 1.$$

Since F is a generalized fuzzy Meir–keeler type function, for $\epsilon = 1 - [M(x,u,t) * M(y,v,t)]$, there exists a $\delta(\epsilon) > 0$ such that

$$1 - k(\epsilon + \delta(\epsilon)) < M(x_0,u_0,t) * M(y_0,v_0,t) \leq 1 - \epsilon \Longrightarrow M(F(x_0,y_0), F(u_0,v_0), kt) > 1 - \epsilon;$$

for all $x_0, y_0, u_0, v_0 \in X$. Now putting $x_0 = x$, $y_0 = y$, $u_0 = u$,$v_0 = v$; we have

$$M(F(x,y), F(u,v), kt) \geq M(x,u,t) * M(y,v,t).$$

□

Now we prove our main result.

Theorem 2.4. *Let $(X, M, *)$ be a complete fuzzy metric space. Let M has n–property and let $a * b \geq ab$ for all $a, b \in [0,1]$. Let $F : X \times X \longrightarrow X$ be a generalized fuzzy Meir–keeler type function. Then, there exists a $(x,y) \in X \times X$ such that $x = F(x,y)$ and $y = F(y,x)$.*

Proof. Let x_0, y_0 be two arbitrary points of X. since $F(X \times X) \subseteq X$, we can choose $x_1, y_1 \in X$ such that $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Again from $F(X \times X) \subseteq X$ we can choose $x_2, y_2 \in X$ such that $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Continuing this process we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n), \ y_{n+1} = F(y_n, x_n) \tag{3}$$

for all $n \geq 0$. Now, we denote

$$\delta_n(t) = M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t). \tag{4}$$

So, $\delta_n(t) \leq 1$. It follows from (3) that

$$M(x_n, x_{n+1}, t) = M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t). \tag{5}$$

Now if $x_{n+1} = x_n$, and $y_{n+1} = y_n$ then from (3), we have

$$F(x_n, y_n) = x_n, F(y_n, x_n) = y_n. \tag{6}$$

Therefore (x_n, y_n) is a couple fixed point of F. If $x_{n-1} \neq x_n$, or $y_{n-1} \neq y_n$ then from (5) and Lemma 2.3 we have

$$M(x_n, x_{n+1}, t) = M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t) > M(x_{n-1}, x_n, \frac{t}{k}) * M(y_{n-1}, y_n, \frac{t}{k}) = \delta_{n-1}(\frac{t}{k}) \tag{7}$$

similarly, from (3), we have

$$M(y_n, y_{n+1}, t) = M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t). \tag{8}$$

If $x_{n-1} \neq x_n$, or $y_{n-1} \neq y_n$ then from (8) and Lemma 2.3 we have

$$M(y_n, y_{n+1}, t) = M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t) > M(y_{n-1}, y_n, \frac{t}{k}) * M(x_{n-1}, x_n, \frac{t}{k}) = \delta_{n-1}(\frac{t}{k}). \tag{9}$$

By (7) and (9) we obtain

$$\delta_n(t) > \delta_{n-1}(\frac{t}{k}) * \delta_{n-1}(\frac{t}{k}) \geq [\delta_{n-1}(\frac{t}{k})]^2.$$

Thus, we have

$$\delta_n(t) > \ [\delta_{n-1}(\frac{t}{k})]^2 > [\delta_{n-2}(\frac{t}{k^2})]^{2^2} > \cdots > [\delta_0(\frac{t}{k^n})]^{2^n}.$$

That is,

$$M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t) \geq [M(x_0, x_1, \frac{t}{k^n}) * M(y_0, y_1, \frac{t}{k^n})]^{2^n}. \tag{10}$$

. We have

$$t(1-k^{m-n}) = t(1-k)(1+k+\cdots+k^{m-n-1}) < t$$

for every $m > n$ and $0 < k < 1$. Then, by Lemma 2.3, the triangle inequality and (10), we have

$$\begin{aligned}
M(x_n, x_m, t) * M(y_n, y_m, t) &\geq M(x_n, x_m, t(1-k)(1+k+\cdots+k^{m-n-1})) \\
&\quad * M(y_n, y_m, t(1-k)(1+k+\cdots+k^{m-n-1})) \\
&\geq M(x_n, x_{n+1}, t(1-k)) * M(y_n, y_{n+1}, t(1-k)) \\
&\quad * M(x_{n+1}, x_{n+2}, t(1-k)k) * M(y_{n+1}, y_{n+2}, t(1-k)k) \\
&\vdots \\
&\quad * M(x_{m-1}, x_m, t(1-k)k^{m-n-1}) * M(y_{m-1}, y_m, t(1-k)k^{m-n-1}) \\
&> [M(x_0, x_1, \frac{t(1-k)}{k^n}) * M(y_0, y_1, \frac{t(1-k)}{k^n})]^{2^n} \\
&\quad * [M(x_0, x_1, \frac{t(1-k)}{k^n}) * M(y_0, y_1, \frac{t(1-k)}{k^n})]^{2^{n+1}} \\
&\vdots \\
&\quad * [M(x_0, x_1, \frac{t(1-k)}{k^n}) * M(y_0, y_1, \frac{t(1-k)}{k^n})]^{2^{m-1}} \\
&\geq [M(x_0, x_1, \frac{t(1-k)}{k^n}).M(y_0, y_1, \frac{t(1-k)}{k^n})]^{(2^n+2^{n+1}+\cdots+2^{m-1})} \\
&= [M(x_0, x_1, \frac{t(1-k)}{k^n}).M(y_0, y_1, \frac{t(1-k)}{k^n})]^{(2^n)(1+2+\cdots+2^{m-n-1})} \\
&= [M(x_0, x_1, \frac{t(1-k)}{k^n})]^{(2^m-2^n)}.[M(y_0, y_1, \frac{t(1-k)}{k^n})]^{(2^m-2^n)} \\
&\geq [M(x_0, x_1, \frac{t(1-k)}{k^n})]^{2^m}.[M(y_0, y_1, \frac{t(1-k)}{k^n})]^{2^m} \\
&\geq [M(x_0, x_1, \frac{t(1-k)}{k^n})]^{n^p}.[M(y_0, y_1, \frac{t(1-k)}{k^n})]^{n^p}
\end{aligned}$$

where $p > 0$ such that $2^m \leq n^p$. Therefore,

$$\lim_{n\to\infty}(M(x_n, x_m, t) * M(y_n, y_m, t)) \geq \lim_{n\to\infty}([M(x_0, x_1, \frac{t(1-k)}{k^n})]^{n^p}.[M(y_0, y_1, \frac{t(1-k)}{k^n})]^{n^p}) = 1$$

i.e., for each $0 < \in < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for $n, m \geq n_0$,

$$M(x_n, x_m, t) * M(y_n, y_m, t) > 1 - \epsilon$$

Hence,

$$M(x_n, x_m, t) > 1 - \epsilon$$

and

$$M(y_n, y_m, t) > 1 - \epsilon.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, then there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad and \quad \lim_{n\to\infty} y_n = y. \tag{11}$$

We now show that $x = F(x, y)$ and $y = F(y, x)$. Since $x_n \longrightarrow x$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$, then from (3) and Lemma 2.3, we get

$$M(x_{n+1}, F(x, y), kt) = M(F(x_n, y_n), F(x, y), kt) > M(x_n, x, t) * M(y_n, y, t).$$

It follows that

$$\lim_{n\to\infty} M(x_{n+1}, F(x, y), kt) \geq lim_{n\to\infty}(M(x_n, x, t) * M(y_n, y, t)).$$

From Lemma 1.9, we have

$$M(x, F(x, y), kt) \geq 1.$$

Hence $x = F(x, y)$. Similarly we can show that $y = F(y, x)$. □

Theorem 2.5. *In addition to the hypotheses of theorem* 2.4, *suppose that for every* $n \in \mathbb{N}$, $x_{n-1} \neq x_n$ *or* $y_{n-1} \neq y_n$. *Then there exists a unique* $x \in X$ *such that* $x = F(x, x)$.

Proof. From theorem 2.4, there exists a $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Now,we prove that $x = y$. Suppose, to the contrary, $x \neq y$. By (3) and Lemma 2.3, we get

$$M(x_{n+1}, y_{n+1}, kt) = M(F(x_n, y_n), F(y_n, x_n), kt) > M(x_n, y_n, t) * M(y_n, x_n, t)$$

so letting $n \longrightarrow \infty$ yields

$$M(x, y, kt) \geq M(x, y, t) * M(x, y, t) \geq [M(x, y, t)]^2.$$

Thus

$$M(x, y, t) \geq [M(x, y, \frac{t}{k})]^2 \geq [M(x, y, \frac{t}{k^2})]^{2^2} \geq \cdots \geq [M(x, y, \frac{t}{k^n})]^{2^n} \longrightarrow 1,$$

as $n \longrightarrow \infty$. Thus we proved that $x = F(x, x)$. Now, we prove the uniqueness property of x. Suppose, there exist $x, z \in X$ such that $x = F(x, x)$ and $z = F(z, z)$. If $x \neq z$, from Lemma 2.3, we have

$$\begin{aligned} M(x, z, kt) &= M(F(x, x), F(z, z), kt) > M(x, z, t) * M(x, z, t) \geq M(x, z, t)^2 \\ &= M(F(x, x), F(z, z), \frac{t}{k})^2 > [M(x, z, \frac{t}{k^2}) * M(x, z, \frac{t}{k^2})]^2 \\ &\geq [M(x, z, \frac{t}{k^2})]^4 > \cdots \end{aligned}$$

i.e.,

$$M(x, z, t) > [M(x, z, \frac{t}{k})]^2 > [M(x, z, \frac{t}{k^2})]^4 > \cdots > [M(x, z, \frac{t}{k^n})]^{2^n} \longrightarrow 1$$

as $n \longrightarrow \infty$. Thus we have $M(x, z, t) = 1$ i.e. $x = z$. This makes end to the proof. □

Example 2.6. *Let* $X = [-1, 1]$,$a * b = ab$ *for all* $a, b \in [0, 1]$. *If* $M(x, y, t) = e^{\frac{-|x-y|}{t}}$ *for every* $x, y \in X$ *and* $t > 0$, *then* $(X, M, *)$ *is a complete fuzzy metric space and from example* 1.7 M *is satisfies the* $n-$*property on* $X^2 \times (0, \infty)$. *Define a map* F *on* $X \times X$ *as follows:*

$$F(x, y) = \frac{x^2}{4} + \frac{y^2}{4} - 1.$$

It is easy to see that

$$F(X \times X) = [-1, \frac{-1}{2}] \subseteq X.$$

Let $x, y, u, v \in X$. *Since* $|\frac{x}{2} + \frac{u}{2}| \leq 1$ *and* $|\frac{y}{2} + \frac{v}{2}| \leq 1$ *and by the triangle inequality, we have*

$$2|\frac{x^2}{4} + \frac{y^2}{4} - \frac{u^2}{4} - \frac{v^2}{4}| \leq 2(|\frac{x^2}{4} - \frac{u^2}{4}| + |\frac{y^2}{4} - \frac{v^2}{4}|) \leq |x - u| + |y - v|$$

i.e.

$$\begin{aligned} M(F(x, y), F(u, v), \frac{t}{2}) &= e^{\frac{-2|\frac{x^2}{4} + \frac{y^2}{4} - \frac{u^2}{4} - \frac{v^2}{4}|}{t}} \\ &\geq e^{\frac{-(|x-u|+|y-v|)}{t}} \\ &= e^{\frac{-|x-u|}{t}}.e^{\frac{-|y-v|}{t}} \\ &= M(x, u, t).M(y, v, t). \end{aligned}$$

So, from proposition 2.2, F *is a generalized fuzzy Meir–keeler type function. Therefore all conditions of Theorem* 2.5 *hold. Thus* F *has a unique fixed point in* X *indeed,* $x = 1 - \sqrt{3}$ *is a unique fixed point of* F.

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ESTIMATING THE APÉRY'S CONSTANT

CRISTINEL MORTICI

ABSTRACT. In this paper we propose new simple sequences approximating the Apéry's constant, by defining the new series $1+\sum \frac{1}{k^3+4k^7}$ which converges to $\zeta(3)$ as n^{-6}.

1. Introduction

In 1979 the French mathematician Roger Apéry first proved that the number $\zeta(3) = 1.202056903...$ (now known as the Apéry's constant) is irrational, where

$$\zeta(s) = \lim_{n\to\infty}\left(1+\frac{1}{2^s}+...+\frac{1}{n^s}\right), \quad s>1, \tag{1.1}$$

denotes the Riemann-zeta function. Other proofs were given later by Beukers [4], Zudilin [13], Sorokin [11], or Nesterenko [10]. Whether $\zeta(3)$ is a transcendental number still remains an open question.

Apéry's proof [3] was based on the fact that a number ω is irrational if there are infinitely many coprime integers p and q such that $|\omega - p/q| < cq^{-\delta}$, for some fixed $c>0$ and $\delta>1$. The sequence (1.1) converges very slowly to its limit, but faster convergences were defined. See, *e.g.,* [1], [2], or [6].

We show in this paper how can be added a convenient term to substantially improve the rate of convergence. More precisely, we introduce the new faster series

$$\zeta(3) = 1+\sum_{k=1}^{\infty}\frac{1}{k^3+4k^7}.$$

Moreover, we can state the following lower and upper bounds for $\zeta(3)$:

Theorem 1. *For every positive integer n, we have:*

$$\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2}+\frac{1}{6n^3}+\frac{1}{12n^4}} < \zeta(3)$$

$$< \sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2}+\frac{1}{6n^3}+\frac{1}{12n^4}-\frac{1}{3n^5}}.$$

Other direction to improve the convergence is to use the continued fractions approximations, as we can see from the following:

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Theorem 2. *For every integer $n \geq 7$, we have*

$$\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2+6n+9-\frac{64}{10n^2+10n+35}}}<\zeta(3)$$

$$<\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2+6n+9-\frac{64}{10n^2+10n+34}}}.$$

(the left hand side inequality holds for every $n \geq 1$).

2. The Results

In the first part of this section we define the sequence $w_n = w_n(a,b,c)$ by

$$w_n = 1+\frac{1}{2^3}+...+\frac{1}{n^3}+\frac{1}{an^2+bn+c}$$

and we search the real parameters a, b, c which provide the best approximation of the form $\zeta(3) \approx w_n$. In this sense, we use the elementary idea that such an approximation is better as $(w_n)_{n\geq 1}$ faster converges to $\zeta(3)$.

An important tool for measuring the rate of convergence is the following:

Lemma 1. *If $(x_n)_{n\geq 1}$ is convergent to x and there exists the limit*

$$\lim_{n\to\infty} n^k(x_n - x_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, then there exists the limit:

$$\lim_{n\to\infty} n^{k-1}(x_n - x) = \frac{l}{k-1}.$$

This Lemma was used in [7]-[9] to improve some convergences, or to construct some asymptotic expansions.

Using Lemma 1 for the sequence

$$x_n = 1+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3},$$

we have $\lim_{n\to\infty} n^3(x_n - x_{n+1}) = -1$ and $\lim_{n\to\infty} n^2(x_n - \zeta(3)) = -1/2$, that means $(x_n)_{n\geq 1}$ converges to $\zeta(3)$ as n^{-2}.

Evidently, the sequences $w_n = w_n(a,b,c)$ with general term

$$w_n = 1+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}+\frac{1}{an^2+bn+c}$$

converge to $\zeta(3)$, indifferently of the real values a, b, c, with $a \neq 0$. By introducing the last fraction, we expect to obtain faster convergences, for some convenient reals a, b, c. The result is given by the following

Theorem 3. *For the sequence $w_n = w_n(a,b,c)$, the following assertions hold true:*
(i) If $a \neq 2$, then the rate of convergence of the sequence $(w_n)_{n\geq 1}$ is n^{-2}, since

$$\lim_{n\to\infty} n^2(w_n - \zeta(3)) = \frac{2-a}{2a} \neq 0.$$

(ii) If $a = 2$ and $b \neq 2$, then the rate of convergence of the sequence $(w_n)_{n\geq 1}$ is n^{-3}, since

$$\lim_{n\to\infty} n^3(w_n - \zeta(3)) = \frac{3(2-b)}{4} \neq 0.$$

(iii) If $a=2$, $b=2$ and $c\neq 1$, then the rate of convergence of the sequence $(w_n)_{n\geq 1}$ is n^{-4}, since

$$\lim_{n\to\infty} n^4\left(w_n-\zeta\left(3\right)\right)=\frac{1-c}{4}\neq 0.$$

(iv) If $a=2$, $b=2$ and $c=1$, then the rate of convergence of the sequence $(w_n)_{n\geq 1}$ is n^{-6}, since

$$\lim_{n\to\infty} n^6\left(w_n-\zeta\left(3\right)\right)=-\frac{1}{24}.$$

Proof. We have

$$\begin{aligned}w_n-w_{n+1}&=-\frac{1}{\left(n+1\right)^3}+\frac{1}{an^2+bn+c}-\frac{1}{a\left(n+1\right)^2+b\left(n+1\right)+c}\\&=\frac{2-a}{an^3}+\frac{3\left(a^2-a-b\right)}{a^2n^4}-\frac{2\left(2ac-3ab-2a^2+3a^3-2b^2\right)}{a^3n^5}+\\&+\frac{5\left(2abc-a^3+2a^4-b^3-2ab^2-2a^2b+2a^2c\right)}{a^4n^6}+O\left(\frac{1}{n^7}\right).\end{aligned}$$

Now we can easily see that

$$\lim_{n\to\infty} n^3\left(w_n-w_{n+1}\right)=\frac{2-a}{a}$$

$$\lim_{n\to\infty} n^4\left(w_n-w_{n+1}\right)=\frac{3\left(a^2-a-b\right)}{a^2}$$

$$\lim_{n\to\infty} n^5\left(w_n-w_{n+1}\right)=\frac{2\left(2ac-3ab-2a^2+3a^3-2b^2\right)}{a^3}$$

$$\lim_{n\to\infty} n^6\left(w_n-w_{n+1}\right)=\frac{5\left(2abc-a^3+2a^4-b^3-2ab^2-2a^2b+2a^2c\right)}{a^4}$$

and so on and the conclusion follows by Lemma 1. In case $a=2$, $b=2$, $c=1$,

$$w_n\left(2,2,1\right)-w_{n+1}\left(2,2,1\right)=-\frac{1}{4n^7}+O\left(\frac{1}{n^8}\right).\square$$

□

Faster convergences can be further constructed. After some tryings, we discovered the sequence

$$a_n=\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2}+\frac{1}{6n^3}+\frac{1}{12n^4}-\frac{1}{3n^5}}$$

that converges to $\zeta\left(3\right)$ as n^{-10}, since

$$a_n-a_{n+1}=\frac{11}{144n^{11}}+O\left(\frac{1}{n^{12}}\right)$$

and by Lemma 1, $\lim_{n\to\infty} n^{10}a_n=\frac{11}{1440}$.

Proof of Theorem 1. As the sequences

$$a_n=\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2}+\frac{1}{6n^3}+\frac{1}{12n^4}}$$

and

$$b_n=\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2}+\frac{1}{6n^3}+\frac{1}{12n^4}-\frac{1}{3n^5}}$$

converges to $\zeta(3)$, it suffices to show that $(a_n)_{n\geq 1}$ is strictly increasing and $(b_n)_{n\geq 1}$ is strictly decreasing. But

$$a_{n+1}-a_n=\frac{P(n)}{(n+1)^3 Q(n) R(n)}>0,$$

where $P(n)=73+540n+1564n^2+2244n^3+1588n^4+432n^5$,

$$Q(n)=1+2n-2n^2+12n^4+24n^5+24n^6,$$

$$R(n)=61+310n+670n^2+768n^3+492n^4+168n^5+24n^6$$

and

$$b_{n+1}-b_n=-\frac{S(n)}{(n+1)^3 T(n) W(n)}<0,$$

where $S(n)=276+1799n+4659n^2+6328n^3+4592n^4+1496n^5+44n^6$,

$$T(n)=24n^7+24n^6+12n^5-2n^3+2n^2+n-4,$$

$$W(n)=57+371n+980n^2+1438n^3+1260n^4+660n^5+192n^6+24n^7$$

and the conclusion follows. □

Proof of Theorem 2. We prove that

$$c_n=\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2+6n+9-\frac{64}{10n^2+10n+35}}}$$

is strictly increasing and

$$d_n=\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2n^2+2n+1-\frac{1}{6n^2+6n+9-\frac{64}{10n^2+10n+34}}}$$

is strictly decreasing. In this sense, we have

$$c_{n+1}-c_n=\frac{324}{(n+1)^3 U(n) V(n)}>0 \tag{2.1}$$

and for every $n\geq 7$,

$$d_{n+1}-d_n=-\frac{4\left(3n^2+6n-166\right)}{(n+1)^3 X(n) Y(n)}<0, \tag{2.2}$$

where $U(n)=18+66n+121n^2+120n^3+85n^4+30n^5+10n^6$,

$$V(n)=450+1218n+1441n^2+960n^3+385n^4+90n^5+10n^6,$$

$$X(n)=662+1800n+2139n^2+1431n^3+576n^4+135n^5+15n^6,$$

$$Y(n)=26+96n+177n^2+177n^3+126n^4+45n^5+15n^6.$$

□

By (2.1)-(2.2), the sequences $c_{n+1}-c_n$ and $d_{n+1}-d_n$ converge to zero as n^{-15}, respective n^{-13}, thus $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ converge to $\zeta(3)$ as n^{-14}, respective n^{-12}. It is to be noticed that already the 20th term of $(c_n)_{n\geq 1}$ provides $\zeta(3)$ with accuracy up to the 19th decimal,

$$c_{20}=\frac{826\,632\,678\,291\,432\,071\,382\,647\,022\,679}{687\,681\,819\,486\,778\,474\,816\,928\,870\,400}=1.\,202\,056\,903\,159\,594\,285\,3...\ .$$

Finally, we introduce a new sequence convergent to $\zeta(3)$, having a simple form. The idea is to rewrite the sequence $(w_n)_{n\geq 1}$ in an equivalent form. Precisely, we have $w_n(2,2,1) =$

$$= 1+\sum_{k=1}^{n}\frac{1}{k^3}+\sum_{k=1}^{n}\left(\frac{1}{2k^2+2k+1}-\frac{1}{2(k-1)^2+2(k-1)+1}\right) = 1+\sum_{k=1}^{n}\left(\frac{1}{k^3}-\frac{4k}{4k^4+1}\right).$$

We get the following interesting representation

$$\zeta(3) = 1+\sum_{k=1}^{\infty}\frac{1}{(4k^4+1)k^3} = 1+\sum_{k=1}^{\infty}\frac{1}{k^3+4k^7},$$

with the rate of convergence n^{-6}.

We are convinced that our results can be improved and our original approach can be used in other problems for accelerating some convergences.

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Mixed type second-order symmetric duality for nonlinear programming problems involving η - bonvex functions *

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Abstract

In this paper, a pair of mixed type second-order symmetric duality in nonlinear programming over arbitrary cones is fomulated. The weak, strong and converse duality theorems are also established for these programs by using η-bonvexity/η-pseudobonvexity assumptions. Our results generalize these existing dual formulations which were discussed by Gulati et al [Second-order symmetric duality with cone constraints, Journal of Computational and Applied Mathematics, 220(2008)347-357].

Key words. nonlinear programming, second-order symmetric duality, cones, η-bonvexity/η-pseudobonvexity.

MR(2000)Subject Classification: 49N15,90C30

1. Introduction

A pair of primal and dual problems in mathematical programming is called symmetric if the dual of the dual is primal problem. The duality in linear programming is symmetric. However, it is not so in nonlinear programming in general. The notation of symmetric dual were first introduced by Dorn [3] for quadratic programming. It is extended to general nonlinear programs for convex\concavity functions by Dantzig [4] and Mond [5]. Later on, another pair of symmetric dual nonlinear programs under weaker convexity assumptions were presented by Mond et al. [8]. Weir et al. [9] as well as Gulati et al.[11] proved multiobjective symmetric duality results. Chandra and Kumar [12] studied Mond-Weir type symmetric duality with cone constraints. His results were extended by Khurana [20] to the case which the objective function has been optimized with respect to a closed convex cone.

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Mangasarian [6] introduced the notation of second-order duality for nonlinear programs. He has indicated a possible computational advantage of the second-order dual over the first order dual. This motivated several authors [2, 7, 10, 13, 14, 15, 16, 17, 18, 19, 21, 22] in this field. Recently, Yang et al.[19, 21] studied second-order multiobjective symmetric dual programs and established the duality results under F-convexity assumptions. Gulati et al.[22] studied Wolfe and Mond-Weir type second-order symmetric duality over arbitrary cones under η-bonvexity\η-pseudobonvexity assumptions. This formulation removes several omissions in an earlier second-order primal dual pair introduced by Devi [13]. Gulati et al.[23] consider a pair of Mond-Weir type second-order symmetric dual programs over arbitrary cones and prove duality results under invexity assumptions. Ahmad et al.[24] formulate a pair of mixed symmetric dual programs over arbitrary cones and establish duality results by using cone-invexity assumptions.

In this paper, motivated by [24], we consider a pair of mixed type second-order nonlinear symmetric dual programs over arbitrary cones and prove weak, strong, converse duality results under η-bonvexity/η-pseudobonvexity assumptions. Our results generalize the work in [22].

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space. For $N = \{1, 2, \cdots, n\}$, let $J_1 \subseteq N$, $I_1 \subseteq N$ and $J_2 = N\backslash J_1$, $I_2 = N\backslash I_1$. Let $\mid J_1 \mid$ denote the number of elements in J_1. The other symbols $\mid J_2 \mid$, $\mid I_1 \mid$ and $\mid I_2 \mid$ are defined similarly.
Let C_1, C_2, C_3 and C_4 be closed convex cone with nonempty interior in R^n, repectively. For $i = 1, 2, 3, 4$, C_i^* called the polar of C_i, is defined as follows:

$$C_i^* = \{z : x^T z \leq 0 \ for \ all \ x \in C_i\}.$$

Suppose that $S_1, S_2 \subseteq R^n$ are open sets such that $C_1 \times C_2 \subseteq S_1 \times S_2$.

Definition 2.1 *[17]A twice differentiable function $K : S_1 \times S_2 \to R$ is said to be η_1-bonvex in the first variable at $u \in S_1$, if there exists a function $\eta_1 : S_1 \times S_1 \to R^n$ such that for $v \in S_2$, $r \in R^n$, $x \in S_1$,*

$$k(x, v) - k(u, v) \geq \eta_1^T(x, u)[\nabla_x k(u, v) + \nabla_{xx} k(u, v)r] - \frac{1}{2} r^T \nabla_{xx} k(u, v)r$$

and $k(x, y)$ is said to be η_2-bonvex in the second variable at $v \in S_2$, if there exists a function $\eta_2 : S_2 \times S_2 \to R^n$ such that for $u \in S_1$, $p \in R^n$, $y \in S_2$,

$$k(u, y) - k(u, v) \geq \eta_2^T(y, v)[\nabla_y k(u, v) + \nabla_{yy} k(u, v)p] - \frac{1}{2} p^T \nabla_{yy} k(u, v)p.$$

Definition 2.2 *[17]A twice differentiable function $K : S_1 \times S_2 \to R$ is said to be η_1-pseudobonvex in the first variable at $u \in S_1$, if there exists a function $\eta_1 : S_1 \times S_1 \to R^n$ such that for $v \in S_2$, $r \in R^n$, $x \in S_1$,*

$$\eta_1^T(x, u)[\nabla_x k(u, v) + \nabla_{xx} k(u, v)r] \geq 0 \Rightarrow k(x, v) \geq k(u, v) - \frac{1}{2} r^T \nabla_{xx} k(u, v)r$$

and $k(x,y)$ is said to be η_2-bonvex in the second variable at $v \in S_2$, if there exists a function $\eta_2 : S_2 \times S_2 \to R^n$ such that for $u \in S_1,\ p \in R^n,\ y \in S_2$,

$$\eta_2^T(y,v)[\nabla_y k(u,v) + \nabla_{yy} k(u,v)p] \geq 0 \Rightarrow k(u,y) \geq k(u,v) - \frac{1}{2}p^T \nabla_{yy} k(u,v)p.$$

3. Mixed type second-order symmetric duality

We now consider the following pair of mixed type second-order symmetric dual nonlinear programming problems over arbitrary cones.
Primal(MP):

$$\begin{array}{ll} Minimize & F(x_1,x_2,y_1,y_2,p_1,p_2) = k_1(x_1,y_1) - {y_1}^T[\nabla_{y_1} k_1(x_1,y_1) + \nabla_{y_1y_1} k_1(x_1,y_1)p_1] \\ & -\frac{1}{2}{p_1}^T \nabla_{y_1y_1} k_1(x_1,y_1)p_1 + k_2(x_2,y_2) - \frac{1}{2}{p_2}^T \nabla_{y_2y_2} k_2(x_2,y_2)p_2 \\ \text{s.t.} & \nabla_{y_1} k_1(x_1,y_1) + \nabla_{y_1y_1} k_1(x_1,y_1)p_1 \in C_3^*, \\ & \nabla_{y_2} k_2(x_2,y_2) + \nabla_{y_2y_2} k_2(x_2,y_2)p_2) \in C_4^*, \\ & {y_2}^T(\nabla_{y_2} k_2(x_2,y_2) + \nabla_{y_2y_2} k_2(x_2,y_2)p_2) \geq 0, \\ & x_1 \in C_1, \quad x_2 \in C_2. \end{array}$$

Dual(MD):

$$\begin{array}{ll} Maximize & G(u_1,u_2,v_1,v_2,r_1,r_2) = k_1(u_1,v_1) - {u_1}^T[\nabla_{x_1} k_1(u_1,v_1) + \nabla_{x_1x_1} k_1(u_1,v_1)r_1] \\ & -\frac{1}{2}{r_1}^T \nabla_{x_1x_1} k_1(u_1,v_1)r_1 + k_2(u_2,v_2) - \frac{1}{2}{r_2}^T \nabla_{x_2x_2} k_2(u_2,v_2)r_2 \\ \text{s.t.} & -\nabla_{x_1} k_1(u_1,v_1) - \nabla_{x_1x_1} k_1(u_1,v_1)r_1 \in C_1^*, \\ & -\nabla_{x_2} k_2(u_2,v_2) - \nabla_{x_2x_2} k_2(u_2,v_2)r_2 \in C_2^*, \\ & {u_2}^T(\nabla_{x_2} k_2(u_2,v_2) + \nabla_{x_2x_2} k_2(u_2,v_2)r_2) \leq 0, \\ & v_1 \in C_3, \quad v_2 \in C_4. \end{array}$$

where

$k_1 : R^{|J_1|} \times R^{|I_1|} \to R$ is a twice differentiable function,

$k_2 : R^{|J_2|} \times R^{|I_2|} \to R$ is a twice differentiable function.

Remark 3..1 *If we set $J_2 = \phi, I_2 = \phi$,then (MP)and (MD)reduce to wolfe type second-order symmetric dual programs in [22] Similarly,for $J_1 = \phi$, and $I_1 = \phi$,we get the mond-weir type second-order symmetric dual programs discussed in [22]*

Theorem 3.1 *(weak duality)*
Let $(x_1,x_2,y_1,y_2,p_1,p_2)$ be feasible for the primal problem (MP) and $(u_1,u_2,v_1,v_2,r_1,r_2)$ be feasible for the dual problem (MD). Let

(i) $k_1(\cdot,v_1)$ be η_1-bonvex in the first variable at $u_1, \eta_1(x_1,u_1) + u_1 \in C_1$ for all $x_1 \in C_1$;

(ii) $-k_1(x_1,\cdot)$ be η_2-bonvex in the second variable at $y_1, \eta_2(v_1,y_1) + y_1 \in C_3$ for all $v_1 \in C_3$;

(iii)$k_2(\cdot,v_2)$ be η_3-pseudobonvex in the first variable at u_2, $\eta_3(x_2,u_2) + u_2 \in C_2$ for all $x_2 \in C_2$;

(iv) $-k_2(x_2,\cdot)$ be η_4-pseudobonvex in the second variable at $y_2, \eta_4(v_2,y_2) + y_2 \in C_3$ for all $v_2 \in C_4$;

Then

$$F(x_1,x_2,y_1,y_2,p_1,p_2) \geq G(u_1,u_2,v_1,v_2,r_1,r_2). \tag{3.1}$$

Proof. By imitating the proof of Theorem 3.1 in [22], we get

$$\begin{aligned} k_1(x_1,y_1) &- y_1^T[\nabla_{y_1}k_1(x_1,y_1)+\nabla_{y_1y_1}k_1(x_1,y_1)p_1]-\tfrac{1}{2}p_1^T\nabla_{y_1y_1}k_1(x_1,y_1)p_1 \\ &\geq k_1(u_1,v_1)-u_1^T(\nabla_{x_1}k_1(u_1,v_1)+\nabla_{x_1x_1}k_1(u_1,v_1)r_1)-\tfrac{1}{2}r_1^T\nabla_{x_1x_1}k_1(u_1,v_1)r_1. \end{aligned} \tag{3.2}$$

Similarly, by imitating the proof of Theorem 4.1 in [22], one gets

$$k_2(x_2,y_2)-\tfrac{1}{2}p_2^T\nabla_{y_2y_2}k_2(x_2,y_2)p_2 \geq k_2(u_2,v_2)-\tfrac{1}{2}r_2^T\nabla_{x_2x_2}k_2(u_2,v_2)r_2 \tag{3.3}$$

The sum of (3.2) and (3.3) contradicts (3.1).

Theorem 3.2 *(Strong Duality)*
Suppose that k_1 and k_2 be thrice differentiable and $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{p_1},\overline{p_2})$ be a local optimal solution for (MP). Let

(i) $\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})$ is nonsingular,

(ii) the vector $\overline{p_1}^T\nabla_{y_1}(\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})\overline{p_1})=0$ implies $\overline{p_1}=0$,

(iii)either $\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})$ is positive definite and $\overline{p_2}^T\nabla_{y_2}k_2(\overline{x_2},\overline{y_2})\geq 0$ or $\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})$ is negative definite and $\overline{p_2}^T\nabla_{y_2}k_2(\overline{x_2},\overline{y_2})\leq 0$,

(iv) $\nabla_{y_2}k_2(\overline{x_2},\overline{y_2})+\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})\overline{p_2}\neq 0$.

Then, $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{r_1}=0,\overline{r_2}=0)$ is feasible for (MD), and the objective function values of (MP) and (MD) are equal. Also, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of (MP) and (MD), then $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{r_1}=0,\overline{r_2}=0)$ is global optimal solution for (MP) and (MD), respectively.

Proof. Since $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{p_1},\overline{p_2})$ be a local optimal solution for (MP), by using the Fritz John type necessary optimality conditions given in [1], there exist $\lambda_1\in R_+,\ \lambda_2\in C_3,\ \lambda_3\in C_4,\ \lambda_4\in R_+$ such that the following conditions are satisfied at $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{p_1},\overline{p_2})$:

$$\begin{aligned} (x_1-\overline{x_1})^T\{\lambda_1\nabla_{x_1}k_1(\overline{x_1},\overline{y_1}) &- \nabla_{y_1x_1}k_1(\overline{x_1},\overline{y_1})(-\lambda_1\overline{y}_1+\lambda_2) \\ &+ \nabla_{x_1}(\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})\overline{p_1})(-\lambda_1\overline{y}_1+\lambda_2-\tfrac{1}{2}\lambda_1\overline{p_1})\}\geq 0, \end{aligned} \tag{3.4}$$

for all $x_1\in C_1$,

$$\begin{aligned} (x_2-\overline{x_2})^T\{\lambda_1\nabla_{x_2}k_2(\overline{x_2},\overline{y_2}) &+ \nabla_{y_2x_2}k_2(\overline{x_2},\overline{y_2})(\lambda_3-\lambda_4\overline{y_2}) \\ &+ \nabla_{x_2}(\nabla_{y_2y_2}k_2\overline{p_2})(\lambda_3-\tfrac{1}{2}\lambda_1\overline{p_2}-\lambda_4\overline{y_2})\}\geq 0, \end{aligned} \tag{3.5}$$

for all $x_2\in C_2$,

$$\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})(-\lambda_1\overline{y}_1+\lambda_2-\lambda_1\overline{p_1})+\nabla_{y_1}(\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})\overline{p_1})(-\lambda_1\overline{y}_1\lambda_2-\tfrac{1}{2}\lambda_1\overline{p_1})=0, \tag{3.6}$$

$$\begin{aligned} (\lambda_1-\lambda_4)\nabla_{y_2}k_2(\overline{x_2},\overline{y_2}) &+ \nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})(\lambda_3-\lambda_4\overline{y_2}-\lambda_4\overline{p_2}) \\ &+ \nabla_{y_2}(\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})\overline{p_2})(\lambda_3-\tfrac{1}{2}\lambda_1\overline{p_2}-\lambda_4\overline{y_2})=0, \end{aligned} \tag{3.7}$$

$$\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})(-\lambda_1\overline{y_1}+\lambda_2-\lambda_1\overline{p_1})=0, \tag{3.8}$$

$$\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})(\lambda_3-\lambda_1\overline{p_2}-\lambda_4\overline{y_2})=0, \tag{3.9}$$

$$\lambda_2^T(\nabla_{y_1}k_1(\overline{x_1},\overline{y_1})+\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})\overline{p_1})=0, \tag{3.10}$$

$$\lambda_3^T(\nabla_{y_2}k_2(\overline{x_2},\overline{y_2})+\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})\overline{p_2})=0, \tag{3.11}$$

$$\lambda_4 y_2^T(\nabla_{y_2}k_2(\overline{x_2},\overline{y_2})+\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})\overline{p_2})=0, \tag{3.12}$$

$$(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)\neq 0. \tag{3.13}$$

In view of hypothesis (i), (3.8) and (3.9) give

$$\lambda_2=\lambda_1(\overline{y_1}+\overline{p_1}),\ \lambda_3=\lambda_1\overline{p_2}+\lambda_4\overline{y_2}. \tag{3.14}$$

We claim that $\lambda_1\neq 0$. indeed, if $\lambda_1\neq 0$, then (3.14) implies $\lambda_2=0,\ \lambda_3=\lambda_4\overline{y_2}$, which together with (3.7) yields $\lambda_4(\nabla_{y_2}k_2(\overline{x_2},\overline{y_2})+\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})\overline{p_2})=0$.

By hypothesis (iv), we get $\lambda_4=0$. Hence $\lambda_3=0$, which contradicts (3.13). Thus, we obtain

$$\lambda_1>0. \tag{3.15}$$

Using (3.14) in (3.6), we get $\nabla_{y_1}(\nabla_{y_1y_1}k_1(\overline{x_1},\overline{y_1})\overline{p_1})(\frac{1}{2}\lambda_1\overline{p_1})=0$, which by hypothesis (ii) and (3.15) yield $\overline{p_1}=0$.

Subtracting (3.12) from (3.11) and using (3.14) and (3.15), we get

$$\overline{p_2}^T(\nabla_{y_2}k_2(\overline{x_2},\overline{y_2})+\nabla_{y_2y_2}k_2(\overline{x_2},\overline{y_2})\overline{p_2})=0,$$

which implies that $\overline{p_2}=0$ by hypothesis (iii).

From (3.14) and $\overline{p_1}=0$, we obtain

$$\lambda_2=\lambda_1\overline{y_1}. \tag{3.16}$$

Using (3.15), (3.16) and $\overline{p_1}=0$ in (3.4), we get

$$(x_1-\overline{x_1})^T\nabla_{x_1}k_1(\overline{x_1},\overline{y_1})(\overline{x_1},\overline{y_1})\geq 0, \tag{3.17}$$

for all $x_1\in C_1$.

Let $x_1\in C_1$, then $\overline{x_1}+x_1\in C_1$, and so (3.17) implies $x_1^T\nabla_{x_1}k_1(\overline{x_1},\overline{y_1})\geq 0$, for all $x_1\in C_1$. Therefore $-\nabla_{x_1}k_1(\overline{x_1},\overline{y_1})\in C_1^*$.

Also, from (3.14), (3.15), $\overline{p_1}=0$ and $\lambda_2\in C_3$, we obtain $\overline{y_1}\in C_3$. (3.14) and $\overline{p_2}=0$ implies

$$\lambda_3=\lambda_4\overline{y_2}. \tag{3.18}$$

This together with (3.7), (3.8), $\overline{p_2}=0$ and hypothesis (iv) yields $\lambda_4=\lambda_1>0$. Therefore, from (3.18), we get $\overline{y_2}=\frac{1}{\lambda_4}\lambda_3\in C_4$.

Further, using $\overline{p_2}=0$ and (3.18) in (3.5), we get

$$(x_2-\overline{x_2})^T\nabla_{x_2}k_2(\overline{x_2},\overline{y_2})\geq 0, \tag{3.19}$$

for all $x_2\in C_2$.

Let $x_2\in C_2$, then $x_2+\overline{x_2}\in C_2$ and (3.19) implies $x_2^T\nabla_{x_2}k_2(\overline{x_2},\overline{y_2})\geq 0$, for all $x_2\in C_1$. Therefore $-\nabla_{x_2}k_2(\overline{x_2},\overline{y_2})\in C_2^*$. $\overline{p_1}=0$, (3.10), (3.16), (3.17) and (3.19) implies $\overline{x_1}^T\nabla_{x_1}k_1(\overline{x_1},\overline{y_1})=0,\ \overline{y_1}^T\nabla_{y_1}k_1(\overline{x_1},\overline{y_1})=0$.

Hence $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{r_1}=0,\overline{r_2}=0)$ is feasible solution for the dual (MD). Moreover,

$$F(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{p_1}=0,\overline{p_2}=0)=G(\overline{u_1},\overline{u_2},\overline{v_1},\overline{v_2},\overline{r_1}=0,\overline{r_2}=0).$$

Now, by Theorem 3.1, $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{p_1}=0,\overline{p_2}=0)$ and $(\overline{x_1},\overline{x_2},\overline{y_1},\overline{y_2},\overline{r_1}=0,\overline{r_2}=0)$ is global optimal solution for (MP) and (MD), repectively.

Theorem 3.3 *(Converse Duality)*
Suppose that k_1 *and* k_2 *be thrice differentiable and* $(\overline{u_1}, \overline{u_2}, \overline{v_1}, \overline{v_2}, \overline{r_1}, \overline{r_2})$ *be a local optimal solution for (MD). Let*

(i) $\nabla_{x_1x_1} k_1(\overline{u_1}, \overline{v_1})$ *is nonsingular,*

(ii) the vector $\overline{r_1}^T \nabla_{x_1}(\nabla_{x_1x_1} k_1(\overline{u_1}, \overline{v_1})\overline{r_1}) = 0$ *implies* $\overline{r_1} = 0$,

(iii)either $\nabla_{x_2x_2} k_2(\overline{u_2}, \overline{v_2})$ *is positive definite and* $\overline{r_2}^T \nabla_{x_2} k_2(\overline{u_2}, \overline{v_2}) \geq 0$ *or* $\nabla_{x_2x_2} k_2(\overline{u_2}, \overline{v_2})$ *is negative definite and* $\overline{r_2}^T \nabla_{x_2} k_2(\overline{u_2}, \overline{v_2}) \leq 0$,

(iv) $\nabla_{x_2} k_2(\overline{u_2}, \overline{v_2}) + \nabla_{x_2x_2} k_2(\overline{u_2}, \overline{v_2})\overline{r_2} \neq 0$.

Then, $(\overline{u_1}, \overline{u_2}, \overline{v_1}, \overline{v_2}, \overline{p_1} = 0, \overline{p_2} = 0)$ *is feasible for* (MD)*, and the objective function values of (MP) and* (MD) *are equal. Also, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of* (MP) *and* (MD)*, then* $(\overline{u_1}, \overline{u_2}, \overline{v_1}, \overline{v_2}, \overline{r_1} = 0, \overline{r_2} = 0)$ *is global optimal solution for* (MP) *and* (MD)*, respectively.*

Proof. Follows on the lines of Theorem 3.2.

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Common fixed points theorems for occasionally weakly compatible pairs on cone metric type spaces

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Abstract. We prove some common fixed point theorems of mappings on cone metric-type spaces in which the cone does not need to be normal. These theorems generalize the recent results of Jungck et al. [10] and references therein. The examples are given to illustrate the usability of our results.

Keywords: Cone metric type space; Common fixed point; Concidence point; Weakly compatible; Occasionally weakly compatible.

AMS Subject Classifications [2000]: Primary 54H25, Secondary 47H10.

1 Introduction

In this paper, using the notion of occasionally weakly compatible mappings (see Definition 1.7.), we generalize some results of [10] to the setting of a cone metric type space. Our results also generalize the results of [1],[5] and [15] without the assumption of normality on the cone. We must mention here that non-convex analysis, especially ordered normed spaces, normal cones and topical functions ([4], [5], [6] [14], [15]) have some applications in optimization theory. In these cases an order is introduced by using vector space cones. Huang and Zang [5] used this approach and they have replaced the real numbers $\mathbb{R}$ by an ordered Banach space E and defined a cone metric space. The following definitions and results will be needed in the sequel. Let E be a real Banach space. A subset P of E is called a cone if and only if:

(**i**) P is closed, nonempty and $P \neq \{\theta\}$;

(**ii**) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;

(**iii**) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$ (interior of P). A cone $P \subset E$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K \|y\|$. The least positive number satisfying the above inequality is called the normal constant of P. It is clear that $K \geq 1$. From [15] we know that there exists ordered Banach space E with cone P which is not normal but with $intP \neq \emptyset$.

Definition 1.1. Let X be a non-empty set and E a real Banach space with cone P. A vector-valued function $d : X \times X \to P$ is said to be a *cone metric type function* on X with the constant $K \geq 1$ if the following conditions are satisfied:

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($\mathbf{d}_1$) $\theta \preceq d(x,y)$, for all $x, y \in X$, and $d(x,y) = \theta$ if and only if $x = y$;
($\mathbf{d}_2$) $d(x,y) = d(y,x)$ for all $x, y \in X$;
($\mathbf{d}_3$) $d(x,z) \preceq K(d(x,y) + d(y,z))$ for all $x, y, z \in X$.

The pair (X,d) is called *the cone metric type space*. Observe that if $K = 1$ then the ordinary triangle inequality in a cone metric space is satisfied, however it does not hold true when $K > 1$. Thus the class of cone metric type spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone metric type space, but the converse need not be true. The following examples illustrates the above remarks.

Example 1.2 Let $X = \{-1, 0, 1\}$, $E = \mathbb{R}^2$, $P = \{(x,y) : x \geq 0, y \geq 0\}$. Define $d : X \times X \to P$ by $d(x,y) = d(y,x)$ for all $x, y \in X$, $d(x,x) = \theta$, $x \in X$ and $d(-1,0) = (3,3)$, $d(-1,1) = d(0,1) = (1,1)$. Then (X,d) is a complete cone metric-type space but the triangle inequality is not satisfied. Indeed, we have that $d(-1,1) + d(1,0) = (1,1) + (1,1) = (2,2) \preceq (3,3) = d(-1,0)$. It is not hard to verify that $K = \frac{3}{2}$.

For the given cone metric type space (X,d) one can construct a metric type space $\left(X, D, K^2\right)$ in sense of [12], [13] and [16] where "symmetric" $D : X \times X \to \mathbb{R}$ is given by $D(x,y) = \|d(x,y)\|$.

Example 1.3 Let X be as in example 1 from [12], $E = C^1_{\mathbb{R}}[0,1]$, $P = \{\varphi \in E : \varphi \geq 0\}$. Define $d : X \times X \to P$ by $d(f,g)(t) := D(f,g) \cdot \varphi(t)$, for each $f, g \in X$, where φ is a fixed function from P, for instance $\varphi(t) = e^t$. Since (X, D, K) is a metric type space (see [12], page 6) so (X,d) is a cone metric type space with $K = 2$.

Definition 1.4. Let (X,d) be a cone metric type space. We say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(ii) convergent sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed $x \in X$. A cone metric-type space X is said to be complete if every Cauchy sequence in X is convergent in X.

If the cone P is normal we have the following:

Proposition 1.5. The sequence $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. It is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.

Remark 1.6. Let (X,d) be a cone metric type space over ordered real Banach space E with a cone P. Then the following properties are often used:

(**1**) If $a \preceq b$ and $b \ll c$, then $a \ll c$.
(**2**) If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.
(**3**) If $c \in intP$, $a_n \in E$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.
(**4**) If E is a real Banach space with cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

For the sake of convenience we mention the following definition.

Definition 1.7. [1], [9] Let f and g be self mappings of a set X and $C(f,g) = \{x \in X : fx = gx\}$. The pair (f,g) is called:

(**i**) weakly compatible [1] if $fgx = gfx$ for all $x \in C(f,g)$, and
(**ii**) occasionally weakly compatible (in brief OWC) [9] if $fgx = gfx$ for some $x \in C(f,g)$.

The following example shows that the weakly compatible maps form a proper subclass of the occasionally weakly compatible maps.

Example 1.8. [18] Let $X = [0, \infty)$ with usual metric. Define $f, g : X \to X$ by $fx = 2x$ and $gx = x^2$ for all $x \in X$. Then $C(f,g) = \{0, 2\}$, $fg0 = gf0$, and $fg2 \neq gf2$. Thus (f,g) is occasionally weakly compatible (in brief OWC) pair but not weakly compatible.

Definition 1.9. [1] Let f and g be self mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 1.10. [9] Let f and g be occasionally weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is a the unique common fixed point of f and g.

2 Main results

In this section we shall prove some fixed point theorems of contractive mappings in the setting of a cone metric type space (in brief KMTS). We generalize some results of [10] and the references therein.

Theorem 2.1. *Let (X, d) be a KMTS with constant $K \geq 1$ and P a cone with a nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X, and that for some constant $\lambda \in (0, \frac{1}{K})$ and for every $x, y \in X$, there exists $u \equiv u_{x,y} \in \left\{Kd(gx, gy), Kd(gx, fx), Kd(gy, fy), \frac{d(gy,fx)+d(gx,fy)}{2}\right\}$, such that*

$$d(fx, fy) \preceq \frac{\lambda}{K}u. \tag{2.1}$$

Then f and g have a unique point of coincidence in X. Moreover if f, g are OWC, then f, g have a unique common fixed point.

Proof. Choose x_0, $x_1 \in X$ such that $gx_1 = fx_0 = y_0$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $gx_{n+1} = fx_n = y_n$. We first show that

$$d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n) \text{ for } n \geq 1. \tag{2.2}$$

Putting $x = x_n$ and $y = x_{n+1}$ in (2.1) we get

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{K}u \tag{2.3}$$

where $u \in \left\{Kd(gx_n, gx_{n+1}), Kd(gx_n, fx_n), Kd(gx_{n+1}, fx_{n+1}), \frac{d(gx_{n+1},fx_n)+d(gx_n,fx_{n+1})}{2}\right\}$ i.e., $u \in \left\{Kd(y_{n-1}, y_n), Kd(y_n, y_{n+1}), \frac{d(y_{n-1},y_{n+1})}{2}\right\}$.

Now we have to consider the following three cases. If $u = Kd(y_{n-1}, y_n)$ in (2.3) then it, clearly, yields (2.2). If $u = Kd(y_n, y_{n+1})$ then by Remark 1.6 (4) with $a = u$ we get $d(y_n, y_{n+1}) = 0$, and (2.2) is immediate. Finally, if $u = \frac{d(y_{n-1},y_{n+1})}{2}$ then we have $d(y_n, y_{n+1}) \preceq \frac{\lambda}{K}\frac{d(y_{n-1},y_{n+1})}{2} \prec \frac{\lambda d(y_{n-1},y_n)}{2} + \frac{d(y_n,y_{n+1})}{2}$ (as $\lambda < 1$).

Hence $d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n)$. So (2.2) follows and yields

$$d(y_n, y_{n+1}) \preceq \lambda^n d(y_0, y_1). \tag{2.4}$$

We now show that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence. Let $m > n$. Then we have

$$\begin{aligned} d(y_n, y_m) &\preceq Kd(y_n, y_{n+1}) + K^2 d(y_{n+1}, y_{n+2}) + ... + K^{m-n} d(y_{m-1}, y_m) \\ &\preceq (K\lambda^n + K^2\lambda^{n+1} + ... + K^{m-n}\lambda^{m-1}) d(y_0, y_1) \\ &\preceq K\lambda^n(1 + K\lambda + K^2\lambda^2 + ...) d(y_0, y_1) \\ &\preceq \frac{K\lambda^n}{1 - K\lambda} d(y_0, y_1) \to \theta, \text{ as } n \to \infty, \end{aligned}$$

and from Remark 1.6 (3) it follows that for $\theta \ll c$ and sufficiently large n, $K\lambda^n(1-K\lambda)^{-1}d(y_0,y_1) \ll c$. Thus, by Remark 1.6 (1), $d(y_n,y_m) \ll c$. Hence, by Definition 1.4 (i), $\{y_n\}$ is a Cauchy sequence. Since $fX \subseteq gX$ and fX or gX is complete, there exists $a \in gX$ such that $y_n = fx_n = gx_{n+1} \to a$ as $n \to \infty$. Consequently, there exists $b \in X$ such that $gb = a$. To show that $fb = a$ we have

$$d(a,fb) \preceq K\left(d(a,fx_n) + d(fx_n,fb)\right) \preceq Kd(a,fx_n) + K\frac{\lambda}{K}\ u_n = Kd(a,fx_n) + \lambda\ u_n$$

where $u_n \in \{Kd(gx_n,gb), Kd(gx_n,fx_n), Kd(gb,fb), \frac{d(gb,fx_n)+d(gx_n,fb)}{2}\}$. Let $\theta \ll c$. Then at least one of the following four cases holds:

1. $d(a,fb) \preceq Kd(a,fx_n) + \lambda K d(gx_n,gb) \ll \lambda K\ \frac{c}{2\lambda K} + K\ \frac{c}{2K} = c$.

2.

$$\begin{aligned} d(a,fb) &\preceq Kd(a,fx_n) + \lambda\ Kd(gx_n,fx_n) \\ &\preceq Kd(a,fx_n) + \lambda K^2\left(d(gx_n,a) + d(a,fx_n)\right) \\ &\preceq K(1+\lambda K)d(a,fx_n) + \lambda K^2 d(gx_n,a) \\ &\ll K(1+\lambda K)\frac{c}{2K(1+\lambda K)} + \lambda K^2\frac{c}{2\lambda K^2} = c. \end{aligned}$$

3. $d(a,fb) \preceq Kd(a,fx_n) + \lambda K d(gb,fb) = Kd(a,fx_n) + \lambda K d(a,fb)$ or

$$d(a,fb) \preceq \frac{K}{1-\lambda K}d(a,fx_n) \ll \frac{K}{1-\lambda K}\frac{1-\lambda K}{K}c = c.$$

4.

$$\begin{aligned} d(a,fb) &\preceq Kd(a,fx_n) + \lambda\frac{d(gb,fx_n)+d(gx_n,fb)}{2} \\ &= \left(K+\frac{\lambda}{2}\right)d(a,fx_n) + \frac{\lambda}{2}d(gx_n,fb) \\ &\preceq \frac{2K+\lambda}{2}d(a,fx_n) + \frac{\lambda K}{2}\left(d(gx_n,a) + d(a,fb)\right) \end{aligned}$$

that is.,

$$\begin{aligned} d(a,fb) &\preceq \frac{2K+\lambda}{2-\lambda K}d(a,fx_n) + \frac{\lambda K}{2-\lambda K}d(gx_n,a) \\ &\ll \frac{2K+\lambda}{2-\lambda K}\frac{2-\lambda K}{2(2K+\lambda)}c + \frac{\lambda K}{2-\lambda K}\frac{2-\lambda K}{2\lambda K}c = c. \end{aligned}$$

Thus in all the cases, we obtain $d(a,fb) \ll c$ for each $c \in intP$. Using Remark 1.6 (2) it follows that $fb = a$. So f and g have a coincidence point $b \in X$, and point of coincidence $a \in X$, i.e., $a = fb = gb$. If q is another point of coincidence, then there is $p \in X$ such that $q = fp = gp$. Now

$$d(a,q) = d(fb,fp) \preceq \frac{\lambda}{K}u \tag{2.5}$$

where $u \in \{Kd(gb,gp), Kd(gb,fb), Kd(gp,fp,), \frac{d(gp,fb)+d(gb,fp)}{2}\} = \{Kd(a,q), d(q,a)\}$. If $u = Kd(a,q)$ or $a = d\,(a,q)$ then (2.5) gives us $d(a,q) = \theta$, i.e., $a = q$. Since $a = fb = gb$ is the unique point of coincidence of f and g, and f and g are OWC, a is the unique common fixed point of f and g by Lemma 1.10. □

In Theorem 2.1 if $K = 1$ then since (X, d) is a cone metric space, we get Theorem 2.1 of [10].

Corollary 2.2. ([10], Theorem 2.1) Let (X, d) be a cone metric space, P a cone having a nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X, and that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists $u \equiv u_{x,y} \in \left\{ d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gy,fx)+d(gx,fy)}{2} \right\}$ such that $d(fx, fy) \preceq \lambda u$. Then f and g have a unique point of coincidence in X. Moreover if f, g are weakly compatible, then f, g have a unique common fixed point.

Following example illustrates Theorem 2.1. From this example it follows that our Theorem 2.1 is a proper extension of results from [10].

Example 2.3 Let $X = \mathbb{R}$ and $E = C([0, 1], \mathbb{R})$ with the positive cone $P = \{\varphi \in E : \varphi \geq 0\}$. Define $d : X \times X \to P$ by $d(x, y)(t) := (x - y)^2 \varphi(t)$, where $\varphi : [0, 1] \to \mathbb{R}$ is defined by $\varphi(t) = e^t$. If $x = 3$, $y = 2$, and $z = 1$, then the usual cone metric triangle inequality fails to hold while the cone metric type triangle inequality does hold. Define $f, g : X \to X$ by $fx = \left(\frac{\lambda}{2K}\right)^{\frac{1}{2}} x$ and $gx = \left[\left(\frac{\lambda}{2K}\right)^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}\right] x$, for $x \in X$ where $\lambda \in (0, \frac{1}{K})$ and $K \geq 1$. Observe that $fX \subseteq gX$ and both fX, gX are complete and for $x, y \in X$, we have

$$d(fx, fy)(t) = \frac{\lambda}{2K} d(x, y)(t) \text{ and } d(gx, gy)(t) = \left(\frac{\lambda}{2K} + \frac{2}{\sqrt{2K}} + \frac{1}{\lambda}\right) d(x, y)(t).$$

According to careful computation we obtain that for each $x, y \in X : u = Kd(gx, gy)$ satisfies the contraction condition (2.1) and so by Theorem 2.1 f and g have a coincidence point 0 and unique point of coincidence which is also 0, and since f, g commute at 0, it is a unique common fixed point of f and g.

In the next theorem we generalize Theorem 2.2 of Jungck et al. [10] and results therein. The proof is similar to the previous one, and so is omitted.

Theorem 2.4. *Let (X, d) be a KMTS with the constant $K \geq 1$ and P* a cone *with a non-empty interior. Suppose that the mappings $f, g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X*, and that for some constant $\lambda \in (0, \frac{1}{K})$ *and for every $x, y \in X$, there exists $u \equiv u_{x,y} \in \left\{ Kd(gx, gy), \frac{d(gx,fx)+d(gy,fy)}{2}, \frac{d(gy,fx)+d(gx,fy)}{2} \right\}$, such that $d(fx, fy) \preceq \frac{\lambda}{K} u$. Then f and g have a unique point of coincidence in X. Moreover if f and g are OWC, f and g have a unique common fixed point.*

Corollary 2.5. *Let (X, d) be a KMTS with the constant $K \geq 1$ and P a cone having a nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X, and that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, we have $d(fx, fy) \preceq \lambda\, d(gx, gy)$. Then f and g have a unique point of coincidence in X. Moreover if f and g are OWC, f and g have a unique common fixed point.*

Corollary 2.6. *Let (X, d) be a KMTS with the constant $K \geq 1$ and P a cone having a nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X, and for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, we have $d(fx, fy) \preceq \lambda \frac{d(gx,fx)+d(gy,fy)}{2}$. Then f and g have a unique point of coincidence in X. Moreover if f and g are OWC, f and g have a unique common fixed point.*

Corollary 2.7. *Let (X, d) be a KMTS with the constant $K \geq 1$ and P a cone having a nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X, and for some constant $\lambda \in (0, \frac{1}{K})$ and for every $x, y \in X$, we have $d(fx, fy) \preceq \frac{\lambda}{K} \frac{d(gy,fx)+d(gx,fy)}{2}$. Then f and g have a unique point of coincidence in X. Moreover if f and g OWC, f and g have a unique common fixed point.*

Corollary 2.8. *Let (X,d) be a KMTS with the constant $K \geq 1$ and P a cone having a nonempty interior. Suppose that the mappings $f,g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X, and that for some constant $\lambda \in (0, \frac{1}{K})$ and for every $x,y \in X$, there exists $u = u(x,y) \in \{Kd(gx,gy), Kd(gx,fx), Kd(gy,fy)\}$ such that $d(fx,fy) \preceq \frac{\lambda}{K}\, u$. Then f and g have a unique point of coincidence in X. Moreover if f and g are OWC, f and g have a unique common fixed point.*

Corollary 2.9. *Let (X,d) be a KMTS with the constant $K \geq 1$ and P a cone having nonempty interior. Suppose that the mappings $f,g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X, and that for some constant $\lambda \in (0, \frac{1}{K})$ and for every $x,y \in X$, there exists $u = u(x,y) \in \{Kd(gx,fx), Kd(gy,fy)\}$ such that $d(fx,fy) \preceq \frac{\lambda}{K}\, u$. Then f and g have a unique point of coincidence in X. Moreover if f and g are OWC, f and g have a unique common fixed point.*

Finally, we prove the theorem of Hardy-Rogers type in the frame of cone metric type spaces.

Theorem 2.10. *Let (X,d) be a KMTS with the constant $K \geq 1$ and P* a cone *having a nonempty interior. Suppose that the mappings $f,g : X \to X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X and that there exists nonnegative constants $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ with $2Ka_1 + (K+1)(a_2+a_3) + \left(K^2+K\right)(a_4+a_5) < 2$ such that, for each $x,y \in X$,*

$$d(fx,fy) \preceq a_1 d(gx,gy) + a_2 d(gx,fx) + a_3 d(gy,fy) + a_4 d(gx,fy) + a_5 d(gy,fx). \tag{2.6}$$

Then f and g have a unique point of coincidence in X. Moreover if f and g are OWC, f and g have a unique common fixed point.

Proof. Define the sequences $\{y_n\}_{n\geq 1}$ and $\{x_n\}_{n\geq 1}$ as in the proof of Theorem 2.1. We show that

$$d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n) \tag{2.7}$$

for $\lambda \in [0, \frac{1}{K})$ and $n \geq 1$. From (2.6) with $x = x_n$, $y = x_{n+1}$, we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\preceq a_1 d(gx_n, gx_{n+1}) + a_2 d(gx_n, fx_n) + a_3 d(gx_{n+1}, fx_{n+1}) \\ &\quad + a_4 d(gx_n, fx_{n+1}) + a_5 d(gx_{n+1}, fx_n) \\ &= a_1 d(y_{n-1}, y_n) + a_2 d(y_{n-1}, y_n) + a_3 d(y_n, y_{n+1}) + a_4 d(y_{n-1}, y_{n+1}) \\ &\preceq (a_1+a_2) d(y_{n-1}, y_n) + a_3 d(y_n, y_{n+1}) + a_4 K d(y_{n-1}, y_n) + a_4 K d(y_n, y_{n+1}), \end{aligned}$$

this implies that

$$d(y_n, y_{n+1}) \preceq (a_1 + a_2 + a_4 K) d(y_{n-1}, y_n) + (a_3 + a_4 K) d(y_n, y_{n+1}),$$

and because of symmetry in (2.6),

$$d(y_{n+1}, y_n) \preceq (a_1 + a_3 + a_5 K) d(y_n, y_{n-1}) + (a_2 + a_5 K) d(y_{n+1}, y_n).$$

Adding we have

$$d(y_n, y_{n+1}) \preceq \frac{2a_1 + a_2 + a_3 + (a_4 + a_5) K}{2 - (a_2 + a_3 + (a_4 + a_5) K)} d(y_{n-1}, y_n) = \lambda d(y_{n-1}, y_n),$$

where $\lambda = \frac{2a_1+a_2+a_3+(a_4+a_5)K}{2-(a_2+a_3+(a_4+a_5)K)} < \frac{1}{K}$, and so (2.7) follows.

Now, from the proof of Theorem 2.1, we know that $\{gx_{n+1}\} = \{fx_n\} = \{y_n\}$ is a Cauchy sequence. Since $fX \subseteq gX$ and fX or gX is complete, there exists $a \in gX$ such that $y_n = fx_n = gx_{n+1} \to a$ as $n \to \infty$. Consequently, there exists $b \in X$ such that $gb = a$. To show that $fb = a$ we have

$$\begin{aligned} d(a,fb) &\preceq K\left(d(a,fx_n)+d(fx_n,fb)\right) \\ &\preceq K\left(d(a,fx_n)+a_1d(gx_n,a)+Ka_2d(gx_n,a)+Ka_2d\left(a,fx_n\right)\right. \\ &\quad \left.+a_3d(a,fb)+a_4d(gx_n,fb)+a_5d(a,fx_n)\right) \\ &= (K+a_2K^2+a_5K)d(a,fx_n)+(a_1K+a_2K^2+a_4K)d(gx_n,a)+a_3Kd(a,fb), \end{aligned}$$

Similarly,

$$\begin{aligned} d(a,fb) &\preceq K\left(d(a,fx_n)+d(fx_n,fb)\right) = K\left(d(a,fx_n)+d(fb,fx_n)\right) \\ &\preceq (K+a_3K^2+a_4K)d(a,fx_n)+(a_1K+a_3K^2+a_5K)d(gx_n,a)+a_2Kd(a,fb). \end{aligned}$$

Adding up, one obtains that,

$$d(a,fb) \preceq \frac{2K+\left(a_2+a_3\right)K^2+\left(a_5+a_4\right)K}{2-K\left(a_2+a_3\right)}d(a,fx_n)+\frac{2a_1K+\left(a_2+a_3\right)K^2+\left(a_4+a_5\right)K}{2-K\left(a_2+a_3\right)}d(gx_n,a).$$

Let $\theta \ll c$. Then for sufficiently large n we have $d(a,fb) \ll \lambda_1\frac{c}{2\lambda_1}+\lambda_2\frac{c}{2\lambda_2} = c$, where $\lambda_1 = \frac{2K+(a_2+a_3)K^2+(a_5+a_4)K}{2-K(a_2+a_3)}$ and $\lambda_2 = \frac{2a_1K+(a_2+a_3)K^2+(a_4+a_5)K}{2-K(a_2+a_3)}$.

Therefore by Remark 1.6 (2) $d(a,fb) = 0$; i.e., $fb = a$. Consequently, f and g have a coincidence point $b \in X$ and a point of coincidence $a \in X$ such that $a = fb = gb$. If q is another point of coincidence, then there is $p \in X$ with $q = fp = gp$. Now from (2.6) with $x = b$, $y = p$, we have

$$\begin{aligned} d(a,q) &= d(fb,fp) \preceq a_1d(gb,gp)+a_2\cdot\theta+a_3\cdot\theta+a_4d(gb,fp)+a_5d(gp,fb) \\ &= (a_1+a_4+a_5)d(a,q). \end{aligned}$$

Since $a_1+a_4+a_5 < 1$, so by Remark 1.6 (4) we get $d(a,q) = 0$, that is, $a = q$. By Lemma 1.10, since f, g are OWC and $a = fb = gb$ is the unique point of coincidence of f and g, we obtain that a is the unique common fixed point of f and g. □

Observe that for the special choices of a_i in Theorem 2.10, all the results from Corollaries 2.5, 2.6 2.7, 2.8 and 2.9 could be obtained.

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A NEW HYBRID APPROXIMATION ALGORITHM BASED ON THE SHRINKING PROJECTION METHOD FOR TWO ASYMPTOTICALLY QUASI-ϕ-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce a new hybrid projection algorithm based on the shrinking projection method for approximating a common element of in the fixed point sets of two asymptotically quasi-ϕ-nonexpansive mappings and the solutions set of a variational inequality corresponding to an inverse-strongly monotone operator. The strong convergence theorem is established in real Banach spaces. Our results improve and extend the corresponding results announced by recent results.

Keywords: Asymptotically quasi-ϕ-nonexpansive mappings, Banach space, Generalzied projection, Shrinking projection method, Variational inequality.

AMS Subject Classification: 47H09, 47H10

1. Introduction

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be an operator. The classical variational inequality problem [19] for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \forall y \in C, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. The solution set of (1.1) is denoted by $VI(A, C)$. Such a problem is connected with convex minimization problem, complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$. First, we recall that a mapping $A : C \to E^*$ is said to be:

(i) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$, for all $x, y \in C$.

(ii) *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \qquad \text{for all } x, y \in C.$$

Let J be the *normalized duality mapping* from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}.$$

It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Some properties of the duality mapping are given in [10, 32, 36].

Recall that a mappings $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

A mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \text{ for all } x \in C, y \in F(T).$$

A mapping T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \text{ for all } x, y \in C.$$

A mapping T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - y\| \leq k_n \|x - y\|, \text{ for all } x \in C, y \in F(T).$$

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A mapping T is called *uniformly L-Lipschitzian continuous* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } x, y \in C.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] in 1972. Since 1972, many authors have studied the weak and strong convergence of iterative processes for such a class of mappings.

Let C be a nonempty closed convex subset of a Hilbert space H and a mapping $P_C : H \to C$ is called the *metric projection* of H onto C if for each $x \in H$, there exists a unique element $Px \in C$ such that $\|x - Px\| = d(x, C)$.

If C is a nonempty closed convex subset of a Hilbert space H and let $P_C : H \to C$ be the *metric projection* of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator $\prod_C$ in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{1.2}$$

for all $x, y \in E$. Observe that, in a Hilbert space H, (1.2) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The *generalized projection* $\Pi_C : E \to C$ is the mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution of the following minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{1.3}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and the strict monotonicity of the mapping J (see, for example, [1, 2, 9, 28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$.
(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$.
(3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$.
(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \text{ if and only if } x = y.$$

For more details see, for example, [10, 32]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the fixed point set of T. A point p in C is said to be an *asymptotic fixed point* of T [30] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T will be denoted by $\hat{F}(T)$. Recall the following definitions:

(i) A mapping $T : C \to C$ is called *relatively nonexpansive* [8, 9, 11] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.
(ii) A mapping $T : C \to C$ is said to be *relatively asymptotically nonexpansive* [1, 26] if $\hat{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leqslant k_n \phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(iii) A mapping $T : C \to C$ is said to be *ϕ-nonexpansive* [25, 28, 38] if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$.
(iv) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* [25, 28, 38] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.
(v) A mapping $T : C \to C$ is said to be *asymptotically ϕ-nonexpansive* [38] if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x, y)$ for all $x, y \in C$.
(vi) A mapping $T : C \to C$ is said to be *asymptotically quasi-ϕ-nonexpansive* [38] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(vii) A mapping $T : C \to C$ is said to be *asymptotically regular* on C if, for any bounded subset D of C, the following equality holds:

$$\lim_{n\to\infty} \sup_{x \in D} \|T^{n+1} x - T^n x\| = 0.$$

(viii) A mapping $T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Remark 1.1. The class of (asymptotically) quasi-ϕ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings which requires the strong restriction that $\hat{F}(T) = F(T)$.

Remark 1.2. In a real Hilbert spaces, the class of (asymptotically) quasi-ϕ-nonexpansive mappings is coincides with the class of (asymptotically) quasi-nonexpansive mappings.

We give some examples which are closed and asymptotically quasi-ϕ-nonexpansive.

Example 1.3. (1). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}J$ is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

(2). Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then Π_C is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (see [4]). More precisely, let $t \in (0, 1)$ and define a contraction $G_t : C \to C$ by $G_t x = tx_0 + (1-t)Tx$ for all $x \in C$, where $x_0 \in C$ is a fixed point in C. Applying Banach's Contraction Principle, there exists a unique fixed point x_t of G_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$ even if T has a fixed point. However, in the case of T having a fixed point, Browder [4] proved that the net $\{x_t\}$ defined by $x_t = tx_0 + (1-t)Tx_t$ for all $t \in (0, 1)$ converges strongly to an element of $F(T)$ which is nearest to x_0 in a real Hilbert space. Motivated by Browder [4], Halpern [15] proposed the following iteration process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad n \geqslant 0 \tag{1.4}$$

and proved the following theorem.

Theorem H. *Let C be a bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping on C. Define a real sequence $\{\alpha_n\}$ in $[0, 1]$ by $\alpha_n = n^{-\theta}, 0 < \theta < 1$. Define a sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges strongly to an element of $F(T)$ which is nearest to u.*

Recently, Martinez-Yanes and Xu [21] has adapted Nakajo and Takahashi's [23] idea to modify the process (1.4) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ y_n = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leqslant \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.5}$$

They proved that if $\{\alpha_n\} \subset (0, 1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}x$.

In [27] (see also [22]), Qin and Su improved the result of Martinez-Yanes and Xu [21] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

Theorem QS. *Let E be a uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E and let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C \text{chosen arbitrary}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, y_n) + (1 - \alpha_n)\phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.6}$$

If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

In [24], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for two relatively nonexpansive mappings:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary }, \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ y_n = J^{-1}(\delta_n Jx_0 + (1-\delta_n)Jz_n), \\ H_n = \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and T, S are relatively nonexpansive mappings. They proved, under appropriate conditions on the parameters, that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a common fixed point of T and S.

Very recently, motivated by the above research work, Qin, Cho and Kang [25] introduced and considered the shrinking projection method which was first introduced by Takahashi et al. [35] in a real Hilbert space for two asymptotically quasi-ϕ-nonexpansive mappings in uniformly smooth and uniformly convex Banach spaces. To be more precise, they proved the following results.

Theorem QCK. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let T be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^T\} \subset [0,1)$ such that $k_n^T \to 1$ as $n \to \infty$ and let S be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^S\} \subset [0,1)$ such that $k_n^S \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in E, \text{ chosen arbitrary}, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n), \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)Jz_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \phi(u, x_n) + (k_n - 1)\theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{1.8}$$

where $k_n = \max\{k_n^T, k_n^S\}$ for all $n \geq 1$, and $\theta_n = \sup\{\phi(z, x_n) : z \in \Omega\}$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are sequences in $[0,1]$ which satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;
(C2) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\lim_{n\to\infty} \alpha_n = 0$;
(C3) $0 \leq \delta_n < 1$ *and* $\limsup_{n\to\infty} \delta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

The problem of finding a solution to variational inequalities for monotone mappings in Hilbert spaces and Banach spaces has been intensively studied by many authors; see, for instance, [5, 16, 20] and the references therein. Iiduka and Takahashi [17] introduced the following algorithm for finding a solution of the variational inequality for an α-inverse-strongly monotone mapping A with $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$ in a 2-uniformly convex and uniformly smooth Banach space E. For an initial point $x_0 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \forall n \geq 0. \tag{1.9}$$

Assume that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$ where $1/c$ is the 2-uniformly convexity constant of E. They proved that if J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} \Pi_{VI(A,C)}(x_n)$.

In this paper, motivated and inspired by the above research works, we introduce a new hybrid projection algorithm based on the shrinking projection method for two of asymptotically quasi-ϕ-nonexpansive mappings to have strong convergence theorems for approximating a common element in the fixed point sets of these two mappings and the solution set of the variational inequality for an inverse-strongly monotone operator in Banach spaces. Our results improve and extend the corresponding results announced by recent results.

2. Preliminaries

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [13, 29, 32, 33]. A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [13, 32, 33] for more details.

We define the function $\delta : [0, 2] \to [0, 1]$ which is called the modulus of convexity of E as following

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in C, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}. \tag{2.1}$$

Then E is said to be 2-uniformly convex if there exists a constant $c > 0$ such that constant $\delta(\varepsilon) > c\varepsilon^2$ for all $\varepsilon \in (0, 2]$. Constant $\frac{1}{c}$ is called the 2-uniformly convexity constant of E. A 2-uniformly convex Banach space is uniformly convex, see [7, 34] for more details. We know the following lemma of 2-uniformly convex Banach spaces:

Lemma 2.1. [3, 6] *Let E be a 2-uniformly convex Banach, then for all x, y from any bounded set of E and $jx \in Jx, jy \in Jy$,*

$$\langle x - y, jx - jy\rangle \geq \frac{c^2}{2}\|x - y\|^2 \tag{2.2}$$

where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.

Now we present some definitions and lemmas which will be applied in the proof of the main result in the next section.

Lemma 2.2 (Kamimura and Takahashi [18])**.** *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.3 (Alber [2])**.** *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for any $y \in C$.*

Lemma 2.4 (Alber [2])**.** *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$$

for all $y \in C$.

Let E be a reflexive strictly convex, smooth and uniformly Banach space and the duality mapping J from E to E^*. Then J^{-1} is also single-valued, one to one, surjective, and it is the duality mapping from E^* to E. We need the following mapping V which studied in Alber [2],

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x\|^2 \tag{2.3}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma:

Lemma 2.5 (Kamimura and Takahashi [18])**.** *Let E be a reflexive, strictly convex and smooth Banach space, and let V be as in (2.3). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 ([12, Lemma 1.4]). *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|), \tag{2.4}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping F of $[0,1)$ into E^* defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \forall y \in C\}.$$

Lemma 2.7. [31] *Let C be a nonempty closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $\Psi \subset E \times E^*$ be an operator defined as follows:*

$$\Psi v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C \ . \end{cases}$$

Then Ψ is maximal monotone and $\Psi^{-1}0 = VI(A, C)$.

3. Main Results

In this section, we prove strong convergence theorem which is our main result.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let T be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^T\} \subset [0,1)$ such that $k_n^T \to 1$ as $n \to \infty$ and S be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^S\} \subset [0,1)$ such that $k_n^S \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S) \cap VI(A, C)$ is nonempty and bounded. Let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in E, \text{ chosen arbitrary}, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n), \\ y_n = J^{-1}(\delta_n Jx_1 + (1-\delta_n)Jz_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)\xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\xi_n = \alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) + (k_n - 1)\theta_n$, $k_n = \max\{k_n^T, k_n^S\}$ for all $n \geq 1$, and $\theta_n = \sup\{\phi(z, x_n) : z \in \Omega\}$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

- (C1) $\lim_{n\to\infty} \delta_n = 0$;
- (C2) $\{r_n\} \subset [a, b]$ *for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;*
- (C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*
 - (a) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ *and*
 - (b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.

Proof. We divide the proof of Theorem 3.1 into four steps.
(I). We show first that the sequence $\{x_n\}$ is well defined. It is easily to seen that $VI(A, C)$ is closed and convex. By the same argument as in the proof of [38, Lemma 2.4], one can show that $F(T) \cap F(S)$ is closed and convex. Hence $\Omega := F(S) \cap F(T) \cap VI(A, C)$ is a nonempty, closed and convex subset of C. Consequently, Π_Ω is well defined.
Next, we prove by induction that C_n is closed and convex for all $n \geq 1$. It is obvious that $C_1 = C$ is

closed and convex. Suppose that C_j is closed and convex for some $j \in \mathbb{N}$. For any $u \in C_j$, we observe that

$$\phi(u, y_j) \leq \delta_j \phi(u, x_1) + (1-\delta_j)[\alpha_j \phi(u, x_{j-1}) + (1-\alpha_j)\phi(u, x_j) + (k_j - 1)\theta_j]$$

is equivalent to

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u, Jx_j\rangle + 2\delta_j\langle u, Jx_1\rangle - 2\langle u, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

It can be seen that C_{j+1} is closed. Next, we prove that C_{j+1} is convex. In fact, $u_1, u_2 \in C_{j+1}$, $\varrho \in (0,1)$. Put $u^* = \varrho u_1 + (1-\varrho)u_2$. From the definition of C_{j+1}, we have

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u_1, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u_1, Jx_j\rangle + 2\delta_j\langle u_1, Jx_1\rangle - 2\langle u_1, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j \end{aligned}$$

and

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u_2, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u_2, Jx_j\rangle + 2\delta_j\langle u_2, Jx_1\rangle - 2\langle u_2, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

Therefor we have

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle \varrho u_1, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle \varrho u_1, Jx_j\rangle + 2\delta_j\langle \varrho u_1, Jx_1\rangle - 2\langle \varrho u_1, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j \end{aligned}$$

and

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle (1-\varrho)u_2, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle (1-\varrho)u_2, Jx_j\rangle \\ & +2\delta_j\langle (1-\varrho)u_2, Jx_1\rangle - 2\langle (1-\varrho)u_2, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

Combining the last two inequalities, we obtain

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u^*, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u^*, Jx_j\rangle + 2\delta_j\langle u^*, Jx_1\rangle - 2\langle u^*, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

The convexity of C_j implies that $u^* \in C_j$. Therefor, we have $u^* \in C_{j+1}$. Then we conclude that C_{j+1} is closed and convex. Hence for each $n \geq 1$, C_n is closed and convex.

(II). Next, we show that $\Omega \subset C_n$ for all $n \geq 1$. In fact, $\Omega \subset C_1 = C$ is obvious. Suppose $\Omega \subset C_n$ for some $n \in \mathbb{N}$. Then, for all $q \in \Omega \subset C_n$, we know from Lemma 2.5 that

$$\begin{aligned} \phi(q, w_n) &= \phi(q, \Pi_C J^{-1}(Jx_n - r_n Ax_n)) \\ &\leq \phi(q, J^{-1}(Jx_n - r_n Ax_n)) \\ &= V(q, Jx_n - r_n Ax_n) \\ &\leq V(q, (Jx_n - r_n Ax_n) + r_n Ax_n) - 2\langle J^{-1}(Jx_n - r_n Ax_n) - q, r_n Ax_n\rangle \\ &= V(q, Jx_n) - 2r_n\langle J^{-1}(Jx_n - r_n Ax_n) - q, Ax_n\rangle \\ &= \phi(q, x_n) - 2r_n\langle x_n - q, Ax_n\rangle + 2\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n\rangle. \end{aligned} \tag{3.2}$$

Since $q \in VI(A, C)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned} -2r_n\langle x_n - q, Ax_n\rangle &= -2r_n\langle x_n - q, Ax_n - Aq\rangle - 2r_n\langle x_n - q, Aq\rangle \\ &\leq -2\alpha r_n\|Ax_n - Aq\|^2. \end{aligned} \tag{3.3}$$

Therefore, from Lemma 2.1 and the assumption that $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$, we obtain that

$$\begin{aligned} 2\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - r_n Ax_n) - J^{-1}(Jx_n), -r_n Ax_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - r_n Ax_n) - J^{-1}(Jx_n)\|\|r_n Ax_n\| \\ &\leq \frac{4}{c^2}\|JJ^{-1}(Jx_n - r_n Ax_n) - JJ^{-1}(Jx_n)\|\|r_n Ax_n\| \\ &= \frac{4}{c^2}\|(Jx_n - r_n Ax_n) - Jx_n\|\|r_n Ax_n\| \\ &= \frac{4}{c^2} r_n^2\|Ax_n\|^2 \end{aligned}$$

$$\leq \frac{4}{c^2} r_n^2 \|Ax_n - Aq\|^2. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) and using the condition that $r_n < c^2\alpha/2$, we get

$$\phi(q, w_n) \leq \phi(q, x_n) + 2r_n \left(\frac{2}{c^2} r_n - \alpha \right) \|Ax_n - Aq\|^2 \leq \phi(q, x_n). \tag{3.5}$$

Using (3.5) and Lemma 2.6, for each $q \in \Omega \subset C_n$, we obtain

$$\begin{aligned}
\phi(q, z_n) &= \phi(q, J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n)) \\
&= \|q\|^2 - 2\alpha_n \langle q, Jx_{n-1}\rangle - 2\beta_n \langle q, JT^n x_n\rangle - 2\gamma_n \langle q, JS^n w_n\rangle \\
&\quad + \|\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_n \langle q, Jx_{n-1}\rangle - 2\beta_n \langle q, JT^n x_n\rangle - 2\gamma_n \langle q, JS^n w_n\rangle \\
&\quad + \alpha_n \|Jx_{n-1}\|^2 + \beta_n \|JT^n x_n\|^2 + \gamma_n \|JS^n w_n\|^2 \\
&= \alpha_n \phi(q, x_{n-1}) + \beta_n \phi(q, T^n x_n) + \gamma_n \phi(q, S^n w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n k_n^T \phi(q, x_n) + \gamma_n k_n^S \phi(q, w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n k_n \phi(q, x_n) + \gamma_n k_n \phi(q, w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n k_n \phi(q, x_n) + \gamma_n k_n \phi(q, x_n) \\
&= \alpha_n \phi(q, x_{n-1}) + (1 - \alpha_n) k_n \phi(q, x_n) \\
&= \alpha_n \phi(q, x_{n-1}) + (1 - \alpha_n) k_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, x_n) - (1 - \alpha_n)\phi(q, x_n) \\
&= [\alpha_n \phi(q, x_{n-1}) + (1 - \alpha_n)\phi(q, x_n)] + (1 - \alpha_n)(k_n - 1)\phi(q, x_n) \\
&= [\alpha_n \phi(q, x_{n-1}) + (1 - \alpha_n)\phi(q, x_n)] + (k_n - 1)\phi(q, x_n).
\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}
\phi(q, y_n) &= \phi(q, J^{-1}(\delta_n Jx_1 + (1 - \delta_n) Jz_n)) \\
&= \|q\|^2 - 2\delta_n \langle q, Jx_1\rangle - 2(1 - \delta_n)\langle q, Jz_n\rangle + \|\delta_n Jx_1 + (1 - \delta_n) Jz_n)\|^2 \\
&\leq \|q\|^2 - 2\delta_n \langle q, Jx_1\rangle - 2(1 - \delta_n)\langle q, Jz_n\rangle + \delta_n \|x_1\|^2 + (1 - \delta_n)\|z_n\|^2 \\
&= \delta_n \phi(q, x_1) + (1 - \delta_n)\phi(q, z_n) \\
&\leq \delta_n \phi(q, x_1) + (1 - \delta_n)([\alpha_n \phi(q, x_{n-1}) + (1 - \alpha_n)\phi(q, x_n)] + (k_n - 1)\theta_n) \\
&= \delta_n \phi(q, x_1) + (1 - \delta_n)\xi_n.
\end{aligned} \tag{3.7}$$

So, $q \in C_{n+1}$. Then by induction, $\Omega \subset C_n$ for all $n \geq 1$ and hence the sequence $\{x_n\}$ generated by (3.60) is well defined.

(III) Next, we show that $\{x_n\}$ is a convergent sequence in C. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - u, Jx_0 - Jx_n\rangle \geq 0, \quad \forall u \in C_n. \tag{3.8}$$

It follows from $\Omega \subset C_n$ for all $n \geq 1$ that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in \Omega. \tag{3.9}$$

From Lemma 2.4, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, x_n) \leq \phi(u, x_0),$$

for each $u \in \Omega \subset C_n$ and for all $n \geq 1$. Therefore the sequence $\{\phi(x_n, x_0)\}$ is bounded. Furthermore, since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \text{for all } n \geq 1.$$

This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing and hence $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Similarly, by Lemma 2.4, we have, for any positive integer m, that

$$\begin{aligned}
\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \\
&\leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+m}, x_0) - \phi(x_n, x_0), \text{ for all } n \geq 1.
\end{aligned} \tag{3.10}$$

The existence of $\lim_{n\to\infty} \phi(x_n, x_0)$ implies that $\phi(x_{n+m}, x_n) \to 0$ as $n \to \infty$. From Lemma 2.2, we have

$$\|x_{n+m} - x_n\| \to 0, \text{ as } n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

(IV). Now, we will show that $p \in \Omega$. Indeed, taking $m = 1$ in (3.10), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.11}$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

This implies that

$$\|x_{n+1} - x_{n-1}\| \le \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty.$$

Then

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_{n-1}\| = 0. \tag{3.13}$$

It follows from the last two inequalities that

$$\phi(x_{n+1}, x_{n-1}) \le \|x_{n+1}\|\|Jx_{n+1} - Jx_{n-1}\| + \|x_{n-1} - x_{n+1}\|\|x_{n-1}\| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

On the other hand, since $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, y_n) \le \delta_n \phi(x_{n+1}, x_n) + (1 - \delta_n)[\alpha_n \phi(x_{n+1}, x_{n-1}) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + (k_n - 1)\theta_n].$$

It follows from the condition (3.11), (3.14) and $\lim_{n\to\infty} k_n = 1$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.15}$$

Combining (3.12) and (3.15), we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.17}$$

On the other hand, noticing

$$\|Jy_n - Jz_n\| = \delta_n \|Jx_1 - Jz_n\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.19}$$

Using (3.12), (3.15) and (3.19) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.20}$$

Taking the constant $r = \sup_{n\ge1}\{\|x_{n+1}\|, \|T^n x_n\|, \|S^n w_n\|\}$, we have, from Lemma 2.6, that there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ satisfying the inequality (2.4) and $g(0) = 0$.

Case I. Assume that (a) holds. Applying (2.4) and (3.5), we can calculate

$$\begin{aligned}
\phi(u, z_n) &= \phi(u, J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n)) \\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT^n x_n\rangle - 2\gamma_n\langle u, JS^n w_n\rangle \\
&\quad + \|\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n\|^2 \\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT^n x_n\rangle - 2\gamma_n\langle u, JS^n w_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT^n x_n\|^2 + \gamma_n\|JS^n w_n\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT^n x_n\|) \\
&\le \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, T^n x_n) + \gamma_n\phi(u, S^n w_n) \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT^n x_n\|) \\
&\le \alpha_n\phi(u, x_{n-1}) + \beta_n k_n\phi(u, x_n) + \gamma_n k_n\phi(u, w_n) \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT^n x_n\|)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\phi(u,x_{n-1})+\beta_n k_n\phi(u,x_n)+\gamma_n k_n\phi(u,x_n)\\
&\quad +2r_n\gamma_n\left(\frac{2}{c^2}r_n-\alpha\right)\|Ax_n-Au\|^2-\alpha_n\beta_n g(\|Jx_{n-1}-JT^nx_n\|)\\
&\leq \alpha_n\phi(u,x_{n-1})+(1-\alpha_n)k_n\phi(u,x_n)+2r_n\gamma_n\left(\frac{2}{c^2}r_n-\alpha\right)\|Ax_n-Au\|^2\\
&\quad -\alpha_n\beta_n g(\|Jx_{n-1}-JT^nx_n\|). \qquad (3.21)
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha_n\beta_n g(\|Jx_{n-1}-JT^nx_n\|) &\leq \alpha_n[\phi(u,x_{n-1})-k_n\phi(u,x_n)]+k_n\phi(u,x_n)-\phi(u,z_n)\\
&\leq \alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\phi(u,x_n)+\phi(u,x_n)-\phi(u,z_n)\\
&\leq \alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n). \qquad (3.22)
\end{aligned}$$

We observe that

$$\begin{aligned}
&\alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n)\\
\leq\ &\alpha_n[\|x_{n-1}\|^2-\|x_n\|^2-2\langle u,Jx_{n-1}-Jx_n\rangle]\\
&+\|x_n\|^2-\|z_n\|^2-2\langle u,Jx_n-Jz_n\rangle+(k_n-1)\theta_n\\
\leq\ &\alpha_n[\|x_{n-1}-x_n\|(\|x_{n-1}\|+\|x_n\|)+2\|u\|\|Jx_{n-1}-Jx_n\|]\\
&+\|x_n-z_n\|(\|x_n\|+\|z_n\|)+2\|u\|\|Jx_n-Jz_n\|+(k_n-1)\theta_n.
\end{aligned}$$

It follows from (3.12), (3.17), (3.18) and (3.20) that

$$\lim_{n\to\infty}\alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n)=0. \qquad (3.23)$$

Applying $\liminf_{n\to\infty}\alpha_n\beta_n>0$ and (3.23) to (3.22), we get

$$\lim_{n\to\infty}g(\|Jx_{n-1}-JT^nx_n\|)=0.$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|Jx_{n-1}-JT^nx_n\|=0. \qquad (3.24)$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|x_{n-1}-T^nx_n\|=\lim_{n\to\infty}\|J^{-1}(Jx_{n-1})-J^{-1}(JT^nx_n)\|=0. \qquad (3.25)$$

From (3.12) and (3.25) , we have

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0. \qquad (3.26)$$

Note that

$$\|T^nx_n-p\|\leq\|T^nx_n-x_n\|+\|x_n-p\|.$$

From (3.26), we obtain

$$\lim_{n\to\infty}\|T^nx_n-p\|=0. \qquad (3.27)$$

On the other hand, we have

$$\|T^{n+1}x_n-p\|\leq\|T^{n+1}x_n-T^nx_n\|+\|T^nx_n-p\|\leq\sup_{x\in\{x_n\}}\|T^{n+1}x-T^nx\|+\|T^nx_n-p\|.$$

The uniformly asymptotically regularity of T and the last inequality imply that

$$\lim_{n\to\infty}\|T^{n+1}x_n-p\|=0. \qquad (3.28)$$

That is $TT^nx_n\to p$ as $n\to\infty$. From the closedness of T, we see that $p\in F(T)$. In the same manner, we can apply the condition $\liminf_{n\to\infty}\alpha_n\gamma_n>0$ to conclude that

$$\lim_{n\to\infty}\|x_n-S^nw_n\|=0. \qquad (3.29)$$

Again, by (C2) and (3.21), we have

$$2\gamma_n\left(\alpha-\frac{2}{c^2}b\right)\|Ax_n-Au\|^2 \leq \frac{1}{a}[\alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]$$

$$+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n)]. \tag{3.30}$$

It follows from (3.23) and $\liminf_{n\to\infty}\gamma_n \geq \liminf_{n\to\infty}\beta_n\gamma_n > 0$ that

$$\liminf_{n\to\infty}\|Ax_n - Au\| \leq 0.$$

Since $\liminf_{n\to\infty}\|Ax_n - Au\| \geq 0$, we have

$$\lim_{n\to\infty}\|Ax_n - Au\| = 0. \tag{3.31}$$

From Lemma 2.4, Lemma 2.5, and (3.4), we have

$$\begin{aligned}
\phi(x_n,w_n) &= \phi(x_n,\Pi_C J^{-1}(Jx_n - r_nAx_n)) \\
&\leq \phi(x_n, J^{-1}(Jx_n - r_nAx_n)) \\
&= V(x_n, Jx_n - r_nAx_n) \\
&\leq V(x_n,(Jx_n - r_nAx_n) + r_nAx_n) \\
&\quad -2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, r_nAx_n\rangle \\
&= \phi(x_n,x_n) + 2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n\rangle \\
&= 2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n\rangle \\
&\leq \frac{4}{c^2}b^2\|Ax_n - Au\|^2.
\end{aligned} \tag{3.32}$$

It follows from (3.31) that

$$\lim_{n\to\infty}\phi(x_n,w_n) = 0. \tag{3.33}$$

Lemma 2.2 implies that

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.34}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|Jx_n - Jw_n\| = 0. \tag{3.35}$$

Combining (3.29) and (3.34), we also obtain

$$\lim_{n\to\infty}\|w_n - S^n w_n\| = 0. \tag{3.36}$$

Moreover

$$\|w_n - w_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|.$$

By (3.34) and (3.12), we have

$$\lim_{n\to\infty}\|w_n - w_{n+1}\| = 0. \tag{3.37}$$

Note that

$$\|S^n w_n - p\| \leq \|S^n w_n - w_n\| + \|w_n - x_n\| + \|x_n - p\|.$$

From (3.26), we obtain

$$\lim_{n\to\infty}\|S^n w_n - p\| = 0. \tag{3.38}$$

On the other hand, we have

$$\begin{aligned}
\|S^{n+1}w_n - p\| &\leq \|S^{n+1}w_n - S^n w_n\| + \|S^n w_n - p\| \\
&\leq \sup_{w\in\{w_n\}}\|S^{n+1}w - S^n w\| + \|S^n w_n - p\|.
\end{aligned}$$

The uniformly asymptotically regularity of S and the last inequality imply that

$$\lim_{n\to\infty}\|S^{n+1}w_n - p\| = 0. \tag{3.39}$$

That is $SS^n w_n \to p$ as $n\to\infty$. From the closedness of S, we see that $p \in F(S)$.

Case II. Assume that (b) holds. Using the inequalities (2.4) and (3.5), we obtain

$$\begin{aligned}
\phi(u,z_n) &= \phi(u, J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n)) \\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT^n x_n\rangle - 2\gamma_n\langle u, JS^n w_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\|\alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n\|^2\\
\leq\ & \|u\|^2-2\alpha_n\langle u, Jx_{n-1}\rangle-2\beta_n\langle u, JT^n x_n\rangle-2\gamma_n\langle u, JS^n w_n\rangle\\
&+\alpha_n\|Jx_{n-1}\|^2+\beta_n\|JT^n x_n\|^2+\gamma_n\|JS^n w_n\|^2\\
&-\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|)\\
\leq\ & \alpha_n\phi(u,x_{n-1})+\beta_n\phi(u,T^n x_n)+\gamma_n\phi(u,S^n w_n)\\
&-\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|)\\
\leq\ & \alpha_n\phi(u,x_{n-1})+\beta_n k_n\phi(u,x_n)+\gamma_n k_n\phi(u,w_n)\\
&-\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|)\\
\leq\ & \alpha_n\phi(u,x_{n-1})+\beta_n k_n\phi(u,x_n)+\gamma_n k_n\phi(u,x_n)\\
&+2r_n\gamma_n(\frac{2}{c^2}r_n-\alpha)\|Ax_n-Au\|^2-\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|)\\
\leq\ & \alpha_n\phi(u,x_{n-1})+(1-\alpha_n)k_n\phi(u,x_n)+2r_n\gamma_n(\frac{2}{c^2}r_n-\alpha)\|Ax_n-Au\|^2\\
&-\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|). \qquad (3.40)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|) &\leq \alpha_n[\phi(u,x_{n-1})-k_n\phi(u,x_n)]+k_n\phi(u,x_n)-\phi(u,z_n)\\
&\leq \alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\phi(u,x_n)+\phi(u,x_n)-\phi(u,z_n)\\
&\leq \alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n). \qquad (3.41)
\end{aligned}
$$

It follows from the condition $\liminf_{n\to\infty}\beta_n\gamma_n>0$ and (3.23) that

$$\lim_{n\to\infty} g(\|JT^n x_n-JS^n w_n\|)=0.$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|JT^n x_n-JS^n w_n\|=0. \qquad (3.42)$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|T^n x_n-S^n w_n\|=\lim_{n\to\infty}\|J^{-1}(JT^n x_n)-J^{-1}(JS^n w_n)\|=0. \qquad (3.43)$$

On the other hand, we can calculate

$$
\begin{aligned}
\phi(T^n x_n,z_n) &= \phi(T^n x_n, J^{-1}(\alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n))\\
&= \|T^n x_n\|^2-2\langle T^n x_n,\alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n)\rangle\\
&\quad+\|\alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n)\|^2\\
&\leq \|T^n x_n\|^2-2\alpha_n\langle T^n x_n, Jx_n\rangle-2\beta_n\langle T^n x_n, JT^n x_n\rangle-2\gamma_n\langle T^n x_n, JS^n w_n\rangle\\
&\quad+\alpha_n\|x_n\|^2+\beta_n\|T^n x_n\|^2+\gamma_n\|S^n w_n\|^2\\
&\leq \alpha_n\phi(T^n x_n,x_n)+\gamma_n\phi(T^n x_n,S^n w_n). \qquad (3.44)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\phi(T^n x_n,S^n w_n) &= \|T^n x_n\|^2-2\langle T^n x_n, JS^n w_n\rangle+\|S^n w_n\|^2\\
&= \|T^n x_n\|^2-2\langle T^n x_n, JT^n x_n\rangle+2\langle T^n x_n, JT^n x_n-JS^n w_n\rangle+\|S^n w_n\|^2\\
&\leq \|S^n w_n\|^2-\|T^n x_n\|^2+2\|T^n x_n\|\|JT^n x_n-JS^n w_n\|\\
&\leq \|S^n w_n-T^n x_n\|(\|S^n w_n\|+\|T^n x_n\|)+2\|T^n x_n\|\|JT^n x_n-JS^n w_n\|.
\end{aligned}
$$

It follows from (3.42) and (3.43) that

$$\lim_{n\to\infty}\phi(T^n x_n,S^n w_n)=0. \qquad (3.45)$$

Applying $\lim_{n\to\infty}\alpha_n=0$ and (3.45) and the fact that $\{\phi(T^n x_n,x_n)\}$ is bounded to (3.44), we obtain

$$\lim_{n\to\infty}\phi(T^n x_n,z_n)=0. \qquad (3.46)$$

From Lemma 2.2, one obtains

$$\lim_{n\to\infty}\|T^n x_n-z_n\|=0. \qquad (3.47)$$

We observe that

$$\|T^n x_n - x_n\| \leq \|T^n x_n - z_n\| + \|z_n - x_n\|.$$

This together with (3.20) and (3.47), we obtain

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.48}$$

From the uniformly asymptotically regularity of T, we obtain that

$$\lim_{n\to\infty} \|T^{n+1} x_n - p\| = 0. \tag{3.49}$$

That is $TT^n x_n \to p$ as $n \to \infty$. From the closedness of T, we see that $p \in F(T)$. By the same proof as in Case I, we obtain that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.50}$$

Hence $w_n \to p$ as $n \to \infty$ for each $i \in I$ and

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0. \tag{3.51}$$

Combining (3.43), (3.48) and (3.50), we also have

$$\lim_{n\to\infty} \|S^n w_n - w_n\| = 0. \tag{3.52}$$

From the uniformly asymptotically regularity of S, we obtain that

$$\lim_{n\to\infty} \|S^{n+1} w_n - p\| = 0. \tag{3.53}$$

That is $SS^n w_n \to p$ as $n \to \infty$. From the closedness of S, we see that $p \in F(S)$.

We next show that $p \in VI(C, A)$.

Let $\Psi \subset E \times E^*$ be an operator defined by :

$$\Psi v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.54}$$

By Lemma 2.7, Ψ is maximal monotone and $\Psi^{-1}0 = VI(A, C)$. Let $(v, w) \in G(\Psi)$, since $w \in \Psi v = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $x_n = \Pi_{C_n} x \in C_n \subset C$, we get

$$\langle v - x_n, w - Av\rangle \quad \geq \quad 0. \tag{3.55}$$

Since A is α-inverse-strong monotone, we have

$$\begin{aligned} \langle v - x_n, w\rangle &\geq \langle v - x_n, Av\rangle \\ &= \langle v - x_n, Av - Ax_n\rangle + \langle v - x_n, Ax_n\rangle \\ &\geq \langle v - x_n, Ax_n\rangle. \end{aligned} \tag{3.56}$$

On other hand, from $w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n)$ and Lemma 2.3, we have $\langle v - w_n, Jw_n - (Jx_n - r_n Ax_n\rangle \geq 0$, and hence

$$\langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\rangle \leq 0. \tag{3.57}$$

Because A is $\frac{1}{\alpha}$ constricted, it holds from (3.56) and (3.57) that

$$\begin{aligned} \langle v - x_n, w\rangle &\geq \langle v - x_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\rangle \\ &= \langle v - w_n, Ax_n\rangle + \langle w_n - x_n, Ax_n\rangle - \langle v - w_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\rangle \\ &= \langle w_n - x_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\rangle \\ &\geq -\|w_n - x_n\|.\|Ax_n\| - \|v - w_n\|.\frac{\|Jx_n - Jw_n\|}{a}, \end{aligned} \tag{3.58}$$

$\forall n \in \mathbb{N} \cup \{0\}$. By taking the limit as $n \to \infty$ in (3.58) and from (3.34) and (3.35), we have $\langle v - p, w\rangle \geq 0$ as $n \to \infty$. By the maximality of Ψ we obtain $p \in \Psi^{-1}0$ and hence $p \in VI(A, C)$. Hence we conclude that

$$p \in \Omega := F(T) \cap F(S) \cap VI(A, C).$$

Finally, we show that $p = \Pi_\Omega x_0$. Indeed, taking the limit as $n \to \infty$ in (3.9), we obtain

$$\langle p - z, Jx_0 - Jp\rangle \geq 0, \quad \forall z \in \Omega \tag{3.59}$$

and hence $p = \Pi_\Omega x_0$ by Lemma 2.3. This complete the proof.

■

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let T be a uniformly L-Lipschitzian continuous and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^T\} \subset [0,1)$ such that $k_n^T \to 1$ as $n \to \infty$ and S be a uniformly L-Lipschitzian continuous and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^S\} \subset [0,1)$ such that $k_n^S \to 1$ as $n \to \infty$. Assume that $\Omega := F(T) \cap F(S) \cap VI(A,C)$ is nonempty and bounded. Let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$. Let $\{x_n\}$ be a sequence generated by (3.60). Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions (C1), (C2) and (C3) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.*

Proof. Following the proof lines of Theorem 3.1, we can obtain the following facts :
(i) Ω is closed and convex;
(ii) $\Omega \subset C_n$ for all $n \geq 1$;
(iii) $\{x_n\}$ is a Cauchy sequence and then $\{x_n\}$ and $\{w_n\}$ are convergent sequences in C;
(iv) $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0 = \lim_{n\to\infty} \|w_n - w_{n+1}\|$;
(v) $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0 = \lim_{n\to\infty} \|w_n - S^n w_n\|$.
It is sufficient to show that $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0 = \lim_{n\to\infty} \|S^{n+1}w_n - S^n w_n\|$. We observe that

$$\begin{aligned}\|T^{n+1}x_n - T^n x_n\| &\leq \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\leq (1+L)\|x_{n+1} - x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_n - T^n x_n\|.\end{aligned}$$

This implies that $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0$. Similarly, we can show that $\lim_{n\to\infty} \|S^{n+1}w_n - S^n w_n\| = 0$. By Theorem 3.1, we have the desired conclusion. This completes the proof.

■

From the definition of quasi-ϕ-nonexpansive mappings, we see that every quasi-ϕ-nonexpansive mapping is asymptotically quasi-ϕ-nonexpansive with the constant sequence $\{1\}$. From the proof of Theorem 3.1, we have the following results immediately.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let T and S be a closed and quasi-ϕ-nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap VI(A,C)$ is nonempty and bounded. Let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in E, \text{ chosen arbitrary}, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_{n-1} + \beta_n JTx_n + \gamma_n JSw_n), \\ y_n = J^{-1}(\delta_n Jx_1 + (1-\delta_n)Jz_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)\xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.60}$$

where $\xi_n = \alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)$, and $\theta_n = \sup\{\phi(z, x_n) : z \in \Omega\}$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

(C1) $\lim_{n\to\infty} \delta_n = 0$;
(C2) $\{r_n\} \subset [a,b]$ *for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;*
(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*
 (a) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ *and*
 (b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$*, where* Π_Ω *is the generalized projection from* E *onto* Ω*.*

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CALCULATING ZEROS OF THE SECOND KIND q-BERNOULLI POLYNOMIALS

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ABSTRACT : In this paper, we observe the behavior of real roots of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $0 < q < 1$. By means of numerical experiments, we demonstrated a remarkably regular structure of the complex roots of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$. The main purpose of this paper is to investigate the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$. Furthermore, we give a table for the solutions of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$.

Key words : Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind q-Bernoulli numbers, the second kind q-Bernoulli polynomials

1. INTRODUCTION

In the recent, several mathematicians have studied the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the q-Bernoulli numbers and polynomials, the second kind q-Bernoulli numbers and polynomials (see [1, 2, 3, 4, 5, 6, 7, 8]). These numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In the 21st century, the computing environment would make more and more rapid progress. The importance of numerical simulation and analysis in mathematics is steadily increasing. Using computer, a realistic study for these numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$ in complex plane. The main motivation of this paper are summarized as follows: In Section 2, using a numerical investigation, we observe the beautiful zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$. Finally, we investigate the structure of the roots of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$. In Section 3, we shall discuss the more general open problems and observations.

Let q be a complex number with $|q| < 1$. Let us define the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$ as follows:

$$F_q(t) = \frac{2te^t}{qe^{2t}-1} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}, \tag{1.1}$$

$$F_q(x,t) = \frac{2te^t}{qe^{2t}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}, \quad \text{(see [7])}. \tag{1.2}$$

We consider the second kind Genocchi polynomials $G_n(x)$ as follows:

$$G(x,t) = \frac{2te^t}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad \text{(cf. [8])}. \tag{1.3}$$

By (1.2) and (1.3), we obtain the following relationships between the second kind q-Bernoulli polynomials $B_{n,q}(x)$ and the second kind Genocchi polynomials $G_n(x)$:

$$\lim_{q\to -1} B_{n,q}(x) = -G_n(x).$$

The following elementary properties of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ are readily derived from (1.1) and (1.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [1]-[4], [7], [8].

Proposition 1. For any positive integer n, we have

$$qB_{n,q}(x+2) - B_{n,q}(x) = 2n(x+1)^{n-1}. \tag{1.4}$$

Proposition 2. For $n \in \mathbb{N}$, we have

$$B_{n,q}(x) = (-1)^n q^{-1} B_{n,q^{-1}}(-x). \tag{1.5}$$

By (1.4) and (1.5), we have

$$B_{n,q}(-1) = (-1)^n B_{n,q^{-1}}(-1).$$

2. ZEROS OF THE SECOND KIND q-BERNOULLI POLYNOMIALS $B_{n,q}(x)$

In this section, we assume that $q \in \mathbb{C}$, with $-1 < q < 0$. We investigate the beautiful zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ by using a computer. We plot the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $n = 10, 15, 25, 30$ and $x \in \mathbb{C}$ (Figure 1).

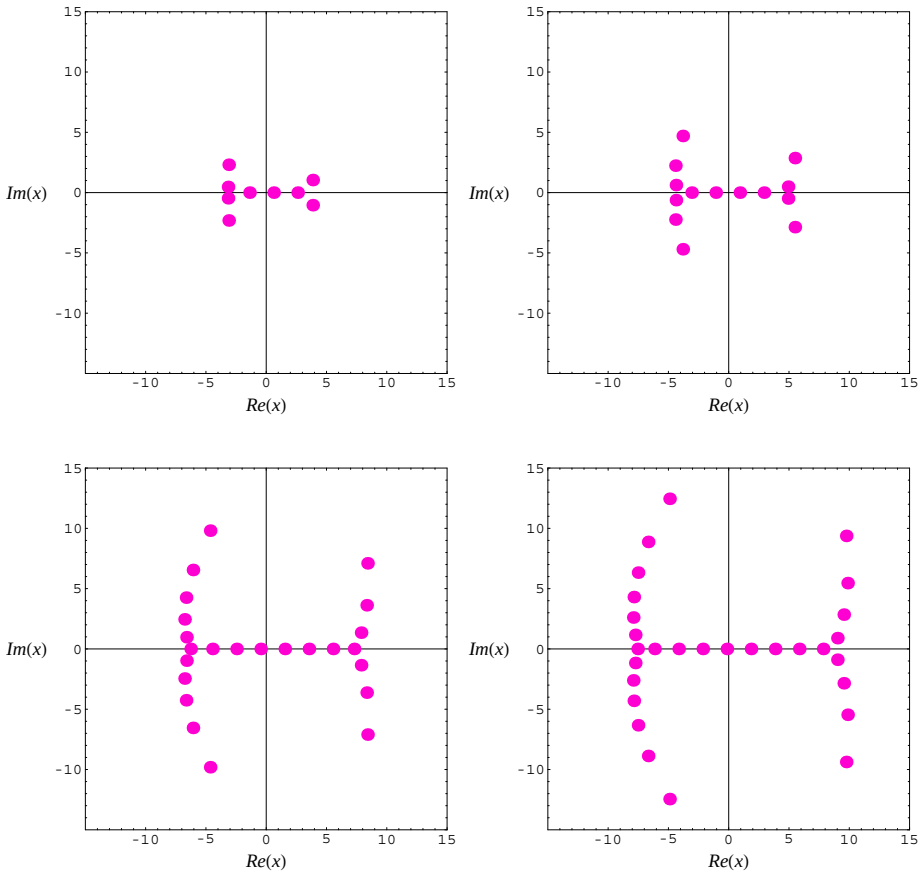


Figure 1: Zeros of $B_{n,q}(x)$ for $n = 10, 15, 25, 30$

In Figure 1(top-left), we choose $n = 10$ and $q = -1/2$. In Figure 1(top-right), we choose $n = 15$ and $q = -1/2$. In Figure 1(bottom-left), we choose $n = 25$ and $q = -1/2$. In Figure 1(bottom-right), we choose $n = 30$ and $q = -1/2$.

We plot the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ for $n = 30$, $q = -1/10$, $-3/10, -7/10, -9/10$ and $x \in \mathbb{C}$(Figure 2).

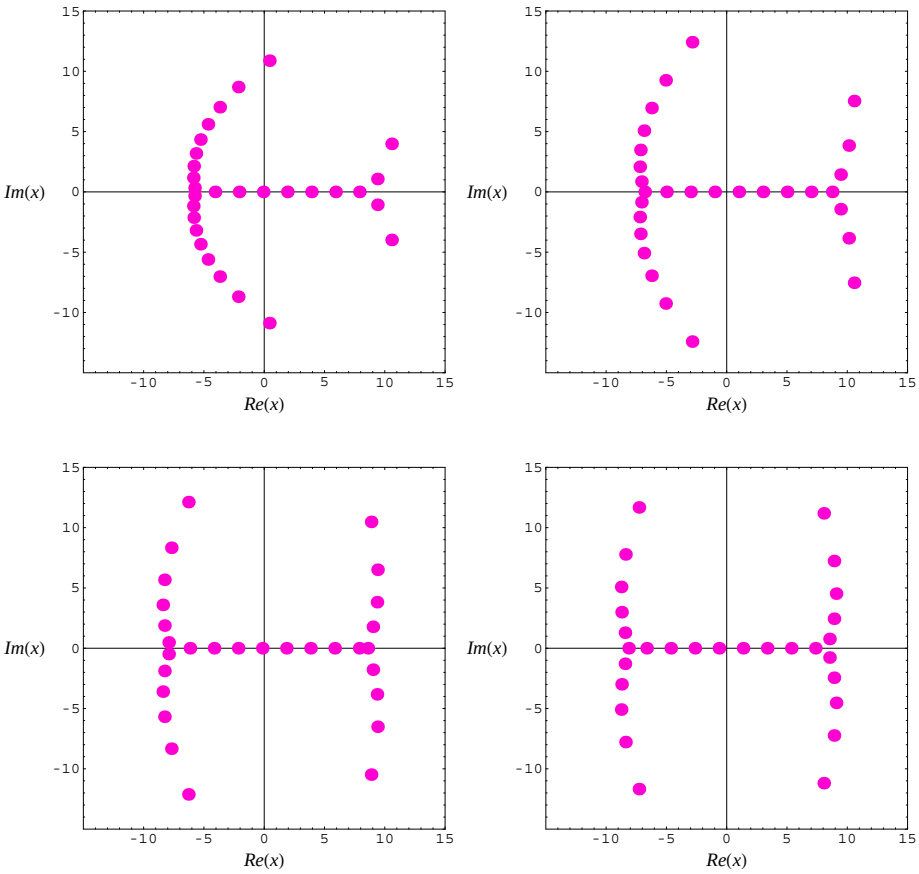


Figure 2: Zeros of $B_{30,q}(x)$ for $q = -1/10, -3/10, -7/10, -9/10$

In Figure 2(top-left), we choose $n = 30$ and $q = -1/10$. In Figure 2(top-right), we choose $n = 30$ and $q = -3/10$. In Figure 2(bottom-left), we choose $n = 30$ and $q = -7/10$. In Figure 2(bottom-right), we choose $n = 30$ and $q = -9/10$.

We plot the zeros of $\lim_{q\to -1} B_{n,q}^{(h)}(x) = -G_n(x)$ for $n = 30$ (Figures 3, 4).

Stacks of zeros of $B_{n,q}(x)$ for $1 \leq n \leq 30$ and $q = -1/2$ from a 3-D structure are presented (Figure 5).

Our numerical results for approximate solutions of real zeros of $B_{n,q}(x)$ are displayed (Tables 1, 2).

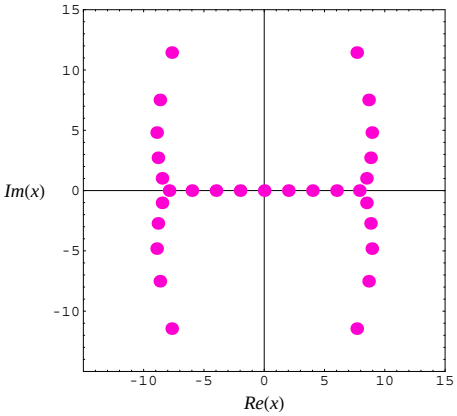


Figure 3: Zeros of $\lim_{q\to -1} B_{n,q}^{(h)}(x)$

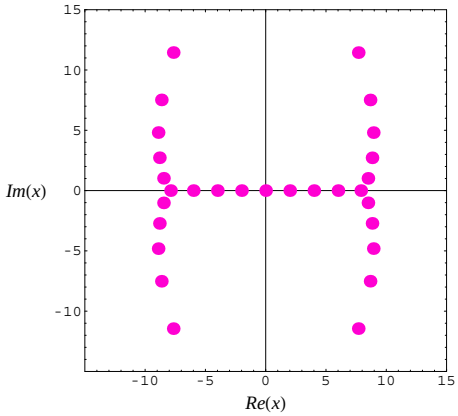


Figure 4: Zeros of $G_n(x)$

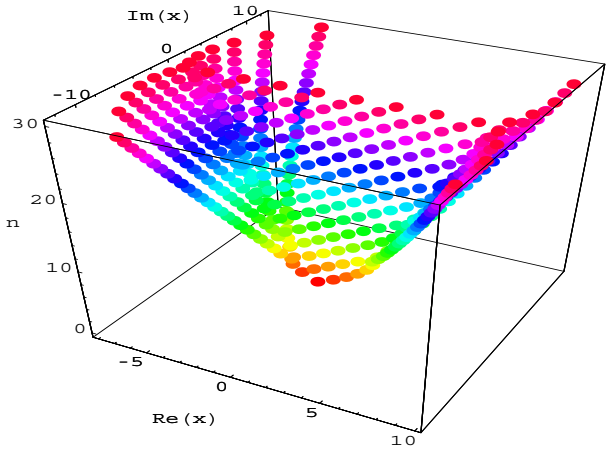


Figure 5: Stacks of zeros of $B_{n,q}(x), 1 \le n \le 30$

Table 1. Numbers of real and complex zeros of $B_{n,q}(x)$

	$q=-1/3$		$q=-1/2$	
degree n	real zeros	complex zeros	real zeros	complex zeros
2	1	0	1	0
3	2	0	2	0
4	3	0	3	0
5	2	2	2	2
6	3	2	3	2
7	4	2	4	2
8	5	2	3	4
9	2	6	4	4
10	3	6	3	6
11	4	6	4	6
12	5	6	5	6

We observe a remarkably regular structure of the complex roots of the second kind q-Bernoulli

polynomials $B_{n,q}(x)$. We hope to verify a remarkably regular structure of the complex roots of the second kind q-Bernoulli polynomials $B_{n,q}(x)$(Table 1). This numerical investigation is especially exciting because we can obtain an interesting phenomenon of scattering of the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$. These results are used not only in pure mathematics and applied mathematics, but also used in mathematical physics and other areas.

Figure 6 shows the distribution of real zeros of $B_{n,q}(x)$ for $1 \leq n \leq 30$.

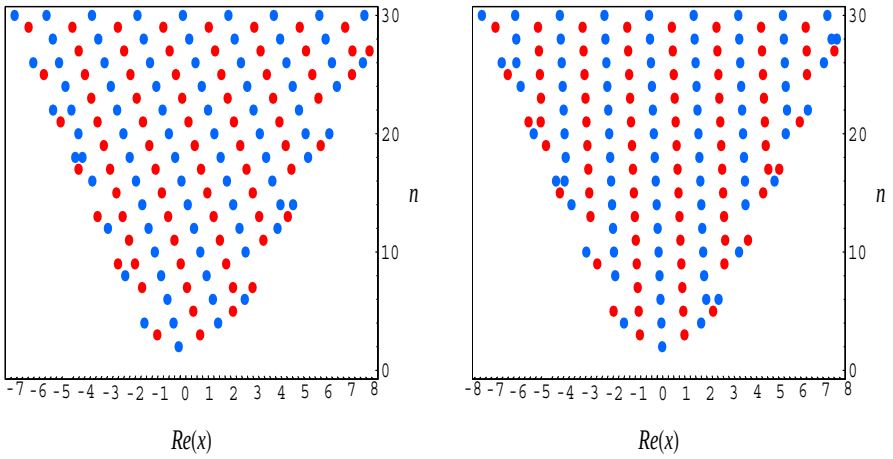


Figure 6: Plot of real zeros of $B_{n,q}(x), 1 \leq n \leq 30$

In Figure 6(left), we choose $q = -5/10$. In Figure 6(right), we choose $q = -9/10$.

Figure 7 presents the distribution of real zeros of the second kind Genocchi polynomials $G_n(x)$ for $1 \leq n \leq 30$.

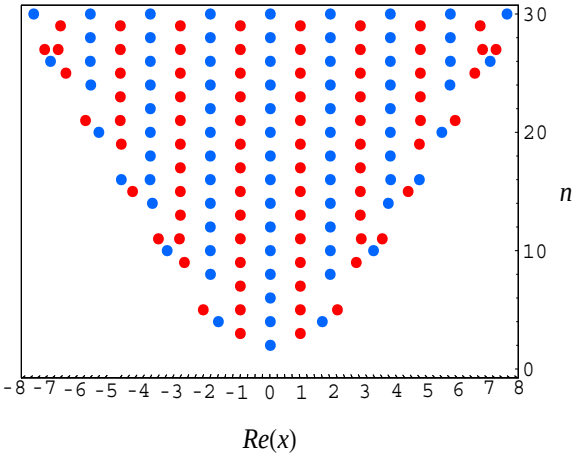


Figure 7: Plot of real zeros of $G_n(x), 1 \leq n \leq 30$

Next, we calculated an approximate solution satisfying the second kind q-Bernoulli polynomials

$B_{n,q}(x)$. The results are given in Table 2.

Table 2. Approximate solutions of $B_{n,q}(x) = 0, q = -1/2, x \in \mathbb{R}$

degree n	x
2	-0.3333333333
3	$-1.276142375, \quad 0.609475708$
4	$-1.84120150, \quad -0.55991792, \quad 1.401119421$
5	$0.309334278, \quad 2.05464597$
6	$-0.82916632, \quad 1.17084623, \quad 2.57319374$
7	$-1.9437980, \quad 0.032019155, \quad 2.05570747, \quad 2.91507272$
8	$-2.6847850, \quad -1.10585266, \quad 0.89414734$
9	$-3.0034250, \quad -2.2550785, \quad -0.244249446, \quad 1.75565741$

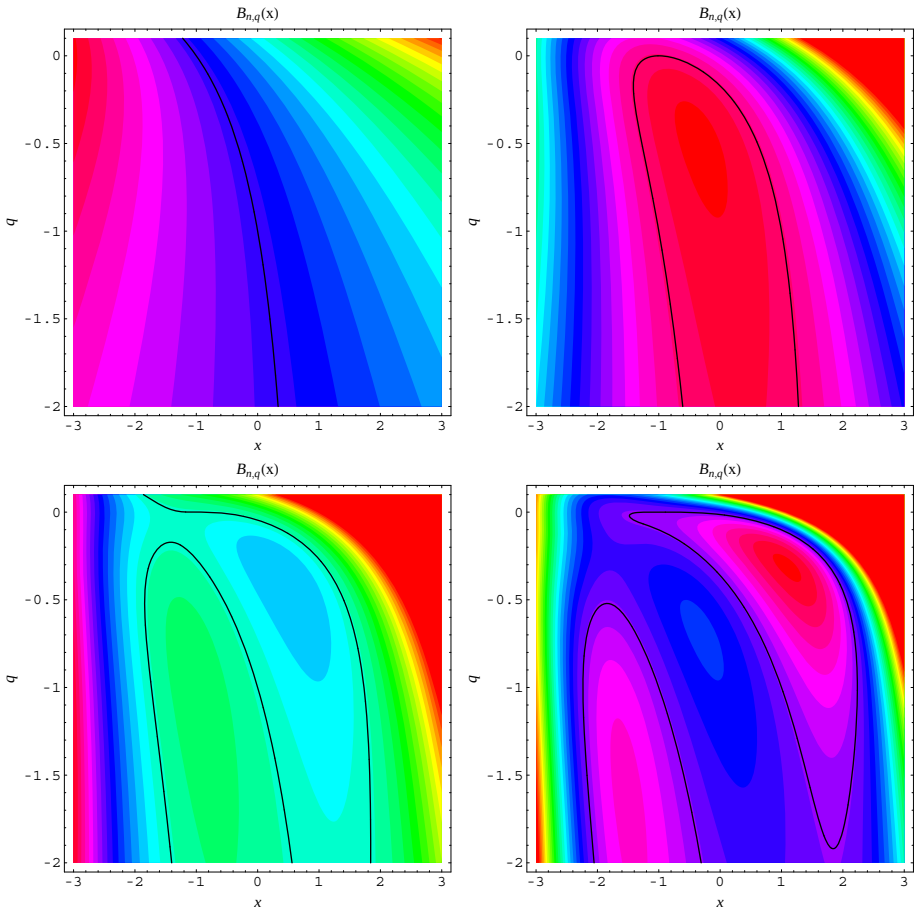


Figure 8: Zero contour of $B_{n,q}(x)$

The plot above shows $B_{n,q}(x)$ for real $-2 \leq q \leq 1/10$ and $-3 \leq x \leq 3$, with the zero contour indicated in black(Figure 8). In Figure 8(top-left), we choose $n = 2$. In Figure 8(top-right), we

choose $n = 3$. In Figure 8(bottom-left), we choose $n = 4$. In Figure 8(bottom-right), we choose $n = 5$.

3. DIRECTIONS FOR FURTHER RESEARCH

We observe the behavior of complex roots of the second kind q-Bernoulli polynomials $B_{n,q}(x) = 0$, using numerical investigation. Prove that $B_{n,q}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions. The obvious corollary is that the zeros of $B_{n,q}(x)$ will also inherit these symmetries. If $B_{n,q}(x) = 0$, then $B_{n,q}(x^*) = 0$, where $*$ denotes complex conjugate (see Figures 1, 2, 3, 5).

Prove that $B_{n,q}(x) = 0$ has $n-1$ distinct solutions, ie., all the zeros are non-degenerate. Find the numbers of complex zeros $C_{B_{n,q}(x)}$ of $B_{n,q}(x), Im(x) \neq 0$. Since $n-1$ is the degree of the polynomial $B_{n,q}(x)$, the number of real zeros $R_{B_{n,q}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{B_{n,q}(x)} = n - 1 - C_{B_{n,q}(x)}$, where $C_{B_{n,q}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_{n,q}(x)}$ and $C_{B_{n,q}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane. Observe that the structure of the zeros of the Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ as $q \to -1$ (see Figures 2, 3, 4, 6, 7). In order to study the second kind q-Bernoulli polynomials $B_{n,q}(x)$, we must understand the structure of the second kind q-Bernoulli polynomials $B_{n,q}(x)$. Therefore, using computer, a realistic study for the second kind q-Bernoulli polynomials $B_{n,q}(x)$ plays an important part. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [2, 3, 4, 5, 6, 7, 8].

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A NUMERICAL INVESTIGATION OF THE STRUCTURE OF THE ROOTS OF THE SECOND KIND q-EULER POLYNOMIALS

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Abstract : In this paper we observe the behavior of complex roots of the second kind q-Euler polynomials $E_{n,q}(x)$, using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the second kind q-Euler polynomials $E_{n,q}(x)$. Finally, we give a table for the solutions of the second kind q-Euler polynomials $E_{n,q}(x)$.

Key words : the second kind Euler numbers and polynomials, the second kind q-Euler numbers and polynomials

1. Introduction

Several mathematicians have studied the Bernoulli numbers and polynomials, Euler numbers and polynomials, q-Bernoulli numbers and polynomials, q-Euler numbers and polynomials, the second kind Euler numbers and polynomials(see [1-14]). These numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In this paper, we introduce the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$. In order to study the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$, we must understand the structure of the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$. Therefore, using computer, a realistic study for the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the second kind q-Euler polynomials $E_{n,q}(x)$ in complex plane. The outline of this paper is as follows. We introduce the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$. In Section 2, we describe the beautiful zeros of the second kind q-Euler polynomials $E_{n,q}(x)$ using a numerical investigation. Finally, we investigate the roots of the second kind q-Euler polynomials $E_{n,q}(x)$. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind q-Euler polynomials $E_{n,q}(x)$. Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

First, we introduce the second kind Euler numbers E_n and polynomials $E_n(x)$. The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \frac{\pi}{2}), \text{ cf. [9, 11]} \tag{1.1}$$

where we use the technique method notation by replacing E^n by $E_n (n \geq 0)$ symbolically. We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x,t) = \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.2}$$

2. The second kind q-Euler numbers and polynomials

In this section, we introduce the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties. Let q be a complex number with $|q| < 1$. By the meaning of (1.1) and (1.2), let us define the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows:

$$F_q(t) = \frac{2e^t}{qe^{2t}+1} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}, \tag{2.1}$$

$$F_q(x,t) = \frac{2e^t}{qe^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{2.2}$$

We have the following remark.

Remark. Note that

(1) $E_{n,q}(0) = E_{n,q}$,

(2) If $q \to 1$, then $E_{n,q}(x) = E_n(x), E_{n,q} = E_n$,

(3) If $q \to 1$, then $F_q(x,t) = F(x,t), F_q(t) = F(t)$.

By using computer, the second kind q-Euler numbers $E_{n,q}$ can be determined explicitly. A few of them are

$$E_{0,q} = \frac{2}{1+q}, \quad E_{1,q} = \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2},$$
$$E_{2,q} = \frac{4}{(1+q)^3} - \frac{8q}{(1+q)^3} + \frac{4q^2}{(1+q)^3} - \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2},$$
$$E_{3,q} = \frac{12}{(1+q)^4} - \frac{36q}{(1+q)^4} + \frac{36q^2}{(1+q)^4} - \frac{12q^3}{(1+q)^4} - \frac{12}{(1+q)^3} + \frac{12q^2}{(1+q)^3} + \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2}.$$

The following elementary properties of the second kind q-Euler polynomials $E_{n,q}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [1]-[4], [6], [9].

The second kind q-Euler polynomials $E_{n,q}(x)$ can be determined explicitly. A few of them are

$$E_{0,q}(x) = \frac{2}{1+q},$$
$$E_{1,q}(x) = \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2} + \frac{2x}{(1+q)},$$
$$E_{2,q}(x) = \frac{4}{(1+q)^3} - \frac{8q}{(1+q)^3} + \frac{4q^2}{(1+q)^3} - \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2} + \frac{4x}{(1+q)^2} - \frac{4qx}{(1+q)^2} + \frac{2x^2}{(1+q)}.$$

Proposition 1 (Differential relation).

$$\frac{d}{dx}E_{n,q}(x) = nE_{n-1,q}(x).$$

Proposition 2 (Integral formula).

$$\int_a^b E_{n-1,q}(x)dx = \frac{1}{n}(E_{n,q}(b) - E_{n,q}(a)).$$

Theorem 3. The second kind q-Euler numbers $E_{n,q}$ are defined by

$$q(E_q+1)^n + (E_q-1)^n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $(E_q)^n$ by $E_{n,q}$ in the binomial expansion.

Theorem 4 (Difference equation). For any positive integer n, we have

$$qE_{n,q}(x+2) + E_{n,q}(x) = 2(x+1)^n.$$

Theorem 5 (Theorem of complement). For any positive integer n, we have

$$E_{n,q}(x) = (-1)^n q^{-1} E_{n,q^{-1}}(-x).$$

2. Zeros of the second kind q-Euler polynomials $E_{n,q}(x)$

In this section, we investigate the zeros of the second q-Euler polynomials $E_{n,q}(x)$. We investigate the beautiful zeros of the $E_{n,q}(x)$ by using a computer. We plot the zeros of the second kind q-Euler polynomials $E_{n,q}(x)$ for $n = 30, q = 1/5, 2/5, 3/5, 4/5$ and $x \in \mathbb{C}$ (Figure 1). In Figure 1

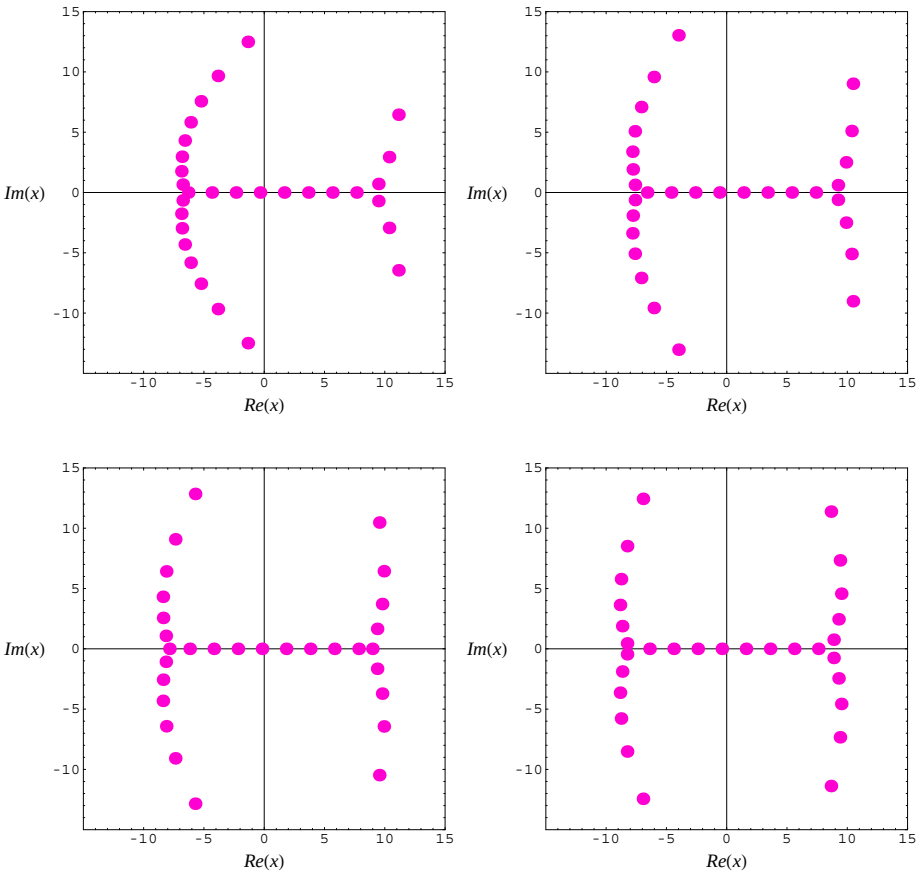


Figure 1: Zeros of $E_{n,q}(x)$ for $q = 1/5, 2/5, 3/5, 4/5$

(top-left), we choose $n = 30$ and $q = 1/5$. In Figure 1 (top-right), we choose $n = 30$ and $q = 2/5$. In Figure 1 (bottom-left), we choose $n = 30$ and $q = 3/5$. In Figure 1 (bottom-right), we choose $n = 30$ and $q = 4/5$. The real zeros of $E_{30,q}(x)$ for $q \to 1$ structure are presented (Figure 1).

Stacks of zeros of $E_{n,1/2}(x)$ for $1 \leq n \leq 30$ from a 3-D structure are presented (Figure 2). Plot of real zeros of $E_{n,q}(x)$ for $1 \leq n \leq 30$ and $q = 1/2, 4/5$ structure are presented (Figure 3). In

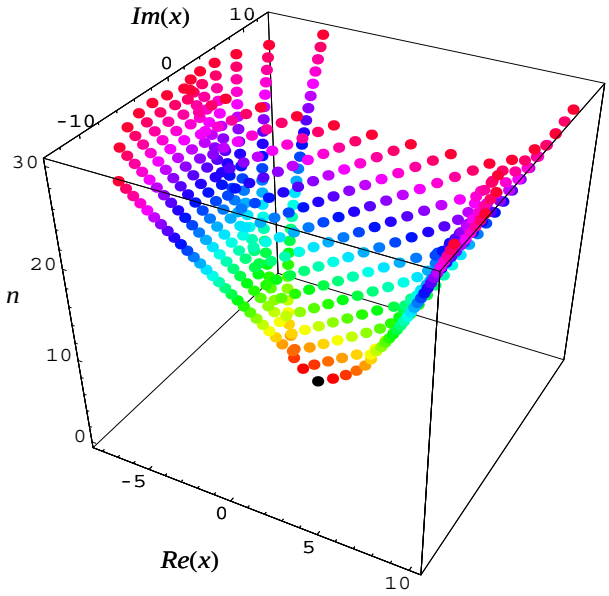


Figure 2: Stacks of zeros of $E_{n,1/2}(x)$ for $1 \leq n \leq 30$

Figure 3(left), we choose $1 \leq n \leq 30$ and $q = 1/2$. In Figure 3(right), we choose $1 \leq n \leq 30$ and $q = 4/5$. Plot of real zeros of $E_{n,q}(x)$ for $q \to 1, 1 \leq n \leq 30$ structure are presented (Figure 4). Our numerical results for approximate solutions of real zeros of $E_{n,q}(x)$ are displayed (Tables 1, 2).

Table 1. Numbers of real and complex zeros of $E_{n,q}(x)$

	$q = 1/2$		$q = 1/3$	
degree n	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	2	2	2	2
5	3	2	3	2
6	4	2	4	2
7	3	4	5	2
8	4	4	2	6
9	3	6	3	6
10	4	6	4	6
11	5	6	5	6
12	6	6	4	8
13	5	8	5	8

We observe a remarkably regular structure of the complex roots of the second kind q-Euler polynomials $E_{n,q}(x)$. We hope to verify a remarkably regular structure of the complex roots of the second kind q-Euler polynomials $E_{n,q}(x)$ (Table 1). Next, we calculated an approximate solution satisfying

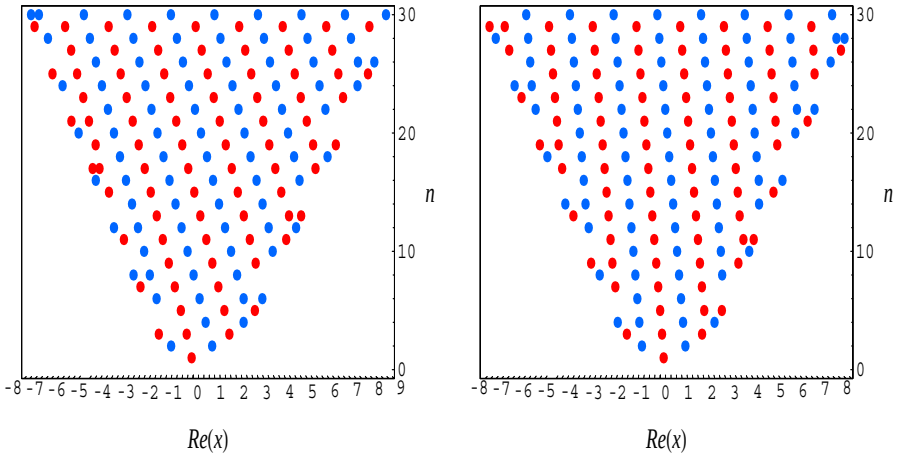


Figure 3: Real zeros of $E_{n,q}(x)$ for $1 \leq n \leq 30$

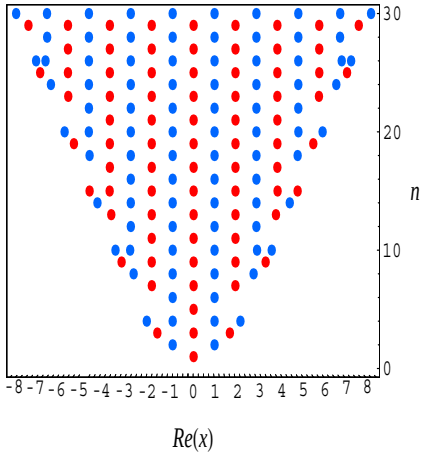


Figure 4: Real zeros of $E_{n,q}(x)$ for $q \to 1$ and $1 \leq n \leq 30$

$E_{n,q}(x), q = 1/2, x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $E_{n,q}(x) = 0, x \in \mathbb{R}$

degree n	x
1	-0.33333
2	$-1.2761, \quad 0.6095$
3	$-1.8412, \quad -0.5599, \quad 1.4011$
4	$0.3093, \quad 2.0546$
5	$-0.8292, \quad 1.1708, \quad 2.5732$
6	$-1.000, \quad 1.000$
7	$-2.685, \quad -1.1059, \quad 0.8941$
8	$-3.00, \quad -2.255, \quad -0.2442, \quad 1.7557$
9	$-1.382, \quad 0.6175, \quad 2.609$

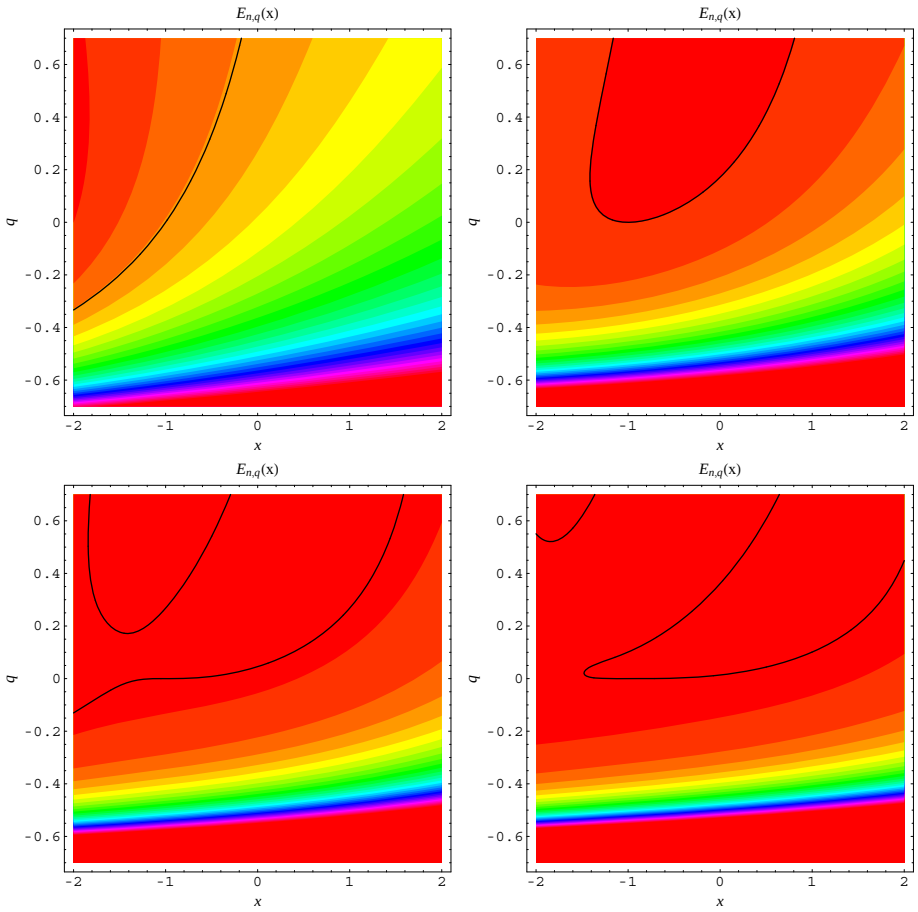


Figure 5: Zero contour of $E_{n,q}(x)$

The plot above shows $E_{n,q}(x)$ for real $-7/10 \leq q \leq 7/10$ and $-2 \leq x \leq 2$, with the zero contour indicated in black (Figure 5). In Figure 5(top-left), we choose $n = 1$. In Figure 5(top-right), we choose $n = 2$. In Figure 5(bottom-left), we choose $n = 3$. In Figure 5(bottom-right), we choose $n = 4$.

3. Directions for Further Research

Finally, we shall consider the more general problems. In general, how many roots does $E_{n,q}(x)$ have? This is open problem. Prove or disprove: $E_{n,q}(x) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q}(x)}$ of $E_{n,q}(x), Im(x) \neq 0$. Since n is the degree of the polynomial $E_{n,q}(x)$, the number of real zeros $R_{E_{n,q}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{E_{n,q}(x)} = n - C_{E_{n,q}(x)}$, where $C_{E_{n,q}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q}(x)}$ and $C_{E_{n,q}(x)}$. Observe that the structure of the zeros of the second Euler polynomials $E_n(x)$ resembles the structure of the zeros of the second kind q-Euler polynomials $E_{n,q}(x)$ as $q \to 1$ (see Figures 1, 2, 3, 4). Find the equation of envelope curves bounding the real zeros lying on the plane. The theoretical prediction on the zeros of $E_{n,q}(x)$ is await for further study. We plot the zeros of $E_{n,q}(x)$, respectively(Figures 1-4). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $E_{n,q}(x)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing

numerical method in the field of research of the second kind q-Euler polynomials $E_{n,q}(x)$ to appear in mathematics and physics. The reader may refer to [10-13] for the details.

Acknowledgment

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Strong differential subordinations obtained by Ruscheweyh operator

Roxana Şendruţiu

Abstract

In this paper we define a class of univalent functions $\mathcal{R}\zeta_n^m(\alpha)$ and we study new strong differential subordinations using Ruscheweyh derivative. The notion of strong differential subordination is developed from the classic notion of differential subordination.

Keywords: analytic function, differential subordination, strong differential subordination, univalent function, convex function, dominant, Ruscheweyh operator.
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1 Introduction and preliminaries

Let U denote the unit disc of the complex plane :

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}.$$

Let $\mathcal{H}(U \times \overline{U})$ denote the class of analytic functions in $U \times \overline{U}$.
In [6],[8] the authors have defined the classes

$$\mathcal{H}\zeta[a,n] = \{f \in \mathcal{H}(U \times \overline{U}) : \ f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots , \ z \in U, \zeta \in \overline{U}\}$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$,

$$\mathcal{H}\zeta_u(U) = \{f \in \mathcal{H}\zeta[a,n] : \ f(z,\zeta) \text{ univalent in U for all } z \in U, \zeta \in \overline{U}\},$$

$$\mathcal{A}\zeta_n = \{f \in \mathcal{H}\zeta[a,n] : \ f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + a_{n+2}(\zeta)z^{n+2} + \cdots, \ z \in U, \zeta \in \overline{U}\}$$

with $\mathcal{A}\zeta_1 = \mathcal{A}\zeta$,

$$K\zeta = \left\{ f \in \mathcal{H}\zeta[a,n] : \ \operatorname{Re} \frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1 > 0, \ z \in U, \text{for all } \zeta \in \overline{U} \right\}.$$

Definition 1.1. [6], [8] Let $h(z,\zeta)$, $f(z,\zeta)$ be analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $h(z,\zeta)$, or $h(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$, if there exists a function ω analytic in U, $\omega(0) = 0$, $|\omega(z)| < 1$, such that $f(z,\zeta) = h[\omega(z),\zeta]$, for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}$.

Remark 1.1. If $h(z,\zeta) \equiv h(z)$ and $f(z,\zeta) \equiv f(z)$, then the strong subordination becomes the usual notion of subordination.

Remark 1.2. The notion of differential subordination was introduced and developed by S.S.Miller and P.T.Mocanu [3], and the concept of strong differential subordination was introduced in [1] by J.A.Antonino and S.Romaguera and developed by G.I. Oros and Gh. Oros [6].

In order to prove the main results we use some lemmas found in [2],[4], and we adapt them to the classes defined in [6],[8]:

Lemma 1.1 (D.J. Hallenbeck, S. Ruscheweyh) Let

$$h(z,\zeta) \in K\zeta$$

with $h(0,\zeta) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}\zeta[a,n]$ and

$$p(z,\zeta) + \frac{1}{\gamma} zp'(z,\zeta) \prec\prec h(z,\zeta)$$

then

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),$$

where

$$q(z,\zeta) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt \quad (z \in U, \zeta \in \overline{U}).$$

The function q belongs to the class $K\zeta$ and is the best (a,n)dominant.

Lemma 1.2 (P.T. Mocanu, T. Bulboacă, G.Şt. Sălăgean) Let $q(z,\zeta) \in K\zeta$ and let

$$h(z,\zeta) = q(z,\zeta) + n\alpha z q'(z,\zeta), \quad z \in U, \zeta \in \overline{U}$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z,\zeta) = q(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \cdots$$

is holomorphic in $U \times \overline{U}$ and

$$p(z,\zeta) + \alpha z p'(z,\zeta) \prec\prec h(z,\zeta),$$

then

$$p(z,\zeta) \prec\prec q(z,\zeta)$$

and this result is sharp.

2 Main results

Definition 2.1. For $f(z,\zeta) \in \mathcal{H}(U \times \overline{U})$, $n \in \mathbb{N}$, we define the differential operator R^n:

$R^0 f(z,\zeta) = f(z,\zeta)$
$R^1 f(z,\zeta) = z f'_z(z,\zeta)$
$2R^2 f(z,\zeta) = z[R^1 f(z,\zeta)]'_z + R^1 f(z,\zeta)$
$\cdots$
$(n+1)R^{n+1} f(z,\zeta) = z[R^n f(z,\zeta)]'_z + n[R^n f(z,\zeta)] \quad (z \in U, \zeta \in \overline{U}).$

Remark 2.1. For $f(z) \in \mathcal{H}(U)$ we have Ruscheweyh differential operator [7].

Property 2.1. For $f(z,\zeta) \in \mathcal{A}\zeta$,

$$f(z,\zeta) = z + a_2(\zeta)z^2 + a_3(\zeta)z^3 + \cdots, \quad z \in U, \zeta \in \overline{U}$$

then:

$$R^n f(z,\zeta)] = z + \sum_{k=2}^{\infty} C_{n+k-1}^n a_k(\zeta) z^k \quad (z \in U, \zeta \in \overline{U}, n \in \mathbb{N}).$$

Property 2.2. For $f(z,\zeta) \in \mathcal{A}\zeta$, $n \in \mathbb{N}$, then the differential operator R^n can be define as

$$R^n f(z,\zeta) = \frac{z}{(1-z)^{n+1}} * f(z,\zeta) = \frac{z(z^{n-1}f(z,\zeta))^{(n)}}{n!} \quad (z \in U, \zeta \in \overline{U}).$$

Definition 2.2. Let $\alpha < 1$ and $m, n \in \mathbb{N}$. We denote by $\mathcal{R}\zeta_n^m(\alpha)$ the set of functions $f \in A\zeta_n$, that satisfies the inequality

$$Re[R^m f(z,\zeta)]'_z > \alpha, \quad z \in U, \zeta \in \overline{U}, \tag{2.1}$$

where R^m is the differential operator defined in Definition 2.1.

Ideas found in paper [5] were used for obtaining the original results of the present paper.

Theorem 2.1. If $\alpha < 1$, and $m, n \in \mathbb{N}$, then

$$\mathcal{R}\zeta_n^{m+1}(\alpha) \subset \mathcal{R}\zeta_n^m(\delta),$$

where

$$\delta = \delta(\alpha,\zeta,n,m) = 2\alpha - \zeta + 2(\zeta-\alpha)\frac{m+1}{n}\sigma(\frac{m+1}{n}) \tag{2.2}$$

and

$$\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t}dt. \tag{2.3}$$

Proof. Let $f(z,\zeta) \in \mathcal{R}\zeta_n^{m+1}(\alpha)$. Using the properties of the $R^m f$ operator, we have

$$(2.4) \quad (m+1)[R^{m+1}f(z,\zeta)] = z[R^m f(z,\zeta)]' + m[R^m f(z,\zeta)],$$

with $z \in U, \zeta \in \overline{U}$. Differentiating (2.4), with respect to z, we obtain

$$(2.5) \quad (m+1)[R^{m+1}f(z,\zeta)]' = z[R^m f(z,\zeta)]'' + (m+1)[R^m f(z,\zeta)]',$$

$z \in U, \zeta \in \overline{U}$.
If we denote by $p(z,\zeta) = [R^m f(z,\zeta)]'$, then $p'(z,\zeta) = [R^m f(z,\zeta)]''$, and after an simple calculus, the equality (2.5) becomes

$$(2.6) \quad [R^{m+1}f(z,\zeta)]' = p(z,\zeta) + \frac{1}{m+1} z p'(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

Since $f(z,\zeta) \in \mathcal{R}\zeta_n^{m+1}(\alpha)$, using Definition 2.2 and the equality (2.6), we obtain

$$(2.7) \qquad Re[p(z,\zeta) + \frac{1}{m+1} z p'(z,\zeta)] > \alpha, \quad z \in U, \zeta \in \overline{U},$$

equivalent to

$$p(z,\zeta) + \frac{1}{m+1} z p'(z,\zeta) \prec\prec \frac{\zeta + (2\alpha - \zeta)z}{1+z} \equiv h(z,\zeta),$$

since the image of the unit disc U through function h is the semiplane $\{\omega \in \mathbb{C} : \ |\omega| > \alpha\}$. Using Lemma 1.1, results

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta)$$

where

$$(2.8) \qquad q(z,\zeta) = \frac{m+1}{n z^{\frac{m+1}{n}}} \int_0^z \frac{\zeta + (2\alpha - \zeta)t}{1+t} t^{\frac{m+1}{n}-1} dt$$

$$= \frac{m+1}{n z^{\frac{m+1}{n}}} \int_0^2 [2\alpha - \zeta + \frac{2(\zeta - \alpha)}{1+t}] t^{\frac{m+1}{n}-1} dt$$

$$= 2\alpha - \zeta + 2(\zeta - \alpha)\frac{m+1}{n}\sigma(\frac{m+1}{n})\frac{1}{z^{\frac{m+1}{n}}}, \quad z \in U, \zeta \in \overline{U},$$

where $\sigma(x)$ is given by (2.3), and the function $q(z,\zeta) \in K\zeta$ and is the best dominant.
With $p(z,\zeta) \prec\prec q(z,\zeta)$ and $q(z,\zeta) \in K\zeta$, and the fact that the image of $U \times \overline{U}$ through $q(z,\zeta)$ is symmetric with respect to the real axis, we deduce that

$$Re\, p(z,\zeta) > q(1,\zeta) = \delta = \delta(\alpha,\zeta,n,m) = 2\alpha - \zeta + 2(\zeta-\alpha)\frac{m+1}{n}\sigma(\frac{m+1}{n}),$$

which means that

$$Re[R^m f(z,\zeta)]' > \delta, \quad z \in U, \zeta \in \overline{U}.$$

Using Definition 2.2 we obtain $f(z,\zeta) \in \mathcal{R}\zeta_n^m(\delta)$. Since $f(z,\zeta) \in \mathcal{R}\zeta_n^{m+1}(\alpha)$, we obtain that

$$\mathcal{R}\zeta_n^{m+1}(\alpha) \subset \mathcal{R}\zeta_n^m(\delta),$$

where δ is given by (2.2). □

Theorem 2.2. Let $q(z,\zeta) \in K\zeta$ a function with $q(0,\zeta) = 1$, and let $h(z,\zeta)$ an analytic function given by

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} zq'(z,\zeta), \quad z \in U, \zeta \in \overline{U}. \tag{2.9}$$

If $f \in A\zeta_n$ and the strong differential subordination

$$[R^{m+1} f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U} \tag{2.10}$$

holds, then

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta)$$

and this is the best result.
Proof. Using the properties of Ruscheweyh operator and the relation (2.6), the strong differential subordination (2.10) becomes

$$p(z,\zeta) + \frac{1}{m+1} zp'(z,\zeta) \prec\prec q(z,\zeta) + \frac{1}{m+1} zq'(z,\zeta) \equiv h(z,\zeta),$$

with $z \in U, \zeta \in \overline{U}$.
Using Lemma 1.2. we obtain $p(z,\zeta) \prec\prec q(z,\zeta)$, i.e.

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta), \quad z \in U, \zeta \in \overline{U},$$

and this is the best result. □

Remark 2.2. For $n = 1$, Theorem 2.2 can be restate under the form of the next Corollary.

Corollary 2.1. Let $q(z,\zeta) \in K\zeta$ a function with $q(0,\zeta) = 1$, and let $h(z,\zeta)$ an analytic function given by

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} z q'(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

If $f \in A\zeta$ and the strong differential subordination

$$[R^{m+1} f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}$$

holds, then

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta).$$

Theorem 2.3. Let $h(z,\zeta)$ an analytic function from $U \times \overline{U}$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$, $z \in U, \zeta \in \overline{U}$, that satisfies the inequality

$$Re[1 + \frac{z h''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2(m+1)}, m \geq 0. \tag{2.11}$$

If $f(z,\zeta) \in A\zeta_n$ and the strong differential subordination

$$[R^{m+1} f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{2.12}$$

holds, then

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta)$$

where

$$q(z,\zeta) = \frac{m+1}{n z^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt, \tag{2.13}$$

$z \in U, \zeta \in \overline{U}$.
The function $q(z,\zeta) \in K\zeta$ and is the best dominant.

Proof. A simple application of the differential subordination technique [1], [3] shows that the function $q(z,\zeta)$ belongs to the class $K\zeta$. Using the properties of the $R^m f$ operator, and after an simple calculus we obtain

$$[R^{m+1}f(z,\zeta)]' = [R^m f(z,\zeta)]' + \frac{1}{m+1} z[R^m f(z,\zeta)]'', \tag{2.14}$$

with $z \in U, \zeta \in \overline{U}$.
If we denote by

$$p(z,\zeta) = [R^m f(z,\zeta)]', \quad z \in U, \zeta \in \overline{U} \tag{2.15}$$

then the equality (2.14) becomes

$$[R^{m+1}f(z,\zeta)]' = p(z,\zeta) + \frac{1}{m+1} zp'(z,\zeta), \quad z \in U, \zeta \in \overline{U}, \tag{2.16}$$

and the strong differential subordination (2.12) is equivalent with

$$p(z,\zeta) + \frac{1}{m+1} zp'(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}. \tag{2.17}$$

Using Lemma 1.1, we have

$$p(z,\zeta) \prec\prec q(z,\zeta) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt,$$

and using (2.15), results

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt,$$

with $z \in U, \zeta \in \overline{U}$.
The function $q(z,\zeta) \in K\zeta$ and is the best dominant. □

Remark 2.3. For $n = 1$, Theorem 2.3 can be restate under the form of the next Corollary.

Corollary 2.2. Let $h(z,\zeta)$ an analytic function from $U \times \overline{U}$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$, $z \in U, \zeta \in \overline{U}$, that satisfies the inequality

$$Re[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2(m+1)}, m \geq 0.$$

If $f(z,\zeta) \in A\zeta$ and the strong differential subordination

$$[R^{m+1}f(z,\zeta)]' \prec\prec h(z,\zeta),$$

holds, then

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta)$$

where

$$q(z,\zeta) = \frac{m+1}{z^{m+1}} \int_0^z h(t,\zeta)t^m dt,$$

$z \in U, \zeta \in \overline{U}$.
The function $q(z,\zeta) \in K\zeta$ and is the best dominant.

Theorem 2.4. Let $q(z,\zeta) \in K\zeta$ be a function with $q(0,\zeta) = 1$ and suppose that

$$h(z,\zeta) = q(z,\zeta) + nzq'(z,\zeta), \quad z \in U, \zeta \in \overline{U}, n \in \mathbb{N}).$$

If $f(z,\zeta) \in A\zeta_n$ and the strong differential subordination

$$[R^m f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U} \tag{2.18}$$

holds, then

$$\frac{[R^m f(z,\zeta)]}{z} \prec\prec q(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

Proof. We denote by

$$p(z,\zeta) = \frac{[R^m f(z,\zeta)]}{z}, \quad z \in U, \zeta \in \overline{U}, \tag{2.19}$$

hence

$$R^m f(z,\zeta) = zp(z,\zeta), \quad z \in U, \zeta \in \overline{U}. \tag{2.20}$$

Differentiating the equality (2.20), with respect to z, we obtain

$$[R^m f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta), \quad z \in U, \zeta \in \overline{U} \tag{2.21}$$

and the strong differential subordination (2.18) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec q(z,\zeta) + nzq'(z,\zeta), \quad z \in U, \zeta \in \overline{U}$$

Using Lemma 1.2, results $p(z,\zeta) \prec\prec q(z,\zeta)$, which means

$$\frac{[R^m f(z,\zeta)]}{z} \prec\prec q(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

□

Remark 2.4. For $n = 1$, Theorem 2.4 can be restate under the form of the next Corollary.

Corollary 2.3. Let $q(z,\zeta) \in K\zeta$ be a function with $q(0,\zeta) = 1$ and suppose that

$$h(z,\zeta) = q(z,\zeta) + zq'(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

If $f(z,\zeta) \in A\zeta$ and the strong differential subordination

$$[R^m f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}$$

holds, then

$$\frac{[R^m f(z,\zeta)]}{z} \prec\prec q(z,\zeta).$$

Theorem 2.5. Let $h(z,\zeta)$ an analytic function from $U \times \overline{U}$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$, $z \in U, \zeta \in \overline{U}$, that satisfies the inequality

$$Re[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2}, z \in U, \zeta \in \overline{U}.$$

If $f(z,\zeta) \in A\zeta$ and the strong differential subordination

$$[R^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{2.22}$$

holds, then

$$\frac{[R^m f(z,\zeta)]}{z} \prec\prec q(z,\zeta)$$

where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt, \quad z \in U, \zeta \in \overline{U}.$$

The function q belongs to the class $K\zeta$ and is the best dominant.

Proof. We denote by

$$p(z,\zeta) = \frac{R^m f(z,\zeta)}{z}, \quad z \in U, \zeta \in \overline{U}, \tag{2.23}$$

hence

$$R^m f(z,\zeta) = zp(z,\zeta), \quad z \in U, \zeta \in \overline{U}. \tag{2.24}$$

Differentiating (2.24), with respect to z, we obtain

$$[R^m f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta), \quad z \in U, \zeta \in \overline{U}, \tag{2.25}$$

and the strong differential subordination (2.22) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

Using Lemma 1.1, we have

$$p(z,\zeta) \prec\prec q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt,$$

i.e.

$$\frac{R^m f(z,\zeta)}{z} \prec\prec q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt.$$

From [3] (p.66-67) we have $h(z,\zeta) \in K\zeta(-\frac{1}{2})$ and the function $q(z,\zeta) \in K\zeta$. From Lemma 1.1. the function q is the best dominant. □

Example 2.1. Let $h(z,\zeta)$ be the function

$$h(z,\zeta) = \frac{\zeta + (2\alpha - \zeta)z}{1+z},$$

where $z \in U, \zeta \in \overline{U}, h(0,\zeta) = 1, \alpha \in \mathbb{R}, \alpha < 1$.
If $f(z,\zeta) \in \mathcal{R}\zeta_1^m(\alpha)$ then, from Theorem 2.5 results

$$\frac{R^m f(z,\zeta)}{z} \prec\prec q(z,\zeta) = \frac{1}{z}\int_0^z \frac{\zeta + (2\alpha - \zeta)t}{1+t}dt,$$

then

$$Re[\frac{R^m f(z,\zeta)}{z}] > q(1,\zeta) = \int_0^1 h(t,\zeta)dt$$

$$= \int_0^1 \frac{\zeta + (2\alpha - \zeta)t}{1+t}dt = 2\alpha - \zeta + 2(\zeta - \alpha)\ln 2,$$

with $\zeta \in \overline{U}, \alpha \in \mathbb{R}, \alpha < 1$. □

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Learning Rates of Tikhonov Regularized Regressions Based on Sample Dependent RKHS

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Abstract

It is known that the learning algorithm with sample hypothesis spaces is essentially different from the algorithms with hypothesis spaces independent of the sample. In the present paper, we consider the error bounds for norm square regularized regressions associated with Lipschitz loss and sample depending reproducing kernel spaces. By giving the unique solution with subgradients of the loss functions, we estimate the learning rates with the regularization parameters λ and the sample number m. The sample error rates obtained are $O(\frac{1}{\lambda\sqrt{m}})$ and the approximation error rates are $O(\frac{1}{\sqrt{m}} + \lambda)$.

Keywords: Sample depending reproducing kernel Hilbert spaces, convex analysis, Lipschitz loss.

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1 Introduction

Recently, with the development of learning theory, considerably attention has been paid for theory analysis of regularized kernel machine learning with sample dependent hypothesis spaces(see [3], [17],[18], [23],[25], etc.).

Let X be a compact metric space, $Y = \Re$. $\rho(x, y)$ is a fixed but unknown Borel probability distribution on $Z := X \times Y$ which describes the relation between variables $x \in X$ and $y \in Y$. It can be factorized into the marginal probability $\rho_X(x)$ and the conditional probability $\rho(y|x)$ of y given x.

Let $V(t) : \Re \to \Re_+$ be a convex loss function. Denote by $\mathcal{E}_{\rho,V}(f) = \int_Z V\big(|y-f(x)|\big)\, d\rho$ the generalization error. The minimizerf_V^* defined by

$$f_V^*(x) := arg \min_f \mathcal{E}_{\rho,V}(f)$$

over all measurable functions is the target which we want to learn. In particular, if $V(t) = t^2$ is the least square loss, f_V^* is exactly the regression function $f_\rho(x) = E(y|x) = \int_Y y\, d\rho(y|x)$ (see [8]). For the pinball loss and the ε-insensitive loss the existence of f_V^* is studied in [13] and [19] respectively. Therefore, throughout this paper we assume that f_V^* always exists and is uniqueness. The task of learning theory is to find, from the sample $z = (z_i)_{i=1}^m = ((x_i,\ y_i))_{i=1}^m \in Z^m$ drawn from independent and identically distributed (i.i.d.) random variable (X_i, Y_i) each with the unknown probability distribution $\rho(x, y)$ on Z with $1 \le i \le m$, a function $f_z(x) : X \to Y$ such that it is a good approximation of f_V^*.

A way of obtaining f_z is by the following empirical Tikhonov regularization regressions (see e.g. [9])

$$f_{z,\lambda,V,\mathcal{H}} = arg \min_{f\in\mathcal{H}} \left\{ \frac{1}{m} \sum_{i=1}^{m} V\Big(\big|y_i - f(x_i)\big|\Big) + \lambda\, \Omega(f) \right\}, \tag{1.1}$$

where λ are positive constants which are called the regularized parameters, $\mathcal{H}$ is a hypothesis space which is, in most of the case, a function space consisting of functions on X with norm $\|\cdot\|_{\mathcal{H}}$, $\Omega(f) : \mathcal{H} \to \Re$ is the penalty function called regularizer.

To show the performance of algorithm (1.1), one needs to consider how fast $f_{\alpha_{z,\lambda,V,\mathcal{H}}}$ approximates f_V^* as m increases. A way is to show the rates for error

$$\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V,\mathcal{H}}}) - \mathcal{E}_{\rho,V}(f_V^*). \tag{1.2}$$

In learning theory based on kernels, the hypothesis spaces $\mathcal{H}$ are usually taken to be the reproducing kernel Hilbert spaces(RKHS).

Let $K(x,y) = K_x(y) : X \times X \to \Re$ be continuous, symmetric and positive semi-definite, i.e., for any finite set of distinct points $\overline{X} = \{x_1, x_2, \cdots, x_l\} \subset X$, the matrix $K_{\overline{X},\overline{X}} = (K(x_i,x_j))_{i,j=1}^l$ is positive semi-definite. Such functions are called Mercer kernels.

The reproducing kernel Hilbert space (RKHS)(see [1]) $\mathcal{H}_K$ associating with a Mercer kernel $K(x,y)$ is defined to be the closure of the linear span of the set of functions $\{K_x := K(x,\cdot) : x \in X\}$ with the inner product $\langle\cdot,\cdot\rangle_K$ satisfying

$$\langle K_x, K_y\rangle_K = K(x,y) = K_x(y), \qquad x,y \in X$$

and the reproducing property takes the form

$$f(x) = \langle f, K_x\rangle_K, \qquad x \in X.$$

Many methods have been used for estimating the convergence rates when $\mathcal{H}$ are RKHS $\mathcal{H}_K$. The capacity dependent (see e.g. [2, 4, 5, 9, 11, 21, 22, 24]) and the capacity independent (see e.g. [6, 10, 15]) are two approaches for this purpose. Based on these discussions, learning algorithms with sample dependent are developed. Let $\overline{X} = \{x_1, x_2, \cdots, x_m\} \subset X$ be data taken from the observations $z = \{(x_i,y_i)\}_{i=1}^m$ and $\mathcal{H}_{K,\overline{X}} = \{f_\alpha(x) = \sum_{j=1}^m \alpha_j K_{x_j}(x) : \alpha = (\alpha_1,\alpha_2,\cdots,\alpha_m)^\top \in \Re^m\}$ be the hypothesis space depending on $\overline{X}$. Then, $\mathcal{H}_{K,\overline{X}} \subset \mathcal{H}_K$. Define an inner product $\langle\cdot,\cdot\rangle_{K,\overline{X}}$ on $\mathcal{H}_{K,\overline{X}}$ by the linear extension from $\langle K_t, K_s\rangle_{K,\overline{X}} = K(t,s)$ for $t,s \in \overline{X}$ and take the completion to obtain the Hilbert space $(\mathcal{H}_{K,\overline{X}}, \|\cdot\|_{K,\overline{X}})$. Then, for any given $\alpha \in \Re^m$ there holds

$$\|f_\alpha\|_K^2 = \alpha^\top K_{\overline{X},\overline{X}}\, \alpha.$$

$\mathcal{H}_{K,\overline{X}}$ provides the reproducing property(see[17])

$$f(x) = \langle f, K_x\rangle_{K,\overline{X}}, \qquad x \in \overline{X}.$$

Choosing $\mathcal{H}_{K,\overline{X}}$ as the hypothesis space of framework (1.1), we have the sample dependent learning regressions

$$\min_{f_\alpha \in \mathcal{H}_{K,\overline{X}}} \Big\{\frac{1}{m}\sum_{i=1}^m V\big(\big|y_i - f_\alpha(x_i)\big|\big) + \lambda\, \Omega(f_\alpha)\Big\}. \tag{1.3}$$

It has been pointed by [23] and [25] that, the error analysis for algorithm (1.3) is essentially different from and more difficult than that for (1.1). For example, when we estimate the error with capacity approach, the sample dependent nature of the algorithm will lead an extra error term called hypothesis error which makes the analysis become more complexity. In the least square loss case, [18] and [25] gave the learning rates for $\Omega(f_\alpha) = \sum_{i=1}^{m} \alpha_i^2$ and $\Omega(f_\alpha) = \sum_{i=1}^{m} |\alpha_i|$ respectively. The former is coefficient regularization with l^2-regularizer and the latter is the l^1-regularizer which leads to some sparse properties. Moreover, the kernels there are not necessarily nonsymmetric. It is known that the least square loss has the strong convexity and its derivatives are easy to be computed. This advantages make it possible for us to show the explicit expression of the solution and give the rates with integral operator approach. In the present paper, we want to show the learning rates for framework (1.3) when $V(t)$ is the Lipschitz loss and $\Omega(f_\alpha) = \|f_\alpha\|_K^2$ is the usual norm square regularizer. In this case, scheme (1.3) becomes

$$f_{z,\lambda,V} = f_{\alpha_{z,\lambda,V}}, \qquad \alpha_{z,\lambda,V} = arg \min_{\alpha\in\Re^m} \big\{\frac{1}{m}\sum_{i=1}^{m} V\big(\big|y_i - f_\alpha(x_i)\big|\big) + \lambda\, \alpha^\top\, K_{\overline{X},\overline{X}}\, \alpha\big\}. \quad (1.4)$$

We now give the definition of Lipschitz loss.

Definition 1. *We call a convex function $V(t) : \Re \to \Re_+$ a Lipschitz loss with rank $L > 0$ if it satisfies $V(0) = 0$ and*

$$|V(t) - V(t')| \le L|t - t'|, \qquad t, t' \in \Re.$$

For example, the absolute value loss $V_a(t) = |t|$, the $\varepsilon-$insensitive loss (see [14])

$$V_{il}(t) = \begin{cases} 0, & \text{if } |t| < \varepsilon, \\ |t| - \varepsilon, & \text{otherwise,} \end{cases}$$

and the pinball loss V_{pb} (see [20]) are all Lipschitz loss functions.

Before stating our discussions, let us introduce some definitions and notations. The general integral regularized framework corresponding to scheme (1.4) is defined by

$$\alpha^{(\rho)} := \alpha_{\lambda,V}^{(\rho)} = \arg\min_{\alpha\in\Re^m} \big\{\mathcal{E}_{\rho,V}(f_\alpha) + \lambda\, \alpha^\top\, K_{\overline{X},\overline{X}}\, \alpha\big\}, \quad (1.5)$$

where $f_\alpha(x) \in \mathcal{H}_{K,\overline{X}}$. The empirical measure $\rho_z(x, y)$ corresponding to the sample z with respect to a bounded $\rho-$measurable function $f(x, y)$ on Z is

$$\int_Z f(x, y)\, d\rho_z = \frac{1}{m}\sum_{i=1}^{m} f(x_i, y_i). \quad (1.6)$$

We can decompose the excess generalization error $\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) - \mathcal{E}_{\rho,V}(f_V^*)$ into two parts. The first is the sample error $\left|\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) - \mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}})\right|$, the second is the approximation error $\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*)$.

We now make some discussions about the setting (1.4) and (1.5):

- The discrete set $\overline{X}$ is taken from the sample, the hypothesis space $\mathcal{H}_{K,\overline{X}}$ is sample dependent and therefore are reproduced presently by kernel $K(x,y)$ and the sample $\overline{X}$. Then, it is convenient for us to design learning algorithm.

- We assume the loss functions $V(t)$ are Lipschitzians, they lack of strong convexity and smoothness. However, we may use sub-gradients to describe the optimality. These facts make it possible for us to use convex analysis skills in our analysis.

- Since $\mathcal{H}_{K,\overline{X}}$ is sample dependent, we know $f_{\alpha_{z,\lambda,V}}$ and $f_{\alpha^{(\rho)}}$ are sample dependent as well. Therefore, both the errors are sample dependent. We need to use possibility to derive the error analysis.

- A direct advantage of framework (1.4) over framework (1.1) in that the former is a finite dimensional optimization problem and, therefore, is easy to design algorithm. Moreover,we shall see in Section 3.2 that it will lead to a better method for us to estimate the approximation error conveniently.

Basing on these features, we shall estimate the sample error by a large number law provided by [15], the sub-gradients of the loss function $V(t)$ and the parallelogram identity for Hilbert spaces. Also, we shall give an estimate for the approximation error by the Chebyshev inequality under the condition that f_V^* has the type of

$$f_V^*(x) = \int_X \varphi(y)\ K(x,y)\ d\rho_X(y), \qquad \varphi \in L^2(\rho_X), \tag{1.7}$$

where $L^2(\rho_X) = \{f(x) : \ \|f\|_{L^2(\rho_X)} = (\int_X |f(x)|^2\ d\rho_X)^{\frac{1}{2}} < +\infty\}$.

2 Notations and Results

To make the statement clearer, we give here some symbols. By $\Re^m (m \geq 1)$ we denote the $m-$dimensional Euclidean space with the usual inner product, i.e., for any

$a = (a_1, a_2, \cdots, a_m)^\top, b = (b_1,\ b_2,\ \cdots,\ b_m)^\top \in \Re^m$, we define $\|a\|^2 = \sum_{i=1}^{m} \big|a_i\big|^2 = a^\top a, \quad \langle a, b\rangle = \sum_{i=1}^{m} a_i\, b_i = a^\top b.$

For a vector function $f(x) = (f_1(x),\ \cdots,\ f_m(x))^\top$ and a real function $\alpha(x)$ on X we define $f(x)\alpha(x) = \big(f_1(x)\alpha(x), \cdots, f_m(x)\alpha(x)\big)^\top$ and

$$\int_X f(x)\ \alpha(x)\ d\rho_X = \big(\int_X f_1(x)\ \alpha(x)\ d\rho_X, \cdots, \int_X f_m(x)\ \alpha(x)\ d\rho_X\big)^\top.$$

Let $\{X_1, X_2, \cdots, X_m\}$ be i.i.d. random variables with the same distribution ρ and for a function $g(X_1,\ X_2,\ \cdots,\ X_m)$ on Z^m we define the mathematical expectation $E(g)$ by

$$E(g) = \int_{Z^m} g(X_1,\ X_2,\ \cdots,\ X_m)\ d\rho(X_1)\ \cdots\ d\rho(X_m).$$

Let f be a real function on a Hilbert space $(X, \|\cdot\|)$ with norm $\|\cdot\|$ being induced by an inner product $\langle\cdot,\cdot\rangle$. Then, the sub-differential $\partial f(x)$ of $f(x)$ at x is defined as (see [7]),

$$\partial f(x) = \{\xi \in X : f(x') - f(x) \geq \langle \xi,\ x' - x\rangle\}, \qquad \forall x' \in X \tag{2.1}$$

or

$$\partial f(x) = \{\xi \in X : f(x) - f(x') \leq \langle \xi, x - x'\rangle\}, \qquad \forall x' \in X, \tag{2.2}$$

and if $\xi \in \partial f(x)$ we call ξ a sub-gradient of f at x.

If $f(x) : \Re^m \to \Re$ is a differentiable function, then, $\partial f(x) = \{\nabla f(x)\}$, where $\nabla f(x) = (\frac{\partial f}{\partial x_1},\ \cdots,\ \frac{\partial f}{\partial x_m})$ is the usual gradient.

A well known result is, if $f(x)$ is a convex function on X, then x_0 is the minimal value point of $f(x)$ on X if and only if $0 \in \partial f(x_0)$(see [7]).

We present our sample error estimate as the following Theorem 1.

Theorem 1. *Let V be a Lipschitz loss with rank L. $\alpha^{(\rho)}$ and $\alpha_{z,\lambda,V}$ are defined in (1.5) and (1.4) respectively. Then, for any $0 < \delta < 1$, with confidence $1 - \delta$, there holds*

$$|\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) - \mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}})| \leq \frac{4kL^2 \log\frac{2}{\delta}}{\lambda\sqrt{m}}(1 + \frac{4\log\frac{2}{\delta}}{\sqrt{m}}), \tag{2.3}$$

where $k = \sup_{(x,y)\in X\times X} |K(x,y)|$.

The approximation error is related to RKHS approximation. Many references(see e.g. [9, 21, 24]) show it by $K-$functional and whose convergence rates are often given by assumptions. In the present paper, we shall give a bound for it when f_V^* satisfies (1.7).

Theorem 2. *Let $\alpha_{z,\lambda,V}$ be defined as in (1.4), f_V^* satisfy (1.7). Then, for any $\delta \in (0,1)$, with confidence $1-\delta$, there holds*

$$\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*) \;\leq\; \frac{1}{\delta}\Big[L\sqrt{\frac{\gamma - \|f_V^*\|^2_{L^2(\rho_X)}}{m}} + \lambda\beta + \frac{\lambda\nu}{m}\Big], \tag{2.4}$$

where $\gamma = \int_X \int_X \varphi(y)^2\ K(x,y)^2\ d\rho_X(x)\ d\rho_X(y), \beta = \int_X \int_X \varphi(x)\varphi(y)K_x(y)d\rho_X(x)d\rho_X(y)$ and $\nu = \int_X \varphi(x)^2 K(x,x)d\rho_X(x)$.

We now make some further analysis on rates (2.3) and (2.4).

- (2.3) shows that, the method used here is different from the integral operator approach (see [15]), however, they give the same error rates.
- The method used here induces a capacity independent estimate. It is a covering number independent approach, avoids the hypothesis error estimate, and, also, there is no need to make additional assumptions on the Mercer kernels $K(x,y)$.
- Define an integral operator L_K given by $L_K(f,x) = \int_X K(x,t)f(t)\ d\rho_X(t)$. Then, (1.7) implies $f_V^*(x)$ belongs to the range of operator L_K. This class of function has, in fact, been studied in [15, 16].

3 Proof of the Results

Theorem 1 will be shown in Section 3.1 and the Theorem 2 will be shown in Section 3.2.

3.1. Proof of Theorem 1

To show Theorem 1, we first give a proposition to show the robustness for the solution of (1.5).

Proposition 1. *Let $V(t)$ be a convex loss function on $\Re$, ρ and γ be given Borel probability distributions on Z. $\alpha^{(\rho)}$ and $\alpha^{(\gamma)}$ are the solutions of scheme (1.5) for ρ and γ respectively. Then, there is an $H(t) \in \partial V(|t|)$ such that*

$$\begin{aligned}\left\|f_{\alpha^{(\rho)}}-f_{\alpha^{(\gamma)}}\right\|_{K,\overline{X}} \;\leq\; & \frac{1}{\lambda}\Big\|\int_Z H\big(\big|y-f_{\alpha^{(\rho)}}(x)\big|\big)\;K_x(\cdot)\;d\rho(x,y)\\ & -\int_Z H\big(\big|y-f_{\alpha^{(\rho)}}(x)\big|\big)\;K_x(\cdot)\;d\gamma(x,y)\Big\|_K. \end{aligned}\tag{3.1}$$

Inequality (3.1) shows how the solution of scheme (1.5) is influenced by distribution ρ. In fact, we have the following clearer one.

If $V(t)$ is an even differentiable convex function on $\Re$, then, $\partial V(t)=\{V'(t)\}$, inequality (3.1) becomes

$$\begin{aligned}\left\|f_{\alpha^{(\rho)}}-f_{\alpha^{(\gamma)}}\right\|_{K,\overline{X}} \;\leq\; & \frac{1}{\lambda}\Big\|\int_Z V'\big(y-f_{\alpha^{(\rho)}}(x)\big)\;K_x(\cdot)\;d\rho(x,y)\\ & -\int_Z V'\big(y-f_{\alpha^{(\rho)}}(x)\big)\;K_x(\cdot)\;d\gamma(x,y)\Big\|_K. \end{aligned}$$

To show Proposition 1 we need some lemmas.

Lemma 1. *Let $V(t)$ be a convex loss function on $\Re$, ρ be a given Borel probability distribution on Z.$\alpha^{(\rho)}$ is the unique solution of scheme (1.5). Then,*

(i). For any $(x,y)\in Z$ and any given $\alpha\in\Re^m$ there holds

$$\partial_\alpha V\big(\big|y-f_\alpha(x)\big|\big)=-\partial V\big(\big|y-f_\alpha(x)\big|\big)\;K_{\overline{X}}(x),\qquad x\in X,\tag{3.2}$$

where $K_{\overline{X}}(x)=\{K_{x_1}(x),K_{x_2}(x),\cdots,K_{x_m}(x)\}$.

(ii). There is an $H(t)\in\partial V(|t|)$ such that for any $f\in\mathcal{H}_{K,\overline{X}}$

$$\big\langle f,2\lambda f_{\alpha^{(\rho)}}\big\rangle_{K,\overline{X}}=\Big\langle f,\int_Z H\big(\big|y-f_{\alpha^{(\rho)}}(x)\big|\big)\;K_x(\cdot)\;d\rho(x,y)\Big\rangle_K.\tag{3.3}$$

Proof. Since $\lambda>0$, we known $\|f_\alpha\|^2_{K,\overline{X}}$ is a strict convex function about α on $\Re^m$. On the other hand, since $V(t)$ is a convex function, we know (1.5) is a strict convex optimization problem about α on $\Re^m$. The solution $\alpha^{(\rho)}$ is thus uniqueness.

Proof of (i). By the Theorem 4.2.1 in [12] we have following results:

If $A:\Re^q\to\Re^q$ is an affine mapping ($Ax=A_0x+b$) with A_0 linear and $b\in\Re^q$ and let g be a finite convex function on $\Re^q$. Then,

$$\partial(g\circ A)(x)=A_0^*\;\partial g(Ax)\tag{3.4}$$

for all $x \in \Re^q$, *where* A_0^* *is the adjoint of* A_0.

Since $V\big(\big|y - f_\alpha(x)\big|\big) = V(|y - K_{\overline{X}}(x)\alpha|)$, by taking $A_0 = K_{\overline{X}}(x)$ and $b = y$ in (3.4), we have (3.2).

Proof of (ii). By Theorem 4.1.1 of [12] we know following results:

If f_i $(i = 1, 2, \cdots, m)$ *are convex functions on* X *and* $\lambda_i (i = 1, 2, \cdots, m)$ *are positive scalars, then,*

$$\partial(\sum_{i=0}^{m} \lambda_i \ f_i(x)) = \sum_{i=0}^{m} \lambda_i \ \partial f_i(x).$$

Since $\alpha^{(\rho)}$ is the minimizer of (1.5) and $\partial_\alpha(\alpha^\top \ K_{\overline{X},\overline{X}} \ \alpha)|_{\alpha=\alpha^{(\rho)}} = 2 \ K_{\overline{X},\overline{X}} \ \alpha^{(\rho)}$, we have

$$\begin{aligned} 0 \ &\in \ \partial_\alpha\big(\int_Z V\big(\big|y - f_\alpha(x)\big|\big) \ d\rho\big)\big|_{\alpha=\alpha^{(\rho)}} + 2\lambda \ \ K_{\overline{X},\overline{X}} \ \alpha^{(\rho)} \\ &= \ \int_Z \partial_\alpha V\big(\big|y - f_\alpha(x)\big|\big)\big|_{\alpha=\alpha^{(\rho)}} \ d\rho + 2\lambda \ K_{\overline{X},\overline{X}} \ \alpha^{(\rho)} \\ &= \ -\int_Z \partial V\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \ K_{\overline{X}}(x)^\top \ d\rho + 2\lambda K_{\overline{X},\overline{X}} \ \alpha^{(\rho)}, \end{aligned}$$

where, in the last deduction, we have used equality (3.2). Hence, there is $H(t) \in \partial V(|t|)$ such that

$$2\lambda \ K_{\overline{X},\overline{X}} \ \alpha^{(\rho)} = \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \ K_{\overline{X}}(x)^\top \ d\rho. \tag{3.5}$$

(3.5) yields

$$\begin{aligned} (2\lambda f_{\alpha^{(\rho)}}(x_1), \cdots, 2\lambda f_{\alpha^{(\rho)}}(x_m)) \ &= \ (\int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \ K_x(x_1) \ d\rho(x,y)), \\ &\qquad \cdots, \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \ K_x(x_m) \ d\rho(x,y)). \end{aligned} \tag{3.6}$$

Since $f_{\alpha^{(\rho)}}(x_i) = \langle f_{\alpha^{(\rho)}}, K_{x_i}\rangle_{K,\overline{X}}$ and $K_x(x_i) = \langle K_x, K_{x_i}\rangle_K$ for $i = 1, 2, \cdots, m$, we have by (3.6) that

$$\begin{aligned} &\Big(2\lambda\langle f_{\alpha^{(\rho)}}, K_{x_1}\rangle_{K,\overline{X}}, \cdots, 2\lambda\langle f_{\alpha^{(\rho)}}, K_{x_m}\rangle_{K,\overline{X}}\Big) \\ = \ &\Big(\langle \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \ K_x(\cdot) \ d\rho(x,y), K_{x_1}\rangle_K, \\ &\cdots, \langle \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \ K_x(\cdot) \ d\rho(x,y), K_{x_m}\rangle_K\Big). \end{aligned}$$

Therefore,

$$\langle 2\lambda f_{\alpha^{(\rho)}}, K_{x_i}\rangle_{K,\overline{X}} = \Big\langle \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ K_x(\cdot)\ d\rho(x,y), K_{x_i}\Big\rangle_K, \qquad i = 1, 2, \cdots, m.$$

(3.3) then holds.

Lemma 3. *For any $f, g \in \mathcal{H}_{K,\overline{X}}$ there holds*

$$\big\|f\big\|^2_{K,\overline{X}} - \big\|g\big\|^2_{K,\overline{X}} = 2\big\langle f - g,\ g\big\rangle_{K,\overline{X}} + \big\|f - g\big\|^2_{K,\overline{X}}. \tag{3.7}$$

Proof. (3.7) is a reformed parallelogram identity.

Proof of Proposition 1. On one hand, the reproducing property yields

$$f_{\alpha^{(\gamma)}}(x) - f_{\alpha^{(\rho)}}(x) = \Big\langle K_x, f_{\alpha^{(\gamma)}} - f_{\alpha^{(\rho)}}\Big\rangle_K, \qquad x \in X.$$

On the other hand, let H be defined as in (3.3). Then, (2.1) gives

$$\begin{aligned} & V\big(\big|y - f_{\alpha^{(\gamma)}}(x)\big|\big) - V\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \\ \geq\ & H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)(f_{\alpha^{(\rho)}}(x) - f_{\alpha^{(\gamma)}}(x)) \\ =\ & \Big\langle f_{\alpha^{(\gamma)}} - f_{\alpha^{(\rho)}},\ -H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ K_x\Big\rangle_K. \end{aligned} \tag{3.8}$$

(3.8) gives

$$\begin{aligned} & \int_Z V\big(\big|y - f_{\alpha^{(\gamma)}}(x)\big|\big)d\gamma - \int_Z V\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ d\gamma \\ \geq\ & \Big\langle f_{\alpha^{(\gamma)}} - f_{\alpha^{(\rho)}},\ -\int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ K_x(\cdot)\ d\gamma(x,y)\Big\rangle_K. \end{aligned}$$

Above equality and (3.7) yields

$$\begin{aligned} & \big(\mathcal{E}_{\gamma,V}(f_{\alpha^{(\gamma)}}) + \lambda\ \big\|f_{\alpha^{(\gamma)}}\big\|^2_{K,\overline{X}}\big) - \big(\mathcal{E}_{\gamma,V}(f_{\alpha^{(\rho)}}) + \lambda\ \big\|f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}}\big) \\ \geq\ & \big\langle f_{\alpha^{(\gamma)}} - f_{\alpha^{(\rho)}},\ -\int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ K_x(\cdot)\ d\gamma(x,y)\big\rangle_K \\ & +2\lambda\ \big\langle f_{\alpha^{(\gamma)}} - f_{\alpha^{(\rho)}},\ f_{\alpha^{(\rho)}}\big\rangle_{K,\overline{X}} + \lambda\ \big\|f_{\alpha^{(\rho)}} - f_{\alpha^{(\gamma)}}\big\|^2_{K,\overline{X}} \\ =\ & \Big\langle f_{\alpha^{(\gamma)}} - f_{\alpha^{(\rho)}},\ 2\lambda\ f_{\alpha^{(\rho)}} - \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ K_x(\cdot)\ d\gamma(x,y)\Big\rangle_K \\ & +\lambda\ \big\|f_{\alpha^{(\rho)}} - f_{\alpha^{(\gamma)}}\big\|^2_{K,\overline{X}} \\ =\ & \big\langle f_{\alpha^{(\gamma)}} - f_{\alpha^{(\rho)}},\ \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ K_x(\cdot)\ d\rho(x,y) \\ & - \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\ K_x(\cdot)\ d\gamma(x,y)\big\rangle_K + \lambda\ \ \big\|f_{\alpha^{(\rho)}} - f_{\alpha^{(\gamma)}}\big\|^2_{K,\overline{X}}, \end{aligned} \tag{3.9}$$

where, in the last deduction, we have used equation (3.3). By the definitions of $\alpha^{(\rho)}$ and $\alpha^{(\gamma)}$ we have

$$\big(\mathcal{E}_{\gamma,V}(f_{\alpha^{(\gamma)}}) + \lambda\big\|f_{\alpha^{(\gamma)}}\big\|^2_{K,\overline{X}}\big) - \big(\mathcal{E}_{\gamma,V}(f_{\alpha^{(\rho)}}) + \lambda\,\big\|f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}}\big) \le 0,$$

which and (3.9) yields

$$\begin{aligned}
\lambda\,\big\|f_{\alpha^{(\rho)}} - f_{\alpha^{(\gamma)}}\big\|^2_{K,\overline{X}} &\le \big\langle f_{\alpha^{(\rho)}} - f_{\alpha^{(\gamma)}},\, \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\, K_x(\cdot)\, d\rho(x,y) \\
&\quad - \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\, K_x(\cdot)\, d\gamma(x,y)\big\rangle_K \\
&\le \big\|f_{\alpha^{(\rho)}} - f_{\alpha^{(\gamma)}}\big\|_{K,\overline{X}} \times \big\| \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) K_x(\cdot)\, d\rho(x,y) \\
&\quad - \int_Z H\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big)\, K_x(\cdot)\, d\gamma(x,y)\big\|_K.
\end{aligned}$$

(3.1) thus holds.

Proof of Theorem 1. By equality (3.7) we have

$$\big\|f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}} - \big\|f_{\alpha_{z,\lambda,V}}\big\|^2_{K,\overline{X}} = 2\big\langle f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}},\, f_{\alpha_{z,\lambda,V}}\big\rangle_{K,\overline{X}} + \big\|f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}}\big\|^2_{K,\overline{X}}. \quad (3.10)$$

(3.10) and the definition of $\alpha^{(\rho)}$ make

$$\begin{aligned}
&\big|\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}})\big| \\
\le\ & \big\{\big(\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) + \lambda\,\|f_{\alpha_{z,\lambda,V}}\|^2_{K,\overline{X}}\big) - \big(\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) + \lambda\|f_{\alpha^{(\rho)}}\|^2_{K,\overline{X}}\big)\big\} \\
&+\lambda\big|\|f_{\alpha^{(\rho)}}\|^2_{K,\overline{X}} - \|f_{\alpha_{z,\lambda,V}}\|^2_{K,\overline{X}}\big| \\
=\ & \big\{\big(\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) + \lambda\|f_{\alpha_{z,\lambda,V}}\|^2_{K,\overline{X}}\big) - \big(\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) + \lambda\|f_{\alpha^{(\rho)}}\|^2_{K,\overline{X}}\big)\big\} \\
&+\lambda\big|2\big\langle f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}},\, f_{\alpha_{z,\lambda,V}}\big\rangle_{K,\overline{X}} + \big\|f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}}\big\|^2_{K,\overline{X}}\big| \\
\le\ & \big\{\big(\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) + \lambda\|f_{\alpha_{z,\lambda,V}}\|^2_{K,\overline{X}}\big) - \big(\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) + \lambda\,\|f_{\alpha^{(\rho)}}\|^2_{K,\overline{X}}\big)\big\} \\
&+2\lambda\|f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}}\|_{K,\overline{X}} \times \|f_{\alpha_{z,\lambda,V}}\|_{K,\overline{X}} + \lambda\big\|f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}}\big\|^2 \\
=\ & A + 2\lambda\,\|f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}}\|_{K,\overline{X}} \times \|f_{\alpha_{z,\lambda,V}}\|_{K,\overline{X}} + \lambda\big\|f_{\alpha^{(\rho)}} - f_{\alpha_{z,\lambda,V}}\big\|^2_{K,\overline{X}}. \quad (3.11)
\end{aligned}$$

A reformulation of (3.8) or (2.2) gives

$$\begin{aligned}
&V\big(\big|y - f_{\alpha_{z,\lambda,V}}(x)\big|\big) - V\big(\big|y - f_{\alpha^{(\rho)}}(x)\big|\big) \\
\le\ & \Big\langle f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}},\, -H\big(\big|y - f_{\alpha_{z,\lambda,V}}(x)\big|\big)\, K_x\Big\rangle_K.
\end{aligned}$$

Therefore,

$$\begin{aligned} & \mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) - \mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) \\ \leq\ & \big\langle -\int_Z H(|y - f_{\alpha_{z,\lambda,V}}(x)|)\ K_x(\cdot)\ d\rho(x,y),\ f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}} \big\rangle_K. \end{aligned} \tag{3.12}$$

By (3.10) we have

$$\big\|f_{\alpha_{z,\lambda,V}}\big\|^2_{K,\overline{X}} - \big\|f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}} = 2\big\langle f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}},\ f_{\alpha_{z,\lambda,V}} \big\rangle_{K,\overline{X}} - \big\|f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}},$$

which and (3.12) give

$$\begin{aligned} A &= \big(\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}}) + \lambda\|f_{\alpha_{z,\lambda,V}}\|^2_{K,\overline{X}}\big) - \big(\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) + \lambda\|f_{\alpha^{(\rho)}}\|^2_{K,\overline{X}}\big) \\ &\leq \big\langle -\int_Z H(|y - f_{\alpha_{z,\lambda,V}}(x)|)\ K_x(\cdot)\ d\rho(x,y), f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}} \big\rangle_K \\ &\quad +2\lambda\langle f_{\alpha_{z,\lambda,V}},\ f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\rangle_{K,\overline{X}} - \lambda\ \big\|f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}} \\ &= \big\langle -\int_Z H(|y - f_{\alpha_{z,\lambda,V}}(x)|)\ K_x(\cdot)\ d\rho(x,y) + 2\lambda\ f_{\alpha_{z,\lambda,V}},\ f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}} \big\rangle_K \\ &\quad -\lambda\big\|f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}}. \end{aligned}$$

By (3.3) and (1.6) we know for any $f \in \mathcal{H}_{K,\overline{X}}$

$$\langle f, 2\lambda f_{\alpha_{z,\lambda,V}}\rangle_{K,\overline{X}} = \Big\langle f, \frac{1}{m}\sum_{i=1}^m\ H(|y_i - f_{\alpha_{z,\lambda,V}}(x_i)|)\ K_{x_i}(\cdot)\Big\rangle_{K,\overline{X}}, \tag{3.13}$$

Therefore,

$$\begin{aligned} A &= \big\langle \frac{1}{m}\sum_{i=1}^m\ H(|y_i - f_{\alpha_{z,\lambda,V}}(x_i)|)\ K_{x_i}(\cdot) - \int_Z\ H(|y - f_{\alpha_{z,\lambda,V}}(x)|)K_x(\cdot)\ d\rho(x,y) \\ &\quad ,\ f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\big\rangle_K - \lambda\big\|f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}} \\ &\leq \big\|\int_Z\ H(|y - f_{\alpha_{z,\lambda,V}}(x)|)\ K_x(\cdot)\ d\rho(x,y) - \frac{1}{m}\sum_{i=1}^m\ H(|y_i - f_{\alpha_{z,\lambda,V}}(x_i)|)\ K_{x_i}(\cdot)\big\|_K \\ &\times\ \big\|f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\big\|_{K,\overline{X}} - \lambda\big\|f_{\alpha_{z,\lambda,V}} - f_{\alpha^{(\rho)}}\big\|^2_{K,\overline{X}}. \end{aligned} \tag{3.14}$$

(3.14) and (3.11) yield

$$\begin{aligned} & \big|\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}})\big| \\ \leq\ & \big\|\int_Z\ H(|y - f_{\alpha_{z,\lambda,V}}(x)|)\ K_x(\cdot)\ d\rho(x,y) - \frac{1}{m}\sum_{i=1}^m H(|y_i - f_{\alpha_{z,\lambda,V}}(x_i)|)\ K_{x_i}(\cdot)\big\|_K \end{aligned}$$

$$
\begin{aligned}
&\times\big\|f_{\alpha_{z,\lambda,V}}-f_{\alpha^{(\rho)}}\big\|_{K,\overline{X}}+2\lambda\,\big\|f_{\alpha^{(\rho)}}-f_{\alpha_{z,\lambda,V}}\big\|_{K,\overline{X}}\times\big\|f_{\alpha_{z,\lambda,V}}\big\|_{K,\overline{X}}\\
\leq\;&\Big(\big\|\int_Z H(|y-f_{\alpha_{z,\lambda,V}}(x)|)\;K_x(\cdot)\;d\rho(x,y)-\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)\;K_{x_i}(\cdot)\big\|_K\\
&+2\lambda\,\big\|f_{\alpha_{z,\lambda,V}}\big\|_{K,\overline{X}}\Big)\big\|f_{\alpha^{(\rho)}}-f_{\alpha_{z,\lambda,V}}\big\|_{K,\overline{X}}\\
=\;&\Big(\big\|\int_Z H(|y-f_{\alpha_{z,\lambda,V}}(x)|)\;K_x(\cdot)\;d\rho(x,y)-\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)\;K_{x_i}(\cdot)\big\|_K\\
+\;&\|\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)\;K_{x_i}(\cdot)\|_{K,\overline{X}}\Big)\big\|f_{\alpha^{(\rho)}}-f_{\alpha_{z,\lambda,V}}\big\|_{K,\overline{X}},
\end{aligned}
$$

where, in the last deduction, we have used equation (3.13). By (3.1) we have

$$
\begin{aligned}
&\big\|f_{\alpha^{(\rho)}}-f_{\alpha_{z,\lambda,V}}\big\|_{K,\overline{X}}\\
&\leq\frac{1}{\lambda}\big\|\int_Z H(|y-f_{\alpha_{z,\lambda,V}}(x)|)\;K_x(\cdot)\;d\rho(x,y)-\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)\;K_{x_i}(\cdot)\big\|_K.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\big|\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}})-\mathcal{E}_{\rho,V}(f_{\alpha_{z,\lambda,V}})\big|\\
\leq\;&\frac{1}{\lambda}\Big(\big\|\int_Z H(|y-f_{\alpha_{z,\lambda,V}}(x)|)\;K_x(\cdot)\;d\rho(x,y)-\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)\;K_{x_i}(\cdot)\big\|_K\\
+\;&\|\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)K_{x_i}(\cdot)\|_{K,\overline{X}}\Big)\times\big\|\int_Z H(|y-f_{\alpha_{z,\lambda,V}}(x)|)\;K_x(\cdot)\;d\rho(x,y)\\
&-\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)\;K_{x_i}(\cdot)\big\|_K. \qquad (3.15)
\end{aligned}
$$

By the Proposition 1.5 in Chapter 2 of [7] we know

If $f(x):\Re\to\Re$ is a Lipschitz function with rank L, then, $\partial f(x)$ is a nonempty, convex, subset of $\Re$, and $\|\xi\|\leq L$ for every $\xi\in\partial f(x)$.

Then, the assumption that $V(t)$ is a Lipschitz loss with rank L yields $|H(|y-f_{\alpha_{z,\lambda,V}}(x)|)|\leq L$ for all $(x,y)\in X\times Y$. Since

$$
\|K_x\|_K=\sqrt{\langle K_x,K_x\rangle_K}=\sqrt{K(x,x)}\leq\sqrt{k},
$$

we have

$$\begin{aligned} & \left\|H\big(\big|y-f_{\alpha_{z,\lambda,V}}(x)\big|\big)\ K_x(\cdot)\right\|_K \\ = \ & |H\big(\big|y-f_{\alpha_{z,\lambda,V}}(x)\big|\big)| \times \left\|K_x(\cdot)\right\|_K \le \sqrt{k}L \end{aligned} \tag{3.16}$$

and

$$\int_Z \left\|H\big(\big|y-f_{\alpha_{z,\lambda,V}}(x)\big|\big)\ K_x(\cdot)\ \right\|_K^2 d\rho \le kL^2. \tag{3.17}$$

Notice that the following large number law(see [15]):

Let $(H,\ \|\cdot\|_H)$ be a Hilbert space and ξ be a random variable on (Z,ρ) with values in H. Assume $\|\xi\|_H \le \widetilde{M} < +\infty$ almost surely. Denoted $\sigma^2(\xi) = E(\|\xi\|_H^2)$. Let $\{\xi_i\}_{i=1}^m$ be independent random drawers of ρ. For any $0<\delta<1$, with confidence $1-\delta$, there holds

$$\left\|\frac{1}{m}\sum_{i=1}^m \big(\xi_i - E(\xi_i)\big)\right\|_H \le \frac{2\widetilde{M}log(2/\delta)}{m} + \sqrt{\frac{2\sigma^2(\xi)\ log(2/\delta)}{m}}. \tag{3.18}$$

Take $\xi(x,y;\cdot) = H(|y-f_{\alpha_{z,\lambda,V}}(x)|)\ K_x(\cdot)$, then, for any given $(x,y)\in Z, \xi(x,y;\cdot)\in \mathcal{H}_K$. (3.16) and (3.17) yield

$$\|\xi\|_K \le \sqrt{k}L \text{ and } E(\|\xi\|_K^2) \le kL^2. \tag{3.19}$$

We know by (3.18) and (3.19) that, with confidence $1-\delta$, there holds

$$\begin{aligned} & \left\| \int_Z H(|y-f_{\alpha_{z,\lambda,V}}(x)|)\ K_x(\cdot)\ d\rho - \frac{1}{m}\sum_{i=1}^m\ H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)\ K_{x_i}(\cdot)\right\|_K \\ \le \ & \sup_{\|\xi\|_K\le\sqrt{k}L} \left\| \int_Z \xi(x,y)\ d\rho - \frac{1}{m}\sum_{i=1}^m \xi(x_i,y_i)\right\|_K \\ \le \ & \sqrt{k}L \times (\frac{2log\frac{2}{\delta}}{m} + \sqrt{\frac{2log\frac{2}{\delta}}{m}}) \le \frac{4\sqrt{k}L}{\sqrt{m}} log\frac{2}{\delta}. \end{aligned} \tag{3.20}$$

By (3.16) we have

$$\left\|\frac{1}{m}\sum_{i=1}^m H(|y_i-f_{\alpha_{z,\lambda,V}}(x_i)|)K_{x_i}(\cdot)\right\|_{K,\overline{X}} \le \sqrt{k}L, \tag{3.21}$$

(3.15),(3.20) and (3.21) yield (2.3).

3.2. Proof of Theorem 2

Proof. Define $\alpha^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_m^*)$ with $\alpha_i = \frac{\varphi(x_i)}{m}, i = 1, 2, \cdots, m$. Then, the definition of $\alpha^{(\rho)}$ gives

$$\begin{aligned}\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*) &\leq \mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*) + \lambda\|f_{\alpha^{(\rho)}}\|_{K,\overline{X}}^2 \\ &= \inf_{\alpha\in\Re^m}\left(\mathcal{E}_{\rho,V}(f_\alpha) - \mathcal{E}_{\rho,V}(f_V^*) + \lambda\|f_\alpha\|_{K,\overline{X}}^2\right) \\ &\leq \mathcal{E}_{\rho,V}(f_{\alpha^*}) - \mathcal{E}_{\rho,V}(f_V^*) + \lambda\|f_{\alpha^*}\|_{K,\overline{X}}^2.\end{aligned}$$

Since $V(t)$ is a Lipschitz loss with rank L, we have

$$\begin{aligned}\mathcal{E}_{\rho,V}(f_{\alpha^*}) - \mathcal{E}_{\rho,V}(f_V^*) &= |\mathcal{E}_{\rho,V}(f_{\alpha^*}) - \mathcal{E}_{\rho,V}(f_V^*)| \\ &\leq \int_Z |V(|f_{\alpha^*}(x) - y|) - V(|f_V^*(x) - y|)|\, d\rho \leq L\|f_{\alpha^*} - f_V^*\|_{L^2(\rho_X)}\end{aligned}$$

and

$$\begin{aligned}E\Big(\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*)\Big) &\leq LE(\|f_{\alpha^*} - f_V^*\|_{L^2(\rho_X)}) + \lambda E(\|f_{\alpha^*}\|_{K,\overline{X}}^2) \\ &\leq L\Big(E(\|f_{\alpha^*} - f_V^*\|_{L^2(\rho_X)}^2)\Big)^{\frac{1}{2}} + \lambda E(\|f_{\alpha^*}\|_{K,\overline{X}}^2), \quad (3.22)\end{aligned}$$

where

$$\begin{aligned}E(\|f_{\alpha^*} - f_V^*\|_{L^2(\rho_X)}^2) &\leq E\Big(\int_X \Big|\sum_{k=1}^m \frac{\varphi(X_k)}{m} \times K(x, X_k) - f_V^*(x)\Big|^2 d\rho_X\Big) \\ &\leq \int_X E\Big(\Big|\sum_{k=1}^m \frac{\varphi(X_k)}{m} \times K(x,\ X_k) - f_V^*(x)\Big|^2\Big)\, d\rho_X \\ &\leq \int_X E\Big(|f_V^*(x)|^2 - 2\sum_{k=1}^m \frac{\varphi(X_k)\ K(x, X_k)}{m} \times f_V^*(x) \\ &\quad + \sum_{k,j=1}^m \frac{\varphi(X_k)\ \varphi(X_j)}{m^2} \times K(x, X_k)\ K(x,\ X_j)\Big)\, d\rho_X.\end{aligned}$$

Since $X_1, X_2, \cdots, X_m$ are independent and $f_V^*(x)$ satisfies (1.7), we have

$$\begin{aligned}&E\Big(|f_V^*(x)|^2 - 2\sum_{k=1}^m \frac{\varphi(X_k)\ K(x,\ X_k)}{m} \times f_V^*(x) \\ &+ \sum_{k,j=1}^m \frac{\varphi(X_k)\varphi(X_j)}{m^2} \times K(x,\ X_k)\ K(x, X_j)\Big)\end{aligned}$$

$$
\begin{aligned}
&= |f_V^*(x)|^2 - \frac{2}{m}\sum_{k=1}^{m} E\big(\varphi(X_k)K(x,X_k)\big) \times f_V^*(x) \\
&\quad + \frac{1}{m^2}\sum_{k,j=1}^{m} E\big(\varphi(X_k)\varphi(X_j)\ K(x,X_k)\ K(x,\ X_j)\big) \\
&= -|f_V^*(x)|^2 + \frac{1}{m^2}\sum_{k=j} E\Big[\varphi(X_k)\varphi(X_j)\ K(x,X_k)K(x,X_j)\Big] \\
&\quad + \frac{1}{m^2}\sum_{k\neq j} E\Big[\varphi(X_k)\ \varphi(X_j)\ K(x,X_k)\ K(x,X_j)\Big] \\
&= -|f_V^*(x)|^2 + \frac{1}{m}\int_X \varphi(y)^2 K(x,y)^2\ d\rho_X(y) + \frac{m(m-1)}{m^2} \times |f_V^*(x)|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_X E\Big(|f_V^*(x)|^2 - 2\sum_{k=1}^{m} \frac{\varphi(X_k)\ K(x,X_k)}{m} \times f_V^*(x) \\
&\quad + \sum_{k,j=1}^{m} \frac{\varphi(X_k)\varphi(X_j)}{m^2} \times K(x,X_k)\ K(x,X_j)\Big)\ d\rho_X \\
&= -\|f_V^*\|^2_{L^2(\rho_X)} + \frac{1}{m}\ \int_X\int_X \varphi(y)^2\ K(x,\ y)^2\ d\rho_X(y)\ d\rho_X(x) \\
&\quad + \frac{m(m-1)}{m^2} \times \|f_V^*(x)\|^2_{L^2(\rho_X)} \\
&= \frac{\int_X\int_X \varphi(y)^2\ K(x,y)^2\ d\rho_X(y)\ d\rho_X(x) - \|f_V^*\|^2_{L^2(\rho_X)}}{m}
\end{aligned}
$$

and

$$
E(\|f_{\alpha^*} - f_V^*\|^2_{L^2(\rho_X)}) \le \frac{\int_X\int_X \varphi(y)^2\ K(x,y)^2\ d\rho_X(y)\ d\rho_X(x) - \|f_V^*\|^2_{L^2(\rho_X)}}{m}. \tag{3.23}
$$

On the other hand, since $\|f_{\alpha^*}\|^2_{K,\overline{X}} = \frac{1}{m^2}\sum\limits_{i,j=1}^{m} \varphi(x_i)K_{x_j}(x_i)\varphi(x_j)$, we have

$$
\begin{aligned}
E(\|f_{\alpha^*}\|^2_{K,\overline{X}}) &= \int_{X^m}\Big(\frac{1}{m^2}\sum_{i,j=1}^{m}\varphi(x_i)\varphi(x_j)K_{x_j}(x_i)\Big)d\rho_X(x_1)\cdots d\rho_X(x_m) \\
&= \frac{m-1}{m}\int_X\int_X \varphi(x)\varphi(y)K_x(y)d\rho_X(x)d\rho_X(y) \\
&\quad + \frac{1}{m}\int_X \varphi(x)^2 K(x,x)d\rho_X(x).
\end{aligned} \tag{3.24}
$$

(3.24),(3.23) and (3.22) gives

$$\begin{aligned} E\Big[\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*)\Big] &\leq L\sqrt{\frac{\gamma - \|f_V^*\|^2_{L^2(\rho_X)}}{m}} \\ &+ \frac{\lambda(m-1)}{m}\int_X\int_X \varphi(x)\varphi(y)K_x(y)d\rho_X(x)d\rho_X(y) \\ &+\frac{\lambda}{m}\int_X \varphi(x)^2K(x,x)d\rho_X(x), \end{aligned}$$

where $\gamma = \int_X\int_X \varphi(y)^2\ K(x,y)^2\ d\rho_X(x)\ d\rho_X(y)$. By the Chebyshev inequality we have for any $\varepsilon > 0$ that

$$\begin{aligned} Prob\{\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*) \geq \varepsilon\} &\leq \frac{E\Big[\mathcal{E}_{\rho,V}(f_{\alpha^{(\rho)}}) - \mathcal{E}_{\rho,V}(f_V^*)\Big]}{\varepsilon} \\ &\leq \frac{1}{\varepsilon}\Big[L\sqrt{\frac{\gamma - \|f_V^*\|^2_{L^2(\rho_X)}}{m}} \\ &+ \frac{\lambda(m-1)}{m}\int_X\int_X \varphi(x)\varphi(y)K_x(y)d\rho_X(x)d\rho_X(y) \\ &\quad +\frac{\lambda}{m}\int_X \varphi(x)^2K(x,x)d\rho_X(x)\Big]. \end{aligned}$$

(2.4) then holds.

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Numerical quenching for a reaction-diffusion system with coupled singular absorption terms *

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Abstract

In this paper, we study numerical approximation for a reaction-diffusion system with coupled singular absorption terms and positive Dirichlet boundary conditions. Our purpose is to find a criterion to identify the simultaneous and non-simultaneous quenching which coincides with the continuous one. We also show that the quenching set and quenching rate converge to the continuous one.

Keywords: Reaction-diffusion system; Numerical Quenching; Non-simultaneous quenching.

Mathematics Subject Classification(2000): 35B25; 35B33; 35K50

1 Introduction and main results

This paper deals with numerical approximations of the following reaction-diffusion system with coupled singular absorption terms and positive Dirichlet boundary conditions

$$\begin{cases} u_t = u_{xx} - v^{-p}, v_t = v_{xx} - u^{-q}, & (x,t) \in (-L,L) \times (0,T), \\ u(-L,t) = v(-L,t) = 1, & t \in (0,T), \\ u(L,t) = v(L,t) = 1, & t \in (0,T), \\ u(x,0) = \phi(x), v(x,0) = \varphi(x), & x \in (-L,L), \end{cases} \tag{1.1}$$

where $p, q > 0$, the initial data satisfy $0 < \phi, \varphi \leq 1$ and compatible conditions with the boundary data.

Since the initial work of Kawarada appeared in 1975 (see [10]), quenching problems have been studied by many authors, see, for instance, [2, 7, 13] and reference therein. From a numerical point of view, in [12, 18, 17] the authors present different algorithms for computing the solution of some problems with quenching. Let us remark that in these

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papers, the authors do not study the properties of the numerical solution near the quenching time.

Some features concerning the phenomenon of quenching are interesting to analyze, such as how rapidly the solutions tend to zero, that is, the quenching rate, see [1]. The extinction set for the solutions and their behavior near these points, that is, the quenching set and the quenching profile, see [11]. The possibility of extending the solution in some weak sense after quenching, that is, complete quenching or incomplete quenching, see [6, 19].

The particular case $p = q$ and $u_0 = v_0$ corresponds to the scalar problem studied by many authors (e.g., see [7] and references therein). For a numerical treatment of this scalar problem, see [3], where the authors use a fixed uniform mesh to approximate the continuous solutions.

The continuous solution of the system (1.1) was analyzed in [15, 20], where the authors find that the so-call finite-time quenching may occur for the model, because of the singular nonlinearity inner absorption terms of (1.1). They say the solution (u, v) of the problem (1.1) quenches, if there exits a time $t = T < \infty$ (T denotes the quenching time, x denotes quenching point), such that

$$\lim_{t\to T^-} \inf \min\{\min_{-L\le x\le L} u(x,t), \min_{-L\le x\le L} v(x,t)\} = 0.$$

For the related problem, in [5, 9], the authors considered a semilinear system coupled at the boundary through a nonlinear flux, and Ferreira and Perez-Llanos considered the numerical solution for this problem (see, [4]), they proved that the solutions of a semidiscretization in space quench in finite time. Moreover, they described the simultaneous versus non-simultaneous quenching phenomena, they also found the numerical quenching sets and the numerical quenching rates. Simultaneous versus non-simultaneous phenomenon has been also considered in the case of blow-up, for instance, see [8].

Now we introduce the numerical scheme for the space discretization, we discretize applying linear finite elements with mass lumping in a uniform mesh for the space variable and keeping t continuous, it is well known that this discretization in space coincides with the classic central finite difference second-order scheme, (see [4]), Mass lumping is widely used in parabolic problems with blow-up and quenching, (see, e.g., [8, 14]).

Let us consider the uniform partition of size h of the interval $[-L, L]$, $(x_i = ih, h = L/N, i = -N, \cdots, 0, \cdots, N)$, and its associated standard piecewise linear finite element space V_h. The semidiscrete approximation $(u_h(t), v_h(t)) \in V_h$ obtained by the finite element method with mass lumping is defined as

$$\int_{-L}^{L} ((u_h)_t w)^I dx = \int_{-L}^{L} (u_h)_x w_x dx - \int_{-L}^{L} ((v_h)^{-p} w)^I dx, \forall w \in V_h, \forall t \in (0, T),$$
$$\int_{-L}^{L} ((v_h)_t w)^I dx = \int_{-L}^{L} (v_h)_x w_x dx - \int_{-L}^{L} ((u_h)^{-q} w)^I dx, \forall w \in V_h, \forall t \in (0, T),$$

where the superindex I denotes the Lagrange interpolation.

We denote with $(U(t), V(t)) = ((u_{-N}, \cdots, u_N), (v_{-N}, \cdots, v_N))$ the values of the numerical approximation at the nodes $x_i = ih$ and the time t. Thus

$$(u_h(x,t), v_h(x,t)) = (\sum_{k=-N}^{N} u_k(t)\psi(x), \sum_{k=-N}^{N} v_k(t)\psi(x)),$$

where $\{\psi_k\}$ is the standard base of V_h. Then $(U(t), v(t))$ satisfies the following ODE system:

$$\begin{cases} MU'(t) = -AU(t) - MV^{-p}(t), \\ MV'(t) = -AV(t) - MU^{-p}(t), \\ u_{-N}(t) = u_N(t) = v_{-N}(t) = v_N(t) = 1, \\ (U(0), V(0)) = (\phi^I, \varphi^I), \end{cases} \tag{1.2}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and (ϕ^I, φ^I) is the Lagrange interpolation of the initial datum $(\phi(x), \varphi(x))$. Writing this system explicitly we get the following ODE system

$$\begin{cases} u_{-N}(t) = u_N(t) = 1, & v_{-N}(t) = v_N(t) = 1, \\ u_k' = \frac{1}{h^2}(u_{k+1} - 2u_k + u_{k-1}) - v_k^{-p}, & v_k' = \frac{1}{h^2}(v_{k+1} - 2v_k + v_{k-1}) - u_k^{-q}, \\ u_k(0) = \phi(x_k), & v_k(0) = \varphi(x_k), \quad -N+1 \le k \le N-1, \end{cases} \tag{1.3}$$

First, we show a convergence result that ensures for any positive τ, the convergence of the above method in sets of the form $[-L.L] \times [0, T-\tau]$. It is easy to see that v^{-p} or u^{-q} bows up when the solutions of (1.1) quenches at the finite time T, it is natural to consider uniform convergence. Since the solution develops a singularity at time $t = T$, we can't expect that the convergence result extends up to T.

Theorem 1.1. Let the pair $(u(x,t), v(x,t)) \in C^{4,1}([-L, L] \times [0, T-\tau]) \times C^{4,1}([-L, L] \times [0, T-\tau])$ be a non-negative solution of (1.1) and $(U(t), V(t))$ the numerical approximation given by (1.3). Then there exists a constant C (C depends on the norm of $(u(x,t), v(x,t))$ in the space $C^{4,1}([-L, L] \times [0, T-\tau])$) such that for every h small enough it holds

$$\max_{t \in [0, T-\tau]} \max_k \{|u(x_k, t) - u_k(t)|, |v(x_k, t) - v_k(t)|\} \le Ch.$$

Theorem 1.2. Assume that $(U(t), V(t))$ be a positive solution of the problem (1.3), we have

(i) If L is small enough, then $(U(t), V(t))$ must be global.

(ii) If L is large enough, then $(U(t), V(t))$ must be quenching in finite time.

Corollary 1.1. Let T be the quenching time for problem (1.1) and T_h be the quenching time for the approximations, then $T_h \to T$ as $h \to 0$.

The above corollary show that the numerical quenching time T_h converges to T as h goes to zero. Next, we propose a criterion to distinguish the simultaneous and non-simultaneous quenching, which is the same with the continuous one. see [15, 20].

Theorem 1.3. If $q < 1$, for every φ, we can find ϕ such that U quenches while V does not.

Theorem 1.4. If $p, q \ge 1$, then quenching in (1.3) is always simultaneous for every initial data.

Theorem 1.5. If $p < 1 \le q$ (respectively, $q < 1 \le p$), then any quenching in (1.3) is non-simultaneous with u (respectively, v)be strictly positive.

Next, we deals with quenching rates for simultaneous and non-simultaneous quenching. Introduce the notation $f \sim g$ means that there exist finite positive constants c_1, c_2 such that $c_1 g \leq f \leq c_2 g$.

Theorem 1.6. (i) If quenching is non-simultaneous and, for instance, v is the quenching variable, then for $t \to T_h$, $v_0(t) \sim (T_h - t)$.

(ii) If quenching is simultaneous, then for $t \to T_h$,

(a) $u_0(t) \sim (T_h - t)^{(p-1)/(pq-1)}, v_0(t) \sim (T_h - t)^{(q-1)/(pq-1)}$ if $p, q > 1$.

(b) $u_0(t), v_0(t) \sim (T_h - t)^{1/2}$ if $p = q = 1$.

(c) $u_0(t) \sim |\log(T_h - t)|^{-1/(q-1)}$, $v_0(t) \sim (T_h - t)|\log(T_h - t)|^{q/(q-1)}$ if $q > p = 1$.

Finally, we study the numerical quenching set, which is define as $Q(U, V) = \{x_i : u_i(t) \to 0 \text{ or } v_i(t) \to 0 \text{ as } t \to T_h\}$, and the distance between two sets A and B is define as $dist(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. We show that the numerical quenching set (denote by $Q(U, V)$) is close to the continuous one (denote by $q(u, v)$).

Theorem 1.7. For all $\varepsilon > 0$ there exists h_0 such that for $h < h_0$ we have

$$dist(Q(U, V), q(u, v)) < \varepsilon.$$

The plan of this paper is organized as follows: In Section 2 we state and prove some auxiliary results that will be used in the rest of the paper, and also we prove Theorem 1.1 and Theorem 1.2. In Section 3 we prove the results of simultaneous and non-simultaneous quenching, and the behavior of the numerical solution near the quenching time. Finally, in Section 4, we present some numerical experiments, which illustrate our results.

2 Preliminaries

Definition 2.1. We say $(\bar{U}, \bar{V})$ is a supersolution of the (1.2) if each of its components verifies

$$\begin{cases} MU'(t) \geq -AU(t) - MV^{-p}(t), \\ MV'(t) \geq -AV(t) - MU^{-p}(t), \\ (U(0), V(0)) \geq (\phi^I, \varphi^I), \end{cases} \tag{2.1}$$

Analogously, $(\underline{U}, \underline{V})$ is a subsolution of problem (1.2) if it satisfies the problem (2.1) with the revise inequalities.

Lemma 2.1. Let $(\bar{U}, \bar{V})$ and $(\underline{U}, \underline{V})$ be a super-solution and a sub-solution of the problem (1.2), respectively. Then

$$(\bar{U}, \bar{V}) \geq (U, V) \geq (\underline{U}, \underline{V}).$$

Proof. Let $(W, Z) = (\bar{U} - \underline{U}, \bar{V} - \underline{V})$. We observe that W satisfies

$$\begin{cases} w_{-N}(t) \geq 0, \quad w_N(t) \geq 0, \\ w_k'(t) \geq \frac{1}{h^2}(w_{k+1}(t) - 2w_k(t) + w_{k-1}(t)) - (\bar{v}_k^{-p}(t) - \underline{v}_k^{-p}(t)), \\ w_k(0) \geq 0, \quad -N + 1 \leq k \leq N - 1. \end{cases}$$

Next, we argue by contradiction. Without loss of generality, we assume that there exists a first time t_0 and a node i such that $w_i(t_0) = \bar{u}_i(t_0) - \underline{u}_i(t_0) < 0$ and $z_i(t_0) = \bar{v}_i(t_0) - \underline{v}_i(t_0) \geq 0$, then $-N+1 < i < N-1$ and $t_0 > 0$. Computing

$$0 \geq \bar{u}_i^{'}(t_0) - u_i^{'}(t_0) = \frac{1}{h^2}(w_{i+1}(t_0) - 2w_i(t_0) + w_{i-1}(t_0)) - (\bar{v}_i^{-p}(t_0) - \underline{v}_i^{-p}(t_0)) > 0,$$

we obtain a contradiction. This result proves that $\bar{U} \geq \underline{U}$ and $\bar{V} \geq \underline{V}$. □

Lemma 2.2. Let $\phi(x)$ and $\varphi(x)$ be a symmetric initial data in $[-L, L]$, that is $\phi(x) = \phi(-x)$ and $\varphi(x) = \varphi(-x)$, consider a symmetric partition x_i, $-N \leq i \leq N$, of this interval, i.e. $x_i = -x_{-i}$. Then, the solution of the problem (1.3) is symmetric, that is, it verifies

$$u_i(t) = u_{-i}(t).$$

Proof. This assertion follows from the uniqueness of the problem (1.3). Indeed, let us define $\hat{u}_i(t) = u_{-i}(t)$ with $i = -N, \cdots, N$. This vector is also a solution of (1.3) and at time $t = 0$ it is equal to $U(0)$. Therefore, by the uniqueness of the ODE, we have $U(t) = \hat{U}(t)$. The proof of Lemma 2.2 is finished. □

Remark. If $\phi(x)$ and $\varphi(x)$ is symmetric, then $U(0)$ and $V(0)$ (therefore $U(t)$ and $V(t)$) is symmetric. Thus, we can restrict ourselves to the half interval $[0, L]$, reducing the size of the system of ODEs to be solved.

Next, we deal with the monotonicity properties of our numerical scheme in $[0, L]$.

Lemma 2.3. Let (U, V) be a solution of (1.3) with $u_i(0) < u_{i+1}(0), v_i(0) < v_{i+1}(0), (i = 0, \cdots, N-1)$, then

$$u_i(t) < u_{i+1}(t), v_i(t) < v_{i+1}(t), 0 \leq i \leq N-1.$$

Proof. We argue by contradiction. Suppose that there exists a first time t_0 and a node x_j such that $u_{j+1}(t_0) = u_j(t_0)$ and $v_j(t_0) \leq v_{j+1}(t_0)$. By the equation (1.3) we obtain

$$0 \geq u_{j+1}^{'}(t_0) - u_j^{'}(t_0) = \frac{1}{h^2}(u_{j+2}(t_0) - u_{j-1}(t_0)) - (v_{j+1}^{-p}(t_0) - v_i^{-p}(t_0)) \geq 0,$$

We get that $u_{j+1}(t_0) = u_{j-1}(t_0)$. Applying the same argument, we get that all the nodes must be equal to $u_N(t_0) = 1$ at time t_0, but this is impossible because the initial satisfies $U(0), V(0) \leq 1$ and hence the solution verifies $u_i(t) < 1$ for all positive t. To this, by the comparison result, we get that $u_i(t) \leq 1$ (because $\bar{u}_i \equiv 1$ is a supersolution). If all the nodes attains its maximum 1 at the same time t_0, then all of them have derivative large or equal than zero, but at that times we have $u_j^{'}(t_0) = -v_j^{-p}(t_0) < 0$, and we conclude a contradiction. □

Remark. This result ensures that the minimum of $U(t)$ is attained at the node $x_0 = 0$ if the initial data ϕ and φ are symmetric and non-decreasing functions, i.e.,

$$\min_k u_k(t) = u_0(t), \min_k v_k(t) = v_0(t) \text{ for every } t \in [0, T - \tau].$$

Proof of Theorem 1.1. Define the error functions as

$$e_k(t) = u_k(t) - u(x_k, t), \varepsilon_k(t) = v_k(t) - v(x_k, t).$$

Let c be a positive constant such that $u(x_k,t), v(x_k,t) \geq c$ for every $t \in [0, T-\tau]$ and

$$t_0 = \max\{t \in [0, T-\tau] \text{ such that } |e_k(t)|, |\varepsilon_k(t)| \leq c/2\}$$

thus, we can ensure that up to time t_0 none of the solutions (neither the approximations solution nor the continuous one) quenching. The following estimations will be performed restricting ourselves to $t \in [0, t_0]$, and we will show at the end that $t_0 = T - \tau$ for every τ small enough.

We just estimate the error function $e_k(t)$, since it is easy to get an analogous estimate for $\varepsilon_k(t)$, due to the similarity of both problem in (1.3). Notice that the error function satisfies that for $-N+1 \leq k \leq N-1$

$$\begin{aligned} e_k^{'}(t) &= \frac{e_{k+1}(t) - 2e_k(t) + e_{k-1}(t)}{h^2} - (v_k^{-p}(t) - v^{-p}(x_k,t)) + O(h^2) \\ &= \frac{e_{k+1}(t) - 2e_k(t) + e_{k-1}(t)}{h^2} + p|\xi|^{-p-1} e_k(t) + O(h^2) \\ &= \frac{e_{k+1}(t) - 2e_k(t) + e_{k-1}(t)}{h^2} + p|\frac{c}{2}|^{-p-1} e_k(t) + O(h^2) \end{aligned}$$

The first and the last nodes verify $e_{-N} = e_N = 0$.
There the error function is a subsolution of the following problem

$$\begin{cases} w_{-N}(t) = w_N(t) = 0, \\ w_k^{'}(t) = \frac{w_{k+1}(t) - 2w_k(t) + w_{k-1}(t)}{h^2} + c_1 w_k(t) + C_2(h^2), \\ w_k(0) = 0, \quad k = -N, \cdots, N, \end{cases} \tag{2.2}$$

and we remark the system has a comparison principle, which can be proved as in Lemma 2.1. Let us look for a supersolution of the problem (2.2),

$$\begin{cases} g^{'}(t) = c_3 g(t) + c_1 h^2, \\ g(0) = c_2 h^2, \end{cases}$$

we obtain that the solution of the above problem has the form $g(t) = h^2(c_3 e^{c_1 t} + c_2)$. Therefore,

$$e_k(t) \leq w_k(t) \leq c_3 h^2 e^{c_1 T}.$$

Arguing in the same way with $-e_k$, we get

$$|e_k(t)| \leq w_k(t) \leq c_3 h^2 e^{c_1 T}.$$

From this inequality it is easy to see that $t_0 = T - \tau$ for every τ small enough. The Theorem 1.1 is complete. □

Proof of Theorem 1.2. Let $\theta = \min(\min_{x\in[-L,L]}(\phi(x), \min_{x\in[-L,L]} \varphi(x))$, and consider

the following system

$$\begin{cases} M\underline{\mathrm{U}}^{'}(t) = -A\underline{\mathrm{U}}(t) - M\underline{\mathrm{V}}^{-p}(t), \\ M\underline{\mathrm{V}}^{'}(t) = -A\underline{\mathrm{V}}(t) - M\underline{\mathrm{U}}^{-p}(t), \\ \underline{\mathrm{u}}_{-N}(t) = \underline{\mathrm{u}}_{N}(t) = \underline{\mathrm{v}}_{-N}(t) = \underline{\mathrm{v}}_{N}(t) = \theta, \\ (\underline{\mathrm{U}}(0), \underline{\mathrm{V}}(0)) = (\theta, \cdots, \theta), \end{cases} \tag{2.3}$$

It is easy to see that the solution $(\underline{\mathrm{U}}(t), \underline{\mathrm{V}}(t))$ is a sub-solution of (1.3) and $(\underline{\mathrm{U}}(t), \underline{\mathrm{V}}(t)) \leq (\theta, \cdots, \theta)$.

Let the vector $\underline{\mathrm{W}}$ given by

$$\begin{cases} \underline{\mathrm{w}}_{-N}(t) = \underline{\mathrm{w}}_{N}(t) = \theta, \\ \underline{\mathrm{w}}_{i-1} = h^2 M_0 + 2\underline{\mathrm{w}}_i - \underline{\mathrm{w}}_{i+1}, i = -N+1, \cdots, N-1. \end{cases}$$

where $M_0 = \max(\theta^{-p}, \theta^{-q})$, we have $\underline{\mathrm{w}}_i = \frac{M_0(|ih|^2 - L^2)}{2} + \theta$ for $i = -N, \cdots, N$, and $\min\limits_{i=-N,\cdots,N}(\underline{\mathrm{w}}_i) = \underline{\mathrm{w}}_0 = \theta - \frac{M_0 L^2}{2}$. Observe that $(\underline{\mathrm{W}}, \underline{\mathrm{W}})$ is a time-independent sub-solution of the problem (2.3), thus, it is also a sub-solution of the problem (1.3). Clearly, if $L < \sqrt{\frac{2\theta}{M_0}}$ the solution of the problem (1.3) must be global.

Next, we prove (ii).Consider the following auxiliary system

$$\begin{cases} \bar{w}_{-N}(t) = \bar{w}_N(t) = 1, \\ \bar{w}_k^{'} = \frac{1}{h^2}(\bar{w}_{k+1} - 2\bar{w}_k + \bar{w}_{k-1}) - 1, \\ \bar{w}_k(0) = 1, -N+1 \leq k \leq N-1, \end{cases} \tag{2.4}$$

Clearly, $1 \geq \bar{w}_i \geq \max(u_i(t), v_i(t))$, there exists a sequence such that $\bar{w}_i(t) \to \bar{w}_i^{\infty}$ for every $i = -N, \cdots, N$, where $\bar{w}^{\infty}$ is a stationary solution of (2.4). This solution is explicit and is given by $\bar{w}_i^{\infty} = \frac{(|ih|^2 - L^2)}{2} + 1$ for $i = -N, \cdots, N$, and $\min\limits_{i=-N,\cdots,N}(\bar{w}_i) = \bar{w}_0 = 1 - \frac{L^2}{2}$. Clearly, if $L > \sqrt{2}$ the solution of the problem (1.3) must be quenching in finite time.The Theorem 1.2 is complete. □

3 Simultaneous and non-simultaneous quenching and quenching rates

In [10, 16] the authors prove that the time-derivative (u_t, v_t) blows up when the continuous solution of (1.1) occurs quenching. Now, we prove the same result happens in our semidiscretization one.

Lemma 3.1. Let $(U(t), V(t))$ be the solution of (1.3). If the initial data satisfies the following condition

$$MU^{'}(0) = -AU(0) - MV^{-p}(0) < 0, MV^{'}(0) = -AV(0) - MU^{-p}(0) < 0, \tag{3.1}$$

then we have

(i) $U^{'}(t), V^{'}(t) < 0$ for every $t \in [0, T_h)$.

(ii) For any $\eta > 0$, there exists a $c > 0$ such that

$$u_i^{'}(t), v_i^{'}(t) < -c, i = -N + z, \cdots, N - z \text{ and } t \in [\eta, T_h), \tag{3.2}$$

where z is a integer and $0 < z < N$.

(iii) There exist $\eta, \theta > 0$ and a integer z with $0 < z < N$ such that

$$u_i^{'}(t) \leq -\theta v_i^{-p}, v_i^{'}(t) \leq -\theta u_i^{-q} \text{ for } i = -N + z, \cdots, N - z \text{ and } t \in [\eta, T_h). \tag{3.3}$$

Proof. First, we claim that both $u^{'}(t)$ and $v^{'}(t)$ are negative. To this, let $f_i(t) = u_i^{'}(t)$ and $g_i(t) = v_i^{'}(t)$. Therefore, by a simple computation, (F, G) verifies

$$\begin{cases} f_{-N}(t) = f_N(t) = 0, & g_{-N}(t) = g_N(t) = 0, \\ f_k^{'} = \frac{1}{h^2}(f_{k+1} - 2f_k + f_{k-1}) + v_k^{-p-1} g_k, & g_k^{'} = \frac{1}{h^2}(g_{k+1} - 2g_k + g_{k-1}) + u_k^{-q-1} f_k, \\ f_k(0) < 0, & g_k(0) < 0, -N + 1 \leq k \leq N - 1. \end{cases} \tag{3.4}$$

By the maximum principle, $F(t), G(t) < 0$.

Next, we prove (ii), consider the following auxiliary system

$$\begin{cases} w_{-N}(t) = w_N(t) = 1, & z_{-N}(t) = z_N(t) = 1, \\ w_k^{'} = \frac{1}{h^2}(w_{k+1} - 2w_k + w_{k-1}) - \phi_k^{-p}, & z_k^{'} = \frac{1}{h^2}(z_{k+1} - 2z_k + z_{k-1}) - \varphi_k^{-q}, \\ w_k(0) = \phi(x_k), & z_k(0) = \varphi(x_k), -N + 1 \leq k \leq N - 1, \end{cases} \tag{3.5}$$

It is easy to see that the (3.5) has a unique global solution and $(W^{'}(t), Z^{'}(t)) < 0$. Let $\bar{\Phi} = U(t) - W(t), \bar{\Psi} = V(t) - Z(t)$. A similar argument shows $\bar{\Phi}_t, \bar{\Psi}_t \leq 0$, i.e., $U_t \leq W_t, V_t \leq Z_t$ in $(0, T_h)$. Choosing $c = \min\{\min\limits_{\substack{-N+z \leq i \leq N-z \\ \eta \leq t \leq T_h}} |w_i^{'}(t)|, \min\limits_{\substack{-N+z \leq i \leq N-z \\ \eta \leq t \leq T_h}} |z_i^{'}(t)|\}$, we deduce $u_i^{'}(t), v_i^{'}(t) < -c$ for $i = -N + z, \cdots, N - z$ and $t \in [\eta, T_h)$.

Finally, in order to prove (iii), let us define $f_k = u_k^{'}(t) + \theta v_k^{-p}(t)$ and $g_k = v_k^{'}(t) + \theta u_k^{-p}(t)$. We want to use the maximum principle to show that f_k is negative. To this end, we observe

that f_k verifies

$$\begin{aligned}
m_k f_k^{'} + \sum_{j=1}^{N} a_{kj} f_j &= m_k(u_k^{''} - \theta p v_k^{-p-1} v_k^{'}) + \sum_{j=1}^{N} a_{kj}(u_j^{'} + \theta v_j^{-p}) \\
&= -m_k \theta p v_k^{-p-1} v_k^{'} + m_k p v_k^{-p-1} v_k^{'} + \theta \sum_{j=1}^{N} a_{kj} v_j^{-p} \\
&= \theta p v_k^{-p-1} (\sum_{j=1}^{N} a_{kj} v_j + m_k u_k^{-q}) + m_k p v_k^{-p-1} v_k^{'} + \theta \sum_{j=1}^{N} a_{kj} v_j^{-p} \\
&= m_k p v_k^{-p-1} g_k + \theta \sum_{j=1}^{N} a_{kj} (p v_k^{-p-1} v_j + v_j^{-p}) \\
&= m_k p v_k^{-p-1} g_k + \theta \{ \sum_{j \neq k} a_{kj} (p v_k^{-p-1} v_j + v_j^{-p}) + a_{kk}(1+p) v_k^{-p} \} \\
&= m_k p v_k^{-p-1} g_k + \theta \{ \sum_{j \neq k} a_{kj} (v_j^{-p} + p v_k^{-p-1}(v_j - v_k) - v_k^{-p}) + \sum_{j=1}^{N} a_{kj}(1+p) v_k^{-p} \}
\end{aligned}$$

Since v^{-p} is convex and from (1.3) we have $a_{ij} \leq 0$ if $i \neq j$, $a_{ii} > 0$ and $\sum\limits_{j=1}^{N} a_{ij} = 0$, it follows that $(F, G) = ((f_1, \cdots, f_n), (g_1, \cdots, g_n))$ verifies

$$\begin{aligned}
MW^{'} &\leq -AW + pV^{-p-1} G, \\
MG^{'} &\leq -AG + qU^{-q-1} F,
\end{aligned} \tag{3.6}$$

In view of (ii), we have the inial data $W(\eta), G(\eta) \leq 0$, the maximum principle implies the $u_i^{'}(t) \leq \theta v_i^{-p}, v_i^{'}(t) \leq \theta u_i^{-q}, i = -N + z, \cdots, N - z$ and $t \in [\eta, T_h)$. □

Corollary 3.1. Under the condition of (3.1), then $(U^{'}(t), V^{'}(t))$ blows up when the quenching happens in (1.3).

Proof. This result is a direct consequence of Lemma 3.2. □

Proof of Corollary 1.1. By direct integration of (3.2), we get, there exists a positive constant such that

$$u_0(t) \geq C(T_h - t), v_0(t) \geq C(T_h - t). \tag{3.7}$$

Since v quenches at times T, then there exist two positive constants such that (see [20])

$$C_1(T - t) \leq v(0, t) \leq C_2(T - t).$$

On the other hand, given $\epsilon > 0$, we can take h small enough to ensure that there exists $t_j < T$ such that

$$v(0, t) \leq \epsilon \text{ and } T - t_j < \frac{1}{C_1}\epsilon.$$

From Theorem 1.1, we have $||v_0(t_j)|| \leq 2\epsilon$. Thus,

$$|T_h - T| \leq |T_h - t_j| + |T - t_j| \leq C\epsilon.$$

Next, we characterize the ranges of parameters to distinguish simultaneous and non-simultaneous quenching. To this end, we collect here the estimates of the time derivatives obtained before (see Lemmas 2.3 and 3.1). At $x = 0$, we have

$$-v_0^{-p}(t) \le u_0^{'}(t) \le -\theta v_0^{-p}(t), \quad t \in (\eta, T). \tag{3.8}$$

$$-u_0^{-q}(t) \le v_0^{'}(t) \le -\theta u_0^{-q}(t), \quad t \in (\eta, T). \tag{3.9}$$

Proof of Theorem 1.3. Let $0 < \varphi \le 1$ be fixed, From Lemma 2.3, we have

$$u_0^{'}(t) \le -\theta v_0^{-p} \le -\theta \varphi_0^{-p}.$$

Integrating over $[0, t]$ it gives $u_0(t) - \phi_0 \le -\varphi_0^{-p} t$, since U quenches we get $T_h \le c\phi_0\varphi_0^p$. On the other hand, From Lemma 3.1 and using Lemma 2.3 again we obtain

$$v_0^{'}(t) \ge -u_0^{-q}(t) \ge -C(T_h - t)^{-q}.$$

Now, integrating in $[0, t]$ we obtain

$$v_0(t) \ge \varphi_0 - C\int_0^t (T_h - s)^{-q} ds \ge \varphi_0 - C\int_0^{T_h} (T_h - s)^{-q} ds = \varphi_0 - \frac{C}{1-q} T_h^{1-q}.$$

Choosing ϕ small in order that the quenching time T_h is so small that $v_0 \ge \frac{1}{4}\varphi_0 > 0$ for every $t \in [0, T_h]$, and non-simultaneous quenching occurs. □

Proof of Theorem 1.4. The proof is by contradiction. Assume that there exists a $c > 0$ such that $u \ge c$ in $[0, a] \times [0, T)$ and v quenching. In view of (3.9), $v_0^{'}(t) \ge -u_0^{-q}(t) \ge -c^{-q}$, integrating from t to T_h we have $v_0(t) \le c^{-q}(T_h - t)$. Combining with (3.8) we get $u_0^{'}(t) \le -\delta c^{pq}(T_h - t)^{-p}$. Integrating from η to T_h we obtain

$$u_0(T_h) - u_0(\eta) \le -\delta c^{pq} \int_\eta^{T_h} (T_h - t)^{-p} dt. \tag{3.10}$$

The convergence of the integral in (3.10) implies $p < 1$. □

Proof of Theorem 1.5. In view of (3.8) and (3.9) we have

$$(1/\theta) v^{-p} v_0^{'}(t) \le u^{-q} u_0^{'}(t) \le \theta v^{-p} v_0^{'}(t), \quad t \in (\eta, T). \tag{3.11}$$

Assume first $p < 1 < q$, integrating the first inequality in (3.11) from η to t

$$v_0^{1-p}(t) \le C_1 - C_2 u_0^{1-q}(t), \tag{3.12}$$

where C_1 and C_2 are positive constants. Notice that $1 - q < 0$ requires u remains positive up to the quenching time. The case $p < 1 = q$ can be treated in an analogous way. □

Proof of Theorem 1.6. The proof is similar to [15] and [20]. We omit it here. □

Proof of Theorem 1.7. We only take the non-simultaneous quenching case, and the other cases can be obtained by a similar argument. Without loss of generality, we consider that v is the quenching variable. We define the function $\omega : (0, \infty) \to (0, \infty)$ given by

$$\omega(d) = \inf\{v(x, t) : dist(x, q(v) \ge d, \forall t \in [0, T))\}.$$

It is easy to see that ω is a continuous, non-decreasing function with $\omega(0+) = 0$ and such that $u(x,t) \geq \omega(dist(x, q(v)))$.

For all $h > 0$, we can find $x_{i_h} \in Q(V_h)$, then we have

$$\omega(dist(x_{i_h}, Q(v))) \leq v(x_{i_h}, t) \leq v_{i_h}(t) + |v(x_{i_h}, t) - v_{i_h}(t)|.$$

From the convergence of the quenching times, Corollary 1.1, for all $h < \hat{h_0}$, there exists $\hat{h_0}$ such that $|T_h - T| \leq \delta/2$. Taking $t < T$ such that $T - t < \delta/2$, thus

$$v_{i_h}(t) \leq C_2(T_h - t) \leq C_2(|T_h - T| + |T - t|) \leq C_2\delta.$$

On the other hand, from Theorem 1.1, we get that, for all $h < \bar{h}_0$, there exists $\bar{h}_0$ such that

$$|u(x_{i_h}, t) - u_{i_h}(t)| \leq C\bar{h}_0.$$

Furthermore, we obtain that

$$dist(x_{i_h}, Q(v)) \leq \omega^{-1}(C_2\delta + Ch_0^2) = \epsilon,$$

where $h_0 = \min(\bar{h}_0, \hat{h}_0)$. □

4 Numerical Experiments

First, we consider the case $p = 0.2, q = 0.2$ and the initial data $\phi(s) = 1 - \frac{1}{2}\sin(\frac{\pi}{4}(s + 2)), \varphi(s) = 1 - \frac{1}{10}\sin(\frac{\pi}{4}(s + 2))$, We observe that the solutions of (1.3) is quenching only at the origin if the symmetric initial data with a unique minimum at $x = 0$ (see Figure 1), and the quenching is non-simultaneous (see Figure 2), then we obtain the result accord with Theorem 1.3 and 1.7.

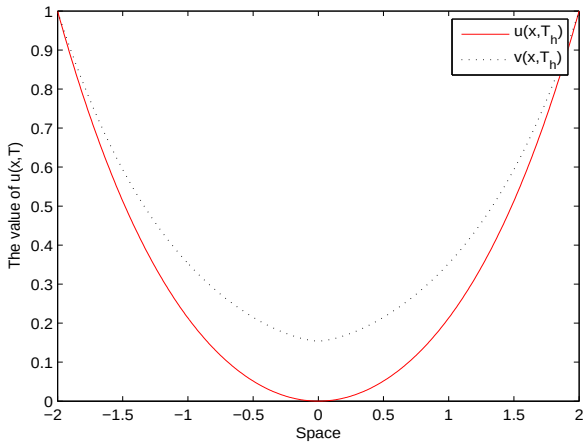


Figure 1: The value of the solution at the quenching time T_h $(p = 0.2, q = 0.2)$.

Next, we taking $p = 0.5, q = 2$ with the same initial data $\phi(s) = \varphi(s) = 1 - \frac{1}{10}\sin(\frac{\pi}{4}(s + 2))$. In this case the quenching in (1.3) is non-simultaneous with v being strictly positive (see Figure 3 and 4), then we can see that our results coincide with Theorem 1.4.

Finally, we choosing $p = 2, q = 3$, In Figure 7 we take the initial data $\phi(s) = \varphi(s) = 1 - \frac{1}{10}\sin(\frac{\pi}{4}(s + 2))$, we can see that simultaneous quenching occurs in (1.3).

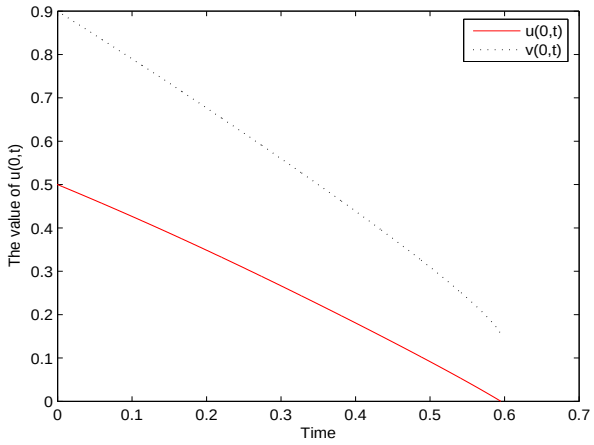


Figure 2: Evolution at the point $x_0 = 0$ of the solution $(p = 0.2, q = 0.2)$.

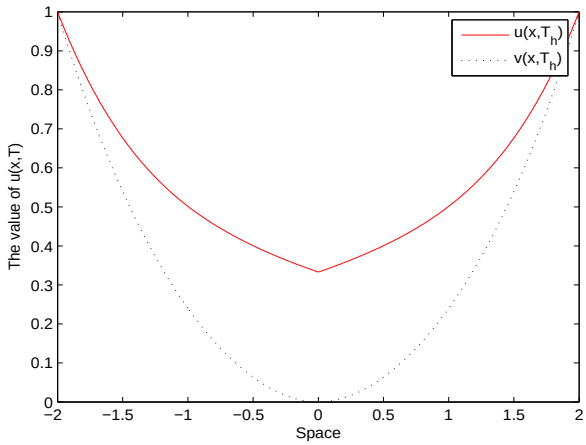


Figure 3: The value of the solution at the quenching time T_h $(p = 0.5, q = 2)$.

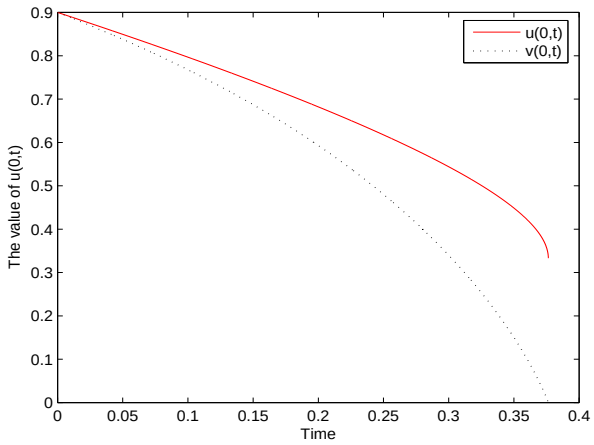


Figure 4: Evolution at the point $x_0 = 0$ of the solution $(p = 0.5, q = 2)$.

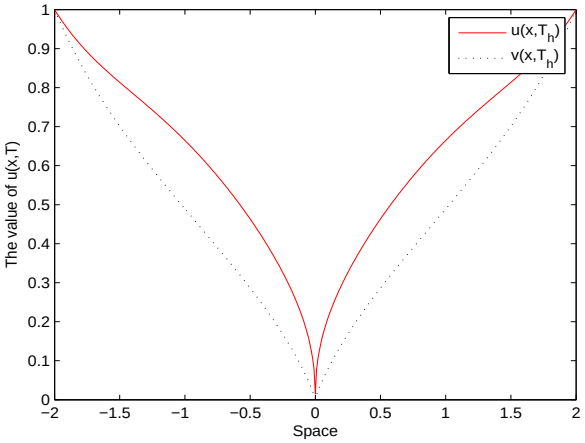


Figure 5: The value of the solution at the quenching time T_h $(p = 2, q = 3)$.

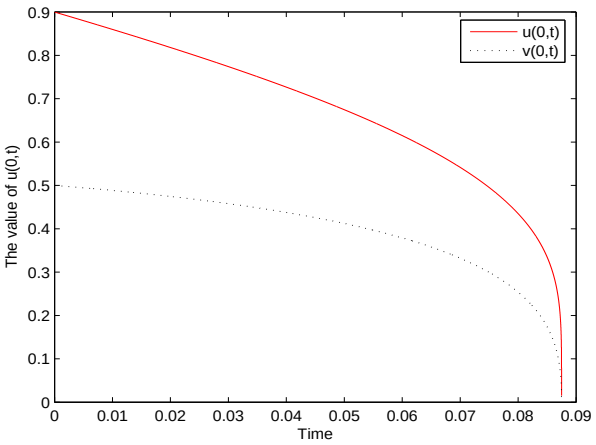


Figure 6: Evolution at the point $x_0 = 0$ of the solution $(p = 2, q = 3)$.

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q-EXTENSION OF BERNSTEIN POLYNOMIALS WITH WEIGHT $(\alpha,\ \beta)$

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Abstract In this paper, we consider a new type of q-Bernstein polynomials with fixed weight (α, β). We call them q-Bernstein polynomials with weight (α, β). We establish many interesting identities and relations satisfied by these new polynomials. These formulae are the arithmetical nature.

1. Introduction

In a recent paper [5] T. Kim proposed q-Bernstein positive linear operators and q-Bernstein polynomials. This paper seems to be a new approach in the area of q-extension of Bernstein polynomials. The main idea of our paper is to study an extension of q-Bernstein operators and polynomials which are introduced by Kim [5]. That is, we considered the extension of Kim's q-Bernstein polynomials and we call them the weighted q-Bernstein polynomials in the viewpoint of the generalization of Kim's q-Bernstein polynomials.

Throught this paper, we assume $0 < q < 1$. Define the q-numbers of x by $[x] = \frac{1-q^x}{1-q}$. Let $\mathcal{C}([0,1],\mathbb{R})$ be the space of continuous functions on $[0,1]$. For $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{C}([0,1],\mathbb{R})$, we consider linear positive operators as follows:

$$\begin{aligned} \mathbb{B}_{n,q}(f,x|\alpha,\beta) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^\alpha}^{k}[1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q|\alpha,\beta). \end{aligned} \tag{1}$$

Here $\mathbb{B}_{n,q}(f,x|\alpha,\beta)$ is called q-Bernstein operator with weight (α,β) of order n. $B_{k,n}(x,q|\alpha,\beta)$ are called the q-Bernstein polynomials with weight (α,β) of degree n.

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2. Weighted q-Bernstein polynomials

From (1) and [1,5], we note that the generating function for $B_{k,n}(x,q|\alpha,\beta)$ is given by

$$\begin{aligned} F^{(k)}_{q,\alpha,\beta}(t,x) &= \sum_{n=0}^{\infty} B_{k,n}(x,q|\alpha,\beta)\frac{t^n}{n!}, \\ &= \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!}, \end{aligned} \tag{2}$$

where $n,k \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}$ and $\alpha,\beta \in \mathbb{R}$.
By (2), we get

$$B_{k,n}(x,q|\alpha,\beta) = \begin{cases} \binom{n}{k}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}}, & \text{if } n \geq k, \\ 0, & \text{if } n < k, \end{cases} \tag{3}$$

Note that $\lim_{q\to 1} B_{k,n}(x,q|\alpha,\beta) = B_{k,n}(x)$ is the ordinary Bernstein polynomial of degree n.
In this section we prove the following interesting properties of weighted q-Bernstein polynomials:

Theorem 1 (Recurrence formula 1)**.** *For $n,k \in \mathbb{Z}_+$, $\alpha,\beta \in \mathbb{R}$ and $x \in [0,1]$, we have*

$$[1-x]_{q^{-\beta}} B_{k,n-1}(x,q|\alpha,\beta) + [x]_{q^\alpha} B_{k-1,n-1}(x,q|\alpha,\beta) = B_{k,n}(x,q|\alpha,\beta),$$

and

$$\frac{d}{dx}B_{k,n}(x,q|\alpha,\beta) = \left(\frac{\alpha\log q}{q^\alpha-1}q^{\alpha x}B_{k-1,n-1}(x,q|\alpha,\beta) - \frac{q^\beta\log q^\beta}{q^\beta-1}q^{-\beta+\beta x}B_{k,n-1}(x,q|\alpha,\beta)\right)n.$$

Proof of Theorem 1 : For $0 \leq k \leq n$, we have

$$\begin{aligned} &[1-x]_{q^{-\beta}} B_{k,n-1}(x,q|\alpha,\beta) + [x]_{q^\alpha} B_{k-1,n-1}(x,q|\alpha,\beta) \\ &= [1-x]_{q^{-\beta}}\binom{n-1}{k}[x]^k_{q^\alpha}[1-x]^{n-k-1}_{q^{-\beta}} + [x]_{q^\alpha}\binom{n-1}{k-1}[x]^{k-1}_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} \\ &= \binom{n-1}{k}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} + \binom{n-1}{k-1}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} \\ &= \binom{n}{k}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} = B_{k,n}(x,q|\alpha,\beta) \end{aligned} \tag{4}$$

and the derivative of the q-Bernstein polynomials with weight (α,β) of degree n are also polynomials of degree $n-1$:

$$\begin{aligned} &\frac{d}{dx}B_{k,n}(x,q|\alpha,\beta) \\ &= k\binom{n}{k}[x]^{k-1}_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}}\frac{\alpha\log q}{q^\alpha-1}q^{\alpha x} + (n-k)\binom{n}{k}[x]^k_{q^\alpha}[1-x]^{n-k-1}_{q^{-\beta}}\frac{q^\beta\log q^\beta}{1-q^\beta}q^{-\beta+\beta x} \\ &= \left(\frac{\alpha\log q}{q^\alpha-1}q^{\alpha x}B_{k-1,n-1}(x,q|\alpha,\beta) - \frac{q^\beta\log q^\beta}{q^\beta-1}q^{-\beta+\beta x}B_{k,n-1}(x,q|\alpha,\beta)\right)n. \end{aligned} \tag{5}$$

Therefore we obtain our desired theorem. □

Theorem 2 (Symmetry and various identities)**.** *For $n, k \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{R}$ and $x \in [0,1]$, we have*

$$B_{n-k,n}(1-x, q^{-1}|\beta, \alpha) = B_{k,n}(x, q|\alpha, \beta)$$

Morover,

$$\mathbb{B}_{n,q}(1, x|\alpha, \beta) = \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^n$$

and

$$\mathbb{B}_{n,q}(x, x|\alpha, \beta) = [x]_{q^\alpha} \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^{n-1}.$$

Proof of Theorem 2 :

Let f be a continuous function on $[0,1]$. Then the q-Bernstein operator with weight (α, β) of degree n for f is defined by

$$\mathbb{B}_{n,q}(f, x|\alpha, \beta) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x, q|\alpha, \beta) \tag{6}$$

where $0 \le x < 1$, $n \in \mathbb{Z}_+$.
By (5) and (6), we get

$$\begin{aligned} \mathbb{B}_{n,q}(1, x|\alpha, \beta) &= \sum_{k=0}^{n} B_{k,n}(x, q|\alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\ &= \left([x]_{q^\alpha} + [1-x]_{q^{-\beta}}\right)^n = \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^n. \end{aligned}$$

Also, we get from (6) that for $f(x) = x$:

$$\begin{aligned} \mathbb{B}_{n,q}(x, x|\alpha, \beta) &= \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=1}^{n} \frac{k}{n} \frac{n(n-1)\cdots(n-k+1)}{k!} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_{q^\alpha}^{k+1} [1-x]_{q^{-\beta}}^{n-k-1} \\ &= [x]_{q^\alpha} \left([x]_{q^\alpha} + [1-x]_{q^{-\beta}}\right)^{n-1} = [x]_{q^\alpha} \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^{n-1}. \end{aligned}$$

Note that the q-Bernstein polynomials with weight (α, β) of degree n are symmetric polynomials as follows:

$$\begin{aligned} B_{n-k,n}(1-x, q^{-1}|\beta, \alpha) &= \binom{n}{n-k} [1-x]_{(q^{-1})^\beta}^{n-k} [1-(1-x)]_{(q^{-1})^{-\alpha}}^k \\ &= \binom{n}{k} [1-x]_{q^{-\beta}}^{n-k} [x)]_{q^\alpha}^k \\ &= B_{k,n}(x, q|\alpha, \beta) \end{aligned} \tag{7}$$

Thus we obtain our theorem. □

Theorem 3 (Explicit formula 1)**.**

$$\mathbb{B}_{n,q}(f, x|\alpha, \beta) = \sum_{m=0}^{n} \binom{n}{m} [x]_{q^\beta}^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f\left(\frac{k}{n}\right) \left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k,$$

where $f \in \mathcal{C}([0,1],\mathbb{R})$, $\alpha,\beta \in \mathbb{R}$ *and* $n \in \mathbb{Z}_+$.

Proof of Thereom 3 : From (6), we note that

$$\begin{aligned}\mathbb{B}_{n,q}(f,x|\alpha,\beta) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q|\alpha,\beta) \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^\alpha}^{k}[1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^\alpha}^{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j[x]_{q^\beta}^{j} \qquad (8)\end{aligned}$$

From the definition of binomial coefficients, we easily get

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{k+j}\binom{k+j}{k}.$$

Let $k+j=m$. Then we have

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{m}\binom{m}{k}. \qquad (9)$$

By (8) and (9), we get

$$\mathbb{B}_{n,q}(f,x|\alpha,\beta) = \sum_{m=0}^{n}\binom{n}{m}[x]_{q^\beta}^{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} f\left(\frac{k}{n}\right)\left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^{k}.$$

This yields our theorem. □

In the special case, $\alpha=\beta$, we have the following corollary.

Corollary 4. *For* $f \in \mathcal{C}([0,1],\mathbb{R})$, $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{Z}_+$, *we have*

$$\mathbb{B}_{n,q}(f,x|\alpha,\alpha) = \sum_{m=0}^{n}\binom{n}{m}[x]_{q^\alpha}^{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} f\left(\frac{k}{n}\right). \qquad (10)$$

It is well known that the second kind Stirling number are defined by

$$\frac{(e^t-1)^k}{k!} = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt} = \sum_{n=0}^{\infty} s(n,k)\frac{t^n}{n!}, \qquad (11)$$

where $k \in \mathbb{N}$ (see[3-5]).

Let Δ be the shift difference operator with

$$\Delta f(x) = f(x+1) - f(x).$$

By iterative process, we get

$$\Delta^n f(0) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} f(k), \quad \text{(see[3-5])}. \qquad (12)$$

By (11) and (12), we get

$$\frac{1}{k!}\Delta^k 0^n = s(n,k). \qquad (13)$$

From (10), (11), (12) and (13), we have the following theorem.

Theorem 5 (Difference operator). *For $f \in \mathcal{C}([0,1],\mathbb{R})$, $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}_+$*

$$\mathbb{B}_{n,q}(f,x|\alpha,\alpha) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k \Delta^k f\left(\frac{0}{n}\right). \tag{14}$$

In the special case, $f(x) = x^m$ $(m \in \mathbb{Z}_+)$, we have the following corollary

Corollary 6. *For $x \in [0,1]$, $m,n \in \mathbb{Z}_+$ and $\alpha,\beta \in \mathbb{R}$ we have*

$$n^m \mathbb{B}_{n,q}(x^m,x|\alpha,\alpha) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k \Delta^k 0^m,$$

and

$$n^m \mathbb{B}_{n,q}(x^m,x|\alpha,\alpha) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k k! s(m,k).$$

Theorem 7 (Recurrence formula 2). *For $k,n \in \mathbb{Z}_+$, $\alpha,\beta \in \mathbb{R}$ and $x \in [0,1]$, we have*

$$\begin{aligned} &\frac{n+1-k}{n+1} B_{k,n+1}(x,q|\alpha,\beta) + \frac{k+1}{n+1} B_{k+1,n+1}(x,q|\alpha,\beta) \\ &= (1 - [x]_{q^\beta} + [x]_{q^\alpha}) B_{k,n}(x,q|\alpha,\beta). \end{aligned} \tag{15}$$

Proof of Theorem 7 :
For $x,t \in \mathbb{C}$ and $n \in \mathbb{Z}_+$ with $n \geq k$, consider

$$\frac{n!}{2\pi i} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}}, \tag{16}$$

where C is a circle around the origin and integration is in the positive direction. We see from the definitin of the weighted q-Bernstein polynomials and the basic theory of complex analysis including Laurent series that,

$$\begin{aligned} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} &= \sum_{m=0}^{\infty} \int_C \frac{B_{k,m}(x,q|\alpha,\beta) t^m}{m!} \frac{dt}{t^{n+1}} \\ &= 2\pi i \frac{B_{k,n}(x,q|\alpha,\beta)}{n!}. \end{aligned} \tag{17}$$

We get from (15) and (16) that

$$\frac{n!}{2\pi i} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} = B_{k,n}(x,q|\alpha,\beta) \tag{18}$$

and

$$\begin{aligned} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} &= \frac{[x]_{q^\alpha}^k}{k!} \sum_{m=0}^{\infty} \left(\frac{[1-x]_{q^{-\beta}}^m}{m!} \int_C t^{m-n-1+k} dt \right) \\ &= 2\pi i \frac{[x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k}}{k!(n-k)!}. \end{aligned} \tag{19}$$

By (15) and (18), we get

$$\frac{n!}{2\pi i} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k}. \tag{20}$$

From (17) and (19), we get

$$B_{k,n}(x,q|\alpha,\beta)=\binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}$$

where $x\in[0,1]$ and $n,k\in\mathbb{Z}_+$, $\alpha,\beta\in\mathbb{R}$.
By the definition of q-Bernstein polynomials, we see that

$$\begin{aligned}&\frac{n-k}{n}B_{k,n}(x,q|\alpha,\beta)+\frac{k+1}{n}B_{k+1,n}(x,q|\alpha,\beta)\\ &= \frac{(n-1)!}{k!(n-k-1)!}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}+\frac{(n-1)!}{k!(n-k-1)!}[x]_{q^\alpha}^{k+1}[1-x]_{q^{-\beta}}^{n-k-1}\\ &= ([1-x]_{q^{-\beta}}+[x]_{q^\alpha})B_{k,n-1}(x,q|\alpha,\beta)\\ &= (1-[x]_{q^\beta}+[x]_{q^\alpha})B_{k,n-1}(x,q|\alpha,\beta).\end{aligned}$$

Hence, we obtain our theorem. □

In the special case, $\alpha=\beta$, we obtain the following corollary.

Corollary 8. *For $k,n\in\mathbb{Z}_+$, $\alpha\in\mathbb{R}$ and $x\in[0,1]$, we have*

$$\frac{n+1-k}{n+1}B_{k,n+1}(x,q|\alpha,\alpha)+\frac{k+1}{n+1}B_{k+1,n+1}(x,q|\alpha,\alpha)=B_{k,n}(x,q|\alpha,\alpha). \tag{21}$$

We get from (3) that for $n,k\in\mathbb{N}$,

$$\begin{aligned}&\frac{n-k+1}{k}\frac{[x]_{q^\alpha}}{[1-x]_{q^{-\beta}}}B_{k-1,n}(x,q|\alpha,\beta)\\ &= \frac{n-k+1}{k}\frac{[x]_{q^\alpha}}{[1-x]_{q^{-\beta}}}\binom{n}{k-1}[x]_{q^\alpha}^{k-1}[1-x]_{q^{-\beta}}^{n-k+1}\\ &= \frac{n!}{k!(n-k)!}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}\\ &= B_{k,n}(x,q|\alpha,\beta).\end{aligned} \tag{22}$$

From (22), we obtain the following corollary.

Corollary 9.

$$\frac{n-k+1}{k}\frac{[x]_{q^\alpha}}{[1-x]_{q^{-\beta}}}B_{k-1,n}(x,q|\alpha,\beta)=B_{k,n}(x,q|\alpha,\beta),$$

where $k,n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$ and $x\in[0,1]$.

Theorem 10 (Explicit formula 2)**.**

$$B_{k,n}(x,q|\alpha,\beta)=\left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}(-1)^{l-k}[x]_{q^\beta}^l.$$

where $\alpha,\beta\in\mathbb{R}$, $n,k\in\mathbb{Z}_+$ and $x\in[0,1]$.

Proof of Theorem 10 : From the definition of the q-Bernstein polynomials and binomial theorem, we note that

$$\begin{aligned}B_{k,n}(x,q|\alpha,\beta) &= \binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}\\ &= \binom{n}{k}[x]_{q^\alpha}^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l[x]_{q^\beta}^l.\end{aligned} \tag{23}$$

It is easy to show that

$$\binom{n}{k}\binom{n-k}{l} = \binom{n}{k+l}\binom{k+l}{l}. \tag{24}$$

By (23) and (24), we get

$$\begin{aligned} B_{k,n}(x,q|\alpha,\beta) &= \left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k \sum_{l=k}^{n} \binom{n}{l}\binom{l}{l-k}(-1)^{l-k}[x]_{q^\beta}^l \\ &= \left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k \sum_{l=k}^{n} \binom{n}{l}\binom{l}{k}(-1)^{l-k}[x]_{q^\beta}^l. \end{aligned} \tag{25}$$

From (25), we obtain our theorem. □
In the special case, $\alpha = \beta$, we obtain the following corollary.

Corollary 11.

$$B_{k,n}(x,q|\alpha,\alpha) = \sum_{l=k}^{n} \binom{n}{l}\binom{l}{k}(-1)^{l-k}[x]_{q^\alpha}^l.$$

where $\alpha, \in \mathbb{R}$, $n, k \in \mathbb{Z}_+$ *and* $x \in [0,1]$.

It is possible to write $[x]_{q^\alpha}^k$ as a linear combination of the q-Bernstein polynomials by using the degree evaluation formula and mathematical induction. We see from the properties of the q-Bernstein polynomials with weight (α,β) that

$$\begin{aligned} \sum_{k=1}^{n} \frac{k}{n} B_{k,n}(x,q|\alpha,\beta) &= \sum_{k=1}^{n} \binom{n-1}{k-1}[x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k}[x]_{q^\alpha}^{k+1}[1-x]_{q^{-\beta}}^{n-k-1} \\ &= [x]_{q^\alpha}\left(1+[x]_{q^\alpha}-[x]_{q^\beta}\right)^{n-1}. \end{aligned}$$

In the special case, $\alpha = \beta$,

$$\sum_{k=1}^{n} \frac{k}{n} B_{k,n}(x,q|\alpha,\alpha) = [x]_{q^\alpha}.$$

By the same method, we get

$$\begin{aligned} \sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}} B_{k,n}(x,q|\alpha,\beta) &= \sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}}\binom{n}{k}[x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=2}^{n} \binom{n-2}{k-2}[x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n-2} \binom{n-2}{k}[x]_{q^\alpha}^{k+2}[1-x]_{q^{-\beta}}^{n-k-2} \\ &= [x]_{q^\alpha}^2\left(1+[x]_{q^\alpha}-[x]_{q^\beta}\right)^{n-2}. \end{aligned}$$

In the special case, $\alpha = \beta$,

$$\sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}} B_{k,n}(x, q|\alpha, \alpha) = [x]_{q^\alpha}^2.$$

By continuing this process, we obtain the following theorem.

Theorem 12.

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x, q|\alpha, \beta) = [x]_{q^\alpha}^j \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^{n-j},$$

where $n, k, j \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{R}$ *and* $x \in [0, 1]$.

In the special case, $\alpha = \beta$, we obtain the following corollary.

Corollary 13.

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x, q|\alpha, \alpha) = [x]_{q^\alpha}^j, \tag{26}$$

where $n, k, j \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$ *and* $x \in [0, 1]$.

In [2], the q-Stirling numbers of the second kind are defined by

$$s_q(n, k) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k - j]_q^n, \tag{27}$$

where $\binom{k}{j}_q = \frac{[k]_q!}{[j]_q![k-j]_q!}$ and $[k]_q! = [k]_q \cdots [2]_q[1]_q$.
For $m \in \mathbb{Z}_+$, we have

$$[x]_{q^\alpha}^m = \sum_{k=0}^{m} q^{\alpha\binom{k}{2}} \binom{x}{k}_{q^\alpha} [k]_{q^\alpha}! s_{q^\alpha}(k, m - k).$$

Thus, we have the following corollary.

Corollary 14.

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x, q|\alpha, \alpha) = \sum_{k=0}^{j} q^{\alpha\binom{k}{2}} \binom{x}{k}_{q^\alpha} [k]_{q^\alpha}! s_{q^\alpha}(k, j - k),$$

where $n, k, j \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$ *and* $x \in [0, 1]$.

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Stability of over-rotation pairs of cycles for the interval maps

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Abstract Let $C^0(I)$ be the set of all continuous self-maps of the compact interval I and $\mathbf{P}(u,v) = \{f \in C^0(I) : f$ has a cycle with over-rotation pair $(u,v)\}$ for any positive integers u and v with $2 \le 2u \le v$. In this paper, we prove that if $(2^m ns, 2^m nt) \succ (\gamma, \lambda)$, then

$$\mathbf{P}(2^m ns, 2^m nt) \subset \text{int } \mathbf{P}(\gamma, \lambda),$$

where $m \ge 0$ is an integer, $n \ge 1$ odd, $2 \le 2s \le t$ with s, t coprime and $2 \le 2\gamma \le \lambda$.
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1. Introduction

Let I be a compact interval and $C^0(I)$ be the set of all continuous maps from I to I. For any $f, g \in C^0(I)$, we define the distance between f and g by

$$d(f,g) = \sup_{x \in I} |f(x) - g(x)|.$$

Then $(C^0(I), d)$ becomes a metric space. For any subset M of $C^0(I)$, we use intM to denote the interior of M. A point $x \in I$ is called a **periodic point** of f with period n if $f^n(x) = x$ and $f^i(x) \ne x$ for $1 \le i \le n-1$, and $\{f^i(x) : 0 \le i \le n-1\}$ is called a **cycle** with period n. Write $F(f) = \{x : f(x) = x\}$, which is called the set of fixed points of f. For any subset $A \subset I$, we use $\#A$ and $[A]$ to denote the cardinal number of A and the smallest closed subinterval of I containing A, respectively. If $A = \{a, b\}$, we write $[A] = [a; b]$ and $(a; b) = [a; b] - \{a, b\}$. For any positive integer n, write $\mathbf{P}(n) = \{f \in C^0(I) : f$ has a cycle with period $n\}$.

One of the remarkable results in one-dimensional dynamics is the **Šharkovskii theorem**. To state it, let us first introduce the **Šharkovskii ordering** for positive integers:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd \cdots \rhd 2^k \cdot 3 \rhd 2^k \cdot 5 \rhd 2^k \cdot 7 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

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Theorem A (see [1]). *For any positive integers m and n, $\mathbf{P}(n) \subset \mathbf{P}(m)$ if $n \rhd m$.*

Block [2] studied stability of cycles in the theorem of Šarkovskii and obtained the following theorem.

Theorem B (see [2]). *For any positive integers m and n, $\mathbf{P}(n) \subset int\ \mathbf{P}(m)$ if $n \rhd m$.*

Blokh and Misiurewicz [3] introduced the following ordering among all pairs of positive integers (k, l) with $2 \leq 2k \leq l$:

(1) If $u/v < 1/2$ and $k/l \in (u/v, 1/2]$, then $(u, v) \succ (k, l)$.

(2) If $u/v = k/l = m/n$, where m, n are coprime, then $(u, v) \succ (k, l)$ if and only if $u/m \rhd k/m$.

They also defined the over-rotation pairs and the over-rotation numbers of cycles with period $n > 1$ for the interval maps.

Definition 1.1 (see [3]) Let $f \in C^0(I)$. Assume that P is a cycle of f with period $n > 1$ and $m = \#\{y \in P : (y - f(y))(f^2(y) - f(y)) > 0\}$. Then $(m/2, n)$ is called the **over-rotation pair** of P and $m/2n$ the **over-rotation number** of P.

For any positive integers v and u with $2 \leq 2u \leq v$, write $\mathbf{P}(u, v) = \{f \in C^0(I) : f$ has a cycle with over-rotation pair $(u, v)\}$.

Theorem C (see [3]). *For any positive integers u, v, k and l with $2 \leq 2u \leq v$ and $2 \leq 2k \leq l$, $\mathbf{P}(u, v) \subset \mathbf{P}(k, l)$ if $(u, v) \succ (k, l)$.*

In this paper, we shall study stability of over-rotation pairs of cycles for the interval maps. Our main result is the following theorem.

Main Theorem . *If $(2^m ns, 2^m nt) \succ (\gamma, \lambda)$, then*

$$\mathbf{P}(2^m ns, 2^m nt) \subset int\ \mathbf{P}(\gamma, \lambda),$$

where $m \geq 0$ is an integer, $n \geq 1$ odd, $2 \leq 2s \leq t$ with s, t coprime and $2 \leq 2\gamma \leq \lambda$.

2. Some Lemmas

We need the following definitions and lemmas for showing main theorem.

Definition 2.1 (see [4]). Let $f \in C^0(I)$ and $p \in F(f)$. A point $y \in I$ is called a centripetal point of f relative to p (or simply a centripetal point if there is no confusion) if $f(y) \in (p; y)$. y is called a striding point of f relative to p if $p \in (y; f(y))$.

Lemma 2.1 (see [5]). *Let $f \in C^0(I)$. If $I_0, I_1, \cdots, I_m$ are compact subintervals of I with $I_m = I_0$ such that $f(I_{k-1}) \supset I_k$ for $1 \leq k < m$, then there exists a point y such that $f^m(y) = y$ and $f^k(y) \in I_k$ for every $0 \leq k < m$.*

Lemma 2.2. *Let $f \in C^0(I)$. If there are points a, b and c such that $f(c) \leq a = f(a) < b < c \leq f(b)$ (resp. $f(c) \geq a = f(a) > b > c \geq f(b)$), then for any integers m and n with*

$2 \le 2m \le n$, there exist a cycle $Q = \{y_1 < y_2 < \cdots < y_n\}$ of f with over-rotation pair (m, n) and $p \in F(f) \cap [Q]$ such that every point of Q is a centripetal point or striding point of f relative to p.

Proof. We only prove the case $f(c) \le a = f(a) < b < c \le f(b)$ (the proof for the case $f(c) \ge a = f(a) > b > c \ge f(b)$ is similar). We may assume that $(a, b) \cap F(f) = \emptyset$. Then $f(x) > x$ for all $x \in (a, b)$. Choose $p \in (b, c) \cap F(f)$. Then there exist points $a < e_1 < e_2 < \cdots < e_{n-2m+1} < b$ such that $f(e_k) = e_{k+1}$ for each $1 \le k \le n - 2m$ and $f(e_{n-2m+1}) = p$. Let

$$\begin{cases} I_k = [e_k, e_{k+1}], & \text{if } k \in \{1, 2, \cdots, n-2m\}, \\ I_{n-2m+2r+1} = [e_{n-2m+1}, b], & \text{if } r \in \{0, \cdots, m-2\}, \\ I_{n-2m+2r+2} = [p, c], & \text{if } r \in \{0, \cdots, m-1\}, \\ I_{n-1} = [b, p]. \end{cases}$$

Then $f(I_i) \supset I_{i+1}$ $1 \le i \le n-1$ and $f(I_n) \supset I_1$. By Lemma 2.1 there exists a cycle $Q = \{x_1, x_2, \cdots, x_n\}$ such that $x_i \in I_i (1 \le i \le n)$. It is easy to verify that every point of Q is a centripetal point or striding point of f relative to p. The proof of Lemma 2.2 is completed.

Lemma 2.3. *Let $f \in \mathbf{P}(m, n)$. Then there exist a cycle $Q = \{y_1 < y_2 < \cdots < y_n\}$ of f with over-rotation pair (m, n) and $p \in F(f) \cap [Q]$ such that every point of Q is a centripetal point or striding point of f relative to p.*

Proof. Let $P = \{x_1 < x_2 < \cdots < x_n\}$ be a cycle of f with over-rotation pair (m, n) and $s = \min\{k : f(x_k) < x_k\}$. Then $s \ge 2$, $(x_{s-1}, x_s) \cap F(f) \ne \emptyset$ and $f(x_i) > x_i$ for each $1 \le i \le s-1$. We may also assume that there exists some $s < j \le n$ such that $f(x_j) > x_j$ since otherwise choose $p \in (x_{s-1}, x_s) \cap F(f)$ and let $Q = P$ which completes the proof of Lemma 2.3.

Let $t = \min\{k : k > s \text{ and } f(x_k) > x_k\}$ and $p = \max\{(x_s, x_t) \cap F(f)\}$. Then $f(x) > x$ for all $x \in (p, x_t)$. Let $j = \min\{k : f^{k+1}(x_t) \le p\}$ and $i = \min\{k : k \le j \text{ and } f^{k+1}(x_t) \ge f^j(x_t)\}$. Then $f^{j+1}(x_t) < p < f^i(x_t) < f^j(x_t) \le f^{i+1}(x_t)$. It follows from Lemma 2.2 that f has a cycle $Q = \{y_1 < y_2 < \cdots < y_n\}$ such that Q with the desirable properties. The proof of Lemma 2.3 is completed.

Definition 2.2 (see [5]). Let $f \in C^0(I)$. A cycle P of f with odd period $n > 1$ is called a cycle of f of **Štefan type** if

$$P = \{f^{n-1}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)\}$$

or

$$P = \{f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-1}(c)\}.$$

Definition 2.3 (see [5,6]). Let $f \in C^0(I)$ and $P = \{x_1 < x_2 < \cdots < x_{n2^m}\}$ be a cycle with period $n2^m$, where $n \ge 1$ is odd and $m \ge 0$ is an integer. For each $0 \le i \le m$ and each $1 \le j \le 2^i$, write $A^j_{2^i} = \{x_{(j-1)2^{m-i}n+1} < x_{(j-1)2^{m-i}n+2} < \cdots < x_{j2^{m-i}n}\}$. We call P a **strongly simple cycle** if one of the following three conditions holds:

(1) If $m = 0$, then either $n = 1$ or $n > 1$ and P is a cycle of f of Štefan type.

(2) If $n = 1$ and $m > 0$, then for each $1 \le i \le m$ and each $1 \le 2k \le 2^i$, $f^{2^{i-1}}(A_{2^i}^{2k-1}) = A_{2^i}^{2k}$ and $f^{2^{i-1}}(A_{2^i}^{2k}) = A_{2^i}^{2k-1}$.

(3) If $n > 1$ and $m > 0$, then the following three conditions holds:

(i) For each $1 \le i \le m$ and each $1 \le 2k \le 2^i$, $f^{2^{i-1}}(A_{2^i}^{2k-1}) = A_{2^i}^{2k}$ and $f^{2^{i-1}}(A_{2^i}^{2k}) = A_{2^i}^{2k-1}$.

(ii) For each $1 \le j \le 2^m$, $A_{2^m}^j$ is a cycle of f^{2^m} of Štefan type.

(iii) f maps each $A_{2^m}^i$ monotonically onto another $A_{2^m}^j$, with one exception.

Lemma 2.4 (see [5,6]). *If $f \in C^0(I)$ has a cycle with period n, then f has a strongly simple cycle with period n.*

Let $P = \{x_1 < x_2 < \cdots < x_n\}$ be a cycle of f with period $n > 1$. Then there is an unique map $g : [x_1, x_n] \longrightarrow [x_1, x_n]$, which is called the **linearization** of P, satisfying

(1) $g(x_i) = f(x_i)$ for all $1 \le i \le n$.

(2) $g|_{[x_i, x_{i+1}]}$ is linear for all $1 \le i \le n - 1$.

Lemma 2.5 (see [5]). *Let $f \in C^0(I)$, P be a cycle f with period $n > 1$ and g be linearization of P. Then f has a cycle $\{x_1 < x_2 < \cdots < x_m\}$ satisfying $f(x_i) = x_j$ $(i \in \{1, 2, \cdots, m\})$ if and only if g has a cycle $\{y_1 < y_2 < \cdots < y_m\}$ satisfying $g(y_i) = y_j$ $(i \in \{1, 2, \cdots, m\})$.*

By Lemma 2.5 we know that if g has a strongly simple cycle with over-rotation pair (p, q), then f has also a strongly simple cycle with over-rotation pair (p, q).

Lemma 2.6 (see [5]). *Let $f \in C^0(I)$. If there exist points $a, b, c, d \in I$ with $a < b \le c < d$ such that $f([a, b]) \cap f([c, d]) \supset [a, d]$, then f has periodic points of all periods.*

For any nonempty subset A and B of real numbers, we write $A < B$ if $x < y$ for any $x \in A$ and $y \in B$.

Lemma 2.7. *Let $f \in \mathbf{P}(ks, kt)$, where s, t are coprime, $k = n2^m$, $n \ge 1$ is odd and $m \ge 0$ is an integer. Then there exist a cycle $P = \{z_1 < z_2 < \cdots < z_{kt}\}$ of f with over-rotation pair (ks, kt) and $p \in F(f) \cap [P]$ satisfying*

(1) *Every point of P is a centripetal point or striding point of f relative to p.*

(2) *f cyclically permutes the sets $B_i = \{z_{(i-1)2^m n+1}, \cdots, z_{i2^m n}\}$ $(i = 1, 2, \cdots, t)$.*

(3) *B_1 is a strongly simple cycle of f^t.*

Proof. By Lemma 2.3 we may assume that $R = \{x_1 < x_2 < \cdots < x_{kt}\}$ is a cycle of f with over-rotation pair (ks, kt) and $q \in F(f) \cap [R]$ such that every point of R is a centripetal point or striding point of f relative to q. By Lemma 2.5 we may assume that f is the linearization of R. Let $I = [x_1, x_{kt}]$ and p be the unique fixed point of f. Obviously, we have that $f(x) > x$ for all $x \in [x_1, p)$ and $f(x) < x$ for all $x \in (p, x_{kt}]$.

If $s/t = 1/2$, then it follows from Lemma 2.4 that the conclusion of Lemma 2.7 holds. Now we assume $s/t < 1/2$.

By Theorem C, f has a cycle $Q = \{y_1 < y_2 < \cdots < y_t\}(t > 2)$ with over-rotation pair (s, t) such that every point of Q is a centripetal point or striding point of f relative to p. We can assume $k \geq 2$ since otherwise there is nothing to prove. Furthermore, we may assume $\#\{x \in Q : x > p\} \geq 2$ (the proof for the case $\#\{x \in Q : x < p\} \geq 2$ is similar). Write $x = \max Q$. Then $f^t(x) = x$.

Claim 2.1. *We may assume that there exists a positive number $\varepsilon > 0$ such that $f^t(y) > y$ for all $y \in (x, x + \varepsilon)$.*

Proof of Claim 2.1. Since $x \notin P$, there exists a positive number $\varepsilon > 0$ such that $(f^t(y) - y)(y-x) > 0$ for all $y \in (x-\varepsilon, x+\varepsilon)-\{x\}$ or $(f^t(y)-y)(y-x) < 0$ for all $y \in (x-\varepsilon, x+\varepsilon)-\{x\}$. If $(f^t(y) - y)(y - x) > 0$ for all $y \in (x - \varepsilon, x + \varepsilon) - \{x\}$, then Claim 2.1 holds. Now we assume $(f^t(y) - y)(y - x) < 0$ for all $y \in (x - \varepsilon, x + \varepsilon) - \{x\}$ and put $u = \max\{y \in (p, x) : f^t(y) = y\}$.

We claim that for all $1 \leq i \leq t$, $(f^i(u) - p)(f^i(x) - p) > 0$. Otherwise, suppose that $(f^i(u) - p)(f^i(x) - p) < 0$ for some $1 \leq i \leq t$. Then there exists a point $v \in (u, x)$ such that $f^i(v) = p$. Thus $f^t(v) = p$, which implies $(u, v) \cap F(f^t) \neq \emptyset$. This is a contradiction.

We also claim $u = \max\{f^i(u) : 0 \leq i \leq t\}$. Otherwise, suppose that $f^i(u) > u$ for some $1 \leq i \leq t - 1$. Then there exists a point $v \in (u, x)$ such that $f^i(v) = v$ since $x = \max Q$. Let $w = \max\{v \in (u, x) : f^i(v) = v\}$. Then $f^{t-i}(w) = f^t(w) > w$. Since $f^{t-i}(x) < x$, there exists a point $e \in (w, x)$ such that $f^{t-i}(e) = e$, which implies $f^i(e) = f^t(e) > e$ and $(e, x) \cap F(f^i) \neq \emptyset$. This is a contradiction. By using u replaces x, we know that Claim 2.1 holds. The proof of Claim 2.1 is completed.

Write $S = \{y : f^t(y) = x\} \cap (x, x_{kt}]$. Let $T = \min S$ if $S \neq \emptyset$ and $T = x_{kt}$ otherwise. Put $J = (x, T)$.

Claim 2.2. *$J, f(J), \cdots, f^{t-1}(J)$ are pairwise disjoint and $p \notin \cup_{i=0}^{t-1} f^i(J)$.*

Proof of Claim 2.2. We first prove that $J, f(J), \cdots, f^{t-1}(J)$ are pairwise disjoint. Suppose that there exist $0 \leq i < j \leq t - 1$ and $u, v \in J$ such that $f^i(u) = f^j(v)$. Then $f^{t-j+i}(u) = f^t(v) > x$. Since $f^{t-j+i}(x) < x$, there exists a point $y \in (x, u)$ such that $f^{t-j+i}(y) = x$, which implies $x > f^{j-i}(x) = f^t(y) > x$. This is a contradiction.

Now we prove $p \notin \cup_{i=0}^{t-1} f^i(J)$. Suppose that there exist some $0 \leq i \leq t - 1$ and $u \in J$ such that $f^i(u) = p$. Then $f^t(u) = p$. Hence $x \in f^t((x, u))$, which contradicts definition of T. The proof of Claim 2.2 is completed.

By definition of T, it follows that $P \cap (\cup_{i=0}^{t-1} f^i(J)) \neq \emptyset$ since otherwise we have $f^t(T) > T$, which is impossible.

If $f^t(J) \subset J$, then $f^t|_J$ has a cycle with period k. It follows from Claim 2.2 and Lemma 2.4 that f has a cycle $P = \{z_1 < z_2 < \cdots < z_{kt}\}$ with over-rotation pair (ks, kt) satisfying conditions (1),(2) and (3).

If $f^t(J) \not\subset J$, then $f^t(T) = x$ and there exists a point $y \in J$ such that $f^t(y) \geq T$. Thus

$f^t([x,y]) \cap f^t([y,T]) \supset [x,T]$. By Lemma 2.6, f^t has a cycle of period 3 on J. It follows from Claim 2.2,Theorem A and Lemma 2.4 that f has a cycle $P = \{z_1 < z_2 < \cdots < z_{kt}\}$ with over-rotation pair (ks, kt) satisfying conditions (1),(2) and (3) . The proof of Lemma 2.7 is completed.

3. Proof of Main Theorem

In this section we shall give the proof of the main theorem.

Proof of Main Theorem. Let $f \in \mathbf{P}(2^m ns, 2^m nt)$. We shall show that there exists a neighourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with over-rotation pair (γ, λ). The proof will be carried out in a number of stages.

Claim 3.1. *If $m \geq 0$ and $n \geq 3$, then there exists a neighourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with over-rotation pair $(2^m(n+2)s, 2^m(n+2)t)$.*

Proof of Claim 3.1. By Lemma 2.7 we know that there exist a cycle $P = \{z_1 < z_2 < \cdots < z_{2^m nt}\}$ of f with over-rotation pair $(2^m ns, 2^m nt)$ and $p \in F(f) \cap [P]$ satisfying

(1) Every point of P is a centripetal point or striding point of f relative to p.

(2) f cyclically permutes the sets $B_i = \{z_{(i-1)2^m n+1}, \cdots, z_{i2^m n}\}$ $(i = 1, 2, \cdots, t)$.

(3) B_1 is a strongly simple cycle of f^t.

For each $1 \leq l \leq 2^m$, let $z_1(l)$ denote the midpoint of the n points in $C_l = \{x_{(l-1)n+1} < \cdots < x_{ln}\}$ and $z_j(l) = f^{2^m t(j-1)}(z_1(l))$ $(1 < j \leq n)$. Then for each $1 \leq l \leq 2^m$, we have either

$$z_n(l) < z_{n-2}(l) < \cdots < z_3(l) < z_1(l) < z_2(l) < \cdots < z_{n-3}(l) < z_{n-1}(l)$$

or

$$z_n(l) > z_{n-2}(l) > \cdots > z_3(l) > z_1(l) > z_2(l) > \cdots > z_{n-3}(l) > z_{n-1}(l).$$

Furthermore, the blocks C_l can be renumbered so that $f^t(z_1(l)) = z_1(l+1)$ for $1 \leq l < 2^m$. Then

$$\begin{cases} f^t(z_j(l)) = z_j(l+1), \text{ if } 1 \leq l < 2^m,\ 1 \leq j \leq n, \\ f^t(z_j(2^m)) = z_{j+1}(1), \quad \text{if } 1 \leq j < n, \\ f^t(z_n(2^m)) = z_1(1). \end{cases}$$

Since $f^t([z_1(l); z_2(l)]) \supset [z_1(l+1); z_2(l+1)]$ for $1 \leq l < 2^m$ and $f^t([z_1(2^m); z_2(2^m)]) \supset [z_3(1); z_2(1)]$, there exist points $z_{-3}, z_{-2}, z_{-1}, z_0$ such that $f^{2^m t}(z_{-i}) = z_{-i+1}$ $(i = 0, 1, 2, 3)$ satisfying

(1) For $1 \leq l \leq 2^m$, either

$$z_1(l) < f^{(l-1)t}(z_{-1}) < f^{(l-1)t}(z_{-3}) < f^{(l-1)t}(z_{-2}) < f^{(l-1)t}(z_0) < z_2(l)$$

or

$$z_1(l) > f^{(l-1)t}(z_{-1}) > f^{(l-1)t}(z_{-3}) > f^{(l-1)t}(z_{-2}) > f^{(l-1)t}(z_0) > z_2(l).$$

(2) For $1 \le l \le 2^m$, $0 \le i < t$ and $j = 0, 1, 2, 3$, we have$f^{(l-1)t+i}(z_{-j}) \in f^i([B_1])$. Write $z_{-i}(l) = f^{l-1}(z_{-i})$ for $i = 0, 1, 2, 3$ and $1 \le l \le 2^m t$. Let

$$\varepsilon = \min\{|f^i(z_{-3}(1)) - f^j(z_{-3}(1))| : 0 \le i < j < 2^m(n+2)t\}/20$$

and $U = \{g \in C^0(I) : d(f^i, g^i) < \varepsilon \text{ for all } 0 \le i \le 2^m(n+2)t\}$. Then for every $g \in U$ and $1 \le i < j \le 2^m(n+2)t - 1$, $f^i(z_{-3}(1)) < f^j(z_{-3}(1))$ if and only if $g^i(z_{-3}(1)) < g^j(z_{-3}(1))$ and $g^{2^m(n+2)t}(z_{-3}(1)) \in [g^{2^m 2t}(z_{-3}(1)); g^{2^m 6t}(z_{-3}(1))]$. Set

$$I_l = [g^{l-1}(z_{-3}(1)), g^{l+1}(z_{-3}(1))] \text{ for } 1 \le l \le 2^m(n+2)t.$$

Then we have

$$\begin{cases} g(I_l) & \supset I_{l+1}, \text{ if } 1 \le l \le 2^m(n+2)t - 1, \\ g(I_{2^m(n+2)t}) & \supset I_1. \end{cases}$$

This yields a cycle $O = \{y, g(y), \cdots, g^{2^m(n+2)t}(y)\}$ such that $g^{i-1}(y) \in I_i$ for $i = 1, 2, \cdots, 2^m(n+2)t$. It is easy to verify that the over-rotation pair of O is $(2^m(n+2)s, 2^m(n+2)t)$. The proof of Claim 3.1 is completed.

Claim 3.2. *If $m \ge 1$ and $n = 1$, then there exists a neighourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with over-rotation pair $(2^{m-1}t, 2^{m-1}s)$.*

Proof of Claim 3.2. By Lemma 2.7 we know that there exist a cycle $P = \{z_1 < z_2 < \cdots < z_{2^m t}\}$ of f with over-rotation pair $(2^m s, 2^m t)$ and $p \in F(f) \cap [P]$ satisfying

(1) Every point of P is a centripetal point or striding point of f relative to p.

(2) f cyclically permutes the sets $B_i = \{z_{(i-1)2^m+1}, \cdots, z_{i2^m}\}$ $(i = 1, 2, \cdots, t)$.

(3) B_1 is a strongly simple cycle of f^t.

Since $f^{2^{m-1}t}(x_1) = x_2$ and $f^{2^{m-1}t}(x_2) = x_1$, there exist points $x_1 \le a < b \le x_2$ such that $f^{2^{m-1}t}(b) = x_1 < x_2 = f^{2^{m-1}t}(a)$ and $f^i(a), f^i(b) \in [f^i(x_1); f^i(x_2)]$ for $0 \le i \le 2^{m-1}t - 1$. Let

$$\varepsilon = \min\{b - a, \min\{|f^i(x_1) - f^j(x_1)| : 0 \le i < j < 2^m t\}\}/20$$

and $U = \{g \in C^0(I) : d(f^i, g^i) < \varepsilon \text{ for all } 0 \le i \le 2^{m-1}t\}$. Then for every $g \in U$, we have $g^{2^{m-1}t}(a) > a$ and $g^{2^{m-1}t}(b) < b$. This yields a cycle $O = \{y, g(y), \cdots, g^{2^{m-1}t-1}(y)\}$ such that $g^{i-1}(y) \in [g^i(a); g^i(b)]$ for $i = 1, 2, \cdots, 2^{m-1}t$. It is easy to verify that the over-rotation pair of O is $(2^{m-1}s, 2^{m-1}t)$. The proof of Claim 3.2 is completed.

Claim 3.3. *If $m = 0$ and $n = 1$, then there exists a neighourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with over-rotation pair (γ, λ).*

Proof of Claim 3.3. By Lemma 2.3 there exist a cycle $Q = \{x_1 < x_2 < \cdots < x_t\}$ of f with over-rotation pair (s, t) and $p \in F(f) \cap [P]$ such that every point of P is a centripetal point or striding point of f relative to p. Choose two coprime integers u, v such that $s/t < u/v < \gamma/\lambda$. Without loss of generality we can assume $f(x_{t-s+1}) < x_{t-s}$. Take $w \in F(f) \cap (x_{t-s}, x_{t-s+1})$.

Put $1 \leq l = tu - sv$. Then there exist points $y_i \in (x_{t-s}, x_{t-s+1})$ $(i = 0, 1, \cdots, 2l-1)$ such that $x_{t-s} < y_{2l-2} < y_{2l-4} < \cdots < y_2 < y_0 < w < y_1 < y_3 < \cdots < y_{2l-3} < y_{2l-1} < x_{t-s+1}$ with $f(y_i) = y_{i+1}$ $(i = 0, 1, \cdots, 2l-2)$ and $f(y_{2l-1}) = x_{t-s}$. Let

$$\varepsilon = \min\{|f^i(y_0) - f^j(y_0)| : 0 \leq i < j < 2l + t\}/20$$

and $U = \{g \in C^0(I) : d(f^i, g^i) < \varepsilon \text{ for all } 1 \leq i \leq (t-2s)v\}$. Then for every $g \in U$, we have $g^{(t-2s)v}(y_0) < y_0 < g(y_0)$ and $g^2(y_0) < g(y_0)$.

Let $z = \max\{x \in I : x < y_0 \text{ and } g^{(t-2s)v}(x) = x\}$, $\alpha = \min\{z, g(z), \cdots, g^{(t-2s)v-1}(z)\}$ and $w_1 \in (y_0, g(y_0) \cap F(g)$.

We claim that for any $i \in \{0, 1, \cdots, (t-2s)v\}$, $(g^i(z) - w_1)(g^i(y_0) - w_1) > 0$. Indeed, if $(g^i(z) - w_1)(g^i(y_0) - w_1) \leq 0$ for some $i \in \{0, 1, \cdots, (t-2s)v\}$. Then there exists a point $c \in [z, y_0)$ such that $g^i(c) = w_1$. Thus $g^{(t-2s)v}(c) = w_1$, which implies $(z, y_0) \cap F(g^{(t-2s)v}) \neq \emptyset$. A contradiction.

If $[\alpha, y_0] \cap F(g) \neq \emptyset$, then $z_0 = \max\{[\alpha, y_0] \cap F(g)\} \in [\alpha, z)$. Since $g(y_0) > y_0$ and $(z, y_0] \cap F(g) = \emptyset$, we have $g(z) > z$. Let $j = \min\{k : f^{k+1}(z) < z_0\}$ and $i = \min\{k \leq j : f^{k+1}(z) \geq f^j(z)\}$. Then $f^{j+1}(z) < z_0 < f^i(z) < f^j(z) \leq f^{i+1}(z)$. It follows from Lemma 2.2 that g has a cycle with over-rotation pair (γ, λ). In the following, we assume that $[\alpha, y_0] \cap F(g) = \emptyset$.

We claim that for any $i \in \{0, 1, \cdots, (t-2s)v\}$, $g^{i+1}(z) \in [g^i(z); w_1]$ or $w_1 \in [g^{i+1}(z); g^i(z)]$. Indeed, if $g^i(z) \in [g^{i+1}(z); w_1]$ for some $i \in \{0, 1, \cdots, (t-2s)v\}\}$, then $g^i(z) > y_0$. Since $g^{i+1}(y_0) \in [g^i(y_0); w_1]$, we have $[g^i(z); g^i(y_0)] \cap F(g) \neq \emptyset$. Let $w_2 \in [g^i(z); g^i(y_0)] \cap F(g)$. Then $w_2 > y_0$ and there exists a point $d \in [z, y_0]$ such that $g^{(t-2s)v}(d) = w_2$. Thus $(d, y_0] \cap F(g^{(t-2s)v}) \neq \emptyset$. A contradiction.

By the above claims, we know that g has a cycle with over-rotation number u/v. It follows from Theorem C that g has a cycle with over-rotation pair (γ, λ). The proof of Claim 3.3 is completed.

From Claim 3.1, 3.2, 3.3 and Theorem C it follows that main theorem holds.

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A family of (h,q)-zeta function associated with (h,q)-Bernoulli numbers and polynomials

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Abstract : In this paper we consider the following (h,q)-zeta function:

$$\zeta_q(s,x \mid h) = \sum_{n=0}^{\infty} \frac{q^{hn+x}}{[n+x]_q^s} + \frac{(q-1)(h-1)}{1-s} \sum_{n=0}^{\infty} \frac{q^{(h-1)n}}{[n+x]_q^{s-1}},$$

where $x \neq 0,-1,-2,\cdots, s \in \mathbb{C} \setminus \{1\}$ and $h \in \mathbb{C}$. Finally, we lead to a useful integral representation for the (h,q)-zeta functions and give the functional equation associated with (h,q)-Bernoulli numbers and polynomials.

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Key words : Bernoulli numbers, Bernoulli polynomials, (h,q)-Bernoulli numbers and polynomials, (h,q)-zeta function

1. Introduction

When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. Throughout this paper we assume that $q \in \mathbb{C}$ with $|q| < 1$. We use the notation of q-number as

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$,(see [6]).

The Bernoulli polynomials in $\mathbb{C}$ are defined by the formula

$$e^{xt}\frac{t}{e^t-1} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

with the usual convention of replacing $B^i(x)$ by $B_i(x)$, (see [1-14]). In the special case, $x = 0, B_n(0) = B_n$ are called the n-th ordinary Bernoulli numbers, (see [1-10]). The Bernoulli numbers are used to express special values of Riemann zeta function, which is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s \in \mathbb{C}, (\text{ see}[7]).$$

That is, $\zeta(2m) = \dfrac{(2\pi)^{2m}(-1)^{m-1}B_{2m}}{2(2m)!}$, for $m \in \mathbb{N}$, and $\zeta(1-2m) = -\dfrac{B_{2m}}{2m}$, (see [7-14]).

First, we consider (h,q)-Bernoulli numbers and polynomials as the q-extension of Bernoulli

numbers and polynomials. From those numbers and polynomials, we derive some (h,q)-zeta functions as the q-extension of Riemann zeta function. That is, the purpose of this paper ir to study the following (h,q)-zeta function:

$$\zeta_q(s,x \mid h) = \sum_{n=0}^{\infty} \frac{q^{hn+x}}{[n+x]_q^s} + \frac{(q-1)(h-1)}{1-s} \sum_{n=0}^{\infty} \frac{q^{(h-1)n}}{[n+x]_q^{s-1}},$$

where $x \neq 0, -1, -2, \cdots$, and $h \in \mathbb{C}$, $s \in \mathbb{C} \setminus \{1\}$.

Finally, we derive to a useful integral representation for the (h,q)-zeta functions and give the functional equation associated with (h,q)-Bernoulli numbers and polynomials. Recently, several authors have studied the q-zeta functions and the q-Bernoulli numbers(see [1-14]). Our q-extensions of Bernoulli numbers and polynomials in this paper are different the q-extension of Bernoulli numbers and polynomials which are treated by several authors in previous papers.

2. (h,q)-Bernoulli numbers and polynomials associated with (h,q)-zeta functions

For $h \in \mathbb{C}$, let us consider (h,q)-Bernoulli polynomials as follows:

$$\begin{aligned} F_q(t,x \mid h) &= -t \sum_{m=0}^{\infty} q^{hm+x} e^{[x+m]_q t} + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} e^{[x+m]_q t} \\ &= \sum_{n=0}^{\infty} \beta_{n,q}^h(x) \frac{t^n}{n!}. \end{aligned} \tag{1}$$

From (1), we note that

$$F_q(t,x \mid h) = e^{\left(\frac{1}{1-q}\right)t} \sum_{l=0}^{\infty} (-1)^l q^{lx} \frac{l+h-1}{[l+h-1]_q} \left(\frac{1}{1-q}\right)^l \frac{t^l}{l!}. \tag{2}$$

By (1) and (2), we get

$$\beta_{n,q}^h(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+h-1}{[l+h-1]_q}. \tag{3}$$

In the special case, $x = 0$, $\beta_{n,q}^h(0) = \beta_{n,q}^h$ are called the n-th (h,q)-Bernoulli numbers.

In (3), it is easy to show that

$$\beta_{n,q}^h(x) = ([x]_q + q^x \beta^h)^n, \tag{4}$$

where we use the usual convention about replacing $(\beta^h)^n = \beta_{n,q}^h$.

Note that

$$\lim_{q \to 1} F_q(t,x \mid h) = \frac{e^{xt} t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Let $F_q(t \mid h) = \sum_{n=0}^{\infty} \beta_{n,q}^h \frac{t^n}{n!}$ be the generating function of (h,q)-Bernoulli numbers. Then we get

$$F_q(t \mid h) = -t \sum_{m=0}^{\infty} q^{hm} e^{[m]_q t} + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} e^{[m]_q t}. \tag{5}$$

From (1), (4), and (5), we can derive the following difference equation:

$$\begin{aligned} F_q(t, x \mid h) &= e^{[x]_q t} F_q(q^x t \mid h) \\ &= -t \sum_{n=0}^{\infty} q^{hn+x} e^{[x+n]_q t} + (h-1)(1-q) \sum_{n=0}^{\infty} q^{(h-1)n} e^{[x+n]_q t}. \end{aligned}$$

Therefore, we obtain the following proposition.

Proposition 1. For $h \in \mathbb{C}$, we have

$$\begin{aligned} \beta_{n,q}^h(x) &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+h-1}{[l+h-1]_q} \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q}^h [x]_q^{n-l}. \end{aligned}$$

Note that

$$\left(\frac{d}{dt}\right)^k F_q(t, x \mid h)\Big|_{t=0} = -k \sum_{n=0}^{\infty} q^{hn+x} [x+n]_q{}^{k-1} + (h-1)(1-q) \sum_{n=0}^{\infty} q^{(h-1)n} [x+n]_q{}^k.$$

Thus, we obtain the following corollary.

Corollary 2. For $k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we have

$$\beta_{k,q}^h(x) = -k \sum_{n=0}^{\infty} q^{hn+x} [x+n]_q^{k-1} + (h-1)(1-q) \sum_{n=0}^{\infty} q^{(h-1)n} [x+n]_q^k,$$

and

$$\beta_{k,q}^h = -k \sum_{n=0}^{\infty} q^{hn} [n]_q^{k-1} + (h-1)(1-q) \sum_{n=0}^{\infty} q^{(h-1)n} [n]_q^k.$$

It is easy to show that

$$q^{(h-1)n} F_q(t, n \mid h) - F_q(t \mid h) = t \sum_{l=0}^{n-1} q^{hl} e^{[l]_q t} - (h-1)(1-q) \sum_{l=0}^{n-1} q^{(h-1)l} e^{[l]_q t}.$$

Thus, we have

$$\beta_{0,q}^h = \frac{h-1}{[h-1]_q}, \text{ and } q^{h-1} \beta_{n,q}^h(1) - \beta_{n,q}^h = \delta_{1n}, \tag{6}$$

where δ_{1n} is kronecker symbol.

Therefore, we obtain the following theorem.

Theorem 3. For $h \in \mathbb{C}, n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, we have

$$q^{(h-1)n}\beta_{m,q}^h(n) - \beta_{m,q}^h = m\sum_{l=0}^{n-1} q^{hl}[l]_q^{m-1} - (h-1)(1-q)\sum_{l=0}^{n-1} q^{(h-1)l}[l]_q^m.$$

In the special case, $n = 1$, we have

$$\beta_{0,q}^h = \frac{h-1}{[h-1]_q}, \text{ and } q^{h-1}\beta_{n,q}^h(1) - \beta_{n,q}^h = \delta_{1n},$$

where δ_{1n} is kronecker symbol.

Now, we consider the following integral representation in complex plane.

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}F_q(-t, x \mid h)dt \\
&= \sum_{n=0}^\infty \frac{q^{hn+x}}{[x+n]_q^s} + \frac{\Gamma(s-1)}{\Gamma(s)}(h-1)(1-q)\sum_{n=0}^\infty \frac{q^{(h-1)n}}{[x+n]_q^{s-1}} \\
&= \sum_{n=0}^\infty \frac{q^{hn+x}}{[x+n]_q^s} + \frac{(h-1)(1-q)}{s-1}\sum_{n=0}^\infty \frac{q^{(h-1)n}}{[x+n]_q^{s-1}},
\end{aligned} \tag{7}$$

where $x \neq 0, -1, -2, \cdots, h \in \mathbb{C}$, and $s \in \mathbb{C} \setminus \{1\}$.

By the definition of (h, q)-Bernoulli polynomials, we see that

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}F_q(-t, x \mid h)dt = \sum_{m=0}^\infty \left(\frac{1}{\Gamma(s)}\int_0^\infty t^{m+s-2}dt\right)\frac{(-1)^m\beta_{m,q}^h(x)}{m!}. \tag{8}$$

In the special case, $s = 1 - k(k \in \mathbb{N})$, we see form (7) and (8) and the basic theory of complex analysis including Laurent series that

$$\sum_{n=0}^\infty q^{hn+x}[x+n]_q^{k-1} - \frac{(h-1)(1-q)}{k}\sum_{n=0}^\infty [x+n]_q^k q^{(h-1)n} = -\frac{\beta_{k,q}^h(x)}{k}. \tag{9}$$

In the viewpoint of (7), we can define the following Hurwitz's type (h, q)-zeta function:

Definition 4. For $h \in \mathbb{C}, s \in \mathbb{C} \setminus \{1\}$, and $x \neq 0, -1, -2, \cdots$, define

$$\zeta_q(s, x \mid h) = \sum_{n=0}^\infty \frac{q^{hn+x}}{[x+n]_q^s} + \frac{(h-1)(1-q)}{s-1}\sum_{n=0}^\infty \frac{q^{(h-1)n}}{[x+n]_q^{s-1}}.$$

Note that $\lim_{q\to 1}\zeta_q(s, x \mid h) = \zeta(s, x)$, where $\zeta(s, x) = \sum_{n=0}^\infty \frac{1}{(n+x)^s}$ is called Hurwitz's zeta function.

Remark. Note that $\zeta_q(s, x \mid h)$ has only simple pole at $s = 1$ and $\zeta_q(s, x \mid h)$ is meromorphic function except for $s = 1$ in complex s-plane.

By (9) and Definition 4, we obtain the following theorem.

Theorem 5. For $k \in \mathbb{N}$, we have

$$\zeta_q(1-k, x \mid h) = -\frac{\beta^h_{k,q}(x)}{k}.$$

In the special case, $x = 1$, we see that

$$\begin{aligned}\zeta_q(s, 1 \mid h) &= \sum_{n=0}^{\infty} \frac{q^{hn+1}}{[n+1]_q^s} + \frac{(h-1)(1-q)}{s-1} \sum_{n=0}^{\infty} \frac{q^{(h-1)n}}{[n+1]_q^{s-1}} \\ &= \sum_{n=1}^{\infty} \frac{q^{hn+1-h}}{[n]_q^s} + \frac{(h-1)(1-q)}{s-1} \sum_{n=1}^{\infty} \frac{q^{(h-1)n+1-h}}{[n]_q^{s-1}} \\ &= q^{-(h-1)} \left(\sum_{n=1}^{\infty} \frac{q^{hn}}{[n]_q^s} + \frac{(h-1)(1-q)}{s-1} \sum_{n=1}^{\infty} \frac{q^{(h-1)n}}{[n]_q^{s-1}} \right).\end{aligned}$$

Now, we define the (h, q)-zeta function as follows:

Definition 6. For $s \in \mathbb{C} \setminus \{1\}$, and $h \in \mathbb{C}$, define

$$\zeta_q(s \mid h) = \sum_{n=1}^{\infty} \frac{q^{hn}}{[n]_q^s} + \frac{(h-1)(1-q)}{s-1} \sum_{n=1}^{\infty} \frac{q^{(h-1)n}}{[n]_q^{s-1}}.$$

Note that

$$\zeta_q(s \mid h) = q^{h-1} \zeta_q(s, 1 \mid h).$$

For $k \in \mathbb{N}$ with $k > 1$, we have

$$\zeta_q(1-k \mid h) = q^{h-1} \zeta_q(1-k, 1 \mid h) = -\frac{q^{h-1} \beta^h_{k,q}(1)}{k}. \tag{10}$$

By (6), (10), and Corollary 2, we obtain the following corollary.

Corollary 7. For $k \in \mathbb{N}$, we have

$$\zeta_q(1-k \mid h) = -\frac{\beta^h_{k,q}}{k}.$$

Let χ be the Dirichlet character with conductor $f \in \mathbb{N}$. Then we define the generalized (h, q)-Bernoulli polynomials attached to χ as follows:

$$\beta^h_{n,\chi,q}(x) = [f]_q^{n-1} \sum_{a=0}^{f-1} \chi(a) q^{(h-1)a} \beta^h_{n,q^f} \left(\frac{x+a}{f} \right). \tag{11}$$

Note that

$$\lim_{q\to 1}\beta^h_{n,\chi,q}(x)=f^{n-1}\sum_{a=0}^{f-1}\chi(a)\beta_n\left(\frac{x+a}{f}\right)=B_{n,\chi}(x),$$

where $B_{n,\chi}(x)$ are the n-th generalized ordinary Bernoulli polynomials attached to χ. In the special case, $x=0$, $\beta^h_{n,\chi,q}(0)=\beta^h_{n,\chi,q}$ are called the n-th generalized (h,q)-Bernoulli numbers attached to χ.

From (3) and (11), we note that

$$\beta^h_{n,\chi,q}(x)=\sum_{a=0}^{f-1}\chi(a)q^{(h-1)a}\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{l(x+a)}\frac{l+h-1}{[f(l+h-1)]_q}. \tag{12}$$

By (12), we easily get

$$\beta^h_{n,\chi,q}(x)=(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}\chi(m)[x+m]^k_q-n\sum_{m=0}^{\infty}q^{hm+x}\chi(m)[x+m]^{n-1}_q.$$

Therefore, we obtain the following theorem.

Theorem 8. For $h\in\mathbb{C}$ and $n\in\mathbb{Z}_+$, we have

$$\beta^h_{n,\chi,q}(x)=(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}\chi(m)[x+m]^k_q-n\sum_{m=0}^{\infty}q^{hm+x}\chi(m)[x+m]^{n-1}_q.$$

Let $F_{\chi,q}(t,x\mid h)=\sum_{n=0}^{\infty}\beta^h_{n,\chi,q}(x)\frac{t^n}{n!}$. Then we see that

$$F_{\chi,q}(t,x\mid h)=(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}\chi(m)e^{[x+m]_qt}-t\sum_{m=0}^{\infty}q^{hm+x}\chi(m)e^{[x+m]_qt}. \tag{13}$$

Therefore, we obtain the following generating function:

Proposition 9. Let $F_{\chi,q}(t,x\mid h)=\sum_{n=0}^{\infty}\beta^h_{n,\chi,q}(x)\frac{t^n}{n!}$. Then we have

$$F_{\chi,q}(t,x\mid h)=(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}\chi(m)e^{[x+m]_qt}-t\sum_{m=0}^{\infty}q^{hm+x}\chi(m)e^{[x+m]_qt}.$$

Let $F_{\chi,q}(t\mid h)=\sum_{n=0}^{\infty}\beta^h_{n,\chi,q}\frac{t^n}{n!}$. Then we also get

$$F_{\chi,q}(t\mid h)=(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}\chi(m)e^{[m]_qt}-t\sum_{m=0}^{\infty}q^{hm}\chi(m)e^{[m]_qt}. \tag{14}$$

From (14) and the definition of $\beta^h_{n,\chi,q}$, we can derive the following functional equation:

$$\begin{aligned}\beta^h_{n,\chi,q}&=\frac{d^n}{dt^n}F_{\chi,q}(t\mid h)\Big|_{t=0}\\&=(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}\chi(m)[m]^n_q-n\sum_{m=0}^{\infty}q^{hm}\chi(m)[m]^{n-1}_q.\end{aligned} \tag{15}$$

By (14) and (15), we obtain the following corollary:

Corollary 10. For $h \in \mathbb{C}, n \in \mathbb{Z}_+$, we have

$$\beta_{n,\chi,q}^{h} = (h-1)(1-q)\sum_{m=0}^{\infty} q^{(h-1)m}\chi(m)[m]_q^n - n\sum_{m=0}^{\infty} q^{hm}\chi(m)[m]_q^{n-1}.$$

For $s \in \mathbb{C} \setminus \{1\}, h \in \mathbb{C}$, and $x \neq 0, -1, -2, \cdots$, we consider complex integral as follows:

$$\begin{aligned} &\frac{1}{\Gamma(s)}\int_0^{\infty} F_{\chi,q}(-t, x \mid h)t^{s-2}dt \\ &= \frac{(h-1)(1-q)}{s-1}\sum_{m=0}^{\infty}\frac{q^{(h-1)m}\chi(m)}{[x+m]_q^{s-1}} + \sum_{m=0}^{\infty}\frac{q^{hm+x}\chi(m)}{[x+m]_q^{s}}. \end{aligned} \tag{16}$$

From (16), we can define Hurwitz's type (h,q)-L-function as follows:

Definition 11. For $s \in \mathbb{C} \setminus \{1\}, h \in \mathbb{C}$, and $x \neq 0, -1, -2, \cdots$, define

$$L_q^h(s, \chi \mid x) = \frac{(h-1)(1-q)}{s-1}\sum_{m=0}^{\infty}\frac{q^{(h-1)m}\chi(m)}{[x+m]_q^{s-1}} + \sum_{m=0}^{\infty}\frac{q^{hm+x}\chi(m)}{[x+m]_q^{s}}.$$

By the definition of the generating function for the generalized (h,q)-Bernoulli polynomials attached to χ, we get

$$\frac{1}{\Gamma(s)}\int_0^{\infty} F_{\chi,q}(-t, x \mid h)t^{s-2}dt = \sum_{n=0}^{\infty}\frac{(-1)^n\beta_{n,\chi,q}^{h}(x)}{n!}\frac{1}{\Gamma(s)}\int_0^{\infty} t^{n+s-2}dt. \tag{17}$$

We see form (16) and (17) and the basic theory of complex analysis including Laurent series that

$$\begin{aligned} &-\frac{(h-1)(1-q)}{k}\sum_{m=0}^{\infty} q^{(h-1)m}\chi(m)[x+m]_q^k + \sum_{m=0}^{\infty} q^{hm+x}\chi(m)[x+m]_q^{k-1} \\ &= -\frac{\beta_{k,\chi,q}^{h}(x)}{k}, \text{ for } k \in \mathbb{N}. \end{aligned} \tag{18}$$

From (18) and Definition 2, we obtain the following functional equation.

Theorem 12. For $k \in \mathbb{N}$, we have

$$L_q^h(1-k, \chi \mid x) = -\frac{\beta_{k,\chi,q}^{h}(x)}{k}.$$

Let χ be non-trivial Dirichlet character with conductor $f \in \mathbb{N}$. Then we can also consider Dirichlet's type (h,q)-L-function as follows:

$$L_q^h(s, \chi) = \frac{(h-1)(1-q)}{s-1}\sum_{m=1}^{\infty}\frac{q^{(h-1)m}\chi(m)}{[m]_q^{s-1}} + \sum_{m=1}^{\infty}\frac{q^{hm}\chi(m)}{[m]_q^{s}}, \tag{19}$$

for $s \in \mathbb{C} \setminus \{1\}, h \in \mathbb{C}$.

By Corollary 10 and (19), we get

$$L_q^h(1-k,\chi) = -\frac{\beta_{k,\chi,q}^h}{k}, \text{ for } k \in \mathbb{N}.$$

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On a p-adic interpolation function for the multiple generalized Euler numbers and its derivatives

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Abstract : We study analytic function interpolating the multiple generalized Euler numbers attached to χ at negative integers in complex plane and we consider the multiple p-adic l-function as the p-adic analog of the above function. Finally, we give the value of the partial derivative of this multiple p-adic l-function at $s = 0$.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. The Euler numbers in $\mathbb{C}$ are defined by the formula

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \text{ for } |t| < \pi, \text{ see [1-6].} \tag{1}$$

It follows from the definition that $E_0 = 1, E_1 = -1/2, E_2 = 0, E_3 = 1/4, \cdots$, and $E_{2k} = 0$ for $k = 1, 2, 3, \cdots$.

Let $r \in \mathbb{N}$. Then the multiple Euler numbers of order r are defined as

$$F^{(r)}(t) = \left(\frac{2}{e^t + 1}\right)^r = \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!}, |t| < \pi.$$

Let x be a variable. Then the multiple Euler polynomials are also defined by the rule

$$F^{(r)}(t, x) = \left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{2}$$

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, assume that χ is a primitive Dirichlet character with conductor f. It is known that the generalized Euler numbers attached to χ, E_n, χ, are defined by the rule

$$F_\chi(t) = 2 \sum_{a=1}^{f} \frac{\chi(a)(-1)^a e^{at}}{e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}, \tag{3}$$

where $|t| < \frac{\pi}{f}$, (see [1]).

In this paper, we consider the multiple generalized Euler numbers attached to χ in the sense of the multiple of Eq.(3). From these numbers, we study an analytic function interpolating the multiple generalized Euler numbers attached to χ at negative integers in complex number field. In the sense of p-adic analog of the above function and give the values of the partial derivative of this multiple p-adic l-function at $s = 0$.

2. Analytic functions associated with Euler numbers and polynomials

We now consider the multiple generalized Euler numbers attached to χ, $E_{n,\chi}^{(r)}$, which are defined by

$$F_{\chi}^{(r)}(t) = 2^r \sum_{a_1,\cdots,a_r=1}^{f} \frac{(-1)^{\sum_{i=1}^{r} a_i} \chi(a_1+\cdots+a_r) e^{t\sum_{i=1}^{r} a_i}}{(e^{ft}+1)^r} = \sum_{n=0}^{\infty} E_{n,\chi}^{(r)} \frac{t^n}{n!}. \tag{4}$$

By (2) and (4), we readily see that

$$E_{n,\chi}^{(r)} = f^n \sum_{a_1,\cdots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^{r} a_i} \chi(a_1+\cdots+a_r) E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{f}\right). \tag{5}$$

For $s \in \mathbb{C}$, we have

$$2^r \sum_{n_1,\cdots,n_r=0}^{\infty} \frac{(-1)^{n_1+\cdots+n_r}}{(x+n_1+\cdots+n_r)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} F^{(r)}(-t,x) t^{s-1} dt, \tag{6}$$

where $x \neq 0, -1, -2, \cdots$. From (6), we can consider the multiple Euler zeta function as follows:

$$\zeta_r(s,x) = 2^r \sum_{n_1,\cdots,n_r=0}^{\infty} \frac{(-1)^{n_1+\cdots+n_r}}{(x+n_1+\cdots+n_r)^s}, \text{ for } x \in \mathbb{C}, x \neq 0, -1, -2, \cdots. \tag{7}$$

By (1), (2), (5), (6) and (7), we easily see that

$$\zeta_r(-n,x) = E_n^{(r)}(x) \text{ for } n \in \mathbb{N}.$$

By using complex integral and (4), we can also obtain the following equation: for $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)} \int_0^{\infty} F_{\chi}^{(r)}(-t) t^{s-1} dt = 2^r \sum_{\substack{n_1,\cdots,n_r=0 \\ n_1+\cdots+n_r \neq 0}}^{\infty} \frac{\chi(n_1+\cdots+n_r)(-1)^{n_1+\cdots+n_r}}{(n_1+\cdots+n_r)^s}, \tag{8}$$

where χ is the primitive Dirichlet character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$.

By (8), we define Dirichlet's type multiple Euler l-function in complex plane as follows: for $s \in \mathbb{C}$,

$$l_r(s,\chi) = 2^r \sum_{\substack{n_1,\cdots,n_r=0 \\ n_1+\cdots+n_r \neq 0}}^{\infty} \frac{\chi(n_1+\cdots+n_r)(-1)^{n_1+\cdots+n_r}}{(n_1+\cdots+n_r)^s}. \tag{9}$$

From (4), (8) and (9), we note that

$$l_r(-n,\chi) = E_{n,\chi}^{(r)} \text{ for } n \in \mathbb{N}. \tag{10}$$

Let s be a complex variable, and let a and F be integer with $0 < a < F$ and $F \equiv 1 \pmod 2$. Then we consider partial zeta function $T_r(s; a_1, \cdots, a_r|F)$ as follows:

$$\begin{aligned} T_r(s; a_1, \cdots, a_r|F) &= 2^r \sum_{\substack{m_1,\cdots,m_r>0 \\ m_i \equiv a_i \pmod F}} \frac{(-1)^{m_1+\cdots+m_r}}{(m_1+\cdots+m_r)^s} \\ &= (-1)^{a_1+\cdots+a_r} F^{-s} \zeta_r\left(s, \frac{a_1+\cdots+a_r}{F}\right). \end{aligned} \tag{11}$$

Let $\chi(\neq 1)$ be the Dirichlet character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$ and let F be the multiple of f with $F \equiv 1 \pmod 2$. Then Dirichlet's type multiple l-function can be expressed as the sum

$$l_r(s,\chi) = \sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r) T_r(s; a_1, \cdots, a_r|F) \text{ for } s \in \mathbb{C}. \tag{12}$$

By simple calculation, we easily see that

$$T_r(-n; a_1, \cdots, a_r|F) = F^n (-1)^{a_1+\cdots+a_r} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right) \text{ for } r, n \in \mathbb{N}. \tag{13}$$

From (5), (12) and (13), we note that

$$l_r(-n,\chi) = E_{n,\chi}^{(r)} \text{ for } n \in \mathbb{N}. \tag{14}$$

By (1) and (2), we have

$$E_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_l^{(r)} x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}^{(r)} x^l. \tag{15}$$

From (11), (13) and (15), we derive the following equation:

$$T_r(s; a_1, \cdots, a_r|F) = (-1)^{a_1+\cdots+a_r} (a_1+\cdots+a_r)^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a_1+\cdots+a_r}\right)^k E_k^{(r)}. \tag{16}$$

It follows from (12), (13) and (16) that

$$\begin{aligned} l_r(s,\chi) = &\sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r}(a_1+\cdots+a_r)^{-s} \\ &\times \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)}. \end{aligned} \tag{17}$$

The values of $l_r(s,\chi)$ at negative integer are algebraic numbers, and hence they can be regarded as numbers belonging to an extension of $\mathbb{Q}_p$. In the next section, we therefore look for a p-adic function that agrees with $l_r(s,\chi)$ at negative integers.

3. On a p-adic interpolation function for the multiple Euler numbers and its derivative

In this section, we study the p-adic analogs of the Dirichlet's type multiple Euler l-function, $l_r(s,\chi)$, which were introduced in previous section. If fact, this function is the p-adic interpolation function for the multiple generalized Euler numbers attached to χ at negative integers. Let w denote the Teichmüller character with the conductor $f_w = p$. For an arbitrary character χ, we set $\chi_n = \chi w^{-n}, n \in \mathbb{Z}$, in the sense of the product of characters. Let

$$< a >= w^{-1}(a)a = \frac{a}{w(a)}.$$

Then we note that $< a > \equiv 1 \pmod{p\mathbb{Z}_p}$. Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j}x^n, a_{n,j} \in \mathbb{C}_p, j = 0, 1, 2, \cdots$$

be a sequence of power series, each convergent on a fixed subset

$$D = \{s \in \mathbb{C}_p \mid |s|_p < p^{1-\frac{1}{p-1}}\},$$

of $\mathbb{C}_p$ such that

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for any n;

(2) for each $s \in D$ and $\epsilon > 0$, there exists an $n_0 = n_0(s,\epsilon)$ such that

$$|\sum_{n \geq n_0} a_{n,j}s^n|_p < \epsilon \text{ for } \forall j.$$

In this case,

$$\lim_{j\to\infty} A_j(s) = A_0(s), \text{ for all } s \in D.$$

This was used by Washington ([6]) to show that each of functions $w^{-s}(a)a^s$ and

$$\sum_{m=0}^{\infty} \binom{s}{m} \left(\frac{F}{a}\right)^m B_m,$$

where F is multiple of p and f and B_m is the m-th Bernoulli numbers, is analytic on D (see [6]).

Let χ be a primitive Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$. Then we consider the multiple Euler p-adic l-function, $l_{p,r}(s,\chi)$, which interpolates the multiple generalized Euler numbers attached to χ at negative integers.

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, let us assume that F is a positive integral multiple of p and $f = f_\chi$. We now define the multiple Euler p-adic l-function as follows:

$$\begin{aligned} l_{p,r}(s,\chi) = &\sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r) < a_1+\cdots+a_r >^{-s} (-1)^{a_1+\cdots+a_r} \\ &\times \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)}. \end{aligned} \tag{18}$$

From (18), we note that $l_{p,r}(s,\chi)$ is analytic for $s \in D$.

For $n \in \mathbb{N}$, we have

$$E_{n,\chi_n}^{(r)} = F^n \sum_{a_1,\cdots,a_r=0}^{F-1} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right)\chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r}. \tag{19}$$

If $\chi_n(p) \neq 0$, then $(p, f_{\chi_n}) = 1$, and thus the ratio F/p is a multiple of f_{χ_n}.

Let

$$I_0 = \left\{ \frac{a_1+\cdots+a_r}{p} \mid a_1+\cdots+a_r \equiv 0 \pmod{p} \text{ for some } a_i \in \mathbb{Z} \text{ with } 0 \le a_i \le F \right\}.$$

Then we have

$$\begin{aligned} & F^n \sum_{\substack{a_1,\cdots,a_r=0 \\ p \mid a_1+\cdots+a_r}}^{F-1} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right)\chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \\ & = p^n \left(\frac{F}{p}\right)^n \chi_n(p) \sum_{\substack{a_1,\cdots,a_r=0 \\ \beta \in I_0}}^{F/p} \chi_n(\beta)(-1)^{\beta} E_n^{(r)}\left(\frac{\beta}{F/p}\right). \end{aligned} \tag{20}$$

From (20), we define the second multiple generalized Euler numbers attached to χ as follows:

$$E_{n,\chi_n}^{*(r)} = \left(\frac{F}{p}\right)^n \sum_{\substack{a_1,\cdots,a_r=0 \\ \beta \in I_0}}^{F/p} \chi_n(\beta)(-1)^{\beta} E_n^{(r)}\left(\frac{\beta}{F/p}\right). \tag{21}$$

By (19), (20) and (21), we easily see that

$$\begin{aligned} & E_{n,\chi_n}^{(r)} - p^n \chi_n(p) E_{n,\chi_n}^{*(r)} \\ & = F^n \sum_{\substack{a_1,\cdots,a_r=1 \\ p \nmid a_1+\cdots+a_r}}^{F} \chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right). \end{aligned} \tag{22}$$

By the definition of the multiple Euler polynomials of order r, we see that

$$E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right) = F^{-n}(a_1+\cdots+a_r)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a_1+\cdots+a_r}\right)^k E_k^{(r)}. \tag{23}$$

From (22) and (23), we have

$$\begin{aligned} E_{n,\chi_n}^{(r)} - p^n \chi_n(p) E_{n,\chi_n}^{*(r)} = & \sum_{\substack{a_1,\cdots,a_r=1 \\ p \nmid a_1+\cdots+a_r}}^{F} (a_1+\cdots+a_r)^n \chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \\ & \times \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a_1+\cdots+a_r}\right)^k E_k^{(r)}. \end{aligned} \tag{24}$$

By (18) and (24), we readily see that

$$\begin{aligned} & l_{p,r}(-n,\chi) \\ &= \sum_{\substack{a_1,\cdots,a_r=1 \\ p\nmid a_1+\cdots+a_r}}^{F} (a_1+\cdots+a_r)^n \chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \\ &\qquad \times \sum_{m=0}^{n} \binom{n}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)} \\ &= E_{n,\chi_n}^{(r)} - p^n \chi_n(p) E_{n,\chi_n}^{*(r)}. \end{aligned} \tag{25}$$

Therefore, we obtain the following theorem.

Theorem 1. Let F be a positive integral of p and $f(=f_{\chi_n})$, and let

$$\begin{aligned} l_{p,r}(s,\chi) = & \sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r) < a_1+\cdots+a_r >^{-s} (-1)^{a_1+\cdots+a_r} \\ & \times \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)}. \end{aligned}$$

Then $l_{p,r}(s,\chi)$ is analytic on D. Furthermore, for each $n \in \mathbb{N}$, we have

$$l_{p,r}(-n,\chi) = E_{n,\chi_n}^{(r)} - p^n \chi_n(p) E_{n,\chi_n}^{*(r)}.$$

Using Taylor expansion at $s=0$, we get

$$\binom{-s}{m} = \frac{(-1)^m}{m} s + \cdots \text{ if } m \geq 1. \tag{26}$$

From (26) and Theorem 1, we obtain the following corollary.

Corollary 2. Let F be a positive integral multiple of p and f. Then we have

$$\begin{aligned} \frac{\partial}{\partial s} l_{p,r}(0,\chi) = & \sum_{\substack{a_1,\cdots,a_r=1 \\ (a_1+\cdots+a_r,p)=1}}^{F} \chi(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \left(1-\log_p(a_1+\cdots+a_r)\right) \\ & + \sum_{\substack{a_1,\cdots,a_r=1 \\ (a_1+\cdots+a_r,p)=1}}^{F} \chi(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)}, \end{aligned}$$

where $\log_p x$ is denoted by the p-adic logarithm.

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A fast algorithm for inversion of real lower triangular Toeplitz matrices *

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An effective fast algorithm of [Fu-Rong Lin et al, Theoretical Computer and Science, 315(2004),511-523] computes the inverse of a lower triangular Toeplitz matrix with computational complexity $O(nlogn)$. In this paper, we present a modified fast algorithm for inversion of lower triangular Toeplitz matrices, using both fast sine transform and fast cosine transform. Our algorithm requires two fast Fourier transforms and one fast sine transform and one fast cosine transform, and so the complexity of our algorithm is $O(nlogn)$. As the theoretical results and the numerical experiments show, the modified version of fast algorithm has more accuracy than Lin's fast algorithm.

Mathematics subject classification: 65F10, 65F50.
Keywords: Lower triangular Toeplitz matrices, Fast sine transform; Fast cosine transform; Fast Fourier transform.

1. Introduction

Special structure of matrices become interesting as soon as they play an important role in various fields, such as rational interpolation problem, signal and image processing etc. Toeplitz systems arise in a variety of applications in mathematics and engineering, e.g. signal and image processing[4]. Applications of triangular Toeplitz matrix inversion include Gauss-Seidel iteration for Toeplitz systems, which can also be used as a smoother in multigrid methods.

There is extensive literature on designing fast inversion algorithms for structured matrices. In this paper, we present a method for fast inversion of a lower triangular Toeplitz matrix

$$T_n = \begin{bmatrix} t_0 & 0 & 0 & 0 \\ t_1 & t_0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ t_{n-1} & \dots & t_1 & t_0 \end{bmatrix},$$

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where $t_j, j = 0, 1, \ldots, n-1$ are real with $t_0 \neq 0$.

There are two types of inversion methods for triangular Toeplitz matrices: exact inversion([1,3])and approximate inversion([2,10,11]).A simple method for obtaining the inverse of a triangular Toeplitz matrix is the forward substitution method which requires about $n(n+1)$ arithmetic operations. Another exact method is the divide-and-conquer method which requires $O(nlogn)$ operations.

Bini[2] proposed a fast algorithm for the approximate inversion of a lower triangular Toeplitz matrix . Let $H_n = [h_{ij}]_{i,j=1}^n$, where all entries of H_n are zero except that $h_{j+1,j} = 1$ for $j = 1, \cdots, n-1$. Then we have

$$T_n = \sum_{j=0}^{n-1} t_j H_n^j.$$

His basic idea is to approximate H_n by

$$H_n^{(\epsilon)} = [h_{ij}^{(\epsilon)}]_{i,j=1}^n$$

where $h_{ij}^{(\epsilon)} = h_{ij}$ when $(i,j) \neq (1,n)$ and $h_{1n}^{(\epsilon)} = \epsilon^n$. Here ϵ is a small positive number.

Let

$$D_n^{(\epsilon)} = diag(1, \epsilon, \cdots, \epsilon^{n-1}).$$

Bini proved that the matrix $D_n^{(\epsilon)} H_n^{(\epsilon)} (D_n^{(\epsilon)})^{-1}$ can be diagonalized by Fourier matrix, which follows that the inverse of

$$T_n^{(\epsilon)} = \sum_{j=0}^{n-1} t_j (H_n^{(\epsilon)})^j$$

can be computed efficiently by using the FFTs and so the computational complexity is $O(nlogn)$.

In paper [9], Lin et al present another fast algorithm for inversion of lower triangular Toeplitz matrices based on trigonometric polynomial interpolation and discrete cosine transform(DCT).

In this paper, we revise the approximate method proposed by Lin[9] to obtain a fast algorithm for inversion of triangular Toeplitz matrices with higher accuracy, by using both discrete sine transform(DST) and discrete cosine transform(DCT). Our algorithm has technical similarity with the algorithm in [9].

The paper is organized as follows: In Section 2, we introduce the definition of discrete sine transform and discrete cosine transform. Also we state some properties of a lower triangular Toeplitz matrices without proof. In Section 3, we present our fast inversion algorithm. Finally, we show some numerical examples.

2. Preliminary

Discrete sine and cosine transform arise in many applications of digital signal and image processing([5–8,12]). The discrete sine transform matrix[5] is defined by

$$[S_{n-1}] = \left[\sqrt{\frac{2}{n}}\sin\frac{ml\pi}{n}\right]_{m,l=1}^{n-1} = \sqrt{\frac{2}{n}}\begin{bmatrix} \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \sin\frac{(n-1)\pi}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{4\pi}{n} & \cdots & \sin\frac{2(n-1)\pi}{n} \\ \vdots & \vdots & \cdots & \vdots \\ \sin\frac{(n-1)\pi}{n} & \sin\frac{2(n-1)\pi}{n} & \cdots & \sin\frac{(n-1)^2\pi}{n} \end{bmatrix} \tag{1}$$

then the following relation holds for the inverse DST matrix:

$$[S_{n-1}]^{-1} = [S_{n-1}]^T = [S_{n-1}] \tag{2}$$

The discrete cosine-II(DCT-II) and the discrete cosine-III(DCT-III) transform matrices[5] are defined by

$$[C_n^{II}] = \sqrt{\frac{2}{n}}\left[k_m\cos\frac{m(2l+1)\pi}{2n}\right]_{m,l=0}^{n-1}, \tag{3}$$

and

$$[C_n^{III}] = \sqrt{\frac{2}{n}}\left[k_l\cos\frac{(2m+1)l\pi}{2n}\right]_{m,l=0}^{n-1}, \tag{4}$$

where in (3)-(4)

$$k_{j(j=m\ or\ l)} = \begin{cases} 1, & if\ j \neq 0, \\ 1/\sqrt{2}, & if\ j = 0. \end{cases} \tag{5}$$

That is, $C_n^{II} = [C_n^{III}]^T$ and

$$[C_n^{II}] = \begin{bmatrix} \sqrt{1/n} & \sqrt{1/n} & \cdots & \sqrt{1/n} \\ \sqrt{\frac{2}{n}}\cos\frac{\pi}{2n} & \sqrt{\frac{2}{n}}\cos\frac{3\pi}{2n} & \cdots & \sqrt{\frac{2}{n}}\cos\frac{(2n-1)\pi}{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{\frac{2}{n}}\cos\frac{(n-1)\pi}{2n} & \sqrt{\frac{2}{n}}\cos\frac{3(n-1)\pi}{2n} & \cdots & \sqrt{\frac{2}{n}}\cos\frac{(2n-1)(n-1)\pi}{2n} \end{bmatrix}, \tag{6}$$

then the following relation holds for the inverse DCT matrix:

$$[C_n^{II}]^{-1} = [C_n^{III}] = [C_n^{II}]^T. \tag{7}$$

As we know, the $n \times n$ downshift matrix $Z = (z_{i,j}), z_{i+1,i} = 1, z_{ij} = 0\ if\ i = 1 \neq j, Z^n = 0$, generates the algebra of lower triangular Toeplitz matrices, $T = \sum_{i=0}^{n-1} t_i Z^i$, which is isomorphic to the ring of polynomial modulo z^n,$t(z) = \sum_{i=0}^{n-1} t_i z^i$. In particular, $T^{-1} = \sum_{i=0}^{n-1} b_i Z^i$ iff $\sum_{i=0}^{n-1} b_i z^i t(z) = 1 \bmod z^n$, that is , the inversion of T is equivalent to computing the reciprocal of $t(z)$ modulo z^n, the latter operation is immediately extend to polynomial division. This leads to the following three properties immediately(see[9]):

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Property 1 For a lower triangular Toeplitz matrix T_n, we define the polynomial

$$\tau_n(z) = \sum_{k=0}^{n-1} t_k z^k. \tag{8}$$

Let the Maclaurin series of $\tau_n^{-1}(z)$ be given by

$$\tau_n^{-1}(z) = \sum_{k=0}^{\infty} b_k z^k, \tag{9}$$

then

$$T_n^{-1} = \begin{bmatrix} b_0 & 0 & 0 & 0 \\ b_1 & b_0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ b_{n-1} & \dots & b_1 & b_0 \end{bmatrix}, \tag{10}$$

So in order to obtain T_n^{-1}, we only need to compute the coefficients b_k for $k = 0, 1, \dots, n-1$.

Property 2 Replacing z in (8) and (9) by εz we get

$$\tau_{n,\varepsilon}(z) = \tau_n(\varepsilon z) = \sum_{k=0}^{n-1} (t_k \varepsilon^k) z^k \quad and \quad \tau_{n,\varepsilon}^{-1}(z) = \tau_n^{-1}(\varepsilon z) = \sum_{k=0}^{\infty} (b_k \varepsilon^k) z^k.$$

Property 3 The inverse of the leading principal sub-matrix $T_n(1:p, 1:p)$ is equal to the leading principal sub-matrix $T_n^{-1}(1:p, 1:p)$ for $1 \le p \le n$.

3. Fast algorithm for inversion

By replacing z by e^{-ix} in (8), where $i = \sqrt{-1}$ and x is a real variable, we see that $\tau_n(e^{-ix})$ is a trigonometric polynomial. Therefore, we consider interpolating τ_n^{-1} by trigonometric polynomial. It is difficult to compute b_k by algebraic polynomial interpolation since the high-order Vandermonde matrices are very ill-conditioned. Thus, we let

$$\tau_\varepsilon(\theta) \equiv \tau_{n\varepsilon}(e^{-i\theta}) = \sum_{k=0}^{n-1} (t_k \varepsilon^k) e^{-ik\theta} = \tau_\varepsilon^{(r)}(\theta) + i\tau_\varepsilon^{(i)}(\theta), \tag{11}$$

where $\tau_\varepsilon^{(r)}(\theta)$ and $\tau_\varepsilon^{(i)}(\theta)$ are the real and imaginary parts of $\tau_\varepsilon(\theta)$, respectively. Similarly, by Property 2, we can have

$$h_\varepsilon(\theta) \equiv \tau_\varepsilon^{-1}(\theta) = \sum_{k=0}^{\infty} (b_k \varepsilon^k) e^{-ik\theta}. \tag{12}$$

In particular, the real and imaginary parts of $h_\varepsilon(\theta)$ are given by,respectively

$$h_\varepsilon^{(r)}(\theta) = \sum_{k=0}^{\infty} (b_k \varepsilon^k) \cos(k\theta) = \frac{\tau_\varepsilon^{(r)}(\theta)}{(\tau_\varepsilon^{(r)}(\theta))^2 + (\tau_\varepsilon^{(i)}(\theta))^2}, \tag{13}$$

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and

$$h_\varepsilon^{(i)}(\theta) = \sum_{k=1}^{\infty}(b_k\varepsilon^k)\sin(-k\theta) = \frac{-\tau_\varepsilon^{(i)}(\theta)}{(\tau_\varepsilon^{(r)}(\theta))^2 + (\tau_\varepsilon^{(i)}(\theta))^2}. \tag{14}$$

In [9], Lin et al interpolated $h_\varepsilon^{(r)}(\theta)$ by the function

$$P(\theta) = \sum_{k=0}^{n-1} c_k cos(k\theta)$$

to obtain approximate values $\hat{b}_k$ for $b_k(k = 1, 2, \ldots, n-1)$,using the following equidistant points

$$\theta_k = \frac{2k-1}{2n}\pi, k = 1, 2, \cdots, n.$$

as the interpolating knots. Then $\hat{b}_k$ can be obtained efficiently by using the fast DCT. In this paper, we interpolate -$h_\varepsilon^{(i)}(\theta)$ using the following interpolating function and interpolating knots,respectively

$$Q(\theta) = \sum_{k=1}^{n-1} s_k sin(k\theta),$$

and

$$\theta_l = \frac{l\pi}{n}, l = 1, 2, \ldots, n-1.$$

Then by using the interpolating conditions $Q(\theta_l) = h_\varepsilon^{(i)}(\theta_l), l = 1, 2, \ldots, n-1$, we have

$$\tilde{S}_{n-1}(s_1, s_2, \ldots, s_{n-1})^T = (h_\varepsilon^{(i)}(\theta_1), h_\varepsilon^{(i)}(\theta_2), \ldots, h_\varepsilon^{(i)}(\theta_{n-1}))^T \tag{15}$$

where

$$\tilde{S}_{n-1} = \begin{bmatrix} \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \sin\frac{(n-1)\pi}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{4\pi}{n} & \cdots & \sin\frac{2(n-1)\pi}{n} \\ \vdots & \vdots & \cdots & \vdots \\ \sin\frac{(n-1)\pi}{n} & \sin\frac{2(n-1)\pi}{n} & \cdots & \sin\frac{(n-1)^2\pi}{n} \end{bmatrix} \tag{16}$$

Note that $\tilde{S}_{n-1} = \sqrt{\frac{n}{2}}S_{n-1}$, where S_{n-1} is the discrete sine transform matrix defined by (1). If the values of $h_\varepsilon^{(i)}(\theta_l), k = 1, 2, \ldots, n-1$ are known, then s_k can be obtained by using fast DST of $(n-1)$-vector. Finally, $\hat{b}_k$ can be computed in terms of $\hat{b}_0 = t_0^{-1}$ and

$$\hat{b}_k = s_k\varepsilon^{-k}, k = 1, 2, \ldots, n-1. \tag{17}$$

For the accuracy of $\hat{b}_k, k = 1, 2, \ldots, n-1$, we have the following theorem.

Theorem 1 *Let the Maclaurin series of $\tau_n^{-1}(\varepsilon z)$ be given by $\sum_{k=0}^{\infty}(b_k\varepsilon^k)z^k$ and $\varepsilon \in (0, 1)$ such that $\sum_{k=0}^{\infty}|b_k\varepsilon^k| < \infty$. Let*

$$Q(\theta) = \sum_{j=1}^{n-1} s_j \sin j\theta$$

be the interpolating polynomial for $-h_\varepsilon^i(\theta)$ *with the interpolating knots* $\theta_k = k\pi/n, k = 1, 2, \ldots, n-1$ *and* $\hat{b}_j = s_j\varepsilon^{-j}, j = 1, 2, \ldots, n-1$. *Then we have*

$$\hat{b}_j = b_j + \sum_{m=1}^{\infty}(b_{2mn+j}\varepsilon^{2mn} - b_{2mn-j}\varepsilon^{2(mn-j)}), j = 1, 2, \ldots, n-1 \tag{18}$$

Proof: Let us consider the interpolating polynomial of $\sin j\theta$ with $\theta_k (k = 1, 2, \ldots, n-1)$ as interpolating knots, where $j \geq 0$ is an integer. Let m and j be integers, we have

$$\sin((2mn \pm j)\theta_k) = \pm\sin(kj\pi/n) = \pm\sin(j\theta_k).$$

So we can get

$$s_j = b_j\varepsilon^j + \varepsilon^j\sum_{m=1}^{\infty}(\varepsilon^{2mn}b_{2mn+j} - \varepsilon^{2(mn-j)}b_{2mn-j}), j = 1, 2, \ldots, n-1.$$

Using $s_j = \hat{b}_j\varepsilon^j$, the theorem follows immediately.

Then, we can give the fast algorithm for inversion as follows:

Algorithm 1:(based on DST)

1. Choose $\epsilon \in (0, 1)$ and compute $\widetilde{t}_j = \epsilon^j t_j$, *for* $j = 0, 1, \cdots, n-1$;

2. Compute $\tau_\varepsilon(\theta_k) = \sum_{k=0}^{n-1}\widetilde{t}_k e^{-ik\theta_k}$ *by FFT, where* $\theta_k = \frac{k\pi}{n}, k = 0, 1, 2, \ldots, n-1$

3. Compute $h_k = h_\epsilon^{(i)}(\theta_k) = \tau_\epsilon^{(i)}(\theta_k)/((\tau_\epsilon^{(i)}(\theta_k))^2 + (\tau_\epsilon^{(r)}(\theta_k))^2), k = 1, \cdots, n-1$;

4. Compute $\widehat{b}_k = S_{n-1}(h_1, h_2, \cdots, h_{n-1})^T$ by one fast DST, where S_{n-1} is a DST matrix defined by (1);

5. Compute $\widehat{b}_k = \sqrt{2/n}\,\widehat{b}_k, k = 1, \cdots, n-1$;

6. Compute $\widehat{b}_0 = (t_0)^{-1}, \widehat{b}_k = \widehat{b}_k\epsilon^{-k}, k = 1, \cdots, n-1$.

It is easy to see that the computational complexity of Algorithm 1 is $O(nlogn)$.

According to Theorem 1, we know that the smaller the value of ϵ, the more accurate the approximate $\widehat{b}_j (j = 1, \cdots, n-1)$ will be. However, we can't set ϵ too small, otherwise the computation of s_j/ϵ_j will bring in very large rounding error for large j. For ϵ is close to 1, $\widehat{b}_j$ can be very accurate for small by (18). However, $\widehat{b}_j$may not be accurate for j close to n. Then, by Property 3, we can interpolate $h_\epsilon^{(i)}$ by trigonometric polynomial of degree $2(n-1)$ to obtain a modification version of Algorithm 1, with the better accuracy and the numerical results.

Algorithm $1'$:

1. Choose $\epsilon \in (0, 1)$ and compute $\widetilde{t}_j = \epsilon^j t_j$, *for* $j = 0, 1, \cdots, n-1$;

2. Compute $\tau_\varepsilon(\theta_k) = \sum_{l=0}^{n-1}\widetilde{t}_l e^{-il\theta_k}$ *by FFT, where* $\theta_k = \frac{k\pi}{2n}, k = 0, 1, 2, \ldots, 2n-1$

3. Compute $h_k = h_\epsilon^{(i)}(\theta_k) = \tau_\epsilon^{(i)}(\theta_k)/((\tau_\epsilon^{(i)}(\theta_k))^2 + (\tau_\epsilon^{(r)}(\theta_k))^2), k = 1, \cdots, 2n-1$;

4. Compute $\widehat{b}_k = S_{2n-1}(h_1, h_2, \cdots, h_{2n-1})^T$ by one fast DST, where S_{2n-1} is a DST matrix of order $2n-1$ defined by (1);

5. Compute $\widehat{b}_k = \sqrt{1/n}\,\widehat{b}_k, k = 1, \cdots, n-1$;

6. Compute $\widehat{b}_0 = (t_0)^{-1}, \widehat{b}_k = \widehat{b}_k\epsilon^{-k}, k = 1, \cdots, n-1$.

We notice that if we can choose a suitable ϵ, then the accuracy of the approximate solution can be improved remarkably, especially for j is not close to n. By large of numerical experiments, we note that $\epsilon = 2^{-20/n}$ is a good choice.

Now we assume that $\widehat{b}_j, j = 1, \cdots, n-1$ is computed by Lin's algorithm[9] and $\widetilde{b}_j, j = 1, \cdots, n-1$ is computed by Algorithm $1^{'}$, then we take the arithmetic mean $\overline{b}_j \equiv (\widehat{b}_j + \widetilde{b}_j)/2$, which has the more accuracy than $\widehat{b}_j$ and $\widetilde{b}_j$.

In fact, by Theorem 1 of this paper and Theorem 1 of [9], we can get

$$\overline{b}_j = b_j + \sum_{m=1}^{\infty}(-b_{2(2m-1)-j}\varepsilon^{2(2m-1)n-2j} + b_{4mn+j}\varepsilon^{4mn}), j = 1, 2, \ldots, n-1 \tag{19}$$

Thus, we can obtain the following fast algorithm for inversion of lower triangular Toeplitz matrices

Algorithm 2:

1. Compute $\widehat{b}_j$ by Lin'algorithm, for $k = 1, \cdots, n-1$.

2. Compute $\widetilde{b}_j$ by Algorithm $1^{'}$, for $k = 1, \cdots, n-1$.

3. Compute $\overline{b}_j = (\widehat{b}_j + \widetilde{b}_j)/2, j = 1, \cdots, n-1$ and $\overline{b}_0 = t_0^{-1}$.

4. Numerical examples

In this section, we will test the accuracy of Bini's algorithm, Lin's algorithm and our algorithms by using the following six sequences of lower triangular Toeplitz matrices

(i) $t_j = 1/(j+1)^3, j = 0, 1, \cdots, n-1$,

(ii) $t_j = 1/(j+1)^2, j = 0, 1, \cdots, n-1$,

(iii) $t_j = 1/(j+1), j = 0, 1, \cdots, n-1$,

(iv) $t_j = 1/log(j+2), j = 0, 1, \cdots, n-1$,

(v) $t_0 = 1, t_1 = 1 \ \ and \ \ t_j = 0, \ \ for \ \ j = 2, \cdots, n-1$,

(vi) $t_0 = 1, t_1 = -2, t_2 = 1 \ \ and \ \ t_j = 0, \ \ for \ \ j = 3, \cdots, n-1$,

In the following tables, we show the relative accuracy of the approximate inverse

$$\frac{||\overline{b} - b||_1}{||b||_1},$$

where $\overline{b}$ is the first column of the approximate inverse and b is the first column of the exact inverse. For (i)-(iv), we set b to the first column of the inverse computed by the divide-and-conquer method.

Example 1. Let $t_j = 1/(j+1)^3, j = 0, 1, \cdots, n-1$,

n	128	256	512	1024	2048	4096
Bini	1.0e-8	1.9e-8	2.5e-8	3.8e-8	5.0e-8	7.0e-8
Lin	2.7e-11	3.1e-11	4.6e-11	7.5e-11	1.2e-10	1.6e-10
Algorithm $1^{'}$	1.3e-11	3.6e-11	5.7e-11	6.6e-11	8.4e-10	5.6e-9
Algorithm 2	2.9e-13	3.3e-13	4.8e-13	8.2e-13	1.8e-12	3.2e-12

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Example 2. Let $t_j = 1/(j+1)^2, j = 0, 1, \cdots, n-1$,

n	128	256	512	1024	2048	4096
Bini	9.5e-9	1.4e-8	2.5e-8	3.3e-8	5.6e-8	7.6e-8
Lin	2.3e-11	2.6e-11	4.4e-11	7.2e-11	8.9e-11	1.5e-10
Algorithm $1^{'}$	3.2e-11	3.9e-11	4.5e-11	5.2e-11	7.8e-11	6.6e-10
Algorithm 2	3.9e-13	4.2e-13	4.9e-13	7.2e-13	9.5e-13	1.2e-12

Example 3. Let $t_j = 1/(j+1), j = 0, 1, \cdots, n-1$,

n	128	256	512	1024	2048	4096
Bini	5.0e-9	2.0e-8	2.9e-8	4.2e-8	5.6e-8	8.2e-8
Lin	2.0e-11	3.7e-11	4.8e-11	6.6e-11	9.8e-11	1.6e-10
Algorithm $1^{'}$	1.7e-11	4.6e-11	5.0e-11	5.7e-11	6.2e-10	7.2e-10
Algorithm 2	2.2e-13	3.7e-13	4.1e-13	6.2e-13	1.8e-12	3.2e-12

Example 4. Let $t_j = 1/log(j+2), j = 0, 1, \cdots, n-1$,

n	128	256	512	1024	2048	4096
Bini	1.5e-8	2.1e-8	3.5e-8	6.4e-8	1.1e-7	1.9e-7
Lin	2.2e-11	3.7e-11	7.6e-11	1.1e-10	1.7e-10	3.4e-10
Algorithm $1^{'}$	3.3e-11	3.6e-11	4.8e-11	8.2e-11	2.2e-10	5.3e-10
Algorithm 2	8.9e-14	3.3e-13	4.8e-13	6.7e-13	7.8e-13	8.2e-12

Example 5. Let $t_0 = 1, t_1 = 1 \ \ and \ \ t_j = 0, \quad for \quad j = 2, \cdots, n-1$,

n	128	256	512	1024	2048	4096
Bini	5.2e-9	5.7e-9	5.3e-9	5.7e-9	5.9e-9	9.2e-9
Lin	1.4e-12	7.0e-12	5.8e-12	8.9e-12	1.1e-11	7.4e-11
Algorithm $1^{'}$	1.0e-12	5.6e-12	6.2e-12	7.1e-12	5.0e-11	7.2e-11
Algorithm 2	1.9e-14	2.7e-14	4.6e-14	5.1e-14	7.0e-14	8.7e-14

Example 6. Let $t_0 = 1, t_1 = -2, t_2 = 1 \ \ and \ \ t_j = 0, \quad for \quad j = 3, \cdots, n-1$,

n	128	256	512	1024	2048	4096
Bini	1.4e-8	1.5e-8	1.7e-8	2.3e-8	6.0e-8	1.1e-7
Lin	6.8e-12	1.7e-11	2.8e-11	1.0e-10	9.7e-10	4.0e-9
Algorithm $1^{'}$	5.3e-12	3.1e-11	4.4e-11	7.1e-11	7.6e-10	9.5e-10
Algorithm 2	3.7e-14	2.6e-13	3.9e-13	5.7e-13	7.2e-13	5.8e-12

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GENERALIZED RIEMANN-STIELTJES OPERATORS BETWEEN DIFFERENT WEIGHTED BERGMAN SPACES

Xiangling Zhu

Abstract. Let $H(B)$ denote the space of all holomorphic functions on the unit ball B of $\mathbb{C}^n$ and $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$ the radial derivative of f. Let m be a nonnegative integer and $g \in H(B)$ such that $g(0) = 0$. In this paper we study the boundedness and compactness of the following operator

$$L_g^{(m)} f(z) = \int_0^1 \Re^m f(tz) g(tz) \frac{dt}{t}, \qquad z \in B,$$

between different weighted Bergman spaces in the unit ball.

1 Introduction

Let B be the unit ball of $\mathbb{C}^n$ and $H(B)$ the space of all holomorphic functions on B. We denote by $d\nu$ the normalized Lebesgue measure of B, i.e. $\nu(B) = 1$. Let $d\nu_\alpha(z) = c_\alpha(1-|z|)^\alpha d\nu(z)$, where $c_\alpha = (\Gamma(n+\alpha+1))/(n!\Gamma(\alpha+1))$. For $f \in H(B)$, let

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

represent the radial derivative of $f \in H(B)$. We write $\Re^m f = \Re(\Re^{m-1} f)$ (see [19]).

Let $p \in (0,\infty)$ and $\alpha > -1$. The weighted Bergman space $A_\alpha^p(B) = A_\alpha^p$ is defined to be the space of all holomorphic functions f on B such that

$$\|f\|_{A_\alpha^p}^p = \int_B |f(z)|^p d\nu_\alpha(z) = c_\alpha \int_B |f(z)|^p (1-|z|^2)^\alpha d\nu(z) < \infty.$$

When $\alpha = 0$, $A_0^p(B) = A^p(B)$ is the standard Bergman space. It is well known that $f \in A_\alpha^p$ if and only if $(1-|z|^2)\Re f(z) \in L^p(B, d\nu_\alpha)$. Moreover

$$\|f\|_{A_\alpha^p}^p \asymp |f(0)|^p + \int_B |\Re f(z)|^p (1-|z|^2)^p d\nu_\alpha(z). \tag{1}$$

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See [18, 19] for some basic facts on Bergman spaces in the unit ball.

For any $r > 0$ and $\zeta \in S$, the boundary of B, the nonisotropic metric ball $Q_r(\zeta)$ is defined by

$$Q_r(\zeta) = \{z \in B : |1 - \langle z, \zeta\rangle| < r\}.$$

Let $\gamma > 0$ and μ be a positive Borel measure on B. If there exists a constant $C > 0$ such that

$$\mu(Q_r(\zeta)) \le Cr^\gamma$$

for all $\zeta \in S$ and $r > 0$, we call μ the γ-Carleson measure. It is well known that μ is a γ-Carleson measure if and only if (see, e.g. [17])

$$\sup_{a\in B} \int_B \left(\frac{1 - |a|^2}{|1 - \langle z, a\rangle|^2}\right)^\gamma d\mu(z) < \infty. \tag{2}$$

Let $g \in H(B)$, the Riemann-Stieltjes(or Extended-Cesàro operator) T_g, which was introduced in [4], with symbol g is defined on $H(B)$ as follows

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \qquad z \in B, \quad f \in H(B). \tag{3}$$

In [5], Li defined the Riemann-Stieltjes operator L_g as follows

$$L_g f(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t}, \qquad z \in B, \quad f \in H(B). \tag{4}$$

Recently, the two operators attracted many attentions. See [1, 3–16, 20, 21] and references therein for the study of the operators T_g and L_g.

In [22], we defined a new integral-type operators as follows.

$$L_g^{(m)} f(z) = \int_0^1 \Re^m f(tz)g(tz)\frac{dt}{t}, \qquad z \in B. \tag{5}$$

When $m = 0$, then we get $L_{\Re g}^{(0)} = T_g$. When $m = 1$, then we get $L_g^{(1)} = L_g$. In [22], we studied the boundedness and compactness of the operator $L_g^{(m)}$ between Hardy spaces H^2 and weighted Bergman spaces.

The purpose of this paper is to study the boundedness and compactness of the operator $L_g^{(m)}$ between different weighted Bergman spaces. Our results generalized the results of [9].

Throughout this paper, C will stand for a positive constant, whose value may differ from one occurrence to the other. The expression $a \asymp b$ means that there is a positive constant C such that $C^{-1}a \le b \le Ca$.

2 Main results and proofs

In this section we formulate and prove the main results of this paper. For this purpose, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 1. *For every $f, g \in H(B)$ such that $g(0) = 0$, it holds*

$$\Re[L_g^{(m)}(f)](z) = \Re^m f(z)g(z).$$

Proof. This identity can be proved similarly as [4]. We omit the details.

Lemma 2. *Let m be a nonnegative integer and $g \in H(B)$ such that $g(0) = 0$. Suppose that $0 < p \le q < \infty$, $\alpha, \beta > -1$, $m \le 1 + \frac{\beta+1+n}{q} - \frac{\alpha+1+n}{p}$. Then the following two conditions are equivalent.*

(a)

$$a_g := \sup_{z\in B} |g(z)|(1-|z|^2)^{\frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p} - (m-1)} < \infty; \tag{6}$$

(b)

$$\sup_{a\in B} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+mp)q/p} |g(z)|^q (1-|z|^2)^q d\nu_\beta(z) < \infty. \tag{7}$$

Proof. We adopt the method of [9]. Let $t = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p} - (m-1)$. For any $r > 0$ and $z \in B$, we write $E(z,r) = \{w \in B : \beta(z,w) < r\}$, where $\beta(z,w)$ is the distance between z and w in the Bergman metric of B. Set $|E(z,r)| = \nu(E(z,r))$. For $w \in E(z,r)$, $r > 0$, we have that (see, e.g., [19])

$$(1-|z|^2)^{n+1} \asymp (1-|w|^2)^{n+1} \asymp |1-\langle z,w\rangle|^{n+1} \asymp |E(z,r)|. \tag{8}$$

By (8) and the subharmonicity of the function $|g|^q$, it follows that

$$\begin{aligned} & (\,|g(z)|(1-|z|^2)^t)^q \\ \le\ & C\frac{(1-|z|^2)^{tq}}{|E(z,r)|}\int_{E(z,r)} |g(w)|^q d\nu(w) \\ \le\ & C\int_{E(z,r)} \left(\frac{1-|z|^2}{|1-\langle z,w\rangle|^2}\right)^{(n+1+\alpha+mp)q/p} |g(w)|^q (1-|w|^2)^q d\nu_\beta(w), \end{aligned} \tag{9}$$

from which we see that (7) implies (6).

Now assume that (6) holds. Then, by (6) and a well-known estimate (see, e.g., Theorem 1.12 in [19]), we have

$$\begin{aligned} & \sup_{a\in B\setminus B(0,1/2)} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+mp)q/p} |g(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ \le\ & a_g^q \sup_{a\in B\setminus B(0,1/2)} (1-|a|^2)^{\frac{(n+1+\alpha+mp)q}{p}} \int_B \frac{(1-|z|^2)^{q-qt+\beta}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+mp)q/p}} d\nu(z) \\ \le\ & C. \end{aligned} \tag{10}$$

On the other hand, since $q \geq p$ we have that

$$\begin{aligned} & \sup_{a\in B(0,1/2)} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+mp)q/p} |g(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ \leq\ & C\int_B (1-|z|^2)^{\frac{q(p+\alpha)+(n+1)(q-p)}{p}} d\nu(z) \\ <\ & \infty. \end{aligned} \tag{11}$$

From (10) and (11) the result follows.

Lemma 3. *Let m be a nonnegative integer and $g \in H(B)$ such that $g(0) = 0$. Suppose that $0 < p \leq q < \infty$, $\alpha, \beta > -1$, $m \leq 1 + \frac{\beta+1+n}{q} - \frac{\alpha+1+n}{p}$. Then $L_g^{(m)} : A_\alpha^p \to A_\beta^q$ is compact if and only if $L_g^{(m)} : A_\alpha^p \to A_\beta^q$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in A_α^p which converges to zero uniformly on compact subsets of B as $k \to \infty$, we have $\|L_g^{(m)} f_k\|_{A_\beta^q} \to 0$ as $k \to \infty$.*

Proof. The proof follows by standard arguments similar to those outlined in Proposition 3.11 of [2]. We omit the details.

Now we are in a position to state and prove our main results in this paper.

Theorem 1. *Let m be a nonnegative integer and $g \in H(B)$ such that $g(0) = 0$. Suppose that $0 < p \leq q < \infty$, $\alpha, \beta > -1$, $m \leq 1 + \frac{\beta+1+n}{q} - \frac{\alpha+1+n}{p}$. Then $L_g^{(m)} : A_\alpha^p \to A_\beta^q$ is bounded if and only if*

$$\sup_{z\in B} |g(z)|(1-|z|^2)^{\frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p} - (m-1)} < \infty. \tag{12}$$

Proof. It is easy to see that $L_g^{(m)} f(0) = 0$. By (1) and Lemma 1, we have

$$\begin{aligned} \|L_g^{(m)} f\|_{A_\beta^q}^q &\asymp \int_B |\Re(L_g^{(m)} f)(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ &= \int_B |g(z)|^q |\Re^m f(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ &= \int_B |\Re^m f(z)|^q d\mu(z), \end{aligned} \tag{13}$$

where

$$d\mu(z) = |g(z)|^q (1-|z|^2)^q d\nu_\beta(z). \tag{14}$$

By Theorem 50 of [18] and (13), we find that $L_g^{(m)} : A_\alpha^p \to A_\beta^q$ is bounded if and only if

$$\mu(Q_r(\zeta)) \leq C r^{(n+1+\alpha+mp)q/p}.$$

From this and (2), we have that

$$\sup_{a\in B}\int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+mp)q/p} |g(z)|^q(1-|z|^2)^q d\nu_\beta(z)<\infty.$$

From Lemma 2 we see that the last inequality is equivalent to (12). The proof is finished.

Using Lemma 3 and combing the proof of Theorem 2 of [9] and the proof of Theorem 1, we can obtain the following result. Since the proof is similar, we omit the details of the proof.

Theorem 2. *Let m be a nonnegative integer and $g\in H(B)$ such that $g(0)=0$. Suppose that $0<p\le q<\infty$, $\alpha,\beta>-1$, $m\le 1+\frac{\beta+1+n}{q}-\frac{\alpha+1+n}{p}$ and $L_g^{(m)}:A_\alpha^p\to A_\beta^q$ is bounded. Then $L_g^{(m)}:A_\alpha^p\to A_\beta^q$ is compact if and only if*

$$\lim_{|z|\to 1}|g(z)|(1-|z|^2)^{\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}-(m-1)}=0. \tag{15}$$

Theorem 3. *Let $g\in H(B)$, $0<q<p<\infty$, $\alpha,\beta>-1$. Suppose that m is a nonnegative integer such that $m<1+\frac{\beta+1}{q}-\frac{\alpha+1}{p}$. Then the following statements are equivalent.*

(a) $L_g^{(m)}:A_\alpha^p\to A_\beta^q$ is bounded;

(b) $L_g^{(m)}:A_\alpha^p\to A_\beta^q$ is compact;

(c) $g\in A_\gamma^r$, where $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$, and $\frac{\gamma}{r}=\frac{\beta}{q}-\frac{\alpha}{p}+1-m$.

Proof. From the proof of Theorem 1 we have that

$$\|L_g^{(m)}f\|_{A_\beta^q}^q \asymp \int_B |\Re^m f(z)|^q d\mu(z),$$

where $d\mu$ is defined by (14). By [18], we know that (a) and (b) are equivalent and both are equivalent to the following condition

$$\int_B \frac{(1-|a|^2)^{n+1+\alpha+mp}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+mp)}}d\mu(z)\in L^{p/(p-q)}(\nu_{\alpha+mp}),$$

which is the same as

$$\int_B |g(z)|^q(1-|z|^2)^q\frac{(1-|a|^2)^{n+1+\alpha+mp}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+mp)}}d\nu_\beta(z)\in L^{p/(p-q)}(\nu_{\alpha+mp}). \tag{16}$$

By the subharmonicity of $|g|^q$, using Lemma 2.24 of [19] and (8),

$$\begin{aligned}
&\int_B |g(z)|^q(1-|z|^2)^q \frac{(1-|a|^2)^{n+1+\alpha+mp}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+mp)}} d\nu_\beta(z)\\
\geq\ & \int_{E(a,r)} |g(z)|^q(1-|z|^2)^q \frac{(1-|a|^2)^{n+1+\alpha+mp}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+mp)}} d\nu_\beta(z)\\
\geq\ & (1-|a|^2)^{q-n-1-\alpha-mp} \int_{E(a,r)} |g(z)|^q d\nu_\beta(z)\\
\geq\ & (1-|a|^2)^{q-n-1-\alpha-mp} |g(a)|^q (1-|a|^2)^{n+1+\beta}\\
\geq\ & (1-|a|^2)^{q+\beta-\alpha-mp} |g(a)|^q. \qquad (17)
\end{aligned}$$

Hence, (16) and (17) imply

$$(1-|a|^2)^{q+\beta-\alpha-mp}|g(a)|^q \in L^{p/(p-q)}(\nu_{\alpha+mp}),$$

which is the same as

$$\int_B |g(a)|^r (1-|a|^2)^\gamma d\nu(a) < \infty.$$

Therefore $g \in A^r_\gamma$. Here $\gamma > -1$ by the assumption.

Conversely, assume $g \in A^r_\gamma$. Using Hölder's inequality, we get

$$\begin{aligned}
\|L^{(m)}_g f\|^q_{A^q_\beta} &\asymp \int_B |\Re^m f(z)|^q |g(z)|^q (1-|z|^2)^q d\nu_\beta(z)\\
&\leq \left(\int_B |g(z)|^r (1-|z|^2)^\gamma d\nu(z)\right)^{\frac{q}{r}} \times\\
&\quad \left(\int_B |\Re^m f(z)|^{\frac{qr}{r-q}} (1-|z|^2)^{(\beta+q-\frac{q\gamma}{r})\frac{r}{r-q}} d\nu(z)\right)^{1-\frac{q}{r}}.
\end{aligned}$$

From this, since

$$\frac{qr}{r-q} = p, \quad 1-\frac{q}{r} = \frac{q}{p},$$

and

$$(\beta+q-\frac{q\gamma}{r})\frac{r}{r-q} = \frac{r\beta-q\gamma+qr}{r-q} = \frac{\frac{\beta}{q}-\frac{\gamma}{r}+1}{\frac{1}{q}-\frac{1}{r}} = \alpha+mp,$$

it follows that

$$\|L^{(m)}_g f\|^q_{A^q_\beta} \leq C\|g\|^q_{A^r_\gamma} \|f\|^q_{A^p_\alpha},$$

which means that the operator $L^{(m)}_g : A^p_\alpha \to A^q_\beta$ is bounded, finishing the proof of the theorem.

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The representations of the Drazin inverse of sums of two matrices

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Abstract

In this paper, we give explicit expressions of $(P+Q)_d$ of two matrices P and Q, in terms of P, Q, P_d and Q_d, under the conditions that $PQ^2 = -P^2Q$, $P^2Q^2 = 0$ and $PQ^2 = P^2Q$, $QPQ = 0$, $P^2Q^2 = 0$.
AMS classification: 15A09

Key words: Drazin inverse, representation

1 Introduction

Recently the representations of the Drazin inverse of matrices have been widely investigated (see, for example, [2, 4, 7, 9, 11, 12, 14] and the literature mentioned below), especially, the representations of the Drazin inverse of sums and differences of two matrices[3, 5, 8, 13, 15]. In [3], M.F. Martnez-Serrano, N. Castro-Gonzlez presented a representation for the Drazin inverse of sums of two matrices under the condition that $PQ^2 = 0$, $Q^2 = 0$. And in [15], later, H. Yang, X. Liu expressed a representation for the Drazin inverse of sums of two matrices under the condition that $PQ^2 = 0$, $PQP = 0$.

These investigations motivate us to deal with a representation for the Drazin inverse of sums of two matrices. In this paper, we apply our result to deducing [3, Theorem 2.2]. The paper is organized as follows. In this section, we will introduce some notions and lemmas. In Section 2, we will present these explicit expressions of sums of two matrices P and Q, in terms of P, Q, P_d and Q_d, under the conditions $PQ^2 = -P^2Q$, $P^2Q^2 = 0$ and $PQ^2 = P^2Q$, $QPQ = 0$, $P^2Q^2 = 0$.

Throughout this paper the symbol $\mathbb{C}^{m\times n}$ stands for the set of $m \times n$ complex matrices, $I \in \mathbb{C}^{n\times n}$ stands for the unit matrix. Let $A \in \mathbb{C}^{n\times n}$, the Drazin inverse,

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denoted by A_d, of matrix A is defined as the unique matrix satisfying

$$A^{k+1}A_d = A^k, \quad A_dAA_d = A_d, \quad AA_d = A_dA$$

where $k = \mathrm{Ind}(A)$ is the index of A. In particular, if $\mathrm{Ind}(A) = 1$, then A_d is called the group inverse, denoted by A_g, of A (see [1, 14, 10]). Apparently, if A is nonsingular, then $\mathrm{Ind}(A) = 0$, otherwise $\mathrm{Ind}(A) \geq 1$, especially $\mathrm{Ind}(0) = 1$. If A is nilpotent, then $A_d = 0$. For convenience, we write $A^\pi = I - AA_d$.

Now we list and deduce some lemmas.

Lemma 1.1. [6, Theorem 3.2] *Let* $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $D \in \mathbb{C}^{m\times m}$, $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, $t = \mathrm{Ind}(A)$ *and* $l = \mathrm{Ind}(D)$. *Then*

$$M_d = \begin{bmatrix} A_d & S \\ 0 & D_d \end{bmatrix}$$

where

$$S = A^\pi \sum_{n=0}^{t-1} A^n B D_d^{n+2} + \sum_{n=0}^{l-1} A_d^{n+2} B D^n D^\pi - A_d B D_d.$$

Lemma 1.2. [1, Theorem 7.8.4] *Let* $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times n}$. *Then* $(AB)_d = A(BA)_d^2 B$.

Lemma 1.3. [7, Theorem 2.1] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *with* $s = \mathrm{Ind}(P)$ *and* $h = \mathrm{Ind}(Q)$. *If* $PQ = 0$, *then*

$$(P+Q)_d = \sum_{n=0}^{s-1} Q_d^{n+1} P^n P^\pi + Q^\pi \sum_{n=0}^{h-1} Q^n P_d^{n+1}.$$

Remark 1: (i)If $Q^r = 0$ in Lemma 1.2, then $(P+Q)_d = \sum_{n=0}^{r-1} Q^n P_d^{n+1}$.
(ii)If $QP = 0$ in Lemma 1.2, then $(P+Q)_d = P_d + Q_d$.

□

2 Main results

In this section, we will investigate how to express $(P \pm Q)_d$ under some conditions. We begin with the following theorem. Now we present the expression of $(P \pm Q)_d$.

Theorem 2.1. *Let* $P, Q \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(P^2) = s$, $\mathrm{Ind}(Q^2) = t$, $\mathrm{Ind}(PQ) = r$, $\mathrm{Ind}(Q^2 - PQ) = s_1$ *and* $\mathrm{Ind}(P^2 - PQ) = s_2$. *If* $PQ^2 = P^2Q$ *and* $P^2Q^2 = 0$, *then* $s_1 \geq t-1$, $s_2 \geq s-1$ *and*

$$\begin{aligned}(P-Q)_d &= -QR_d + QR_d(P-Q)T_dP - Q[R^\pi \sum_{n=0}^{s_1-1} R^n(P-Q)T_d^{n+2} \\ &\quad + \sum_{n=0}^{s_2-1} R_d^{n+2}(P-Q)T^nT^\pi]P + T_dP.\end{aligned}$$

where $R = (Q^2 - PQ)$ *and* $T = (P^2 - PQ)$,

$$R_d = (Q^2 - PQ)_d = \sum_{n=0}^{r-1} Q_d^{2(n+1)}(-PQ)^n(PQ)^\pi + Q^\pi \sum_{n=0}^{t-1} Q^{2n}(-PQ)_d^{n+1},$$

$$T_d = (P^2 - PQ)_d = \sum_{n=0}^{s-1}(-PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi \sum_{n=0}^{r-1}(-PQ)^n P_d^{2(n+1)}.$$

Proof. Let $R = (Q^2 - PQ)$ and $T = (P^2 - PQ)$.
Since $\mathrm{Ind}(T) = s_2$ and $P^2T = P^4$, we obtain $T^{s_2+1}R_d = T^{s_2}$, $P^2T^{s_2+1}T_d = P^2T^{s_2}$, $P^2T^{s_1} = P^4R^{s_2-1} = P^6T^{s_2-2} = \cdots = P^{2s_2+2}$, $P^2T^{s_2+1}T_d = P^{2s_2+4}T_d = P^{2s_1+4}P_d^2$, $P^{2s_1+4}Q_d^2 = P^{2s_1+2}$, $(P^2)^{s_1+2}P_d^2 = (P^2)^{s_1+1}$.
Therefore

$$s_2 \geq s - 1. \tag{2.1}$$

Since $\mathrm{Ind}(R) = s_1$ and $Q^2R = Q^4$, we get $R^{s_1+1}R_d = R^{s_1}$, $Q^2R^{s_1+1}R_d = Q^2R^{s_1}$, $Q^2R^{s_1} = QP^4R^{s_1-1} = Q^6R^{s_1-2} = \cdots = Q^{2s_1+2}$, $Q^2R^{s_1+1}R_d = Q^{2s_2+4}R_d = Q^{2s_1+4}Q_d^2$, $Q^{2s_1+4}Q_d^2 = Q^{2s_1+2}$, $(Q^2)^{s_1+2}Q_d^2 = (Q^2)^{s_1+1}$.
Hence

$$s_1 \geq t - 1. \tag{2.2}$$

Assume that $A = \begin{bmatrix} -Q & I \end{bmatrix}$, $B = \begin{bmatrix} I \\ P \end{bmatrix}$, then $P - Q = AB$. By lemma 1.2,

$$(P - Q)_d = (AB)_d = A(BA)_d^2 B. \tag{2.3}$$

$$BA = \begin{bmatrix} I \\ P \end{bmatrix} \begin{bmatrix} -Q & I \end{bmatrix} = \begin{bmatrix} -Q & I \\ -PQ & P \end{bmatrix}.$$

Since $PQ^2 = P^2Q$, we have

$$(BA)^2 = \begin{bmatrix} Q^2 - PQ & P - Q \\ PQ^2 - P^2Q & P^2 - PQ \end{bmatrix} = \begin{bmatrix} Q^2 - PQ & P - Q \\ 0 & P^2 - PQ \end{bmatrix}.$$

By Lemma 1.1,

$$(BA)_d^2 = \begin{bmatrix} (Q^2 - PQ)_d & S \\ 0 & (P^2 - PQ)_d \end{bmatrix} = \begin{bmatrix} R_d & S \\ 0 & T_d \end{bmatrix}.$$

where

$$S = R^\pi \sum_{n=0}^{s_1-1} R^n(P - Q)T_d^{n+2} + \sum_{n=0}^{s_2-1} R_d^{n+2}(P - Q)T^nT^\pi - R_d(P - Q)T_d.$$

Since $PQ^2 = P^2Q$ and $P^2Q^2 = 0$, we can deduce $PQ^3 = P^3Q = 0$.
By lemma 1.3, we have

$$R_d = (Q^2 - PQ)_d = \sum_{n=0}^{r-1} Q_d^{2(n+1)}(-PQ)^n(PQ)^\pi + Q^\pi \sum_{n=0}^{t-1} Q^{2n}(-PQ)_d^{n+1}.$$

$$T_d = (P^2 - PQ)_d = \sum_{n=0}^{s-1}(-PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi \sum_{n=0}^{r-1}(-PQ)^n P_d^{2(n+1)}.$$

By(2.3), we obtain

$$\begin{aligned}
(P-Q)_d &= (AB)_d = A(BA)_d^2 B = \begin{bmatrix} -Q & I \end{bmatrix} \begin{bmatrix} (Q^2-PQ)_d & S \\ 0 & (P^2-PQ)_d \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} \\
&= -Q(Q^2-PQ)_d - QSP + (P^2-PQ)_d P \\
&= -QR_d - Q[R^\pi \sum_{n=0}^{s_1-1} R^n(P-Q)T_d^{n+2} + \sum_{n=0}^{s_2-1} R_d^{n+2}(P-Q)T^nT^\pi - R_d(P-Q)T_d]P + T_dP \\
&= -QR_d + QR_d(P-Q)T_dP - Q[R^\pi \sum_{n=0}^{s_1-1} R^n(P-Q)T_d^{n+2} \\
&\quad + \sum_{n=0}^{s_2-1} R_d^{n+2}(P-Q)T^nT^\pi]P + T_dP.
\end{aligned}$$

□

Theorem 2.2. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(P^2) = s$, $\mathrm{Ind}(Q^2) = t$, $\mathrm{Ind}(PQ) = r$, $\mathrm{Ind}(Q^2 + PQ) = s_1$ and $\mathrm{Ind}(P^2 + PQ) = s_2$. If $PQ^2 = -P^2Q$ and $P^2Q^2 = 0$, then $s_1 \geq t-1$, $s_2 \geq s-1$ and*

$$\begin{aligned}
(P+Q)_d &= QR_d - QR_d(P+Q)T_dP + Q[R^\pi \sum_{n=0}^{s_1-1} R^n(P+Q)T_d^{n+2} \\
&\quad + \sum_{n=0}^{s_2-1} R_d^{n+2}(P+Q)T^nT^\pi]P\} + T_dP. \qquad (2.4)
\end{aligned}$$

where $R = (Q^2 + PQ)$ and $T = (P^2 + PQ)$,

$$R_d = (Q^2 + PQ)_d = \sum_{n=0}^{r-1} Q_d^{2(n+1)}(PQ)^n(PQ)^\pi + Q^\pi \sum_{n=0}^{t-1} Q^{2n}(PQ)_d^{n+1},$$

$$T_d = (P^2 + PQ)_d = \sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi \sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)}.$$

Proof. Substituting $-P$ for P in Theorem 2.1, (2.4) holds. □

whether $P^2Q = 0$ and $Q^2 = 0$ or $PQ^2 = 0$ and $P^2 = 0$, we can deduce $P^2Q = PQ^2 = 0$ and $P^2Q^2 = 0$. Using Theorem 2.2, so we have the following result.

Corollary 2.1. [3, Theorem 2.2] *(i) Let* $P, Q \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(P^2) = s$, $\mathrm{Ind}(PQ) = r$. *If* $P^2Q = 0$ *and* $Q^2 = 0$, *then*

$$(P+Q)_d = \sum_{n=0}^{r-1}[Q(PQ)^\pi(PQ)^n P_d + (PQ)^\pi(PQ)^n]P_d^{2n+1} + \sum_{n=0}^{s-1}[Q(PQ)_d^{n+1} + (PQ)_d^{n+1}P]P^{2n}P^\pi.$$

(ii) Let $P, Q \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(Q^2) = t$, $\mathrm{Ind}(PQ) = r$. *If* $PQ^2 = 0$ *and* $P^2 = 0$, *then*

$$(P+Q)_d = \sum_{n=0}^{r-1} Q_d^{2n+1}[(PQ)^\pi(PQ)^n + Q_d(PQ)^\pi(PQ)^n P] + \sum_{n=0}^{t-1} Q^\pi Q^{2n}[Q(PQ)_d^{n+1} + (PQ)_d^{n+1}P].$$

Proof. (i)Assume that $R = Q^2 + PQ$ and $T = P^2 + PQ$.
Since $P^2Q = 0$ and $Q^2 = 0$, we can deduce $P^2Q = PQ^2 = 0$, $P^2Q^2 = 0$, $R_d = (Q^2+PQ)_d = (PQ)_d$ and $\mathrm{Ind}(R) = \mathrm{Ind}(PQ) = r$, by Lemma 1.3,

$$T_d = (P^2+PQ)_d = \sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi\sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)}.$$

$$\begin{aligned} T^\pi &= I - (P^2+PQ)(P^2+PQ)_d \\ &= I - (P^2+PQ)[\sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi\sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)} \\ &= P^\pi - PQ\sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi\sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)}. \end{aligned} \tag{2.5}$$

Since $P^2Q = PQ^2 = 0$, we have

$$\begin{aligned} R(P+Q)T^n &= PQ(P+Q)(P^2+PQ)(P^2+PQ)^{n-1} = PQP(P^2+PQ)(P^2+PQ)^{n-1} \\ &= PQP^3(P^2+PQ)^{n-1} = PQP^5(P^2+PQ)^{n-2} = \cdots = (PQ)P^{2n+1}. \end{aligned} \tag{2.6}$$

By (2.1), (2.5) and (2.6), we can deduce $s_2 \geq s-1$ and

$$\begin{aligned} & Q\sum_{n=0}^{s_2-1} R_d^{n+2}(P+Q)T^nT^\pi P \\ =\ & Q\sum_{n=0}^{s_2-1} R_d^{n+3}R(P+Q)T^nT^\pi P \\ =\ & Q\sum_{n=0}^{s_2-1}(PQ)_d^{n+3}(PQ)P^{2n+1}\{P^\pi - PQ\sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi\sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)}\}P \\ =\ & Q\sum_{n=0}^{s_2-1}(PQ)_d^{n+2}P^{2n+2}P^\pi = Q\sum_{n=1}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi. \end{aligned} \tag{2.7}$$

Therefore

$$\begin{aligned}
& (PQ)^\pi \sum_{n=1}^{r-1}(PQ)^n(P+Q)T_d^{n+2} + (PQ)^\pi(P+Q)T_d^2 \\
= \; & (PQ)^\pi \sum_{n=1}^{r-1}(PQ)^n PT_d^{n+2} + (PQ)^\pi P_d^3 + (PQ)^\pi QT_d^2 \\
= \; & (PQ)^\pi \sum_{n=1}^{r-1}(PQ)^n P[P_d^2 + (PQ)_d]T_d^{n+1} + (PQ)^\pi P_d^3 + QT_d^2 \\
= \; & (PQ)^\pi \sum_{n=1}^{r-1}(PQ)^n P_d T_d^{n+1} + (PQ)^\pi P_d^3 + QT_d^2 \\
= \; & (PQ)^\pi \sum_{n=1}^{r-1}(PQ)^n P_d[P_d^2 + (PQ)_d]T_d^n + (PQ)^\pi P_d^3 + QT_d^2 \\
= \; & (PQ)^\pi \sum_{n=1}^{r-1}(PQ)^n P_d^3 T_d^n + (PQ)^\pi P_d^3 + QT_d^2 \\
= \; & \cdots \quad \cdots \quad \cdots \\
= \; & (PQ)^\pi \sum_{n=0}^{r-1}(PQ)^n P_d^{2n+3} + QT_d^2. \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}
& QR_d - QR_d(P+Q)T_dP \\
= \; & Q(PQ)_d - Q(PQ)_d(P+Q)[\sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi \sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)}]P \\
= \; & Q(PQ)_d - Q(PQ)_d PP_d^2 P - Q(PQ)_d Q)[\sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi \sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)}]P \\
= \; & Q(PQ)_d - Q(PQ)_d PP_d = Q(PQ)_d P^\pi. \qquad (2.9)
\end{aligned}$$

By Theorm2.2, (2.7), (2.8)and (2.9), we obtain

$$\begin{aligned}
(P+Q)_d &= QR_d - QR_d(P+Q)T_dP + QR^\pi \sum_{n=0}^{r-1} R^n(P+Q)T_d^{n+2}P \\
&\quad + Q\sum_{n=0}^{s_2-1} R_d^{n+2}(P+Q)T^nT^\pi P + T_dP \\
&= Q(PQ)_d P^\pi + Q(PQ)^\pi \sum_{n=0}^{r-1}(PQ)^n P_d^{2n+2} + Q\sum_{n=0}^{s_2-1}(PQ)_d^{n+2}P^{2n+2}P^\pi \\
&\quad + [\sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi + (PQ)^\pi \sum_{n=0}^{r-1}(PQ)^n P_d^{2(n+1)}]P \\
&= Q(PQ)_d P^\pi + \sum_{n=0}^{r-1} Q(PQ)^\pi (PQ)^n P_d^{2n+2} + \sum_{n=1}^{s-1} Q(PQ)_d^{n+1}P^{2n}P^\pi \\
&\quad + \sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}PP^\pi + \sum_{n=0}^{r-1}(PQ)^\pi (PQ)^n P_d^{2n+1}
\end{aligned}$$

$$= \sum_{n=0}^{r-1}[Q(PQ)^{\pi}(PQ)^{n}P_d + (PQ)^{\pi}(PQ)^{n}]P_d^{2n+1} + \sum_{n=0}^{s-1}[Q(PQ)_d^{n+1} + (PQ)_d^{n+1}P]P^{2n}P^{\pi}.$$

In similar way of (i), (ii) holds. □

Theorem 2.3. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(P^2) = s$, $\mathrm{Ind}(Q^2) = t$, $\mathrm{Ind}(PQ) = r$, $\mathrm{Ind}(Q^2+PQ) = s_1$ and $\mathrm{Ind}(P^2+PQ) = s_2$. If $PQ^2 = P^2Q$, $QPQ = 0$ and $P^2Q^2 = 0$, then $0 < r \le 2$, $s_1 \ge t-1$, $s_2 \ge s-1$ and*
(i)If $r = 1$, then

$$(P+Q)_d = \sum_{n=0}^{2s-1} Q_d^{n+1}P^nP^{\pi} + Q^{\pi}\sum_{n=0}^{2t-1} Q^nP_d^{n+1}.$$

(ii)If $r = 2$.

(a)When $s_1 = t-1$, $s_2 = s-1$, then

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{t-2} Q^{\pi}Q^{2n+1}P_d^{2n+2} + \sum_{n=0}^{t-1} Q^{\pi}Q^{2n}P_d^{2n+1} + \sum_{n=0}^{s-1} Q_d^{2n+1}P^{2n}P^{\pi} + \sum_{n=0}^{s-2} Q_d^{2n+2}P^{2n+1}P^{\pi} \\ &\quad + PQP_d^3 + 2P^2QP_d^4. \end{aligned} \tag{2.10}$$

(b)When $s_1 = t-1$, $s_2 \ge s$, then

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{t-2} Q^{\pi}Q^{2n+1}P_d^{2n+2} + \sum_{n=0}^{t-1} Q^{\pi}Q^{2n}P_d^{2n+1} + \sum_{n=0}^{s-1} Q_d^{2n+1}P^{2n}P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+2}P^{2n+1}P^{\pi} \\ &\quad + PQP_d^3 + 2P^2QP_d^4. \end{aligned} \tag{2.11}$$

(c)When $s_1 \ge t$, $s_2 = s-1$, then

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{t-1} Q^{\pi}Q^{2n+1}P_d^{2n+2} + \sum_{n=0}^{t-1} Q^{\pi}Q^{2n}P_d^{2n+1} + \sum_{n=0}^{s-1} Q_d^{2n+1}P^{2n}P^{\pi} + \sum_{n=0}^{s-2} Q_d^{2n+2}P^{2n+1}P^{\pi} \\ &\quad + PQP_d^3 + 2P^2QP_d^4. \end{aligned} \tag{2.12}$$

(d)When $s_1 \ge t$, $s_2 \ge s$, then

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{t-1} Q^{\pi}Q^{2n+1}P_d^{2n+2} + \sum_{n=0}^{t-1} Q^{\pi}Q^{2n}P_d^{2n+1} + \sum_{n=0}^{s-1} Q_d^{2n+1}P^{2n}P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+2}P^{2n+1}P^{\pi} \\ &\quad + PQP_d^3 + 2P^2QP_d^4. \end{aligned} \tag{2.13}$$

Proof. If $r = 0$, then PQ and $(PQ)^2$ are nonsingular. However, $QPQ = 0$ which implies $(PQ)^2 = 0$. Thus we can deduce $r > 0$. Since $(PQ)^2 = 0$, $(PQ)^3(PQ)_d = (PQ)^2 = 0$, then $r \le 2$. In a word, $0 < r \le 2$.
Now we consider r.

(i) If $r = 1$, then $PQ = (PQ)^2(PQ)_d = 0$. Since $\mathrm{Ind}(P^2) = s$, $\mathrm{Ind}(Q^2) = t$, then $\mathrm{Ind}(P) \leq 2s$, $\mathrm{Ind}(Q) \leq 2t$. By Lemma 1.3, we have

$$(P+Q)_d = \sum_{n=0}^{2s-1} Q_d^{n+1} P^n P^\pi + Q^\pi \sum_{n=0}^{2t-1} Q^n P_d^{n+1}.$$

(ii) If $r = 2$.

Suppose $A = \begin{bmatrix} Q & I \end{bmatrix}$, $B = \begin{bmatrix} I \\ P \end{bmatrix}$, then $P + Q = AB$. By Lemma 1.2, we have

$$(P+Q)_d = (AB)_d = A(BA)_d^2 B. \tag{2.14}$$

$$BA = \begin{bmatrix} I \\ P \end{bmatrix} \begin{bmatrix} Q & I \end{bmatrix} = \begin{bmatrix} Q & I \\ PQ & P \end{bmatrix}.$$

Since $PQ^2 = P^2Q$, then

$$\begin{aligned} (BA)^2 = \begin{bmatrix} Q^2+PQ & P+Q \\ PQ^2+P^2Q & P^2+PQ \end{bmatrix} &= \begin{bmatrix} Q^2+PQ & P+Q \\ 0 & P^2+PQ \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ PQ^2+P^2Q & 0 \end{bmatrix} \\ &=: X + Y \end{aligned}$$

By Lemma 1.1, it is so easy to obtain

$$X_d = \begin{bmatrix} (Q^2+PQ)_d & S \\ 0 & (P^2+PQ)_d \end{bmatrix}. \tag{2.15}$$

where $s_2 = \mathrm{Ind}(P^2+PQ)$, $s_1 = \mathrm{Ind}(Q^2+PQ)$.

$$\begin{aligned} S &= (Q^2+PQ)^\pi \sum_{n=0}^{s_1-1} (Q^2+PQ)^n (P+Q)(P^2+PQ)_d^{n+2} \\ &\quad + \sum_{n=0}^{s_2-1} (Q^2+PQ)_d^{n+2} (P+Q)(P^2+PQ)^n (P^2+PQ)^\pi \\ &\quad - (Q^2+PQ)_d (P+Q)(P^2+PQ)_d. \end{aligned}$$

Since $PQ^2 = P^2Q$, $QPQ = 0$ and $P^2Q^2 = 0$, we obtain $PQ^3 = P^3Q = 0$ and $(PQ^2+P^2Q)(Q^2+PQ) = PQ^4 + P^2Q^3 + PQ^2PQ + P^2QPQ = 0$. Further,

$$(PQ^2+P^2Q)(Q^2+PQ)_d = (PQ^2+P^2Q)(Q^2+PQ)(Q^2+PQ)_d^2 = 0$$

$$\begin{aligned} (PQ^2+P^2Q)S &= (PQ^2+P^2Q)(Q^2+PQ)^\pi (P+Q)(P^2+PQ)_d^2 \\ &= (PQ^2+P^2Q)(P+Q)(P^2+PQ)_d^2. \\ &= (PQ^2+P^2Q)P(P^2+PQ)_d^2. \\ &= 2P^2QP(P^2+PQ)_d^2. \end{aligned}$$

$$\begin{aligned} YX_d^2 &= \begin{bmatrix} 0 & 0 \\ PQ^2+P^2Q & 0 \end{bmatrix} \begin{bmatrix} (Q^2+PQ)_d^2 & (Q^2+PQ)_dS+S(P^2+PQ)_d \\ 0 & (P^2+PQ)_d^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ (PQ^2+P^2Q)(Q^2+PQ)_d^2 & (PQ^2+P^2Q)[(Q^2+PQ)_dS+S(P^2+PQ)_d] \end{bmatrix}. \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (PQ^2+P^2Q)S(P^2+PQ)_d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2P^2QP(P^2+PQ)_d^3 \end{bmatrix}. \end{aligned} \tag{2.16}$$

Since $PQ^2=P^2Q$, $QPQ=0$ and $P^2Q^2=0$. we can deduce $XY=0$ and $Y^2=0$.
By Remark 1(i) and (2.15), (2.16), we have

$$\begin{aligned} (BA)_d^2 &= (X+Y)_d = X_d+YX_d^2 \\ &= \begin{bmatrix} (Q^2+PQ)_d & S \\ 0 & (P^2+PQ)_d+2P^2QP(P^2+PQ)_d^3 \end{bmatrix}. \end{aligned}$$

By (2.14), it is easy to know

$$\begin{aligned} (P+Q)_d &= (AB)_d = A(BA)_d^2B \\ &= \begin{bmatrix} Q & I \end{bmatrix} \begin{bmatrix} (Q^2+PQ)_d & S \\ 0 & (P^2+PQ)_d+2P^2QP(P^2+PQ)_d^3 \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} \\ &= Q(Q^2+PQ)_d+QSP+[(P^2+PQ)_d+2P^2QP(P^2+PQ)_d^3]P. \end{aligned} \tag{2.17}$$

Let $R=(Q^2+PQ)$ and $T=(P^2+PQ)$.
Since $PQ^2=P^2Q$, $P^2Q^2=0$ and $QPQ=0$, we can deduce $Q^2PQ=PQ^3=P^3Q=0$ and $(PQ)_d=(PQ)^2(PQ)_d^3=P(QPQ)(PQ)_d^3=0$. So for $n\geq 1$, we obtain

$$\begin{aligned} R^n &= R^2R^{n-2}=Q^2R^{n-1}=Q^4R^{n-2}=\cdots=Q^{2(n-1)}R, \\ T^n &= T^{n-2}T^2=T^{n-1}P^2=T^{n-2}P^4=\cdots=TP^{2(n-1)}. \end{aligned}$$

By Remark 1(ii) and Lemma 1.3, we get

$$\begin{aligned} R_d &= (Q^2+PQ)_d = Q_d^2+(PQ)_d = Q_d^2, \\ R^\pi &= I-(Q^2+PQ)(Q^2+PQ)_d = Q^\pi. \end{aligned}$$

$$\begin{aligned} T_d &= (P^2+PQ)_d = \sum_{n=0}^{s-1}(PQ)_d^{n+1}P^{2n}P^\pi+(PQ)^\pi\sum_{n=0}^{r-1}(PQ)^nP_d^{2(n+1)} \\ &= \sum_{n=0}^{1}(PQ)^nP_d^{2(n+1)} = P_d^2+PQP_d^4, \end{aligned} \tag{2.18}$$

$$T^\pi = I-TT_d = P^\pi-PQP_d^2. \tag{2.19}$$

Since $QPQ=0$ and $P^3Q=0$, then $P_dQ=P_d^4P^3Q=0$ and

$$\begin{aligned} QPT_d^{n+2} &= QP(P_d^2+PQP_d^4)^{n+2} = QP(P_d^2+PQP_d^4)(P_d^2+PQP_d^4)^{n+1} \\ &= QPP_d^2(P_d^2+PQP_d^4)^{n+1} = QPP_d^4(P_d^2+PQP_d^4)^n \\ &= \cdots = QPP_d^{2n+4} = QP_d^{2n+3}. \end{aligned} \tag{2.20}$$

$$\begin{aligned} QPT^nT^\pi &= QP(P^2+PQ)(P^2+PQ)^{n-1}T^\pi = QP^3(P^2+PQ)^{n-1}T^\pi \\ &= QP^5(P^2+PQ)^{n-2}T^\pi = \cdots = QP^{2n+1}T^\pi = QP^{2n+1}P^\pi. \quad (2.21) \end{aligned}$$

$$QR = Q^3, QR^n = Q^{2n+1}, QR_d = Q_d, QR_d^{n+2} = Q_d^{2n+3}. \quad (2.22)$$

$$QT = QP^2, QT^n = QP^{2n}, QT_d = QP_d^2, QT_d^{n+2} = QP_d^{2n+4}. \quad (2.23)$$

By (2.18), (2.19), (2.20), (2.21), (2.22)and (2.23), we have

$$\begin{aligned} QSP &= QR^\pi \sum_{n=0}^{s_1-1} R^n(P+Q)T_d^{n+2}P + Q\sum_{n=0}^{s_2-1} R_d^{n+2}(P+Q)T^nT^\pi P - QR_d(P+Q)T_dP \\ &= Q^\pi \sum_{n=0}^{s_1-1} QR^n(P+Q)T_d^{n+2}P + \sum_{n=0}^{s_2-1} QR_d^{n+2}(P+Q)T^nT^\pi P - QR_d(P+Q)T_dP \\ &= Q^\pi \sum_{n=0}^{s_1-1} Q^{2n+1}(P+Q)T_d^{n+2}P + \sum_{n=0}^{s_2-1} Q_d^{2n+3}(P+Q)T^nT^\pi P - Q_d(P+Q)T_dP \\ &= \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+1}PT_d^{n+2}P + \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+1}QT_d^{n+2}P \\ &\quad + \sum_{n=0}^{s_2-1} Q_d^{2n+3}PT^nT^\pi P + \sum_{n=0}^{s_2-1} Q_d^{2n+3}QT^nT^\pi P - Q_dPT_dP - Q_dQT_dP \\ &= \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+1}PT_d^{n+2}P + \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+1}QP_d^{2n+4}P \\ &\quad + \sum_{n=0}^{s_2-1} Q_d^{2n+3}PT^nT^\pi P + \sum_{n=0}^{s_2-1} Q_d^{2n+3}QP^{2n}T^\pi P - Q_dPT_dP - QQ_dP_d \\ &= \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+1}PT_d^{n+2}P + \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+2}P_d^{2n+3} + \sum_{n=0}^{s_2-1} Q_d^{2n+3}PT^nT^\pi P \\ &\quad + \sum_{n=0}^{s_2-1} Q_d^{2n+2}P^{2n}T^\pi P - Q_dPT_dP - QQ_dP_d \\ &= \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+1}P_d^{2n+2} + \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+2}P_d^{2n+3} + \sum_{n=0}^{s_2-1} Q_d^{2n+3}P^{2n+2}P^\pi \\ &\quad + \sum_{n=1}^{s_2-1} Q_d^{2n+2}P^{2n}P^\pi P + T^\pi P - Q_dPP_d - QQ_dP_d \\ &= \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+1}P_d^{2n+2} + \sum_{n=0}^{s_1-1} Q^\pi Q^{2n+2}P_d^{2n+3} + \sum_{n=0}^{s_2-1} Q_d^{2n+3}P^{2n+2}P^\pi \\ &\quad + \sum_{n=0}^{s_2-1} Q_d^{2n+2}P^{2n+1}P^\pi - Q_dPP_d - QQ_dP_d. \quad (2.24) \end{aligned}$$

By (2.1) and (2.2), it is so easy to get $s_1 \geq t-1$ and $s_2 \geq s-1$.
(a)When $s_1 = t-1$, $s_2 = s-1$, by (2.24), then

$$\begin{aligned} QSP &= \sum_{n=0}^{t-2} Q^{\pi} Q^{2n+1} P_d^{2n+2} + \sum_{n=1}^{t-1} Q^{\pi} Q^{2n} P_d^{2n+1} + \sum_{n=1}^{s-1} Q_d^{2n+1} P^{2n} P^{\pi} + \sum_{n=0}^{s-2} Q_d^{2n+2} P^{2n+1} P^{\pi} \\ &\quad -Q_d P P_d - Q Q_d P_d. \end{aligned} \tag{2.25}$$

Putting (2.19),(2.20) and (2.25) in (2.17) yields (2.10).
(b)When $s_1 = t-1$, $s_2 \geq s$, by (2.24), then

$$\begin{aligned} QSP &= \sum_{n=0}^{t-2} Q^{\pi} Q^{2n+1} P_d^{2n+2} + \sum_{n=1}^{t-1} Q^{\pi} Q^{2n} P_d^{2n+1} + \sum_{n=1}^{s-1} Q_d^{2n+1} P^{2n} P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+2} P^{2n+1} P^{\pi} \\ &\quad -Q_d P P_d - Q Q_d P_d. \end{aligned} \tag{2.26}$$

Putting (2.19),(2.20) and (2.26) in (2.17) yields (2.11).
(c)When $s_1 \geq t$, $s_2 = s-1$, by (2.24), then

$$\begin{aligned} QSP &= \sum_{n=0}^{t-1} Q^{\pi} Q^{2n+1} P_d^{2n+2} + \sum_{n=1}^{t-1} Q^{\pi} Q^{2n} P_d^{2n+1} + \sum_{n=1}^{s-1} Q_d^{2n+1} P^{2n} P^{\pi} + \sum_{n=0}^{s-2} Q_d^{2n+2} P^{2n+1} P^{\pi} \\ &\quad -Q_d P P_d - Q Q_d P_d \end{aligned} \tag{2.27}$$

Putting (2.19),(2.20) and (2.27) in (2.17) yields (2.12).
(d)When $s_1 \geq t$, $s_2 \geq s$, by (2.24), then

$$\begin{aligned} QSP &= \sum_{n=0}^{t-1} Q^{\pi} Q^{2n+1} P_d^{2n+2} + \sum_{n=1}^{t-1} Q^{\pi} Q^{2n} P_d^{2n+1} + \sum_{n=1}^{s-1} Q_d^{2n+1} P^{2n} P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+2} P^{2n+1} P^{\pi} \\ &\quad -Q_d P P_d - Q Q_d P_d \end{aligned} \tag{2.28}$$

Putting (2.19),(2.20) and (2.28) in (2.17) yields (2.13). □

Substituting $PQP = 0$ for $QPQ = 0$ in Theorem 2.3, we have the following result.

Theorem 2.4. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(P^2) = s$, $\mathrm{Ind}(Q^2) = t$, $\mathrm{Ind}(PQ) = r$, $\mathrm{Ind}(Q^2 + PQ) = s_1$ and $\mathrm{Ind}(P^2 + PQ) = s_2$. If $PQ^2 = P^2Q$, $PQP = 0$ and $P^2Q^2 = 0$, then $0 < r \leq 2$, $s_1 \geq t-1$, $s_2 \geq s-1$ and*
(i)If $r = 1$, then

$$(P+Q)_d = \sum_{n=0}^{2s-1} Q_d^{n+1} P^n P^{\pi} + Q^{\pi} \sum_{n=0}^{2t-1} Q^n P_d^{n+1}.$$

(ii)If $r = 2$,
(a)When $s_1 = t-1$, $s_2 = s-1$, then

$$\begin{aligned} (P+Q)_d &= \sum_{n=0}^{s-2} Q_d^{2n+2} P^{2n+1} P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+1} P^{2n} P^{\pi} + \sum_{n=0}^{t-1} Q^{\pi} Q^{2n} P_d^{2n+1} + \sum_{n=0}^{t-2} P^{\pi} Q^{2n+1} P_d^{2n+2} \\ &\quad +Q_d^3 PQ + 2Q_d^4 PQ^2. \end{aligned} \tag{2.29}$$

(b)When $s_1 = t-1$, $s_2 \geq s$, *then*

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{s-2} Q_d^{2n+2}P^{2n+1}P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+1}P^{2n}P^{\pi} + \sum_{n=0}^{t-1} Q^{\pi}Q^{2n}P_d^{2n+1} + \sum_{n=0}^{t-1} P^{\pi}Q^{2n+1}P_d^{2n+2} \\ &\quad + Q_d^3PQ + 2Q_d^4PQ^2. \end{aligned} \tag{2.30}$$

(c)When $s_1 \geq t$, $s_2 = s-1$, *then*

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{s-1} Q_d^{2n+2}P^{2n+1}P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+1}P^{2n}P^{\pi} + \sum_{n=0}^{t-1} Q^{\pi}Q^{2n}P_d^{2n+1} + \sum_{n=0}^{t-2} P^{\pi}Q^{2n+1}P_d^{2n+2} \\ &\quad + Q_d^3PQ + 2Q_d^4PQ^2. \end{aligned} \tag{2.31}$$

(d)When $s_1 \geq t$, $s_2 \geq s$, *then*

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{s-1} Q_d^{2n+2}P^{2n+1}P^{\pi} + \sum_{n=0}^{s-1} Q_d^{2n+1}P^{2n}P^{\pi} + \sum_{n=0}^{t-1} Q^{\pi}Q^{2n}P_d^{2n+1} + \sum_{n=0}^{t-1} P^{\pi}Q^{2n+1}P_d^{2n+2} \\ &\quad + Q_d^3PQ + 2Q_d^4PQ^2. \end{aligned} \tag{2.32}$$

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Strong Convergence Theorems of a General Iterative Process for Two Nonexpansive Mappings in Banach Spaces*

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Abstract

In this paper, we introduce a general iterative process for approximating common fixed point of two nonexpansive mappings in Banach spaces. We establish some strong convergence theorems of the general iteration scheme under some mild conditions. Our results improve and extend the corresponding results of many others form Hilbert spaces to Banach spaces.

Keywords: Common fixed points; Nonexpansive mapping; Strong convergence; Smooth Banach spaces; Iterative method.

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1 Introduction

Throughout this paper, let X be a real Banach space, and let C be a nonempty closed convex subset of X. A mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. We denote $F(T)$ as the set of fixed points of T. We know that $F(T)$ is nonempty if C is bounded; for more detail see [2]. Recall that a self mapping $f : C \to C$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for each $x, y \in C$. We use Π_c denote the collection of all contraction on C. That is $\Pi_C = \{f : f : C \to C \text{ a contraction}\}$.

Iterative methods for nonexpansive mappings have recently been applied to solve minimization problems; see, e.g. [4, 15, 17, 19, 20]. A typical problem is to minimize a quadratic function over the set of the fixed

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points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F}\frac{1}{2}\langle Ax,x\rangle-\langle x,u\rangle, \tag{1.1}$$

where F is the fixed point set of a nonexpansive mapping T on H, and u is a given point in H.

Assume A is strongly positive; that is, there is a constant $\bar{\gamma}$ with the property such that

$$\langle Ax,x\rangle\geq\bar{\gamma}\|x\|^2 \tag{1.2}$$

for all $x\in H$. In 2003, Xu [15] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1}=\alpha_n u+(I-\alpha_n A)Tx_n,\quad \forall n\geq 0, \tag{1.3}$$

converges strongly to a unique solution of the minimization problem (1.1), provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [11] introduced the viscosity approximation method for nonexpansive mappings (see [18] for further developments in both Hilbert and Banach spaces). Starting with an arbitrary initial $x_0\in H$, defined the sequence $\{x_n\}$ recursively by

$$x_{n+1}=\sigma_n f(x_n)+(1-\sigma_n)Tx_n,\quad \forall n\geq 0, \tag{1.4}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved in [11, 18] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.4) strongly converges to a unique solution x^* of the variational inequality

$$\langle(f-I)x^*,x-x^*\rangle\leq 0,\quad \forall x\in F(T). \tag{1.5}$$

Recently, Marino and Xu [9] combined the iterative method (1.3) with the viscosity approximation method (1.4) considering the following general iterative process:

$$x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_n A)Tx_n,\quad \forall n\geq 0, \tag{1.6}$$

where $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. They proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a unique solution x^* of the variational inequality

$$\langle(\gamma f-A)x^*,x-x^*\rangle\leq 0,\quad \forall x\in F(T). \tag{1.7}$$

On the other hand, Kim and Xu [7] studied the sequence generated by the algorithm:

$$\begin{cases} x_1=x\in C \ \text{ chosen arbitrarily;}\\ y_n=\alpha_n x_n+(1-\alpha_n)Tx_n;\\ x_{n+1}=\beta_n u+(1-\beta_n)y_n,\ \ \forall n\geq 1, \end{cases} \tag{1.8}$$

and proved strong convergence of scheme (1.8) in the framework of a uniformly smooth Banach space.

Recently Yao, et al. [21] introduced a new iteration process by combining the modified Mann iteration [7] and the viscosity method introduced by Moudafi [11]. Let C be a closed convex subset of a Banach space, $T:C\to C$ a nonexpansive mapping such that $F(T)\neq\emptyset$; and $f\in\Pi_C$. Define $\{x_n\}$ in the following way:

$$\begin{cases} x_1=x\in C \ \text{ chosen arbitrarily;}\\ y_n=\alpha_n x_n+(1-\alpha_n)Tx_n;\\ x_{n+1}=\beta_n f(x_n)+(1-\beta_n)y_n,\ \ \forall n\geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. They proved under different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ converge strongly to a fixed point of T.

Very recently, Yao, et al. [22] studied the sequence generated by the algorithm in Hilbert spaces H, as follows:

$$\begin{cases} x_1 \in H \ \text{ chosen arbitrarily;} \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n; \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \ \ \forall n \geq 1, \end{cases} \tag{1.10}$$

and they proved strong convergence for special case of sequence (1.10).

Question 1.1. *Can Theorem of Yao et al. [22] be extend from Hilbert spaces to a general Banach space? such as uniformly convex and uniformly smooth Banach spaces.*

Question 1.2. *Can we prove strong convergence theorem of more general than algorithm* (1.10).

The purpose of this paper is to give affirmative answer to these questions mentioned above. We introduce a general iterative approximation method for finding a common fixed point of nonexpansive mappings which is a unique solution of some variational inequality and prove the strong convergence theorems of such iterative scheme in a uniformly convex and uniformly smooth Banach spaces which admits a weakly continuous duality mapping. As applications, at the end of the paper, we also apply our results to the optimization problems. The results in this paper extend and improve the main results Marino and Xu [9] and some others from Hilbert spaces to Banach spaces.

2 Preliminaries

Let X be a real Banach space, $J : X \to 2^{X^*}$ be a normalized duality mapping by

$$J(x) = \{\ f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality paring. In the following, the notation $\rightharpoonup$ and $\to$ denote the weak and strong convergence, respectively. Also, a mapping $I : C \to C$ denotes the identity mapping. If a Banach space X admits a sequentially continuous duality mapping J from weak topology to weak star topology, then by Lemma 1 of [5], we have that duality mapping J is a single value. In this case, the duality mapping J is said to be a weakly sequentially continuous duality mapping, i.e. for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, we have $J(x_n) \rightharpoonup^* J(x)$ (see [5, 6, 12] for more details).

The norm of a Banach space X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x,\ y \in C$ on the unit sphere $S(X)$ of X. In this case X is smooth. In a smooth Banach space, we always assume that A is strongly positive (see [3]), that is, a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J(x) \rangle \geq \bar{\gamma}\|x\|^2, \quad \|aI - bA\| = \sup_{\|x\| \leq 1} \|\langle (aI - bA)x, J(x) \rangle\|, \ \ a \in [0,1],\ b \in [-1,1]. \tag{2.2}$$

Moreover, if for each y in $S(X)$ the limit (2.1) is uniformly attained for $x \in S(X)$, we say that the norm X is *uniformly Gâteaux differentiable*. The norm of X is said to be *Frêchet differentiable*, if for each

$x \in S(X)$, the limit (2.1) is attained uniformly for $y \in S(X)$. The norm of X is said to be *uniformly Frêchet differentiable* (or X is said to be uniformly smooth), the limit (2.1) is attained uniformly for $(x, y) \in S(X) \times S(X)$. A Banach space X is said to be *strictly convex* if $\|x\| = \|y\| = 1,\ x \neq y$ implies $\frac{\|x+y\|}{2} < 1$; *uniformly convex* if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$, where $\delta_X(\epsilon)$ is *modulus of convexity* of X defined by $\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x+y\| \geq \varepsilon\right\},\ \ \forall \epsilon \in [0, 2]$. A uniformly convex Banach space X is reflexive and strictly convex (see Theorem 4.1.6, Theorem 4.1.2 of [14]) and every uniformly smooth Banach space X is a reflexive Banach with uniformly Gâteaux differentiable norm (see Theorem 4.3.7, Theorem 4.1.6 of [14]) (also see [10]).

The following lemmas will be needed in proving our main results.

Lemma 2.1. (Cai and Hu [3]) *Assume that A is a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.2. (Suzuki [13]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3. (Liu [8]) *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

for all $j(x + y) \in J(x + y)$ with $x \neq y$.

Lemma 2.4. (Cai and Hu [3]) *Let C be a closed convex subset of a reflexive, smooth Banach space X which admits a weakly sequentially continuous duality mapping J from X to X^*. Let $W : C \to C$ be a nonexpansive mapping with $F := F(W) \neq \emptyset$ and $f \in \Pi_C$, A be a strongly positive linear bounded operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Then the sequence x_t defined by $x_t = t\gamma f(x_t) + (I - tA)Wx_t$ converges strongly as $t \to 0$ to a point x^* of W which solves the variational inequality $\langle \gamma f(x^*) - Ax^*, J(x - x^*)\rangle \leq 0,\ \ $ for $x \in F$.*

Lemma 2.5. (Xu [16]) *Let X be a uniformly convex Banach space. Then for each $r > 0$, there exits a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|),$$

for all $x, y \in \{z \in X : \|z\| \leq r\}$ and $\lambda \in [0, 1]$.

Lemma 2.6. (Aoyama et al. [1]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{a_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} a_n = \infty$, $\{c_n\}$ be a sequence of nonnegative real number with $\sum_{n=1}^{\infty} c_n < \infty$, and $\{b_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} b_n \leq 0$. Suppose that $s_{n+1} = (1 - a_n)s_n + a_n b_n + c_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n = 0$.*

3 Main results

In this section, we prove our main results.

Theorem 3.1. *Let X be a uniformly smooth, uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from X into X^*, C be a closed convex subset of X such that $C \pm C \subset C$. Let S, T*

be two nonexpansive mappings from C into itself such that $F := F(ST) = F(S) \cap F(T) \neq \emptyset$, f be a contraction mapping with the coefficient $\alpha \in (0,1)$ and A be a linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} < \frac{\gamma}{\alpha}$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be the sequences in $(0,1)$. Assume the following conditions are hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \beta_n = 0$;

(C3) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;

(C4) $\gamma_n = O(\alpha_n)$.

If $\lim_{n\to\infty} \|x_n - y_n\| = 0$, then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily;} \\ y_n = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)Tx_n; \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)Sy_n, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

converges strongly to common fixed point x^* where x^* is the unique solution in F of the variational inequality

$$\langle \gamma f(x^*) - Ax^*, J(x - x^*) \rangle \leq 0, \quad \text{for } x \in F. \tag{3.2}$$

Proof. By the control condition $(C1)$, we may assume, with no loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \geq 0$. Since A is a linear bounded self-adjoint operator on X, by (2.1), we have $\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in X,\ \|u\| = 1\}$. Observe that

$$\begin{aligned} \langle ((1-\beta_n)I - \alpha_n A)u, J(u)\rangle &= 1 - \beta_n - \alpha_n \langle Au, J(u)\rangle \\ &\geq 1 - \beta_n - \alpha_n \|A\| \\ &\geq 0, \end{aligned}$$

that is $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle ((1-\beta_n)I - \alpha_n A)u, J(u)\rangle : u \in X,\ \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n \langle Au, J(u)\rangle : u \in X,\ \|u\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n \bar{\gamma}. \end{aligned}$$

Firstly, we show that $\{x_n\}$ is bounded. Indeed, for any $p \in F$, we compute

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(Tx_n - p)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|Tx_n - p\| \\ &\leq \alpha_n \big[\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\|\big] + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - p\| \\ &\leq \alpha_n \big[\gamma\alpha \|x_n - p\| + \|\gamma f(p) - Ap\|\big] + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - p\| \\ &= [1 - (\bar{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|x_{n+1} - p\| &= \|\gamma_n(x_n - p) + (1-\gamma_n)(Sy_n - p)\| \\ &\leq \gamma_n \|x_n - p\| + (1-\gamma_n)\|Sy_n - p\| \\ &\leq \gamma_n \|x_n - p\| + (1-\gamma_n)\|y_n - p\|. \end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4), we get

$$\begin{aligned}\|x_{n+1}-p\| &\leq \gamma_n\|x_n-p\|+(1-\gamma_n)\big[1-(\bar{\gamma}-\gamma\alpha)\alpha_n\|x_n-p\|+(1-\gamma_n)\alpha_n\|\gamma f(p)-Ap\|\big]\\ &= [1-(\bar{\gamma}-\gamma\alpha)\alpha_n(1-\gamma_n)]\|x_n-p\|+(\bar{\gamma}-\gamma\alpha)\alpha_n(1-\gamma_n)\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\\ &\leq \max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\right\}.\end{aligned}$$

By mathematical induction, we obtain

$$\|x_{n+1}-p\| \leq \max\left\{\|x_1-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\right\},$$

for all $n\geq 1$. Thus $\{x_n\}$ is bounded, so are $\{ATx_n\}$ and $\{f(x_n)\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From definition of $\{y_n\}$, we observing that

$$\begin{aligned}y_{n+1}-y_n &= \alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}x_{n+1}+((1-\beta_{n+1})I-\alpha_{n+1}A)Tx_{n+1}-\alpha_n\gamma f(x_n)-\beta_n x_n\\ &\quad -((1-\beta_n)I-\alpha_n A)Tx_n\\ &= \alpha_{n+1}\gamma(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)\gamma f(x_n)+\beta_{n+1}(x_{n+1}-x_n)+(\beta_{n+1}-\beta_n)x_n\\ &\quad +((1-\beta_{n+1})I-\alpha_{n+1}A)(Tx_{n+1}-Tx_n)-(\beta_{n+1}-\beta_n)Tx_n-(\alpha_{n+1}-\alpha_n)ATx_n\\ &= \alpha_{n+1}\gamma(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)(\gamma f(x_n)-ATx_n)+\beta_{n+1}(x_{n+1}-x_n)\\ &\quad +(\beta_{n+1}-\beta_n)(x_n-Tx_n)+((1-\beta_{n+1})I-\alpha_{n+1}A)(Tx_{n+1}-Tx_n).\end{aligned}$$

It follows that

$$\begin{aligned}\|y_{n+1}-y_n\| &\leq \alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)-ATx_n\|+\beta_{n+1}\|x_{n+1}-x_n\|\\ &\quad +|\beta_{n+1}-\beta_n|\|x_n-Tx_n\|+(1-\beta_{n+1}-\alpha_{n+1}\bar{\gamma})\|Tx_{n+1}-Tx_n\|\\ &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\gamma\|f(x_n)\|+\|ATx_n\|)+\beta_{n+1}\|x_{n+1}-x_n\|\\ &\quad +|\beta_{n+1}-\beta_n|\|x_n-Tx_n\|+(1-\beta_{n+1}-\alpha_{n+1}\bar{\gamma})\|x_{n+1}-x_n\|\\ &= [1-(\bar{\gamma}-\gamma\alpha)\alpha_{n+1}]\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\gamma\|f(x_n)\|+\|ATx_n\|)+|\beta_{n+1}-\beta_n|\|x_n-Tx_n\|.\end{aligned}$$

We note that

$$\begin{aligned}\|Sy_{n+1}-Sy_n\| &\leq \|y_{n+1}-y_n\|\\ &\leq [1-(\bar{\gamma}-\gamma\alpha)\alpha_{n+1}]\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\gamma\|f(x_n)\|+\|ATx_n\|)\\ &\quad +|\beta_{n+1}-\beta_n|\|x_n-Tx_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|Sy_{n+1}-Sy_n\|-\|x_{n+1}-x_n\| &\leq (\gamma\alpha-\bar{\gamma})\alpha_{n+1}\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\gamma\|f(x_n)\|+\|ATx_n\|)\\ &\quad +|\beta_{n+1}-\beta_n|\|x_n-Tx_n\|.\end{aligned}\tag{3.5}$$

Moreover, since $\{x_n\}$, $\{f(x_n)\}$ and $\{ATx_n\}$ are all bounded. Now, we taking limit superior as $n\to\infty$ and by conditions $(C1)$ and $(C2)$, then inequality (3.5) implies that

$$\limsup_{n\to\infty}(\|Sy_{n+1}-Sy_n\|-\|x_{n+1}-x_n\|)\leq 0.\tag{3.6}$$

Hence, by Lemma 2.2, we get

$$\lim_{n\to\infty}\|Sy_n - x_n\| = 0. \tag{3.7}$$

Consequently, it follows from (3.7) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\gamma_n)\|Sy_n - x_n\| = 0. \tag{3.8}$$

Setting $W = ST$. Since S, T are nonexpansive mappings, obviously W is a nonexpansive mapping from C into itself.

Next, we show that $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$. Observing that

$$\|x_{n+1} - Sy_n\| \leq \|x_{n+1} - x_n\| + \|x_n - Sy_n\|.$$

By (3.7) and (3.8), we have

$$\lim_{n\to\infty}\|x_{n+1} - Sy_n\| = 0. \tag{3.9}$$

Moreover, we note that

$$\begin{aligned}
\|x_{n+1} - STx_{n+1}\| &\leq \|x_{n+1} - Sy_n\| + \|Sy_n - STx_n\| + \|STx_n - STx_{n+1}\| \\
&\leq \|x_{n+1} - Sy_n\| + \|y_n - Tx_n\| + \|x_n - x_{n+1}\| \\
&\leq \|x_{n+1} - Sy_n\| + \alpha_n(\gamma\|f(x_n)\| + \|ATx_n\|) + \beta_n\|x_n - Tx_n\| + \|x_n - x_{n+1}\|.
\end{aligned}$$

By conditions $(C1), (C2)$, (3.8) and noting (3.9), we obtain $\lim_{n\to\infty}\|x_{n+1} - STx_{n+1}\| = 0$, hence

$$\lim_{n\to\infty}\|x_n - Wx_n\| = 0. \tag{3.10}$$

Next, we show that $\limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, J(y_n - x^*)\rangle \leq 0$. For each $t \in (0,1)$, let x_t be a unique point of C such that $x_t = t\gamma f(x_t) + (I - tA)Wx_t$. By Lemma 2.3, we have

$$\begin{aligned}
\|x_t - x_n\|^2 &\leq \|(I - tA)(Wx_t - x_n)\|^2 + 2t\langle\gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&\leq \|I - tA\|^2\|Wx_t - x_n\|^2 + 2t\langle\gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&\leq (1 - t\bar{\gamma})^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 + 2t\langle\gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&\leq (1 - 2\bar{\gamma}t + \bar{\gamma}^2t^2)\|x_t - x_n\|^2 + f_n(t) + 2t\langle\gamma f(x_t) - Ax_t, J(x_t - x_n)\rangle + 2t\langle Ax_t - Ax_n, J(x_t - x_n)\rangle,
\end{aligned} \tag{3.11}$$

where $f_n(t) := \big(2\|x_t - x_n\| + \|x_n - Wx_n\|\big)\|x_n - Wx_n\|$.

On the other hand, observe that A is a strongly positive linear bounded operator, it follows from (2.2), we have

$$\langle Ax_t - Ax_n, J(x_t - x_n)\rangle = \langle A(x_t - x_n), J(x_t - x_n)\rangle \geq \bar{\gamma}\|x_t - x_n\|^2. \tag{3.12}$$

Combining (3.12) with (3.11), we have

$$\begin{aligned}
2t\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle &\leq (\bar{\gamma}^2t^2 - 2\bar{\gamma}t)\|x_t - x_n\|^2 + f_n(t) + 2t\langle Ax_t - Ax_n, J(x_t - x_n)\rangle \\
&\leq (\bar{\gamma}t^2 - 2t)\langle Ax_t - Ax_n, J(x_t - x_n)\rangle + 2t\langle Ax_t - Ax_n, J(x_t - x_n)\rangle + a_n(t) \\
&= \bar{\gamma}t^2\langle Ax_t - Ax_n, J(x_t - x_n)\rangle + f_n(t).
\end{aligned}$$

It follows that

$$\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle \leq \tfrac{\bar{\gamma}t}{2}\langle Ax_t - Ax_n, J(x_t - x_n)\rangle + \tfrac{1}{2t}f_n(t). \tag{3.13}$$

Now, taking limit superior as $n \to \infty$ and by (3.10), we have

$$\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle \leq \frac{t}{2}M, \tag{3.14}$$

where $M > 0$ is a constant such that $M \geq \bar{\gamma}\langle Ax_t - Ax_n, J(x_t - x_n)\rangle$ for all $t \in (0,1)$ and $n \geq 1$. Now, taking limit superior as $t \to 0$ in (3.14). Hence, we get

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle \leq 0. \tag{3.15}$$

Moreover, we note that

$$\begin{aligned}
\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle &= \langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle - \langle \gamma f(x^*) - Ax^*, J(x_n - x_t)\rangle \\
&\quad + \langle \gamma f(x^*) - Ax^*, J(x_n - x_t)\rangle - \langle \gamma f(x^*) - Ax_t, J(x_n - x_t)\rangle \\
&\quad + \langle \gamma f(x^*) - Ax_t, J(x_n - x_t)\rangle - \langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle \\
&\quad + \langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle \\
&= \langle \gamma f(x^*) - Ax^*, J(x_n - x^*) - J(x_n - x_t)\rangle + \langle Ax_t - Ax^*, J(x_n - x_t)\rangle \\
&\quad + \langle \gamma f(x^*) - \gamma f(x_t), J(x_n - x_t)\rangle + \langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{3.16}$$

Taking limit superior as $n \to \infty$ in (3.16), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle &\leq \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*) - J(x_n - x_t)\rangle \\
&\quad + \|A\|\|x_t - x^*\| \limsup_{n\to\infty}\|x_n - x_t\| \\
&\quad + \|\gamma f(x^*) - \gamma f(x_t)\| \limsup_{n\to\infty}\|x_n - x_t\| \\
&\quad + \limsup_{n\to\infty}\langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*) - J(x_n - x_t)\rangle \\
&\quad + (\|A\| + \gamma\alpha)\|x_t - x^*\| \limsup_{n\to\infty}\|x_n - x_t\| \\
&\quad + \limsup_{n\to\infty}\langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{3.17}$$

By Lemma 2.4, $x_t \to x^* \in F$ as $t \to 0$. Since X is a uniformly smooth Banach space, imply that J is norm-to-norm uniformly continuous on bounded subset of C (see, e.g., [14], Lemma 1), we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*) - J(x_n - x_t)\rangle = 0.$$

Therefore, from (3.17), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle \\
&\leq \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle \\
&\leq 0.
\end{aligned}$$

By assumption $\lim_{n\to\infty}\|x_n - y_n\| = 0$, hence, we get

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(y_n - x^*)\rangle \leq 0. \tag{3.18}$$

Finally, we show that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. In (3.3), replace p with x^*, to obtain that

$$\|y_n - x^*\| \leq \|x_n - x^*\| + \alpha_n\|f(x_n) - Ax^*\|. \tag{3.19}$$

By Lemma 2.3, we calculate

$$\begin{aligned}\|y_n - x^*\|^2 &= \|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(Tx_n - x^*)\|^2 \\ &\leq \|\beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(Tx_n - x^*)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*, J(y_n - x^*)\rangle \\ &\leq \left[\beta_n\|x_n - x^*\| + (1-\beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\|\right]^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*, J(y_n - x^*)\rangle \\ &= (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*, J(y_n - x^*)\rangle.\end{aligned}$$

On the other hand, by Lemma 2.5, we compute

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\gamma_n(x_n - x^*) + (1-\gamma_n)(Sy_n - x^*)\|^2 \\ &\leq \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n)\|Sy_n - x^*\|^2 \\ &\leq \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n)\|y_n - x^*\|^2 \\ &\leq \gamma_n\|x_n - x^*\|^2 + \|y_n - x^*\|^2 \\ &\leq \gamma_n\|x_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*, J(y_n - x^*)\rangle \\ &= (1 + \gamma_n - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2)\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), J(y_n - x^*)\rangle \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(y_n - x^*)\rangle \\ &\leq (1 + \gamma_n - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2)\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\|x_n - x^*\|\|y_n - x^*\| \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(y_n - x^*)\rangle.\end{aligned}$$

It follows from (3.19), that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq (1 + \gamma_n - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2)\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\|x_n - x^*\|^2 + 2\alpha_n^2\gamma\alpha\|\gamma f(x_n) - Ax^*\|\|x_n - x^*\| \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(y_n - x^*)\rangle \\ &\leq [1 - 2\alpha_n(\bar{\gamma} - \gamma\alpha)]\|x_n - x^*\|^2 + (\gamma_n + \alpha_n^2\bar{\gamma}^2)\|x_n - x^*\|^2 + 2\alpha_n^2\gamma\alpha\|\gamma f(x_n) - Ax^*\|\|x_n - x^*\| \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(y_n - x^*)\rangle.\end{aligned} \tag{3.20}$$

Then (3.20) reduced to formula

$$\|x_{n+1} - x^*\|^2 \leq (1 - a_n)\|x_n - x^*\|^2 + a_n b_n,$$

where $a_n := 2\alpha_n(\bar{\gamma} - \gamma\alpha)$ and

$$b_n := \frac{1}{2(\bar{\gamma} - \gamma\alpha)}\left[\left(\frac{\gamma_n}{\alpha_n} + \alpha_n\bar{\gamma}^2\right)\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\|\gamma f(x_n) - Ax^*\|\|x_n - x^*\| + 2\langle \gamma f(x^*) - Ax^*, J(y_n - x^*)\rangle\right].$$

By conditions $(C1)$, $(C4)$ and noting (3.18), it easy to seen that $\sum_{n=1}^{\infty} a_n = \infty$ and $\limsup_{n\to\infty} b_n \leq 0$. By Lemma 2.6, we conclude that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Corollary 3.2. *Let X be a uniformly smooth, uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from X into X^*, C be a closed convex subset of X such that $C \pm C \subset C$. Let S be nonexpansive mappings from C into itself such that $F := F(S) \neq \emptyset$, f be a contraction mapping with the coefficient $\alpha \in (0,1)$ and A be a linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} < \frac{\gamma}{\alpha}$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be the sequences in $(0,1)$. Assume the following conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\lim_{n\to\infty}\beta_n = 0$;

(C3) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$;

(C4) $\gamma_n = O(\alpha_n)$.

Then the sequence $\{x_n\}$ *defined by*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily;} \\ y_n = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)x_n; \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)Sy_n, \quad \forall n \geq 1, \end{cases}$$

converges strongly to fixed point x^* *where* x^* *is the unique solution in* F *of the variational inequality* (3.2).

Proof. Taking $S \equiv I$, in (3.2), we get desired conclusion easily. □

Remark 3.3. By the conditions $(C1)$ and $(C2)$, in Corollary 3.2, we get the assumption $\lim_{n\to\infty}\|x_n - y_n\| = 0$ is obvious. Therefore, we can be dropped.

Corollary 3.4. *Let* X *be a uniformly smooth, uniformly convex Banach space which admits a weakly sequentially continuous duality mapping* J *from* X *into* X^*, C *be a closed convex subset of* X *such that* $C \pm C \subset C$. *Let* T *be nonexpansive mappings from* C *into itself such that* $F(T)$ *is a nonempty compact subset,* f *be a contraction mapping with the coefficient* $\alpha \in (0,1)$ *and* A *be a linear bounded self-adjoint operator with the coefficient* $\bar{\gamma} > 0$ *such that* $0 < \bar{\gamma} < \frac{\gamma}{\alpha}$. *Let* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *be the sequences in* $(0,1)$. *Assume the following conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\lim_{n\to\infty}\beta_n = 0$;

(C3) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$;

(C4) $\gamma_n = O(\alpha_n)$.

Then the sequence $\{x_n\}$ *defined by*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily;} \\ y_n = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)Tx_n; \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)y_n, \quad \forall n \geq 1, \end{cases}$$

converges strongly to fixed point x^* *where* x^* *is the unique solution in* F *of the variational inequality* (3.2).

Proof. Taking $T \equiv I$, in (3.2), we obtain desired conclusion easily. □

Remark 3.5. By the conditions $(C1)$ and $(C2)$, in Corollary 3.4, we get the assumption $\lim_{n\to\infty}\|x_n - y_n\| = 0$ is obvious. Therefore, we can be dropped.

4 Applications to Optimization Problem

Let X be a uniformly smooth which admits a weakly sequentially continuous duality mapping J from X into X^*, C be a closed convex subset of X, A be a strongly positive linear bounded operator with a constant $\bar{\gamma} > 0$ and $T : C \to C$ be a nonexpansive mapping. In this section we will utilize the results presented in Section 3 to study the following *optimization problem*:

$$\min_{x \in F} \frac{1}{2}\big(\langle Ax, J(x)\rangle - h(x)\big), \tag{4.1}$$

where $F(T)$ is the set of fixed points of T in C and $h : C \to R$ is a potential function for γf (i.e., $h'(x) = \gamma f(x), x \in C$), where $f : C \to C$ is a contraction mapping with coefficient $\alpha \in (0, 1)$. We have the following theorem.

Theorem 4.1. *Let X be a uniformly smooth which admits a weakly sequentially continuous duality mapping J from X into X^*, C be a closed convex subset of X such that $C \pm C \subset C$. Let T be nonexpansive mappings from C into itself such that $F := F(T) \neq \emptyset$, f be a contraction mapping with the coefficient $\alpha \in (0, 1)$ and A be a linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} < \frac{\gamma}{\alpha}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be the sequences in $(0, 1)$. Assume the following conditions are hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \beta_n = 0$;

Then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily;} \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)Tx_n, \end{cases}$$

converges strongly to fixed point x^* where x^* is the unique solution minimal point of optimization problem (4.1).

Proof. Taking $\gamma_n = 0$ in Corollary 3.4. Hence, we get desired conclusion easily. □

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STRUCTURE OF THE ROOTS OF (h,q)-EULER POLYNOMIALS

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Abstract: Recently several authors studied the q-extension of Euler numbers and polynomials(see [1-12]). In this paper we observe the behavior of complex roots of the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$, using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$. Finally, we give a table for the solutions of the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$.

2000 Mathematics Subject Classification - 11B68, 11S40, 11S80

Key words : Euler numbers, Euler polynomials, (h,q)-Euler numbers, (h,q)-Euler polynomials

1. Introduction

In the 21st century, the computing environment would make more and more rapid progress. Using computer, a realistic study for new analogs of q-Euler numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$. The outline of this paper is as follows. In Section 2, we study (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$. In Section 3, we describe the beautiful zeros of the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$ using a numerical investigation. Also we display distribution and structure of the zeros of the the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$ by using computer. Finally, we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$. First, we introduce the Euler numbers and Euler polynomials. The Euler numbers E_n are defined by the generating function:

$$F(t)=\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\ \text{cf. [1-12]} \tag{1.1}$$

where we use the technique method notation by replacing E^n by $E_n(n\geq 0)$ symbolically. We consider the Euler polynomials $E_n(x)$ as follows:

$$F(x,t)=\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1.2}$$

2. The (h, q)-Euler numbers and polynomials

Let q be a complex number with $|q| < 1$. By the meaning of (1.1) and (1.2), let us define the (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$ as follows:

$$F_q^{(h)}(t) = \frac{2}{q^h e^t + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!}, \tag{2.1}$$

$$F_q^{(h)}(x,t) = \frac{2}{q^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{2.2}$$

The following elementary properties of (h, q)-Euler numbers $E_{n,q,w}^{(h)}$ and polynomials $E_{n,q,w}^{(h)}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved.

Proposition 1. For any positive integer n, we have

$$E_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q}^{(h)} x^{n-k}.$$

Proposition 2(The several values).

$$E_{0,q}^{(h)}(x) = \frac{2}{1+q^h}, E_{1,q}^{(h)}(x) = \frac{2(-q^h + x + q^h x)}{(1+q^h)^2},$$
$$E_{2,q}^{(h)}(x) = \frac{2(-q^h + q^{2h} - 2q^h x - 2q^{2h} x + x^2 + 2q^h x^2 + q^{2h} x^2)}{(1+q^h)^3}.$$

Proposition 3(Differential relation).

$$\frac{\partial}{\partial x} E_{n,q}^{(h)}(x) = n E_{n-1,q}^{(h)}(x).$$

Proposition 4(Integral formula).

$$\int_a^b E_{n-1,q}^{(h)}(x) dx = \frac{1}{n} (E_{n,q}^{(h)}(b) - E_{n,q}^{(h)}(a)).$$

Theorem 5(Addition theorem).

$$E_{n,q}^{(h)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q}^{(h)}(x) y^{n-k}.$$

Theorem 6(Distribution relation). For m is an odd positive integer, we have

$$E_{n,q}(x) = m^n \sum_{a=0}^{m-1} (-1)^a q^{ha} E_{n,q^m}^{(h)} \left(\frac{a+x}{m} \right).$$

Theorem 7(Difference equation).

$$q^h E_{n,q}^{(h)}(x+1) + E_{n,q}^{(h)}(x) = 2x^n.$$

Theorem 8(Theorem of complement).

$$E_{n,q}^{(h)}(1+x) = (-1)^n q^{-h} E_{n,q^{-1}}^{(h)}(-x), \quad E_{n,q}^{(h)}(1-x) = (-1)^n q^{-h} E_{n,q^{-1}}^{(h)}(x).$$

3. Distribution and Structure of the zeros

In this section, we investigate the zeros of (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$ by using computer. Let q be a complex number with $0 < q < 1$. We plot the zeros of $E_{n,q}^{(h)}(x), x \in \mathbb{C}$ for $q = 1/2$(Figures 1, 2, 3, and 4).

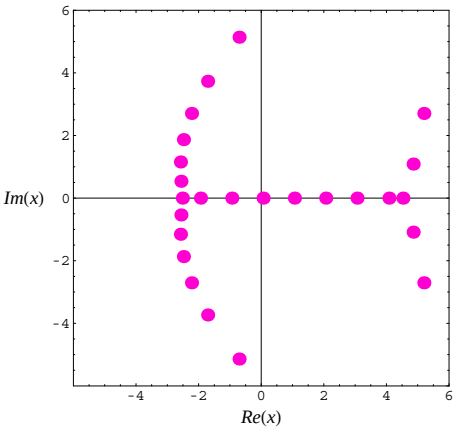


Figure 1: Zeros of $E_{25,q}^{(2)}(x)$

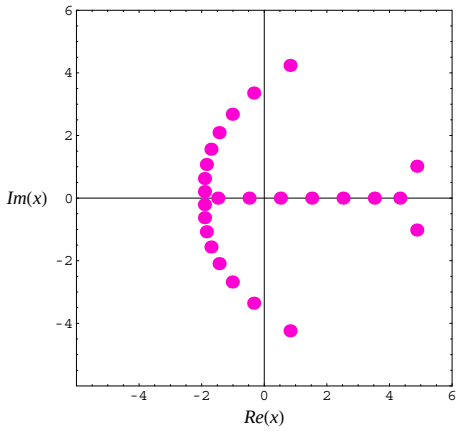


Figure 2: Zeros of $E_{25,q}^{(4)}(x)$

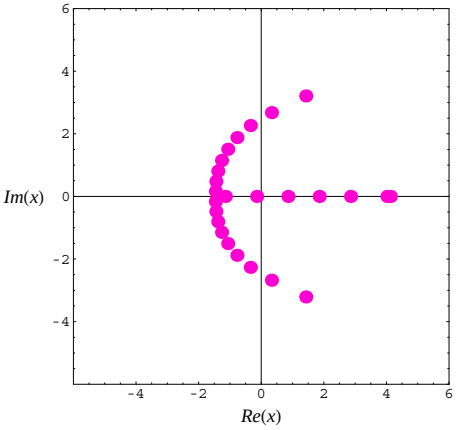


Figure 3: Zeros of $E_{25,q}^{(6)}(x)$

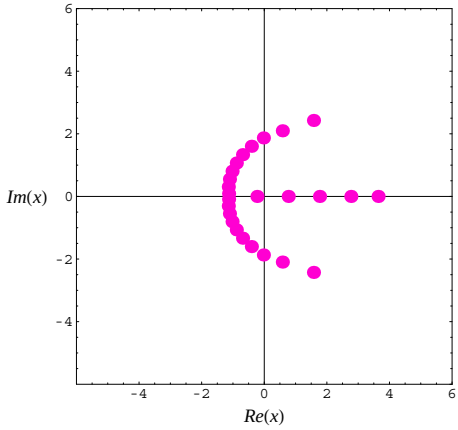


Figure 4: Zeros of $E_{25,q}^{(8)}(x)$

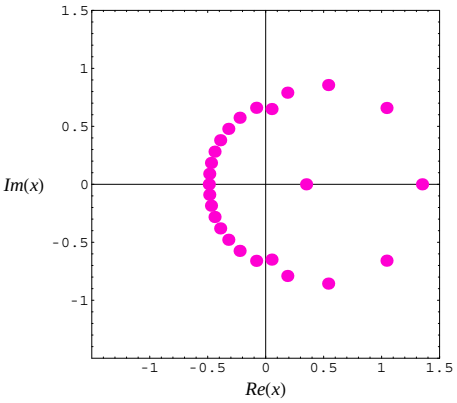


Figure 5: Zeros of $E_{25,q}^{(20)}(x)$

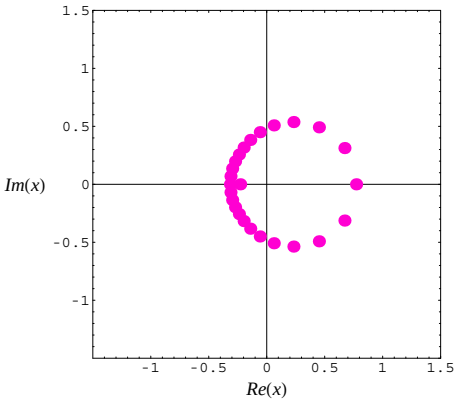


Figure 6: Zeros of $E_{25,q}^{(30)}(x)$

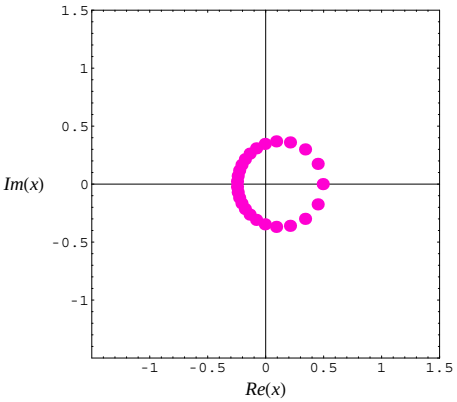


Figure 7: Zeros of $E_{25,q}^{(40)}(x)$

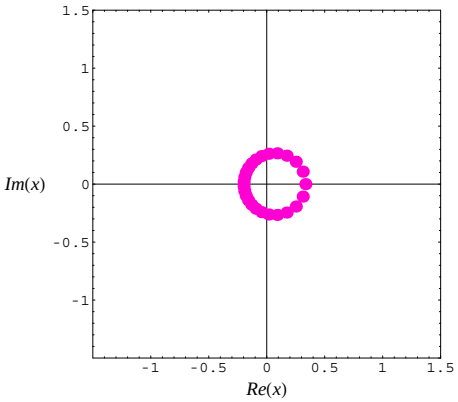


Figure 8: Zeros of $E_{25,q}^{(50)}(x)$

Next, we plot the zeros of $E_{25,q}^{(h)}(x), x \in \mathbb{C}$ for $q = 1/2, h = 20, 30, 40, 50$. (Figures 5, 6, 7, and 8).

Stacks of zeros of $E_{n,q}^{(3)}(x), q = 1/2$ for $1 \leq n \leq 25$ from a 3-D structure are presented(Figure 9). In Figures 1-9, $E_{n,q}^{(h)}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. This translates to the following open problem: Prove or disprove: $E_{n,q}^{(h)}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. Plot of real zeros of $E_{n,q}^{(h)}(x)$ for $1 \leq n \leq 25, h = 5, q = 1/2, 4/5$ structure is presented(Figure 10). Plot of real zeros of $E_{n,q}^{(h)}(x)$ for $q \to 1, 1 \leq n \leq 25$ structure is presented(Figure 11). In Figure 10(left), we choose $q = 1/2$. In Figure 10(right), we choose $q = 4/5$. Observe that the structure of the zeros of the Euler polynomials $E_n(x)$ resembles the structure of the zeros of the (h, q)-Euler polynomials $E_{n,q}^{(h)}(x)$ as $q \to 1$ (see Figures 10, 11). Our numerical results for numbers of real and complex zeros of $E_{n,q}^{(h)}(x)$ are displayed in Table 1.

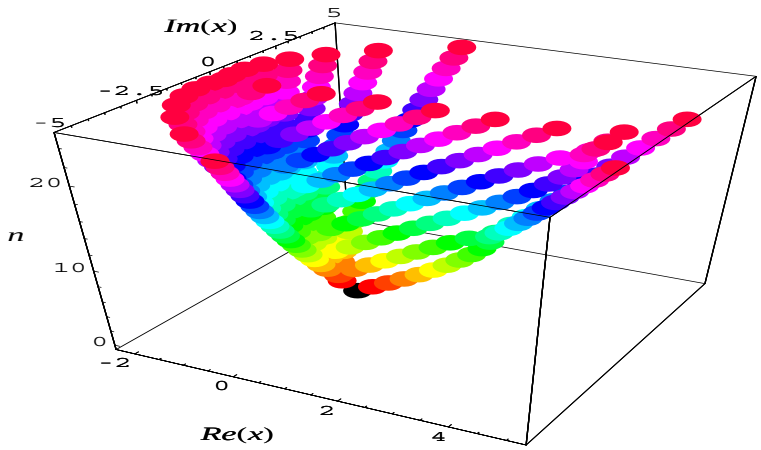


Figure 9: Stacks of zeros $E_{n,q}^{(3)}(x), q = 1/2$ for $1 \leq n \leq 25$

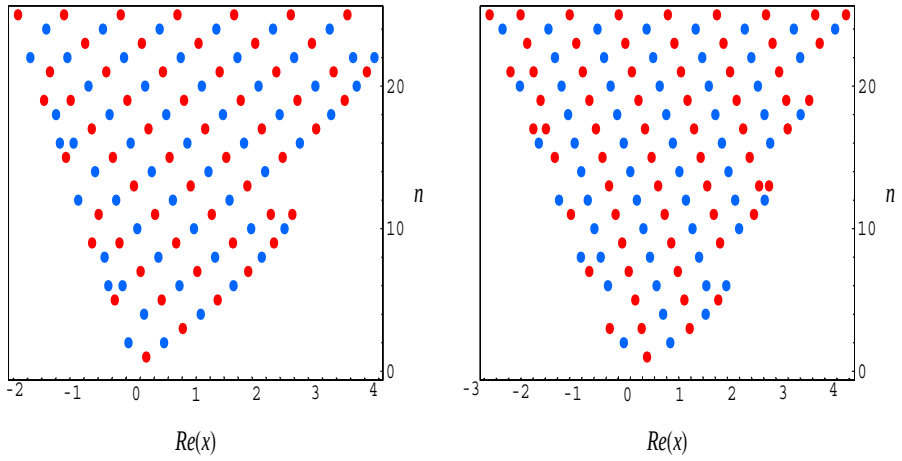


Figure 10: Real zeros of $E_{n,q}^{(3)}(x)$ for $q = 1/2, 4/5, 1 \leq n \leq 25$

Table 1. Numbers of real and complex zeros of $E_{n,q}^{(3)}(x)$

	$q = 1/2$		$q = 1/5$	
degree n	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	1	2	1	2
4	2	2	2	2
5	3	2	1	4
6	4	2	2	4
7	3	4	3	4
8	4	4	2	6
9	5	4	3	6
10	4	6	2	8

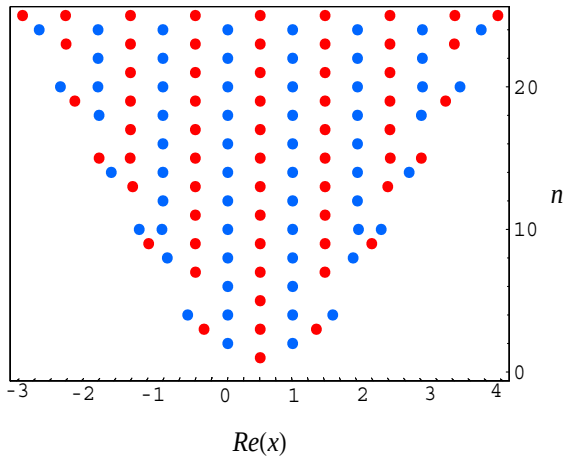


Figure 11: Real zeros of $E_{n,q}^{(h)}(x)$ for $q \to 1$ and $1 \leq n \leq 25$

We calculated an approximate solution satisfying $E_{n,q}^{(h)}(x), q = 1/2, x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $E_{n,q}^{(3)}(x) = 0, q = 1/2$

degree n	x
1	0.11111
2	$-0.20316, \quad 0.42538$
3	0.75568
4	$0.07253, \quad 1.0724$
5	$-0.4442, \quad 0.38304, \quad 1.3709$
6	$-0.5578, \quad -0.3070, \quad 0.6968, \quad 1.6501$
7	$0.011118, \quad 1.0111, \quad 1.9096$

We shall consider the more general open problem. In general, how many roots does $E_{n,q}^{(h)}(x)$ have? Prove or disprove: $E_{n,q}^{(h)}(x)$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q}^{(h)}(x)}$ of $E_{n,q}^{(h)}(x), Im(x) \neq 0$. Prove or give a counterexample: *Conjecture*: Since n is the degree of the polynomial $E_{n,q}^{(h)}(x)$, the number of real zeros $R_{E_{n,q}^{(h)}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{E_{n,q}^{(h)}(x)} = n - C_{E_{n,q}^{(h)}(x)}$, where $C_{E_{n,q}^{(h)}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q}^{(h)}(x)}$ and $C_{E_{n,q}^{(h)}(x)}$. We plot the $E_{n,q}^{(h)}(x)$, respectively (Figures 1-11). These figures give mathematicians an unbounded capacity to create visual mathematical investigations

of the behavior of the roots of the $E_{n,q}^{(h)}(x)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the (h,q)-Euler polynomials $E_{n,q}^{(h)}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [7-12].

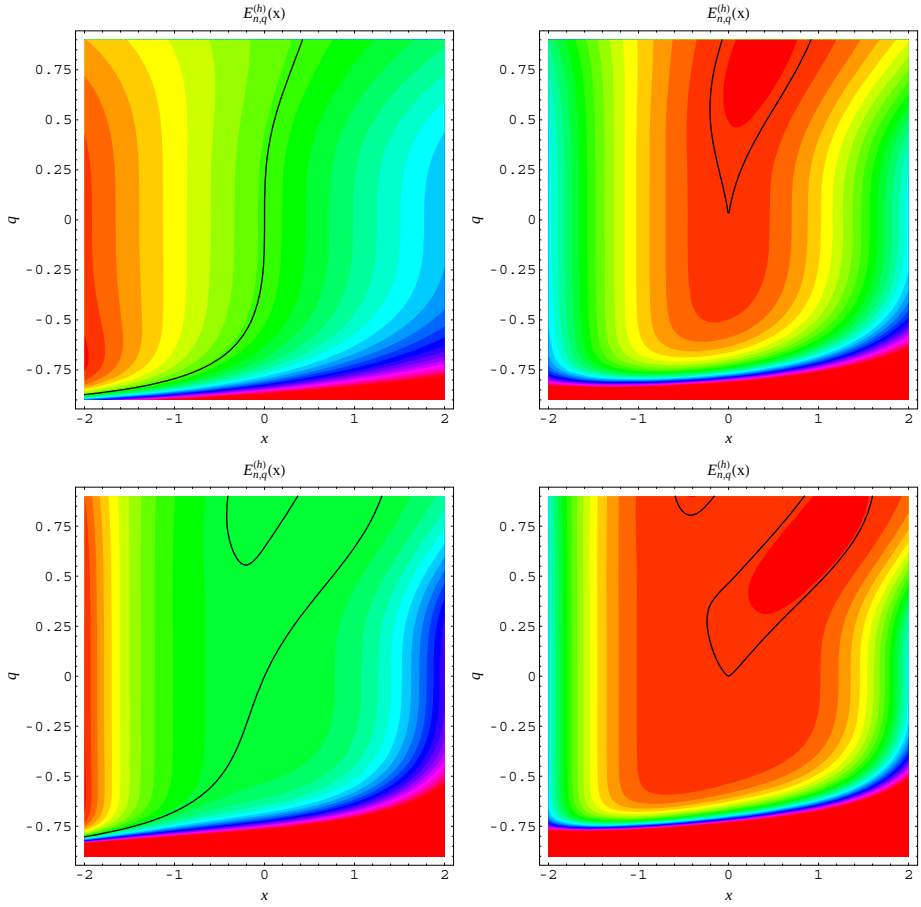


Figure 12: Zero contour of $E_{n,q}^{(h)}$

The plot above shows $E_{n,q}^{(h)}$ for real $-9/10 \leq q \leq 9/10$ and $-2 \leq x \leq 2$, with the zero contour indicated in black (Figure 12). In Figure 12(top-left), we choose $n = 1$ and $h = 3$. In Figure 12(top-right), we choose $n = 2$ and $h = 3$. In Figure 12(bottom-left), we choose $n = 3$ and $h = 3$. In Figure 12(bottom-right), we choose $n = 4$ and $h = 3$.

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Some conditions for subsequential convergence and ordinary convergence

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February 8, 2011

Abstract

In this paper, we obtain certain conditions in terms of the regularly generated sequences under which (u_n) converges subsequentially or converges.

Keywords: Slow oscillation, moderate oscillation, regularly generated sequence, general control modulo.

1 Introduction

Throughout this paper c_0 and l_∞ denote the space of sequences converging to zero and the space of bounded sequences, respectively. Let (u_n) be a sequence of real numbers. If (u_n) converges, then $(\Delta u_n) = (u_n - u_{n-1}) \in c_0$. That the converse is not necessarily true is clear from the example of the sequence $(\log n)$. Since $(\Delta u_n) \in c_0$ is a necessary condition for convergence of a sequence (u_n), we pose the following question: Under which condition(s) may we obtain some information on the behavior of (u_n) satisfying $(\Delta u_n) \in c_0$ or its generalizations? In the case where (u_n) is bounded together with $(\Delta u_n) \in c_0$, we only

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have convergence of some subsequences of (u_n). This suggests a new kind of convergence.

A sequence $u = (u_n)$ is subsequentially convergent [1] if there exists a finite interval $I(u)$ such that all accumulation points of $u = (u_n)$ are in $I(u)$ and every point of $I(u)$ is an accumulation point of $u = (u_n)$. Equivalently, for every $r \in I(u)$ there exists a subsequence $(u_{n(r)})$ of (u_n) such that $\lim_{n(r)} u_{n(r)} = r$.

Bolzano-Weierstrass theorem [2] states that every bounded sequence possesses at least one accumulation point. However there are some bounded sequences such as $(\sin(\log n))$ whose accumulation points lie on a finite interval and all points in this interval are accumulation points of the sequence. It is clear that subsequential convergence implies boundedness by definition of subsequential convergence. However the converse is not generally true. The example of a bounded sequence which is not a subsequential convergent is $((-1)^n)$.

Let $\mathcal{L}$ be any linear space of sequences and $\mathcal{A}$ be a subclass of $\mathcal{L}$. If

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{\alpha_k}{k}, \tag{1}$$

for some $\alpha = (\alpha_n) \in \mathcal{A}$, we say that the sequence (u_n) is regularly generated by the sequence (α_n) and (α_n) is called a generator of (u_n). The class of all sequences regularly generated by the sequences (α_n) in $\mathcal{A}$ is denoted by $U(\mathcal{A})$. For each integer $m \geq 1$, define the class $\mathcal{A}^{(m)} = \{(\alpha_n^{(m)}) | \alpha_n^{(m)} = \sum_{k=1}^{n} \frac{\alpha_k^{(m-1)}}{k}\}$, where $(\alpha_n^{(0)}) := (\alpha_n) \in \mathcal{A}$. Let $u = (u_n) \in \mathcal{L}$. If

$$u_n = \alpha_n^{(m)} + \sum_{k=1}^{n} \frac{\alpha_k^{(m)}}{k} \tag{2}$$

for some $\alpha^{(m)} = (\alpha_n^{(m)}) \in \mathcal{A}^{(m)}$, we say that the sequence (u_n) is regularly generated by the sequence $(\alpha_n^{(m)})$ and $(\alpha_n^{(m)})$ is called a generator of (u_n). The class of all sequences regularly generated by the sequences $(\alpha_n^{(m)})$ in $\mathcal{A}^{(m)}$ is denoted by $U(\mathcal{A}^{(m)})$. The class $U(\mathcal{A}_\Delta^{(m)})$ is the class of all sequences regularly generated by the sequence $(\alpha_n^{(m)})$ where $\Delta\alpha = (\Delta\alpha_n) \in \mathcal{A}$.

The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$, where $\Delta u_n = u_n - u_{n-1}$ and $u_{-1} = 0$. The general control

modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (u_n) is defined inductively in [1, 3] by $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n(\omega^{(m-1)}(u))$, where $\sigma_n(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_k$.

The Kronecker identity

$$u_n - \sigma_n(u) = V_n^{(0)}(\Delta u), \tag{3}$$

where $V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k$ is well known and used in the various steps of proofs. Since arithmetic means of (u_n) can be also expressed as

$$\sigma_n(u) = u_0 + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k},$$

we may write (3) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0.$$

For a sequence (u_n) and for each integer $m \geq 1$ define

$$(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1}u_n),$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$. It is proved in [4] that for each integer $m \geq 1$, $\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u)$, where $V_n^{(m-1)}(\Delta u) = \sigma_n(V^{(m-2)}(\Delta u))$.

A sequence $u = (u_n)$ is slowly oscillating [1, 3] if

$$\lim_{\lambda \to 1^+} \limsup_{n \to \infty} \max_{n+1 \leq k \leq [\lambda n]} |u_k - u_n| = 0,$$

where $[\lambda n]$ denotes the integer part of λn. The class of all slowly oscillating sequences is denoted by $\mathcal{S}$.

Dik [3] proved that necessary and sufficient conditions for slow oscillation of (u_n) are $(V_n^{(0)}(\Delta u)) \in \mathcal{S}$ and $(V_n^{(0)}(\Delta u)) \in l_\infty$.

The fact that the bounded slowly oscillating sequences are subsequentially convergent follows from the definition. The sequence $(\sin(\sum_{k=1}^{n} \frac{\log k}{k}))$ are subsequential convergent, but it is not slowly oscillating.

The following definition is a generalization of the concept of slow oscillation.

A sequence $u = (u_n)$ is moderately oscillating [1, 3] if for $\lambda > 1$,

$$\limsup_{n\to\infty} \max_{n+1\leq k\leq [\lambda n]} |u_k - u_n| < \infty,$$

where $[\lambda n]$ denotes the integer part of λn. The class of all moderately oscillating sequences is denoted by $\mathcal{M}$.

Dik [3] proved that $(V_n^{(0)}(\Delta u)) \in l_\infty$ and $(\sigma_n(u)) \in \mathcal{S}$ for a moderately oscillating sequence $u = (u_n)$.

The example of a moderately oscillating sequence that is not slowly oscillating is the sequence (u_n) defined by

$$u_n = \begin{cases} 1, & n = 2^k, k = 1, 2, ... \\ -1, & n = 2^k + 1, k = 1, 2, ... \\ 0, & \text{for other value of } n \end{cases}.$$

Stanojević [5] showed that if $(S_n(\alpha)) = (\sum_{k=1}^n \alpha_k) \in \mathcal{S}$, then $(\sum_{k=1}^n \frac{\alpha_k}{k})$ converges. Weakening the sufficient condition for convergence of $(\sum_{k=1}^n \frac{\alpha_k}{k})$ in Stanojević's theorem [5], Çanak et al [6] proved that (u_n) converges subsequentially or converges depending on the conditions on the generator sequence (α_n) if (u_n) is regularly generated by (α_n) and $(S_n(\alpha)) \in \mathcal{M}$. Recently, Totur and Çanak [9] generalized that Çanak et al [6] given above-mentioned conditions and proved that (u_n) converges subsequentially or converges depending on the conditions on the generator sequence $(\alpha_n^{(m)})$ if (u_n) is regularly generated by $(\alpha_n^{(m)})$ and $(S_n(\alpha^{(m)})) \in \mathcal{M}$.

The purpose of this paper is to obtain sufficient conditions for (u_n) regularly generated by a sequence $(\alpha_n^{(m)})$ in the class $\mathcal{M}^{(m)}$ or $\mathcal{M}_\Delta^{(m)}$ to be subsequential convergence or convergence.

2 Results

The following theorems provide sufficient conditions in terms of regularly generated sequences for subsequential convergence and convergence of (u_n).

Theorem 2.1 *Let $(u_n) \in U(\mathcal{M}_\Delta^{(m)})$ be regularly generated by $(\alpha_n^{(m)})$ for some integer $m \geq 0$. If $(S_n(\alpha^{(m)})) \in \mathcal{M}$, then (u_n) converges subsequentially.*

Theorem 2.2 *Let $(u_n) \in U(\mathcal{M}^{(m)})$ be regularly generated by $(\alpha_n^{(m)})$ for some integer $m \geq 0$. If $(S_n(\alpha^{(m)})) \in \mathcal{M}$, then (u_n) converges.*

We notice that if (u_n) is regularly generated by a sequence $(\alpha_n^{(m)})$ and $(S_n(\alpha^{(m)})) \in \mathcal{M}$, we only recover subsequential convergence of $(\sigma_n(u))$ instead of subsequential convergence of (u_n). It follows from the definition of subsequential convergence that if (u_n) is subsequential convergent, then so is $(\sigma_n(u))$. But the converse is not true in general. For example, the sequence (u_n) defined by $u_n = \sum_{k=0}^{n}(-1)^k$ is not subsequential convergent, but $(\sigma_n(u))$ is subsequential convergent.

We need the following lemmas to prove Theorem 2.1 and Theorem 2.1.

Lemma 2.3 *[1] If $(u_n) \in l_\infty$ and $(\Delta u_n) \in c_0$, then (u_n) converges subsequentially.*

Lemma 2.4 *[7] Let $(\sigma_n(u))$ converge to s. If $(u_n) \in \mathcal{S}$, then (u_n) converges to s.*

Lemma 2.5 *[6] If $(S_n(\alpha)) \in \mathcal{M}$, then $(\sum_{n=1}^{\infty} \frac{\alpha_n}{n})$ converges.*

Lemma 2.6 *[6] If $(\sum_{k=1}^{n} \frac{\alpha_k}{k})$ converges, then $(\sigma_n(\alpha)) \in c_0$*

The following Lemma is a slightly altered version of Lemma 3.1 in [8].

Lemma 2.7 *Let $\mathcal{L}$ be any linear space of sequences and let $(u_n) \in \mathcal{L}$. If $(u_n) \in U(\mathcal{A}^{(m)})$ for $m \geq 0$ integer, then $(n\Delta)_m V_n^{(0)}(\Delta u) = \alpha_n$, where $(\alpha_n) \in \mathcal{A}$.*

Proof. If $(u_n) \in U(\mathcal{A}^{(m)})$, it follows that

$$u_n = \alpha_n^{(m)} + \sum_{k=1}^{n} \frac{\alpha_k^{(m)}}{k}, \qquad m \geq 0 \quad \textit{integer}, \tag{4}$$

for some $(\alpha_n^{(m)}) \in \mathcal{A}^{(m)}$. We obtain from (4) that

$$n\Delta u_n = n\Delta\alpha_n^{(m)} + \alpha_n^{(m)}. \tag{5}$$

Subtracting (5) from arithmetic means of (5), we have $n\Delta u_n - V_n^{(0)}(\Delta u) = n\Delta\alpha_n^{(m)}$. From the last identity we obtain that $n\Delta V_n^{(0)}(\Delta u) = \alpha_n^{(m-1)}$. By repeating the same reasoning, we have $(n\Delta)_m V_n^{(0)}(\Delta u) = \alpha_n^{(0)} = \alpha_n$.

3 Proofs

Now we are ready to prove the main theorems.

Proof of Theorem 2.1.

We know from [3] that $(S_n(\alpha^{(m)})) \in \mathcal{M}$ implies

$$(V_n^{(0)}(\Delta S_n(\alpha^{(m)}))) = (V_n^{(0)}(\alpha^{(m)})) \in l_\infty. \tag{6}$$

By writing $S_n(\alpha^{(m)})$ in place of u_n in (3) and then taking the backward difference of both sides of the resulting identity, we get

$$\alpha_n^{(m)} - \frac{V_n^{(0)}(\alpha^{(m)})}{n} = \Delta V_n^{(0)}(\alpha^{(m)}). \tag{7}$$

From (6) and (7), we obtain that $(\alpha_n^{(m)}) \in l_\infty$. From Lemma 2.5 applied to $(S_n(\alpha^{(m)})) \in \mathcal{M}$ and from (4), we have

$$(u_n) \in l_\infty. \tag{8}$$

Since $(u_n) \in U(\mathcal{M}_\Delta^{(m)})$ for some positive integer $m \geq 0$, we obtain $(\Delta(n\Delta)_m V_n^{(0)}(\Delta u)) \in \mathcal{M}$ by Lemma 2.7. Taking arithmetic means of both sides of last result, we obtain

$$(\Delta(n\Delta)_{m-1} V_n^{(0)}(\Delta u)) \in \mathcal{S}.$$

Again taking arithmetic means of both sides of the last result we obtain

$$(\Delta(n\Delta)_{m-2} V_n^{(0)}(\Delta u)) \in \mathcal{S}.$$

Continuing in this vein, we get $(\Delta(n\Delta)_0 V_n^{(0)}(\Delta u)) = (\Delta V_n^{(0)}(\Delta u)) \in \mathcal{S}$. Taking arithmetic means in the last result gives

$$(\sigma_n(\Delta V^{(0)}(\Delta u))) = \left(\frac{1}{n}\sum_{k=1}^{n} \Delta V_k^{(0)}(\Delta u)\right) = \left(\frac{V_n^{(0)}(\Delta u)}{n}\right) \in c_0$$

by (8). From Lemma 2.4 it follows that $(\Delta V_n^{(0)}(\Delta u)) \in c_0$. From the identity

$$\Delta u_n = \frac{V_n^{(0)}(\Delta u)}{n} + \Delta V_n^{(0)}(\Delta u)$$

and (8), we have

$$(\Delta u_n) \in c_0. \tag{9}$$

From (8) and (9), we obtain by Lemma 2.3 that (u_n) converges subsequentially.

Corollary 3.1 *Let $(u_n) \in U(\mathcal{S}_\Delta^{(m)})$ be regularly generated by $(\alpha_n^{(m)})$ for some integer $m \geq 0$. If $(S_n(\alpha^{(m)})) \in \mathcal{M}$, then (u_n) converges subsequentially.*

Proof. Assume that $(u_n) \in U(\mathcal{S}_\Delta^{(m)})$. From this it can be obtained that $(\Delta(n\Delta)_m V_n^{(0)}(\Delta u)) \in \mathcal{S}$ by Lemma 2.7. Taking arithmetic means of both sides of the last result, we obtain $(\Delta(n\Delta)_{m-1} V_n^{(0)}(\Delta u)) \in \mathcal{S}$. The rest of the proof is as in the proof in Theorem 2.1.

Corollary 3.2 *Let $(u_n) \in U(\mathcal{S}_\Delta)$ be regularly generated by (α_n). If $(S_n(\alpha)) \in \mathcal{M}$, then (u_n) converges subsequentially.*

Proof. Take $m = 0$ in Corollary 3.1.

Corollary 3.2 is Theorem 2.1 in [6].

Proof of Theorem 2.2.

Since $(u_n) \in U(\mathcal{M}^{(m)})$, we obtain $((n\Delta)_m V_n^{(0)}(\Delta u)) \in \mathcal{M}$ by Lemma 2.7. This implies that

$$((n\Delta)_m V_n^{(m)}(\Delta u)) \in \mathcal{S}. \tag{10}$$

Since $(S_n(\alpha^{(m)})) \in \mathcal{M}$, $(\sum_{k=1}^n \frac{\alpha_k^{(m)}}{k})$ converges by Lemma 2.5. From Lemma 2.6 it follows that $(\sigma_n(\alpha^{(m)})) \in c_0$. Therefore from (2), we obtain that $(\sigma_n(u))$ converges. By Kronecker identity applied to the sequence $(\sigma_n(u))$, we have $\sigma_n(u) - \sigma_n(\sigma(u)) = V_n^{(1)}(\Delta u)$ and from this identity we find that $(V_n^{(1)}(\Delta u)) \in c_0$. It can be easily concluded that

$$((n\Delta)_m V_n^{(m+1)}(\Delta u)) = (\sigma_n(\sigma(\omega^{(m)}(u)))) \in c_0 \tag{11}$$

for each $m \geq 0$. Taking into consideration (10) and (11), we have $((n\Delta)_m V_n^{(m)}(\Delta u)) \in c_0$ by Lemma 2.4. By the identity

$$(n\Delta)_m V_n^{(m)}(\Delta u) = (n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) - (n\Delta)_{m-1} V_n^{(m)}(\Delta u),$$

we have $((n\Delta)_{m-1} V_n^{(m-1)}(\Delta u)) \in c_0$. Continuing in this vein, we see that $(n\Delta V_n^{(1)}(\Delta u)) \in c_0$. Since $(V_n^{(1)}(\Delta u)) \in c_0$ and $n\Delta V_n^{(1)}(\Delta u) = V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)$, it follows that $(V_n^{(0)}(\Delta u)) \in c_0$. By the Kronecker identity we obtain $(u_n) \in \mathcal{S}$. Since $(\sigma_n(u))$ converges, we have by Lemma 2.4 that (u_n) converges.

Corollary 3.3 *Let $(u_n) \in U(\mathcal{S}^{(m)})$ be regularly generated by $(\alpha_n^{(m)})$ for some integer $m \geq 0$. If $(S_n(\alpha^{(m)})) \in \mathcal{M}$, then (u_n) converges.*

Proof. Assume that $(u_n) \in U(\mathcal{S}^{(m)})$. From this it can be easily obtained that $((n\Delta)_m V_n^{(0)}(\Delta u)) \in \mathcal{S}$ by Lemma 2.7. Therefore it is seen that $((n\Delta)_m V_n^{(m)}(\Delta u)) \in \mathcal{S}$. The rest of the proof is as in the proof in Theorem 2.2.

Corollary 3.4 *Let $(u_n) \in U(\mathcal{S})$ be regularly generated by (α_n). If $(S_n(\alpha)) \in \mathcal{M}$, then (u_n) converges.*

Proof. Take $m = 0$ in Corollary 3.3.

Corollary 3.4 is Corollary 3.2 in [6].

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APPLICATION OF SUMUDU AND DOUBLE SUMUDU TRANSFORMS TO CAPUTO-FRACTIONAL DIFFERENTIAL EQUATIONS

F. JARAD AND K. TAS

Abstract. The definition, properties and applications of the Sumudu transform to ordinary differential equations are described in [1-3]. In this manuscript we derive the formulae for the Sumudu and double Sumudu transforms of ordinary and partial fractional derivatives and apply them in solving Caputo-fractional differential equations. Our purpose here is to show the applicability of this new transform and its efficiency in solving such problems.

1. Introduction and Preliminaries

The Sumudu transform introduced by the Watugala [1, 2] can be used for solving ordinary and partial differential equations both in ordinary and in fractional sense.For more of Sumudu properties and applications, see F.B.M.Belgacem et al [5] and D.Loonker et al [6].
Some advantages of the Sumudu transform over the Laplace transform are listed in the following;

- the ability to use the same physical units to measure $f(x,t)$ and $F(u,v) = S\{f(x,t)\}$
- the equality of $f(x,t)$ and $S\{f(x,t)\}$ for the constant functions
- the similarity between the function $f(x,t)$ in the (x,t) domain and the resulting function $S\{f(x,t)\}$ in the (u,v) domain
- the scaling property, that is $S\{f(bx,ct)\} = F(bu,cv)$ in the (u,v) domain.
- the limit of $f(t)$ as t tends to zero is equal to the limit of $F(u)$ as u tends to zero.

In the paper of the Weerakoon [3] , Sumudu transform applications to partial differential equations are given and Watugala [2] extended the transform to two variables with emphasis on solutions to partial differential equations.
For a function of one variable $f(t)$ which is of exponential order,

$$|f(t)| < \begin{cases} Me^{\frac{-t}{\tau_1}} & \text{if } t \le 0 \\ Me^{\frac{t}{\tau_2}} & \text{if } t > 0 \end{cases} \tag{1.1}$$

the Sumudu transform, is defined by the integral equation

$$F(u) = \int_0^\infty f(ut)e^{-t}dt, \quad \tau_1 \le u \le \tau_2, \tag{1.2}$$

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where M is a real finite number and τ_1 and τ_2 can be finite or infinite [2]. Before we prove the main results we provide the following lemma for the existence of Sumudu transform of one variable.

Lemma 1. *(cf. [4]) If f is of exponential order, then its Sumudu transform $F(u) = S\{f(t)\}$ exists and is given by*

$$F(u) = \int_0^\infty f(ut)e^{-t}dt = \int_0^\infty f(t)e^{-\frac{t}{u}}dt, \quad \tau_1 \le u \le \tau_2, \tag{1.3}$$

where M is a real finite number and τ_1 and τ_2 can be finite or infinite.

Below, we present theorems which we use

Theorem 2. [7] *If $f(t)$ and $g(t)$ are of exponential order (1.1)and $F(u)$ and $G(u)$ are the Sumudu tranform of $f(t)$ and $g(t)$ respectively, then*

$$S\{f(t) * g(t)\}(u) = S\left\{\int_0^t f(\tau)g(t-\tau)d\tau)\right\}(u) = uF(u)G(u). \tag{1.4}$$

Theorem 3. [4, 8] *If $f^{(n)}(t)$ is of exponential order (1.1), then the Sumudu transform of the n'th derivative of f is given by*

$$S\{f^{(n)}(t)\}(u) = u^{-n}[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)]. \tag{1.5}$$

The Sumudu transforms of functions of two variables were also discussed, and the Sumudu transforms of their partial derivatives are given in [2, 3, 9].

J.M.Tchuenche and N.S.Mbare [9] defined *the double Sumudu transform of a function of two variables* as;

Let $f(x,t); x,t \in \mathbb{R}^+$ be a function which can be expressed as a convergent infinite series, then, its double Sumudu transform is given by

$$\begin{aligned} F(u,v) &= S_t S_x\{f(x,t)\}(u,v) = S_t\{S_x\{f(x,t); x \to u\}; t \to v\} \\ &= S_t\{\frac{1}{u}\int_0^\infty e^{-\frac{x}{u}} f(x,t)dx; x \to u\} \\ &= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{t}{u}+\frac{x}{v})} f(x,t)dxdt \end{aligned}$$

We also have to mention the following results of [9]:

Let $f(x,t); x,t \in \mathbb{R}^+$ be a function which can be expressed as a convergent infinite series, then

$S_t S_x\{f(ax)g(bt); (u,v)\} = F(au) - G(bv)$ and

$S_t S_x\{f(ax,bt); (u,v)\} = F(au,bv)$.

The Fractional calculus which is the generalization of ordinary differentiation and integration to arbitrary is as old as the usual calculus. The fractional derivatives have been used in many fields of science in the last decades [10, 11, 12, 13].

Below we present some basic definitions related to fractional derivatives.

The Left Riemann-Liouville Fractional Integral and *The Right Riemann-Liouville Fractional Integral* are defined respectively by

$$ {}_aI^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{1.6} $$

$$ I_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau, \tag{1.7} $$

where $\alpha > 0,\ n-1 < \alpha < n$. Here and in the following $\Gamma(\alpha)$ represents the Gamma function.

The Left Riemann-Liouville Fractional Derivative is defined by

$$ {}_aD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{1.8} $$

The Right Riemann-Liouville Fractional Derivative is defined by

$$ D_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau, \tag{1.9} $$

The fractional derivative of a constant takes the form

$$ {}_aD^{\alpha} C = \ C\ \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{1.10} $$

and the fractional derivative of a power of t has the following form

$$ {}_aD^{\alpha} (t-a)^{\beta} = \ \frac{\Gamma(\alpha+1)(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, \tag{1.11} $$

for $\beta > -1, \alpha \geq 0$.

The Caputo's fractional derivatives are defined as follows:

The Left Caputo Fractional Derivative

$$ {}_a^C D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{1.12} $$

and

The Right Caputo Fractional Derivative

$$ {}^C D_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{1.13} $$

where α represents the order of the derivative such that $n-1 < \alpha < n$. By definition the Caputo fractional derivative of a constant is zero.

The left Riemann fractional integral on the half-axis is defined by

$$ I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{1.14} $$

The Left Riemann-Liouville Fractional Derivative on the half-axis is defined by

$$ D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. \tag{1.15} $$

The Left Caputo Fractional Derivative on the half-axis is defined by

$$ {}^C D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau. \tag{1.16} $$

Similarly, the partial fractional integrals and derivatives of $f(x,t)$, on the half-axes can be defined as

$$ {}_{0+}I_x^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s,t) d\tau, \tag{1.17} $$

$$ {}_{0+}I_t^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau) d\tau, \tag{1.18} $$

$$ {}_{0+}I_t^{\beta} \ {}_{0+}I_x^{\alpha} f(x,t) = $$

$$ \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^t \int_0^x (t-\tau)^{\beta-1} (x-s)^{\alpha-1} f(s,\tau) ds d\tau, \tag{1.19} $$

$$ {}_{0+}D_x^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial x}\right)^n \int_0^x (x-s)^{n-\alpha-1} f(s,t) ds, \tag{1.20} $$

$$ {}_{0+}D_t^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial t}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(x,\tau) d\tau, \tag{1.21} $$

$$ {}_{0+}D_t^{\beta} \ {}_{0+}D_x^{\alpha} f(x,t) = $$

$$ \frac{1}{\Gamma(m-\beta)\Gamma(n-\alpha)} \frac{\partial^{m+n}}{\partial t^m \partial x^n} \int_0^t \int_0^x (t-\tau)^{m-\beta-1} (x-s)^{n-\alpha-1} f(s,\tau) ds d\tau, \tag{1.22} $$

$$ {}^{C}_{0+}D_x^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} \left(\frac{\partial}{\partial s}\right)^n f(s,t) ds, \tag{1.23} $$

$$ {}^{C}_{0+}D_t^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \left(\frac{\partial}{\partial \tau}\right)^n f(x,\tau) d\tau, \tag{1.24} $$

$$ {}^{C}_{0+}D_t^{\beta} \ {}^{C}_{0+}D_x^{\alpha} f(x,t) = $$

$$ \frac{1}{\Gamma(m-\beta)\Gamma(n-\alpha)} \int_0^t \int_0^x (t-\tau)^{m-\beta-1} (x-s)^{n-\alpha-1} \frac{\partial^{m+n}}{\partial \tau^m \partial s^n} f(s,\tau) ds d\tau, \tag{1.25} $$

where $m-1 < \beta \leq m$.

2. The Sumudu Transform of Fractional Integrals and Derivatives

Before we present the existence of the Sumudu transform, we give the following lemmas:

Lemma 4. *Let $f(t) \in L_1(0,b)$, $b > 0$, such that f satisfies (1.1). Then $I_{0+}^{\alpha} f(t)$ also satisfies (1.1).*

For the proof we refer to lemma 7.1 [11]. The Sumudu transform of the fractional integral and derivatives are presented in the following theorems;

Theorem 5. *Let f satisfy the conditions of the previous lemma 4. Then the Sumudu transform of $I_{0+}^{\alpha} f(t)$ exists and*

$$ S\left\{I_{0+}^{\alpha} f(t)\right\}(u) = u^{\alpha} F(u). \tag{2.1} $$

Proof. From (1.14), clearly we have

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t). \tag{2.2}$$

Thus, by using theorem 2 and the definition of the gamma function

$$\begin{aligned} S\left\{I_{0+}^{\alpha} f(t)\right\}(u) &= \frac{1}{\Gamma(\alpha)} S\{t^{\alpha-1}\} S\{f(t)\} \\ &= \frac{1}{\Gamma(\alpha)} u^{\alpha}\Gamma(\alpha) F(u) = u^{\alpha} F(u). \end{aligned} \tag{2.3}$$

□

Theorem 6. *Let f satisfy (1.1) and $f \in AC^n([0,b))$. Then*

$$S\left\{D_{0+}^{\alpha} f(t)\right\}(u) = u^{-\alpha} F(u) - \sum_{k=0}^{n-1} u^k D_{0+}^{\alpha-k-1} f(t)\Big|_{t=0} \tag{2.4}$$

Proof. Using (1.15), theorem 3 and theorem 5, we have

$$\begin{aligned} S\left\{D_{0+}^{\alpha} f(t)\right\}(u) &= S\left\{\frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t)\right\}(u) \\ &= u^{-n}\left[u^{n-\alpha} F(u) - \sum_{k=0}^{n-1} u^k \frac{d^k}{dt^k}\left(I_{0+}^{n-\alpha} f(t)\right)\Big|_{t=0}\right] \\ &= u^{-\alpha} F(u) - \sum_{k=0}^{n-1} u^{k-n} \frac{d^k}{D}^{-n+\alpha+k}_{0+} f(t)\Big|_{t=0} \\ &= u^{-\alpha} F(u) - \sum_{k=0}^{n-1} u^{-k-1} D_{0+}^{\alpha-k-1} f(t)\Big|_{t=0}. \end{aligned} \tag{2.5}$$

□

Theorem 7. *Let $f \in C^n(\mathbb{R}^+)$, $f^{(n)} \in L_1(0,b)$ and f satisfy (1.1). Then*

$$S\{{}^{C}D_{0+}^{\alpha} f(t)\}(u) = u^{-\alpha}\left[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right]. \tag{2.6}$$

Proof. The proof can be easily done by using theorem 3 and the fact that ${}^{C}D_{0+}^{\alpha} f(t) = I_{0+}^{n-\alpha} f^{(n)}(t)$. □

Before we extend Theorem 7 to give the Sumudu transform of the partial Caputo fractional derivatives of a function of two variables $f(x,t)$, we present the generalization of the Sumudu transform of the partial derivatives given in [2].

Theorem 8. *Let $\dfrac{\partial^{i+j} f(x,t)}{\partial t^j \partial x^i}$, $i = 0,1,..,n$, $j = 0,1,...,m$ be of exponential order; i.e $\left|\dfrac{\partial^{i+j} f(x,t)}{\partial t^j \partial x^i}\right| < M e^{\frac{x}{\tau_1} + \frac{t}{\tau_2}}$, for some $M > 0$, $\tau_1 > 0$, $\tau_2 > 0$. Then*

$$S_t S_x\{\frac{\partial^n f(x,t)}{\partial x^n}\} = u^{-n}\left[S_t S_x\{f(x,t)\} - \sum_{i=1}^{n-1} u^i S_t S_x\{\frac{\partial^i f}{\partial x^i}(0,t)\}\right], \tag{2.7}$$

$$S_tS_x\{\frac{\partial^m f(x,t)}{\partial t^m}\} = u^{-m}\left[S_tS_x\{f(x,t)\} - \sum_{j=1}^{m-1} v^j S_tS_x\{\frac{\partial^j f}{\partial t^j}(x,0)\}\right], \tag{2.8}$$

$$\begin{aligned} S_tS_x\{\frac{\partial^{m+n} f(x,t)}{\partial t^m \partial x^n}\} &= v^{-m}u^{-n}\Bigg[S_tS_x\{f(x,t)\} \\ &- \sum_{i=1}^{n-1} u^i S_tS_x\{\frac{\partial^i f(0,t)}{\partial x^i}\} - \sum_{j=1}^{m-1} v^j S_tS_x\{\frac{\partial^j f(x,0)}{\partial t^j}\} \\ &+ \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} v^j u^i \frac{\partial^{i+j} f}{\partial t^j \partial x^i}(0,0)\Bigg] \end{aligned} \tag{2.9}$$

In the following theorem we present the Sumudu transform of the partial fractional integrals.

Theorem 9. *If $f(x,t) \in L_1\Big((0,a)\times(0,b)\Big), a,b>0$ and $|f(x,t)| < Me^{\frac{x}{\tau_1}+\frac{t}{\tau_2}}$, for some $M>0,\ \tau_1>0,\ \tau_2>0$, then*
$S_tS_x\{\ {}_{0+}I_x^\alpha f(x,t)\},\ S_tS_x\{\ {}_{0+}I_t^\beta f(x,t)\}$ and
$S_tS_x\{\ {}_{0+}I_t^\beta\ {}_{0+}I_x^\alpha f(x,t)\}$ exist and

$$S_tS_x\{\ {}_{0+}I_x^\alpha f(x,t)\} = u^\alpha S_tS_x\{f(x,t)\}, \tag{2.10}$$

$$S_tS_x\{\ {}_{0+}I_t^\beta f(x,t)\} = v^\beta S_tS_x\{f(x,t)\}, \tag{2.11}$$

$$S_tS_x\{\ {}_{0+}I_t^\beta\ {}_{0+}I_x^\alpha f(x,t)\} = u^\alpha v^\beta S_tS_x\{f(x,t)\}. \tag{2.12}$$

Proof. The proof of existence is analogous of the proof of lemma 4. To derive the formulas (2.10), (2.11) and (2.12) we use the Sumudu transform of the convolution with respect to x for the first, Sumudu transform of the convolution with respect to t for the second and the Sumudu transform for the double convolution for the third. □

Finally the Sumudu transform of the partial Caputo fractional derivatives are given in the following theorem

Theorem 10. *Let $f(x,t)$ be a function with exponential order such that it has continuous partial derivatives on $\mathbb{R}^+\times\mathbb{R}^+$ and these derivatives are in $L_1\Big((0,a)\times(0,b)\Big)$. Then*

$$S_tS_x\left\{\ {}_{0+}^{C}D_x^\alpha f(x,t)\right\} = u^{-\alpha}\left[F(u,v) - \sum_{i=0}^{n-1} u^i F_i(0,v)\right], \tag{2.13}$$

$$S_tS_x\left\{\ {}_{0+}^{C}D_t^\beta f(x,t)\right\} = v^{-\beta}\left[F(u,v) - \sum_{j=0}^{m-1} v^j F_j(u,0)\right], \tag{2.14}$$

$$\begin{aligned} S_tS_x\left\{\ {}_{0+}^{C}D_t^\beta\ {}_{0+}^{C}D_x^\alpha f(x,t)\right\} &= u^{-\alpha}v^{-\beta}\Bigg[F(u,v) - \sum_{i=0}^{n-1} u^i F_i(0,v) \\ &- \sum_{j=0}^{m-1} u^j F_j(u,0) + \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} u^i v^j \frac{\partial^{i+j} f}{\partial t^j \partial x^i}(0,0)\Bigg] \end{aligned} \tag{2.15}$$

where $F(u,v)$ is the Sumudu transform of $f(x,t)$,
$F_i(0,v) = S_tS_x\{\frac{\partial^i f}{\partial x^i}(0,t)\}$ *and* $F_j(u,0) = S_tS_x\{\frac{\partial^j f}{\partial t^j}(x,0)\}$.

Proof. The proof can be done by using theorems 9 and 10. □

3. The Sumudu Transform Method for Solving Differential Equations with Caputo Fractional Derivatives

Before we discuss the solutions of fractional differential equations, we give the Sumudu transform of the Mittag-Leffler and the Wright functions which are used to describe the solutions of such equations. The Mittag-Leffler function is defined [10, 11] by :

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)} \; t,\beta \in \mathbb{C}; Re(\alpha) > 0. \tag{3.1}$$

The Sumudu transform of the function $t^{\beta-1}E_{\alpha,\beta}(\lambda t)$ can be easily obtained as

$$S\{t^{\beta-1}E_{\alpha,\beta}(\lambda t)\}(u) = \frac{u^{\beta-1}}{1-\lambda u^{\alpha}} \tag{3.2}$$

The Wright function is defined [12] by

$$W(t;\alpha,\beta) = \sum_{k=0}^{\infty} \frac{t^k}{k!\Gamma(\alpha k+\beta)} \; t,\beta \in \mathbb{C}; Re(\alpha) > 0. \tag{3.3}$$

It is not difficult to check that

$$S\{t^{k-\alpha/2}W(-\frac{|x|}{\lambda}t^{-\alpha/2}; -\alpha/2, k+1-\alpha/2)\}(u) = u^{k-\alpha/2}e^{-\frac{|x|}{\lambda}u^{-\alpha/2}}. \tag{3.4}$$

In this section we consider the following equation:

$$^{C}D^{\alpha}_{0+}y(t) - \lambda y(t) = f(t), \; n-1 < \alpha \leq n$$
$$y^{(k)}(0) = d_k, k = 0,1,...,n-1, \tag{3.5}$$

where d_k and λ are constant. Applying the Sumudu transform to equation (3.5), we get

$$Y(u) = \sum_{k=0}^{n-1} \frac{d_k u^k}{1-\lambda u^{\alpha}} + u\frac{u^{\alpha-1}F(u)}{1-\lambda u^{\alpha}}, \tag{3.6}$$

where $Y(u)$ and $F(u)$ are the Sumudu transform of $y(t)$ and $f(t)$ respectively. By using the convolution theorem 2 and formula (3.2), one gets the solution

$$y(t) = \sum_{k=0}^{n-1} d_k t^k E_{\alpha,k+1}(\lambda t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})f(\tau))d\tau. \tag{3.7}$$

Now, we consider the following 2-dimensional diffusion-wave equation and solve it by using both Sumudu and Fourier transforms

$$^{C}_{0+}D^{\alpha}_{t}z(x,t) = \lambda^2\frac{\partial^2 z(x,t)}{\partial x^2} \; x \in \mathbb{R}, \; t > 0, \; 0 < \alpha < 2, \tag{3.8}$$

where

$$\frac{\partial^k z}{\partial t^k}(x,0) = g_k(x) \; k = 0,1,...,n-1 \; n-1 < \alpha \leq n. \tag{3.9}$$

Applying the Sumudu transform with respect to and using (2.6), one obtains

$$u^{-\alpha}Z(x,u)-\lambda^2\frac{\partial^2 Z(x,u)}{\partial x^2}=\sum_{k=0}^{n-1}u^{k-\alpha}g_k(x). \tag{3.10}$$

Applying the Fourier transform with respect to x, one gets

$$\mathfrak{F}_x\{Z\}(\sigma,u)=\frac{u^{\alpha/2}}{2\lambda}\sum_{k=0}^{n-1}u^{k-\alpha}\frac{\frac{2}{\lambda u^{\alpha/2}}}{\frac{1}{\lambda^2u^\alpha}+|\sigma|^2}\mathfrak{F}_x\{g_k(x)\}$$

$$=\frac{u^{\alpha/2}}{2\lambda}\sum_{k=0}^{n-1}u^{k-\alpha}\mathfrak{F}_x\{e^{-\frac{|x|}{\lambda u^{\alpha/2}}}\}\mathfrak{F}_x\{g_k(x)\}. \tag{3.11}$$

Thus, by using the known convolution theorem for Fourier transform we have

$$Z(x,u)=\frac{1}{2\lambda}\sum_{k=0}^{n-1}u^{k-\alpha/2}e^{-\frac{|x|}{\lambda u^{\alpha/2}}}*_x g_k(x). \tag{3.12}$$

Finally by using (3.4), we obtain

$$z(x,t)=\frac{1}{2\lambda}\sum_{k=0}^{n-1}t^{k-\alpha/2}W(-\frac{|x|}{\lambda t^{\alpha/2}};-\alpha/2,k+1-\alpha/2)*_x g_k(x)$$

$$=\frac{1}{2\lambda}\sum_{k=0}^{n-1}t^{k-\alpha/2}\int_{-\infty}^{\infty}W(-\frac{|x-s|}{\lambda t^{\alpha/2}};-\alpha/2,k+1-\alpha/2)g_k(s)ds. \tag{3.13}$$

For the following initial-boundary value problem, we utilize the double Sumudu transform. We consider the equation

$${}_{0+}^{C}D_x^{\alpha}z(x,t)=\frac{\pi^\alpha}{\lambda}\,{}_{0+}^{C}D_t^{\beta}z(x,t)\ x>0,\ t>0,\ 1<\alpha\le 2,\ 0<\beta\le 1, \tag{3.14}$$

with the following conditions

$$z(0,t)=c\ ,\ \frac{\partial z(0,t)}{\partial x}=\pi E_{\beta,1}(-\lambda t^\beta)\ t>0 \tag{3.15}$$
$$z(x,0)=\pi x E_{\alpha,2}(-\pi^\alpha x^\alpha)+c\ x>0\ ,$$

where c is a constant. The Sumudu transform of the conditions in (3.15) are as follows

$$\begin{gathered}S_tS_x\{z(0,t)\}=S_{\{}c\}=c,\\ S_tS_x\{\tfrac{\partial z(0,t)}{\partial x}\}=S_tS_x\{\pi E_{\beta,1}(-\lambda t^\beta)\}=\tfrac{\pi}{1+\lambda v^\beta}\\ S_tS_x\{z(x,0)\}=S_tS_x\{\pi x E_{\alpha,2}(-\pi^\alpha x^\alpha)\}=\tfrac{\pi u}{1+\pi^\alpha u^\alpha}+c.\end{gathered} \tag{3.16}$$

If we apply the double Sumudu transform to (3.15), then by using the formulas (2.13), (2.14) and (3.16) we get

$$u^{-\alpha}\left[Z(u,v)-c-\frac{\pi u}{1+\lambda v^\beta}\right]=\frac{\pi^\alpha v^{-\beta}}{\lambda}\left[Z(u,v)-\frac{\pi u}{1+\pi^\alpha u^\alpha}-c\right]. \tag{3.17}$$

Thus,

$$Z(u,v)=c+\frac{\pi u}{1+\pi^\alpha u^\alpha}\frac{1}{1+\lambda v^\beta} \tag{3.18}$$

Applying the inverse Sumudu transform, we get the solution

$$z(x,t)=c+\pi x E_{\alpha,2}(-\pi^\alpha x^\alpha)E_{\beta,1}(-\lambda y^\beta). \tag{3.19}$$

It is clearly observed that

$$\lim_{(\alpha,\beta)\to(2,1)} z(x,t) = c + \sin(\pi x)e^{-\lambda y}. \tag{3.20}$$

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Locally Convex Valued Rectangular Metric Spaces and The Kannan's Fixed Point Theorem

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Abstract

Rectangular TVS-cone metric spaces are introduced and Kannan's fixed point theorem is proved in these spaces. Two approaches are followed for the proof. At first we prove the theorem by a direct method using the structure of the space itself. Secondly, we use the nonlinear scalarization used recently by Wei-Shih Du in [A note on cone metric fixed point theory and its equivalence, Nonlinear Analysis,72(5),2259-2261 (2010).] to prove the equivalence of the Banach contraction principle in cone metric spaces and usual metric spaces. The proof is done without any normality assumption on the cone of the locally convex topological vector space, and hence generalizing several previously obtained results.

Keywords: TVS-cone metric space, rectangular TVS-cone metric space, Kannan's fixed point theorem.

1 Introduction and Preliminaries

Many authors attempted to generalize the notion of the metric space. In 2007, Huang and Zhang [9] announced the notion of cone metric spaces (CMS) by using the same idea, namely, by replacing real numbers with an ordering real Banach space. In that paper, they also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces: Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, has a unique fixed point. Lately, many results on fixed point theorems have been extended to cone metric spaces (see e.g.[9],[11],[5],[13],[3],[10],[1],[2],

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[12]). For Kannan's fixed point theorem in rectangular metric spaces (R-MS) we refer to [17] and for the contraction principle and Kannan's fixed point theorem in rectangular cone metric space (R-CMS) see [15] and [16], respectively.

Recently, Du [8] gave the definition of generalized cone metric space, namely topological vector space-cone metric space (TVS-CMS), and proved some fixed point theorems on that class. The author showed also that Banach contraction principles in usual metric spaces and in TVS-CMS are equivalent.

In this manuscript, we first introduce the notion of rectangular TVS-cone metric spaces (R-TVS-CMS) and then prove Kannan's fixed point theorem in this class of spaces. The obtained result generalizes those in [16] and [17] and hence the classical Kannan's fixed point theorem. Two proofs are presented and the proofs are done without any normality assumption.

Throughout this paper, (E, S) stands for real Hausdorff locally convex topological vector space (t.v.s.) with S its generating system of seminorms. A non-empty subset P of E is called cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. The cone P will be assumed to be closed and has nonempty interior as well. For a given cone P, one can define a partial ordering (denoted by $\leq$ or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$ while $x << y$ will show $y - x \in intP$, where $intP$ denotes the interior of P. Continuity of the algebric operations in a topological vector space and the properties of the cone imply the relations:

$$intP + intP \subseteq intP \text{ and } \lambda intP \subseteq intP \ (\lambda > 0).$$

We appeal to these relations in the following.

Definition 1.1. [4] A cone P of a topological vector space (X, τ) is said to be normal whenever τ has a base of zero consisting of $P-$ full sets. Where a subset of A of an order vector space via a cone P is said to be $P-$full if for each $x, y \in A$ we have $\{a \in E : x \leq a \leq y\} \subset A$.

Theorem 1.2. *[4] (a) A cone P of a topological vector space (X, τ) is normal if and only if whenever $\{x_\alpha\}$ and $\{y_\alpha\}$, $\alpha \in \Delta$ are two nets in X with $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha \in \Delta$ and $y_\alpha \to 0$, then $x_\alpha \to 0$.*

(b) The cone of an ordered locally convex space (X, τ) is normal if and only if τ is generated by a family of monotone $\tau-$ continuous seminorms. Where a seminorm q on X is called monotone if $q(x) \leq q(y)$ for all $x, y \in X$ with $0 \leq x \leq y$.

In particular, if P is a cone of a real Banach space E, then it is called *normal* if there is a number $K \geq 1$ such that for all $x, y \in E$: $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least positive integer K, satisfying this inequality, is called the normal constant of P. Also, P is said to be *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$

such that $\lim_{n\to\infty} \|x_n - x\| = 0$. For more details about cones in locally convex topological vector spaces we may refer the reader to [4].

Definition 1.3. (See [6], [7], [8]) For $e \in intP$, the nonlinear scalarization function $\xi_e : E \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{t \in \mathbb{R} : y \in te - P\}, \text{ for all } y \in E.$$

Lemma 1.4. *(See [6], [7], [8]) For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:*

(i) $\xi_e(y) \leq t \Leftrightarrow y \in te - P$,

(ii) $\xi_e(y) > t \Leftrightarrow y \notin te - P$,

(iii) $\xi_e(y) \geq t \Leftrightarrow y \notin te - intP$,

(iv) $\xi_e(y) < t \Leftrightarrow y \in te - intP$,

(v) $\xi_e(y)$ *is positively homogeneous and continuous on* E,

(vi) *if* $y_1 \in y_2 + P$, *then* $\xi_e(y_2) \leq \xi_e(y_1)$,

(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$, *for all* $y_1, y_2 \in E$.

Definition 1.5. Let X be a non-empty set and E as usual a Hausdorff locally convex topological space. Suppose a vector-valued function $p : X \times X \to E$ satisfies:

$(M1)$ $0 \leq p(x, y)$ for all $x, y \in X$,

$(M2)$ $p(x, y) = 0$ if and only if $x = y$,

$(M3)$ $p(x, y) = p(y, x)$ for all $x, y \in X$

$(M4)$ $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$.

Then, p is called TVS-cone metric on X, and the pair (X, p) is called a TVS-cone metric space (in short, TVS-CMS).

Note that in [9], the authors considered E as a real Banach space in the definition of TVS-CMS. Thus, a cone metric space (in short, CMS) in the sense of Huang and Zhang [9] is a special case of TVS-CMS.

Lemma 1.6. *(See [8]) Let (X, p) be a TVS-CMS. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p = \xi_e \circ p$ is a metric.*

Remark 1.7. *Since a cone metric space (X,p) in the sense of Huang and Zhang [9], is a special case of TVS-CMS, then $d_p : X \times X \to [0,\infty)$ defined by $d_p = \xi_e \circ p$ is also a metric.*

Definition 1.8. (See [8]) Let (X,p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X.

(*i*) $\{x_n\}_{n=1}^{\infty}$ TVS-cone converges to $x \in X$ whenever for every $0 << c \in E$, there is a natural number M such that $p(x_n, x) << c$ for all $n \geq M$ and denoted by $cone - \lim_{n\to\infty} x_n = x$ (or $x_n \overset{cone}{\to} x$ as $n \to \infty$),

(*ii*) $\{x_n\}_{n=1}^{\infty}$ TVS-cone Cauchy sequence in (X,p) whenever for every $0 << c \in E$, there is a natural number M such that $p(x_n, x_m) << c$ for all $n, m \geq M$,

(*iii*) (X,p) is TVS-cone complete if every sequence TVS-cone Cauchy sequence in X is a TVS-cone convergent.

Lemma 1.9. *(See [8]) Let (X,p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Set $d_p = \xi_e \circ p$. Then the following statements hold:*

(*i*) *If $\{x_n\}_{n=1}^{\infty}$ converges to x in TVS-CMS (X,p), then $d_p(x_n, x) \to 0$ as $n \to \infty$,*

(*ii*) *If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in TVS-CMS (X,p), then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence (in usual sense) in (X, d_p),*

(*iii*) *If (X,p) is a complete TVS-CMS, then (X, d_p) is a complete metric space.*

Proposition 1.10. *(See [8]) Let (X,p) be a complete TVS-CMS and $T : X \to X$ satisfy the contractive condition*

$$p(Tx, Ty) \leq kp(x, y) \tag{1}$$

for all $x, y \in X$ and $0 \leq k < 1$. Then, T has a unique fixed point in X. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to fixed point.

Definition 1.11. Let X be a nonempty set. A vector-valued function $p : X \times X \to E$ is said to be a rectangular $TVS-$ cone metric, if the following conditions hold:

$(RC1)$ $0 \leq p(x,y)$ for all $x, y \in X$ and $p(x,y) = 0$ if and only if $x = y$,

$(RC2)$ $p(x,y) = p(y,x)$ for all $x, y \in X$,

$(RC3)$ $p(x,z) \leq p(x,y) + p(y,w) + p(w,z)$ for all $x, y \in X$ and for all distinct points $z, w \in X$ each of them different from x and y.

The pair (X, p) is then called a rectangular TVS-cone metric space (R-TVS-CMS). When E is Banach space (X, p) is called rectangular cone metric space (R-CMS). When $E = \mathbb{R}$ and $P = [0, \infty)$, (X, p) is called rectangular metric space (R-MS).

Every TVS-CMS is R-TVS-CMS. However, the converse need not be true.

Example 1.12. *([15], see also [14]) Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$. Define $d : X \times X \to E$ as follows:*

$$d(1,2) = d(2,1) = (3,6), \quad d(2,3) = d(3,2) = d(1,3) = d(3,1) = (1,2),$$

$$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = (2,4).$$

Then (X, d) is a R-CMS which is not a CMS, because

$$(3,6) = d(1,2) > d(1,3) + d(3,2) = (1,2) + (1,2) = (2,4).$$

Definition 1.13. Let (X, p) be a rectangular TVS-cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X.

(i) $\{x_n\}$ is said to be a Cauchy sequence if for any $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $n \geqslant n_0$, one has $p(x_n, x_{n+m}) \ll c$.

(ii)$\{x_n\}$ is said to converge to x if for any $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, one has $p(x_n, x) \ll c$.

(iii) (X, p) is called **complete** if every Cauchy sequence in X is convergent in X.

Let $T : X \to X$ be a mapping where X is a R-TVS-CMS. For each $x \in X$, let

$$\mathbf{O}(x) = \{x, Tx, T^2x, T^3x, \dots\}.$$

Definition 1.14. A cone metric space X is said to be T-orbitally complete if every Cauchy sequence which is contained in $\mathbf{O}(x)$ for some $x \in X$ converges in X.

2 Kannan's Fixed Point Theorem in R-TVS-CMS

In order to realize the difference between TVS-CMS and R-TVS-CMS, we first prove Kannan's fixed point theorem in TVS-CMS.

Theorem 2.1. *Let (X, d) be a TVS-CMS and the mapping $T : X \to X$ satisfy the contractive condition*

$$d(Tx, Ty) \leqslant \beta[d(x, Tx) + d(y, Ty)] \tag{2}$$

holds for all $x, y \in X$ *where* $0 < \beta < \frac{1}{2}$. *If* X *is* T*-orbitally complete then* T *has a unique fixed point in* X.

Proof Let $x \in X$.

$$\begin{aligned} d(Tx, T^2x) &\leqslant \beta[d(x, Tx) + d(Tx, T^2x)] \\ i.e., \quad d(Tx, T^2x) &\leqslant \tfrac{\beta}{1-\beta} d(x, Tx) \end{aligned}$$

Again,

$$\begin{aligned} d(T^2x, T^3x) &\leqslant \beta[d(Tx, T^2x) + d(T^2x, T^3x)] \\ i.e., \quad d(T^2x, T^3x) &\leqslant \tfrac{\beta}{1-\beta} d(Tx, T^2x) \leqslant \left(\tfrac{\beta}{1-\beta}\right)^2 d(x, Tx) \end{aligned}$$

Similarly,

$$d(T^3x, T^4x) \leqslant \left(\frac{\beta}{1-\beta}\right)^3 d(x, Tx)$$

Thus in general, if n is a positive integer, then

$$d(T^n x, T^{n+1}x) \leqslant r^n d(x, Tx) \tag{3}$$

where $r = \dfrac{\beta}{1-\beta}$. Since $0 < \beta < \dfrac{1}{2}$, clearly $0 < r < 1$.

Now, our aim is to show that $\{T^n x\}$ is a Cauchy sequence. Assume $m \in \mathbb{N}$ and $n > m$, then we have

$$\begin{aligned} d(T^n x, T^m x) &\leqslant d(T^n x, T^{n-1}x) + d(T^{n-1}x, T^{n-2}x) + \ldots + d(T^{m+1}x, T^m x) \\ &\leqslant (r^{n-1} + r^{n-2} + \ldots + r^m) d(x, Tx) \\ &\leqslant \tfrac{r^m}{1-r} d(x, Tx) \end{aligned}$$

Let $0 \ll c$ be given. Find $\delta > 0$ and $q \in S$ such that $q(b) < \delta$ implies $b \ll c$. Now, since

$$\frac{r^m}{1-r} d(x, Tx) \to 0 \ \ as \ \ m \to \infty$$

then find n_0 such that :

$$q\left(\frac{r^m}{1-r} d(x, Tx)\right) < \delta \quad \forall m \geqslant n_0$$

Hence, $\dfrac{r^m}{1-r} d(x, Tx) \ll c$, $\forall m \geqslant n_0$.

Thus, $d(T^n x, T^m x) \ll c$ for $n > m \geq n_0$. Therefore, $\{T^n x\}$ is a Cauchy sequence in (X, d). Since (X, d) is T-orbitally complete, there exists $x^* \in X$ such that $T^n x \to x^*$.
Choose a natural number n_1 such that $d(T^{n-1}x, T^n x) \ll \frac{c}{2}$ and $d(T^n x, x^*) \ll \frac{c}{2}$, for all $n \geqslant n_1$. Hence, for $n > n_1$ we have

$$\begin{aligned} d(Tx^*, x^*) &\leqslant d(TT^{n-1}x, Tx^*) + d(T^n x, x^*) \\ &\leqslant \beta[d(T^{n-1}x, T^n x) + d(x^*, Tx^*)] + d(T^n x, x^*) \\ &= \beta d(T^{n-1}x, T^n x) + \beta d(x^*, Tx^*) + d(T^n x, x^*) \\ &\leqslant \tfrac{c}{2} + \beta d(x^*, Tx^*) + \tfrac{c}{2} \end{aligned}$$

So,

$$(1-\beta)d(Tx^*, x^*) \ll c$$

Hence,

$$(1-\beta)d(Tx^*, x^*) \ll \frac{c}{m} \quad \forall m \geqslant 1$$

Hence, $\frac{c}{m} - (1-\beta)d(Tx^*, x^*) \in P$, for all $m \geqslant 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$ and P is closed;

$$-(1-\beta)d(Tx^*, x^*) \in P \quad \textit{and} \quad (1-\beta)d(Tx^*, x^*) \in P$$

from the cone properties, $(1-\beta)d(Tx^*, x^*) = 0$. Since $(1-\beta)$ never be equal to zero, then $d(Tx^*, x^*) = 0$. Thus $Tx^* = x^*$.

Now, if y^* is another fixed point of T then $Tx^* = x^*$ and $Ty^* = y^*$. Then, we have

$$0 \leqslant d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant \beta d(x^*, Tx^*) + \beta d(y^*, Ty^*) = 0$$

Hence, $d(x^*, y^*) = 0$ and so $x^* = y^*$. Therefore, the fixed point of T is unique.

Now, we prove Kannan's fixed point theorem in R-TVS-CMS.

Theorem 2.2. *Let $T : X \to X$ be a mapping where (X, d) is a T-orbitally complete R-TVS-CMS such that*

$$d(Tx, Ty) \leqslant \beta[d(x, Tx) + d(y, Ty)] \tag{4}$$

holds for all $x, y \in X$ and $0 < \beta < \frac{1}{2}$. Then, T has a unique fixed point in X.

Proof. As in the proof of Theorem 2.1, for a fixed $x \in X$, we have for all $n \in \mathbb{N}$

$$d(T^n x, T^{n+1}x) \leqslant r^n d(x, Tx) \tag{5}$$

where $r = \dfrac{\beta}{1-\beta}$. Since $0 < \beta < \dfrac{1}{2}$, clearly $0 < r < 1$.
Since we are not able to use the triangle inequality, we divide the proof into two cases so that we can make use of the rectangle inequality.

Case I: First assume that $T^m x \neq T^n x$ for $m, n \in N, m \neq n$. Then, for $n \in N$. Clearly,

$$\begin{aligned}
d(T^n x, T^{n+1}x) &\leqslant r^n d(x,Tx) < \left(\tfrac{r^n}{1-r}\right) d(x,Tx) \\
\texttt{and}\ \ d(T^n x, T^{n+2}x) &\leqslant \beta[d(T^{n-1}x, T^n x) + d(T^{n+1}x, T^{n+2}x)] \\
&\leqslant \beta\left[\left(\tfrac{\beta}{1-\beta}\right)^{n-1} d(x,Tx) + \left(\tfrac{\beta}{1-\beta}\right)^{n+1} d(x,Tx)\right] \qquad \textbf{(by 5)} \\
&\leqslant \left(\tfrac{\beta}{1-\beta}\right)^{n} d(x,Tx) + \left(\tfrac{\beta}{1-\beta}\right)^{n+1} d(x,Tx) \\
&\leqslant \left(\tfrac{r^n}{1-r}\right) d(x,Tx)
\end{aligned}$$

since $0 < \beta < \dfrac{1}{2}$, $\beta \leqslant \dfrac{\beta}{1-\beta}$.

Now if $m > 2$ is odd then writing $m = 2k+1$, $k \geqslant 1$ and using the fact that $T^p x \neq T^r x$ for $p, r \in N$, $p \neq r$ we can easily show that by the rectangular inequality

$$\begin{aligned}
d(T^n x, T^{n+m}x) &\leqslant d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \ldots + d(T^{n+2k}x, T^{n+2k+1}x) \\
&\leqslant r^n d(x,Tx) + r^{n+1} d(x,Tx) + \ldots + r^{n+2k} d(x,Tx) \qquad \textbf{(by 5)} \\
&\leqslant \left(\tfrac{r^n}{1-r}\right) d(x,Tx)
\end{aligned} \tag{6}$$

Again if $m > 2$ is even then writing $m = 2k$, $k \geqslant 2$ and using the same arguments as before we can get by the rectangular inequality

$$\begin{aligned}
d(T^n x, T^{n+m}x) &\leqslant d(T^n x, T^{n+2}x) + d(T^{n+2}x, T^{n+3}x) + \ldots + d(T^{n+2k-1}x, T^{n+2k}x) \\
&\leqslant [r^n + r^{n+1} + r^{n+3} + \ldots + r^{n+2k-1}] d(x,Tx) \qquad \textbf{(by 5, 6)}
\end{aligned}$$

Thus combining all the cases we have

$$d(T^n x, T^{n+m}x) \leqslant \left(\frac{r^n}{1-r}\right) d(x,Tx) \tag{7}$$

for all $m, n \in N$. Since $0 < r < 1$, $r^n \to 0$ as $n \to \infty$ and so by following a similar argument as in the proof of Theorem 2.1, $\{T^n x\}$ is a Cauchy sequence. Since X is T-orbitally complete, $\{T^n x\}$ is convergent. Let u is defined as:

$$u = \lim_{n\to\infty} T^n x \tag{8}$$

We shall now show that $Tu = u$. Without any loss of generality we assume that $T^n x \neq u$ and $T^n x \neq Tu$ for any $n \in N$. Then by (4) and the rectangular inequality, we obtain

$$\begin{aligned} d(u, Tu) &\leqslant d(u, T^n x) + d(T^n x, T^{n+1} x) + d(T^{n+1} x, Tu) \\ &\leqslant d(u, T^n x) + d(T^n x, T^{n+1} x) + \beta[d(T^n x, T^{n+1} x) + d(u, Tu)] \\ i.e., \quad d(u, Tu) &\leqslant \tfrac{1}{1-\beta}[d(u, T^n x) + (1+\beta) d(T^n x, T^{n+1} x)] \end{aligned}$$

Since $T^n x \to u$ and $\{T^n x\}$ is Cauchy then we obtain $0 \leq d(u, Tu) \ll c$ for all $c \gg 0$. Then closeness of the cone P implies that $u = Tu$.

Case II: Let $T^m x = T^n x$ for some $m, n \in N$, $m \neq n$. Let $m > n$. Then $T^{m-n}(T^n x) = T^n x$ i.e.,$T^k y = y$ where $k = m - n, y = T^n x$. Now if $k > 1$

$$d(y, Ty) = d(T^k y, T^{k+1} y) \leqslant \left(\frac{\beta}{1-\beta}\right)^k d(y, Ty) \qquad \textbf{(by 5)}$$

Since $0 < \dfrac{\beta}{1-\beta} < 1$, $d(y, Ty) = 0$ i.e., $Ty = y$. That the fixed point of T is unique easily follows from (4). This completes the proof of the theorem. □

Theorem 2.2 above generalizes the results obtained in [16], where Kannan's fixed point theorem was proved in CMS and under the normality assumption. However, the proofs in this article are done without any normality type assumption.

3 The nonlinear scalarization and Kannan's fixed point theorem

In this section, we use the nonlinear scalarization function to obtain a simpler shorter proof for the Kannan's fixed point theorem in R-TVS-CMS.

Theorem 3.1. *Let (X, p) be a rectangular TVS-cone metric space. Then (X, d_p), where $d_p := \xi_e \circ p$, is a rectangular metric space (R-MS).*

Proof. By $RC1$, the definition of ξ_e and that $P \cap -P = \{0\}$ we have $d_p(x, y) \geq 0$ for all $x, y \in X$. By $RC2$, $d_p(x, y) = d_p(y, x)$ for all $x, y \in X$. If

$x = y$, then by $RC1$ $d_p(x, y) = \xi_e(0) = 0$. Conversely, if $d_p(x, y) = 0$, then by Lemma 1.4, $RC1$ and that $P \cap -P = \{0\}$, we conclude that $p(x, y) = 0$ and hence by $RC2$, $x = y$. Finally the rectangular inequality follows by Lemma 1.4 (vi), (vii) and $RC3$. □

Lemma 3.2. *Let (X, p) be a R-TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Set $d_p = \xi_e \circ p$. Then the following statements hold:*

(i) *$\{x_n\}_{n=1}^{\infty}$ converges to x in the R-TVS-CMS (X, p) if and only if $d_p(x_n, x) \to 0$ as $n \to \infty$,*

(ii) *$\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in the R-TVS-CMS (X, p) if and only if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the rectangular metric space (X, d_p),*

(iii) *(X, p) is a complete R-TVS-CMS if and only if (X, d_p) is a complete rectangular metric space.*

Proof. Applying Theorem 3.1, d_p is a rectangular metric on X. Regarding (i) First, assume $\{x_n\}$ converges to x in the R-TVS-CMS (X, p) and let $\epsilon > 0$ be given. Find n_0 such that $p(x_n, x) \ll \epsilon e$ for all $n > n_0$. Therefore, by Lemma 1.4 (iv), $d_p(x_n, x) = \xi_e \circ p(x_n, x) < \epsilon$, for all $n > n_0$. Conversely, we prove that if $x_n \to x$ in (X, d_p) then $x_n \to x$ in the R-TVS-CMS (X, p). To this end, let $c >> 0$ be given, then find $q \in S$ and $\delta > 0$ such that $q(b) < \delta$ implies that $b << c$. Since $\frac{e}{n} \to 0$ in (E, S) find $\epsilon = \frac{1}{n_0}$ such that $\epsilon q(e) = q(\epsilon e) < \delta$ and hence $\epsilon e << c$. Now, find n_0 such that $d_p(x_n, x) = \xi_e \circ p(x_n, x) < \epsilon$ for all $n \geq n_0$. Hence, by Lemma 1.4 (iv) $p(x_n, x) << \epsilon e << c$ for all $n \geq n_0$. The proof of (ii) is similar to the proof of (i). Finally, (iii) is immediate from (i) and (ii). □

Now the proof of Theorem 2.1 can be achieved by Lemma 3.2, Theorem 3.1 and by Kannan's fixed point theorem (see [17]) applied to the R-MS (X, d_p).

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Delay Integral inequalities of a coupled system on time scales with applications

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Abstract

By means of a new linear Gronwall-Bellman inequality on time scales and a elementary inequality, the bounds of a class of new power nonlinear two-dimensional inequalities of a delay on time scales are obtained. An application to the qualitative analysis of delay dynamic system is discussed.

Key words: Gronwall-Bellman inequalities; delay; time scales; Dynamic

2000 Mathematics Subject Classification: 34A40; 39A10

1 INTRODUCTION

To unify and extend the continuous and discrete analysis, the theory of time scales was introduced by Hilger [1] in his Ph.D. thesis in 1988. Since then, the theory has been growing up and applied to many fields of mathematics [2, 3, 4]. For all the basic definition, notation and theorems on time scales, we refer the readers to the excellent monographes [3, 4] and references given therein.

It is well known that Gronwall type integral inequalities and their discrete analogues play a dominant role in the study quantitative properties of solutions of differential, integral and difference equations. The literature on the such inequalities and their applications is vast; see [5-7] and the references given therein. The very recent results we refer [8-13]. In addition, as the theory of calculus on time scales develops, some Gronwall type integral inequalities on

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time scales have been established by many authors during the last few years. For example, we refer readers by the papers [14-20] and references therein. However, there are few results obtained by authors in two-dimensional delay integral inequalities systems on time scales. The only one result concerning the linear situation is due to E. Akin Bohner, M. Bohner and F. Akin [6]. In this paper, we investigate some nonlinear delay integral inequalities of the coupled system on time scales, which provide explicit bounds on unknown functions and can be used as handy and effective tools in the study differential equations and integral equations on time scales. We also have extended the main results of Li [18, 19] to two-dimensional case. An application of our results is given at the end of this paper.

2 Main results

In what follows, R denotes the set of real numbers, $R_+ = [0, +\infty)$, $C(M, S)$ denotes the class of all continuous functions defined on the set M with range in the set S; T is an arbitrary time scale, C_{rd} denotes the set of rd-continuous functions. Throughout this paper, we always assume that $t \in \mathrm{T}, \mathrm{T}_0 = [\mathrm{t}_0, +\infty) \cap \mathrm{T}$.

Before giving our main results, we cite some useful lemmas:

Lemma 2.1. **[21]** *Let $c \geq 0, m \geq n \geq 0$ and $m \neq 0$, then*

$$c^{\frac{n}{m}} \leq \frac{n}{m} K^{\frac{n-m}{m}} c + \frac{m-n}{m} K^{\frac{n}{m}}$$

for any $K > 0$.

Lemma 2.2 (Generalized Gronwall's inequality [6], Corollary 3.11) *Suppose that $u, a \in C_{rd}$ and $u \geq 0$. Let $k(t, s)$ be defined as in Ref. [3] (Theorem 1.117) such that $k(\sigma(t), t) \geq 0$ and $k^{\Delta}(t, s) \geq 0$ for $s, t \in \mathrm{T}$ with $s \leq t$, a is nondecreasing. Then*

$$u(t) \leq a(t) + \int_{t_0}^{t} k(t, s)u(s)\Delta s \quad for\ all\ t \in \mathrm{T}_0$$

implies

$$u(t) \leq a(t)e_{\bar{a}}(t, t_0) \quad for\ all\ t \in \mathrm{T}_0,$$

where

$$\bar{a}(t) = k(\sigma(t), t) + \int_{t_0}^{t} k^{\Delta}(t, s)\Delta s.$$

Lemma 2.3 *Suppose that $u, a \in C_{rd}$ and $u \geq 0$. Let $f(t, s), h(t, s)$ and $k(t, s)$ be defined as in Ref. [3] (Theorem 1.117) such that $f(\sigma(t), t), k(\sigma(t), t) \geq 0$ and $f^{\Delta}(t, s), k^{\Delta}(t, s) \geq 0$ for $s, t \in \mathrm{T}$ with $s \leq t$, a is nondecreasing. If*

$$u(t) \leq a(t) + \int_{t_0}^{t} f(t, s)u(s)\Delta s + \int_{t_0}^{t} h(t, s)\left(\int_{t_0}^{s} k(s, \xi)u(\xi)\Delta\xi\right)\Delta s,\ t \in \mathrm{T}_0 \tag{2.1}$$

then we have

$$u(t) \leq e_F(t, t_0)\left(a(t) + \int_{t_0}^{t} h(t, s)b(s)e_c(s, t_0)\Delta s\right),\ t \in \mathrm{T}_0 \tag{2.2}$$

where

$$F(t) = f(\sigma(t), t) + \int_{t_0}^{t} f^{\Delta}(t, s)\Delta s, \tag{2.3}$$

$$b(t) = \int_{t_0}^{t} k(t, s)e_F(s, t_0)a(s)\Delta s \tag{2.4}$$

and

$$c(t) = k(\sigma(t), t)e_F(t, t_0)\int_{t_0}^{t} h(t, \xi)\Delta\xi + \int_{t_0}^{t} k^{\Delta}(t, s)e_F(s, t_0)\left(\int_{t_0}^{s} h(s, \xi)\Delta\xi\right)\Delta s. \tag{2.5}$$

Proof. Taking

$$a_1(t) = a(t) + \int_{t_0}^{t} h(t, s)\left(\int_{t_0}^{s} k(s, \xi)u(\xi)\Delta\xi\right)\Delta s,\ t \in \mathrm{T}_0,$$

then (2.1) can be written as

$$u(t) \le a_1(t) + \int_{t_0}^{t} f(t, s)u(s)\Delta s.$$

Obviously, $a_1 \in C_{rd}(T_0, R_+)$ and is nondecreasing. Using Lemma 2.2, we get

$$u(t) \le a_1(t)e_F(t, t_0) = e_F(t, t_0)\left[a(t) + \int_{t_0}^{t} h(t, s)\left(\int_{t_0}^{s} k(s, \xi)u(\xi)\Delta\xi\right)\Delta s\right], \tag{2.6}$$

where

$$F(t) = f(\sigma(t), t) + \int_{t_0}^{t} f^{\Delta}(t, s)\Delta s.$$

Define

$$v(t) = \int_{t_0}^{t} k(t, s)u(s)\Delta s, \tag{2.7}$$

then

$$u(t) \le e_F(t, t_0)\left[a(t) + \int_{t_0}^{t} h(t, s)v(s)\Delta s\right].$$

From (2.7) and the last inequality we have

$$\begin{aligned} v(t) &\le \int_{t_0}^{t} k(t, s)e_F(s, t_0)\left(a(s) + \int_{t_0}^{s} h(s, \xi)v(\xi)\Delta\xi\right)\Delta s \\ &= \int_{t_0}^{t} k(t, s)e_F(s, t_0)a(s)\Delta s + \int_{t_0}^{t} k(t, s)e_F(s, t_0)\left(\int_{t_0}^{s} h(s, \xi)v(\xi)\Delta\xi\right)\Delta s \\ &\le \int_{t_0}^{t} k(t, s)e_F(s, t_0)a(s)\Delta s + \int_{t_0}^{t} k(t, s)e_F(s, t_0)\left(\int_{t_0}^{s} h(s, \xi)\Delta\xi\right)v(s)\Delta s \end{aligned}$$

because $v(t)$ is nondecreasing.

Using Lemma 2.2 again into the last inequality we get

$$v(t) \le b(t)e_c(t, t_0), \tag{2.8}$$

where $b(t)$ and $c(t)$ are defined as in (2.4) and (2.5), respectively.

Using (2.6) , (2.7) and (2.8) , we obtain the desired estimate (2.2). □

Consider delay inequalities system on time scales

$$\begin{cases} u^p(t) \le f_1(t) + \displaystyle\int_{t_0}^{t} k_{11}(t, s)u^{r_{11}}(\tau(s))\Delta s + \int_{t_0}^{t} k_{12}(t, s)v^{r_{12}}(\tau(s))\Delta s, \\ v^q(t) \le f_2(t) + \displaystyle\int_{t_0}^{t} k_{21}(t, s)u^{r_{21}}(\tau(s))\Delta s + \int_{t_0}^{t} k_{22}(t, s)v^{r_{22}}(\tau(s))\Delta s, \end{cases} \quad t \in \mathrm{T}_0 \tag{2.9}$$

with the initial condition

$$\begin{cases} u(t)=\varphi_1(t), \quad v(t)=\varphi_2(t) \qquad \text{for } t\in[\alpha,t_0]\cap \mathrm{T}_0 \ \text{ with} \\ \\ \varphi_1(\tau(t))\le f_1^{1/p}(t),\ \varphi_2(\tau(t))\le f_2^{1/q}(t) \ \text{ for every } t\in \mathrm{T}_0 \ \text{ with } \tau(t)\le t_0 \end{cases} \tag{I}$$

where $\tau:\mathrm{T}_0\to\mathrm{T}, \tau(\mathrm{t})\le \mathrm{t}, -\infty<\alpha=\inf\{\tau(\mathrm{t}),\mathrm{t}\in\mathrm{T}_0\}\le \mathrm{t}_0$, and $\varphi_1,\varphi_2\in C_{rd}([\alpha,t_0]\cap \mathrm{T},\mathrm{R}_+)$, p,q and r_{ij} are constants with $p\ge r_{i1}>0, q\ge r_{i2}>0$ $(i=1,2)$.

Theorem 2.4. *Suppose that $u,v,f_i\in C_{rd}(\mathrm{T}_0,\mathrm{R}_+)$ with f_i being nondecreasing for $t\in \mathrm{T}_0$ $(\mathrm{i}=1,2)$. Let $k_{ij}(t,s)$ be defined as in Ref. [3] (Theorem 1.117) such that $k_{ij}(\sigma(t),t)\ge 0$ and $k_{ij}^{\Delta}(t,s)\ge 0$ for $s,t\in\mathrm{T}$ with $s\le t$ $(i,j=1,2)$, then for any $m>0$ the inequalities systems (2.9) with the initial condition* (I) *implies*

$$\begin{cases} u(t)\le e_{K_{11}}^{\frac{1}{p}}(t,t_0)\left(\tilde{A}_1(t)+\displaystyle\int_{t_0}^{t}\bar{k}_{12}(t,s)e_{K_{22}}(s,t_0)\tilde{B}(s)e_{\tilde{D}_1}(s,t_0)\Delta s\right)^{\frac{1}{p}}, \\ \\ v(t)\le e_{K_{22}}^{\frac{1}{q}}(t,t_0)\left(A_1(t)+\displaystyle\int_{t_0}^{t}\bar{k}_{21}(t,s)e_{K_{11}}(s,t_0)B(s)e_{D_1}(s,t_0)\Delta s\right)^{\frac{1}{q}}, \end{cases} \tag{2.10}$$

for $t\in\mathrm{T}_0$, where

$$\tilde{A}_1(t)=\Phi_1(t)+\int_{t_0}^{t}\bar{k}_{12}(t,s)e_{K_{22}}(s,t_0)\Phi_2(s)\Delta s, \tag{2.11}$$

$$A_1(t)=\Phi_2(t)+\int_{t_0}^{t}\bar{k}_{21}(t,s)e_{K_{11}}(s,t_0)\Phi_1(s)\Delta s, \tag{2.12}$$

$$\Phi_1(t)=f_1(t)+\frac{p-r_{11}}{p}m^{\frac{r_{11}}{p}}\int_{t_0}^{t}k_{11}(t,s)\Delta s+\frac{q-r_{12}}{q}m^{\frac{r_{12}}{q}}\int_{t_0}^{t}k_{12}(t,s)\Delta s, \tag{2.13}$$

$$\Phi_2(t)=f_2(t)+\frac{p-r_{21}}{p}m^{\frac{r_{21}}{p}}\int_{t_0}^{t}k_{21}(t,s)\Delta s+\frac{q-r_{22}}{q}m^{\frac{r_{22}}{q}}\int_{t_0}^{t}k_{22}(t,s)\Delta s, \tag{2.14}$$

$$\tilde{B}_1(t)=\int_{t_0}^{t}\bar{k}_{21}(t,s)e_{K_{11}}(s,t_0)\tilde{A}_1(s)\Delta s, \quad B_1(t)=\int_{t_0}^{t}\bar{k}_{12}(t,s)e_{K_{22}}(s,t_0)A_1(s)\Delta s \tag{2.15}$$

$$\begin{aligned}\tilde{D}_1(t)=&\bar{k}_{21}(\sigma(t),t)e_{K_{11}}(t,t_0)\int_{t_0}^{t}\bar{k}_{12}(t,s)e_{K_{22}}(s,t_0)\Delta s\\ &+\int_{t_0}^{t}\bar{k}_{21}^{\Delta}(t,s)e_{K_{11}}(s,t_0)\left(\int_{t_0}^{s}\bar{k}_{12}(s,\xi)e_{K_{22}}(s,\xi)\Delta\xi\right)\Delta s,\end{aligned} \tag{2.16}$$

$$\begin{aligned}D_1(t)=&\bar{k}_{12}(\sigma(t),t)e_{K_{22}}(t,t_0)\int_{t_0}^{t}\bar{k}_{21}(t,s)e_{K_{11}}(s,t_0)\Delta s\\ &+\int_{t_0}^{t}\bar{k}_{12}^{\Delta}(t,s)e_{K_{22}}(s,t_0)\left(\int_{t_0}^{s}\bar{k}_{21}(s,\xi)e_{K_{11}}(\xi,t_0)\Delta\xi\right)\Delta s,\end{aligned} \tag{2.17}$$

$$K_{11}(t)=\bar{k}_{11}(\sigma(t),t)+\int_{t_0}^{t}\bar{k}_{11}^{\Delta}(t,s)\Delta s, \quad K_{22}(t)=\bar{k}_{22}(\sigma(t),t)+\int_{t_0}^{t}\bar{k}_{22}^{\Delta}(t,s)\Delta s, \tag{2.18}$$

and

$$\bar{k}_{i1}(t,s)=\frac{r_{i1}}{p}m^{\frac{r_{i1}-p}{p}}k_{i1}(t,s), \quad \bar{k}_{i2}(t,s)=\frac{r_{i2}}{q}m^{\frac{r_{i2}-q}{q}}k_{i2}(t,s), \quad i=1,2. \tag{2.19}$$

Proof. Define

$$z_1^p(t) = f_1(t) + \int_{t_0}^t k_{11}(t,s)u^{r_{11}}(\tau(s))\Delta s + \int_{t_0}^t k_{12}(t,s)v^{r_{12}}(\tau(s))\Delta s \tag{2.20}$$

and

$$z_2^q(t) = f_2(t) + \int_{t_0}^t k_{21}(t,s)u^{r_{21}}(\tau(s))\Delta s + \int_{t_0}^t k_{22}(t,s)v^{r_{22}}(\tau(s))\Delta s, \tag{2.21}$$

then z_1 and z_2 are nondecreasing and

$$u(t) \le z_1(t), \quad v(t) \le z_2(t), \tag{2.22}$$

on T_0.

Therefore, for every $t \in \mathrm{T}_0$ with $\tau(t) > t_0$, we have

$$u(\tau(t))) \le z_1(\tau(t)) \le z_1(t) \tag{2.23}$$

and

$$v(\tau(t))) \le z_2(\tau(t)) \le z_2(t) \tag{2.24}$$

since $\tau(t) \le t$. On other hand, for $t \in \mathrm{T}_0$ with $\tau(t) \le t_0$, by the initial condition (I), we have

$$u(\tau(t))) = \varphi_1(\tau(t)) \le (f_1(t))^{1/p} \le z_1(t) \tag{2.25}$$

and

$$v(\tau(t))) = \varphi_2(\tau(t)) \le (f_2(t))^{1/q} \le z_2(t) \tag{2.26}$$

for $t \in \mathrm{T}_0$

From (2.12) to (2.15), for $t \in \mathrm{T}_0$ we have the relations

$$u(\tau(t))) \le z_1(t)$$

and

$$v(\tau(t))) \le z_2(t)$$

for $t \in \mathrm{T}_0$.

Substituting the last relations into (2.20) and (2.21) we obtain

$$\begin{cases} z_1^p(t) \le f_1(t) + \displaystyle\int_{t_0}^t k_{11}(t,s)z_1^{r_{11}}(s)\Delta s + \int_{t_0}^t k_{12}(t,s)z_2^{r_{12}}(s)\Delta s, \\ \\ z_2^q(t) \le f_2(t) + \displaystyle\int_{t_0}^t k_{21}(t,s)z_1^{r_{21}}(s)\Delta s + \int_{t_0}^t k_{22}(t,s)z_2^{r_{22}}(s)\Delta s. \end{cases} \quad t \in \mathrm{T}_0 \tag{2.27}$$

Setting

$$\begin{cases} x(t) = f_1(t) + \displaystyle\int_{t_0}^t k_{11}(t,s)z_1^{r_{11}}(s)\Delta s + \int_{t_0}^t k_{12}(t,s)z_2^{r_{12}}(s)\Delta s, \\ \\ y(t) = f_2(t) + \displaystyle\int_{t_0}^t k_{21}(t,s)z_1^{r_{21}}(s)\Delta s + \int_{t_0}^t k_{22}(t,s)z_2^{r_{22}}(s)\Delta s, \end{cases} \quad t \in \mathrm{T}_0 \tag{2.28}$$

then from (2.16) we have

$$\begin{cases} z_1^p(t) \le x(t), \\ \\ z_2^q(t) \le y(t) \end{cases} \quad t \in \mathrm{T}_0$$

or

$$\begin{cases} z_1(t) \le x^{1/p}(t), \\ \\ z_2(t) \le y^{1/q}(t). \end{cases} \qquad t \in \mathrm{T}_0 \qquad (2.29)$$

By Lemma 2.1 and (2.29), for any real number $m > 0$, we can get

$$z_1^{r_{i1}}(t) \le \frac{p - r_{i1}}{p} m^{\frac{r_{i1}}{p}} + \frac{r_{i1}}{p} m^{\frac{r_{i1}-p}{p}} x(t)$$

and

$$z_2^{r_{i2}}(t) \le \frac{q - r_{i2}}{q} m^{\frac{r_{i2}}{q}} + \frac{r_{i2}}{q} m^{\frac{r_{i2}-q}{q}} y(t), \quad i = 1, 2.$$

Substituting the last inequalities into (2.28) we get

$$\begin{cases} x(t) \le \Phi_1(t) + \displaystyle\int_{t_0}^{t} \frac{r_{11}}{p} m^{\frac{r_{11}-p}{p}} k_{11}(t,s) x(s) \Delta s + \int_{t_0}^{t} \frac{r_{12}}{q} m^{\frac{r_{12}-q}{q}} k_{12}(t,s) y(s) \Delta s, \\ \\ y(t) \le \Phi_2(t) + \displaystyle\int_{t_0}^{t} \frac{r_{21}}{p} m^{\frac{r_{21}-p}{p}} k_{21}(t,s) x(s) \Delta s + \int_{t_0}^{t} \frac{r_{22}}{q} m^{\frac{r_{22}-q}{q}} k_{22}(t,s) y(s) \Delta s, \end{cases} \qquad (2.30)$$

for $t \in \mathrm{T}_0$, where $\Phi_1(t)$ and $\Phi_2(t)$ are defined in (2.13) and (2.14), respectively.

Setting

$$\overline{f}_1(t) = \Phi_1(t) + \int_{t_0}^{t} \frac{r_{12}}{q} m^{\frac{r_{12}-q}{q}} k_{12}(t,s) y(s) \Delta s, \qquad (2.31)$$

from (2.30-1) we have

$$x(t) \le \overline{f}_1(t) + \int_{t_0}^{t} \frac{r_{11}}{p} m^{\frac{r_{11}-p}{p}} k_{11}(t,s) x(s) \Delta s.$$

Obviously, $\overline{f}_1 \in C_{rd}(T_0, R_+)$ and is nondecreasing. Using Lemma 2.2, we get

$$x(t) \le \overline{f}_1(t) e_{K_{11}}(t, t_0) = e_{K_{11}}(t, t_0) \left[\Phi_1(t) + \int_{t_0}^{t} \frac{r_{12}}{q} m^{\frac{r_{12}-q}{q}} k_{12}(t,s) y(s) \Delta s \right]$$

where

$$K_{11}(t) = \frac{r_{11}}{p} m^{\frac{r_{11}-p}{p}} k_{11}(\sigma(t), t) + \int_{t_0}^{t} \frac{r_{11}}{p} m^{\frac{r_{11}-p}{p}} k_{11}^{\Delta}(t,s) \Delta s.$$

Substituting the last inequality into (2.30-2), we have

$$y(t) \le \Phi_2(t) + \int_{t_0}^{t} \frac{r_{21}}{p} m^{\frac{r_{21}-p}{p}} k_{21}(t,s) e_{K_{11}}(s, t_0) \Phi_1(s) \Delta s + \int_{t_0}^{t} \frac{r_{22}}{q} m^{\frac{r_{22}-q}{q}} k_{22}(t,s) y(s) \Delta s$$

$$+ \int_{t_0}^{t} \frac{r_{21}}{p} m^{\frac{r_{21}-p}{p}} k_{21}(t,s) e_{K_{11}}(s, t_0) \left(\int_{t_0}^{s} \frac{r_{12}}{q} m^{\frac{r_{12}-q}{q}} k_{12}(s,\xi) y(\xi) \Delta \xi \right) \Delta s.$$

Rewriting the last inequality as

$$y(t) \le A_1(t) + \int_{t_0}^{t} \bar{k}_{22}(t,s) y(s) \Delta s$$

$$+ \int_{t_0}^{t} \bar{k}_{21}(t,s) e_{K_{11}}(s, t_0) \left(\int_{t_0}^{s} \bar{k}_{12}(s,\xi) y(\xi) \Delta \xi \right) \Delta s, \quad t \in \mathrm{T}_0 \qquad (2.32)$$

where $A_1(t), \bar{k}_{21}(t,s)$ and $\bar{k}_{i2}(t,s) (i = 1, 2.)$ are defined in (2.12) and (2.19), respectively.

Using Lemma 2.3 into (2.32), we get

$$y(t) \le e_{K_{22}}(t,t_0)\left(A_1(t) + \int_{t_0}^{t} \bar{k}_{21}(t,s)e_{K_{11}}(s,t_0)B_1(s)e_D(s,t_0)\Delta s\right), \tag{2.33}$$

where$K_{22}(t), B_1(t)$ and $D_1(t)$ are given in (2.18), (2.15) and (2.17), respectively.

By exchanging $x(t)$ and $y(t)$ in (2.29) and taking the same procedures as from (2.31) to (2.33), we can also get

$$x(t) \le e_{K_{11}}(t,t_0)\left(\tilde{A}_1(t) + \int_{t_0}^{t} \bar{k}_{12}(t,s)e_{K_{22}}(s,t_0)\tilde{B}_1(s)e_{\tilde{D}_1}(s,t_0)\Delta s\right), \tag{2.33}$$

where $K_{11}(t), \tilde{B}_1(t)$ and $\tilde{D}_1(t)$ are defined in (2.18), (2.15) and (2.16), respectively.

Combining (2.33) and (2.34) with (2.22) and (2.29), we get the desired (2.10). □

Remark 1. When the functions $f_i(t)(i = 1, 2)$ in Theorem 2.4 do not have the monotone property required, we can replace them by $\widetilde{f}_i(t) := \max\limits_{t_0 \le s \le t}\{f_i(t)\}(i = 1, 2)$ respectively.

Next consider delay inequalities systems on time scales

$$\begin{cases} u^p(t) \le f_1(t) + \displaystyle\int_{t_0}^{t} k_{11}(t,s)N_{11}(s,u^{r_{11}}(\tau(s)))\Delta s + \int_{t_0}^{t} k_{12}(t,s)v^{r_{12}}(\tau(s))\Delta s, \\ \\ v^q(t) \le f_2(t) + \displaystyle\int_{t_0}^{t} k_{21}(t,s)u^{r_{21}}(\tau(s))\Delta s + \int_{t_0}^{t} k_{22}(t,s)N_{22}(s,v^{r_{22}}(\tau(s)))\Delta s, \end{cases} \quad t \in \mathrm{T}_0 \tag{2.35}$$

with the initial condition

$$\begin{cases} u(t) = \varphi_1(t), \quad v(t) = \varphi_2(t) \qquad \text{for } t \in [\alpha, t_0] \cap \mathrm{T}_0 \ \text{ with} \\ \\ \varphi_1(\tau(t)) \le f_1^{1/p}(t), \ \varphi_2(\tau(t)) \le f_2^{1/q}(t) \ \text{ for every } t \in \mathrm{T}_0 \ \text{ with } \tau(t) \le t_0 \end{cases} \tag{I}$$

where $\tau : \mathrm{T}_0 \to \mathrm{T}, \tau(\mathrm{t}) \le \mathrm{t}, -\infty < \alpha = \inf\{\tau(\mathrm{t}), \mathrm{t} \in \mathrm{T}_0\} \le \mathrm{t}_0$, and $\varphi_1, \varphi_2 \in C_{rd}([\alpha, t_0] \cap \mathrm{T}, \mathrm{R}_+)$, p, q and r_{ij} are constants with $p \ge r_{i1} > 0, q \ge r_{i2} > 0$ $(i = 1, 2)$.

Theorem 2.5. *Suppose that u, v, f_i and k_{ij} $(i, j = 1, 2)$ are defined as in Theorem 2.4. If $N_{ij} : \mathrm{T}_0 \times \mathrm{R}_+ \to \mathrm{R}_+$ is continuous function and satisfies*

$$0 \le N_{ii}(t,x) - N_{ii}(t,y) \le M_{ii}(t,y)(x - y) \tag{H}$$

for $x \ge y \ge 0$, where $M_{ii} : \mathrm{T}_0 \times \mathrm{R}_+ \to \mathrm{R}_+$ is continuous $(i = 1, 2)$, then for any $m > 0$ the inequalities systems (2.35) with the initial condition (I) *implies*

$$\begin{cases} u(t) \le e^{\frac{1}{p}}_{\widetilde{K}_{11}}(t,t_0)\left(\tilde{A}_2(t) + \displaystyle\int_{t_0}^{t} \bar{k}_{12}(t,s)e_{\widetilde{K}_{22}}(s,t_0)\tilde{B}_2(s)e_{\tilde{D}_2}(s,t_0)\Delta s\right)^{\frac{1}{p}}, \\ \\ v(t) \le e^{\frac{1}{q}}_{\widetilde{K}_{22}}(t,t_0)\left(A_2(t) + \displaystyle\int_{t_0}^{t} \bar{k}_{21}(t,s)e_{\widetilde{K}_{11}}(s,t_0)B_2(s)e_{D_2}(s,t_0)\Delta s\right)^{\frac{1}{q}}, \end{cases} \tag{2.36}$$

for $t \in \mathrm{T}_0$, where

$$\tilde{A}_2(t) = \Psi_1(t) + \int_{t_0}^{t} \bar{k}_{12}(t,s)e_{\widetilde{K}_{22}}(s,t_0)\Psi_2(s)\Delta s, \tag{2.36}$$

$$A_2(t) = \Psi_2(t) + \int_{t_0}^{t} \bar{k}_{21}(t,s)e_{\widetilde{K}_{11}}(s,t_0)\Psi_1(s)\Delta s, \tag{2.38}$$

$$\Psi_1(t) = f_1(t) + \int_{t_0}^{t} \left[N_{11}(s, \frac{p-r_{11}}{p} m^{\frac{r_{11}}{p}}) k_{11}(t,s) + \frac{q-r_{12}}{q} m^{\frac{r_{12}}{q}} k_{12}(t,s) \right] \Delta s, \tag{2.39}$$

$$\Psi_2(t) = f_2(t) + \int_{t_0}^{t} \left[\frac{p-r_{21}}{p} m^{\frac{r_{21}}{p}} k_{21}(t,s) + N_{22}(s, \frac{q-r_{22}}{q} m^{\frac{r_{22}}{q}}) k_{22}(t,s) \right] \Delta s, \tag{2.40}$$

$$\tilde{B}_2(t) = \int_{t_0}^{t} \bar{k}_{21}(t,s) e_{K_{11}}(s,t_0) \tilde{A}_2(s) \Delta s, \quad B_2(t) = \int_{t_0}^{t} \bar{k}_{12}(t,s) e_{K_{22}}(s,t_0) A_2(s) \Delta s \tag{2.41}$$

$$\begin{aligned} \tilde{D}_2(t) &= \bar{k}_{21}(\sigma(t),t) e_{K_{11}}(t,t_0) \int_{t_0}^{t} \bar{k}_{12}(t,s) e_{K_{22}}(s,t_0) \Delta s \\ &\quad + \int_{t_0}^{t} \bar{k}_{21}^{\Delta}(t,s) e_{\widetilde{K}_{11}}(s,t_0) \left(\int_{t_0}^{s} \bar{k}_{12}(s,\xi) e_{K_{22}}(s,\xi) \Delta \xi \right) \Delta s, \end{aligned} \tag{2.42}$$

$$\begin{aligned} D_2(t) &= \bar{k}_{12}(\sigma(t),t) e_{K_{22}}(t,t_0) \int_{t_0}^{t} \bar{k}_{21}(t,s) e_{\widetilde{K}_{11}}(s,t_0) \Delta s \\ &\quad + \int_{t_0}^{t} \bar{k}_{12}^{\Delta}(t,s) e_{\widetilde{K}_{22}}(s,t_0) \left(\int_{t_0}^{s} \bar{k}_{21}(s,\xi) e_{K_{11}}(\xi,t_0) \Delta \xi \right) \Delta s, \end{aligned} \tag{2.43}$$

$$\widetilde{K}_{11}(t) = \tilde{k}_{11}(\sigma(t),t) + \int_{t_0}^{t} \tilde{k}_{11}^{\Delta}(t,s) \Delta s, \quad \widetilde{K}_{22}(t) = \tilde{k}_{22}(\sigma(t),t) + \int_{t_0}^{t} \tilde{k}_{22}^{\Delta}(t,s) \Delta s, \tag{2.44}$$

and

$$\tilde{k}_{11}(t,s) = \frac{r_{11}}{p} m^{\frac{r_{11}-p}{p}} k_{11}(t,s) M_{11}(s, \frac{p-r_{11}}{p} m^{\frac{r_{11}}{p}}), \quad \bar{k}_{12}(t,s) = \frac{r_{12}}{q} m^{\frac{r_{12}-q}{q}} k_{12}(t,s)$$

$$\bar{k}_{21}(t,s) = \frac{r_{21}}{p} m^{\frac{r_{21}-p}{p}} k_{21}(t,s), \quad \tilde{k}_{22}(t,s) = \frac{r_{22}}{q} m^{\frac{r_{22}-q}{q}} k_{22}(t,s) M_{22}(s, \frac{q-r_{22}}{q} m^{\frac{r_{22}}{q}}). \tag{2.45}$$

Proof. Define

$$z_1^p(t) = f_1(t) + \int_{t_0}^{t} k_{11}(t,s) N_{11}(s, u^{r_{11}}(\tau(s))) \Delta s + \int_{t_0}^{t} k_{12}(t,s) N_{12}(s, v^{r_{12}}(\tau(s))) \Delta s \tag{2.46}$$

and

$$z_2^q(t) = f_2(t) + \int_{t_0}^{t} k_{21}(t,s) N_{21}(s, u^{r_{21}}(\tau(s))) \Delta s + \int_{t_0}^{t} k_{22}(t,s) N_{22}(s, v^{r_{22}}(\tau(s))) \Delta s, \tag{2.47}$$

then z_1 and z_2 are nondecreasing and

$$u(t) \le z_1(t), \quad v(t) \le z_2(t), \tag{2.48}$$

on T_0.

By taking the similar procedure in the proof of Theorem 2.4, it follows that

$$\begin{cases} z_1^p(t) \le f_1(t) + \displaystyle\int_{t_0}^{t} k_{11}(t,s) N_{11}(s, z_1^{r_{11}}(s)) \Delta s + \int_{t_0}^{t} k_{12}(t,s) z_2^{r_{12}}(s) \Delta s, \\ z_2^q(t) \le f_2(t) + \displaystyle\int_{t_0}^{t} k_{21}(t,s) z_1^{r_{21}}(s) \Delta s + \int_{t_0}^{t} k_{22}(t,s) N_{22}(s, z_2^{r_{22}}(s)) \Delta s. \end{cases} \quad t \in \mathrm{T}_0 \tag{2.49}$$

Setting

$$\begin{cases} x(t) = f_1(t) + \displaystyle\int_{t_0}^{t} k_{11}(t,s)N_{11}(s, z_1^{r_{11}}(s))\Delta s + \int_{t_0}^{t} k_{12}(t,s)N_{12}(s, z_2^{r_{12}}(s))\Delta s, \\ \\ y(t) = f_2(t) + \displaystyle\int_{t_0}^{t} k_{21}(t,s)N_{21}(s, z_1^{r_{21}}(s))\Delta s + \int_{t_0}^{t} k_{22}(t,s)N_{22}(s, z_2^{r_{22}}(s))\Delta s, \end{cases} \quad t \in \mathrm{T}_0 \quad (2.50)$$

then from (2.49) we have

$$\begin{cases} z_1^p(t) \le x(t), \\ \\ z_2^q(t) \le y(t) \end{cases} \quad t \in \mathrm{T}_0$$

or

$$\begin{cases} z_1(t) \le x^{1/p}(t), \\ \\ z_2(t) \le y^{1/q}(t). \end{cases} \quad t \in \mathrm{T}_0 \qquad (2.51)$$

By Lemma 2.1 and (2.51), for any real number $m > 0$, we can get

$$z_1^{r_{i1}}(t) \le \frac{p - r_{i1}}{p} m^{\frac{r_{i1}}{p}} + \frac{r_{i1}}{p} m^{\frac{r_{i1}-p}{p}} x(t)$$

and

$$z_2^{r_{i2}}(t) \le \frac{q - r_{i2}}{q} m^{\frac{r_{i2}}{q}} + \frac{r_{i2}}{q} m^{\frac{r_{i2}-q}{q}} y(t), \quad i = 1, 2.$$

By the hypotheses (H) and the last inequalities, it follows that

$$N_{11}(s, z_1^{r_{11}}(s)) \le \frac{r_{11}}{p} m^{\frac{r_{11}-p}{p}} M_{11}(s, \frac{p - r_{11}}{p} m^{\frac{r_{11}}{p}}) x(s) + N_{11}(s, \frac{p - r_{i1}}{p} m^{\frac{r_{11}}{p}})$$

and

$$N_{22}(s, z_2^{r_{22}}(s)) \le \frac{r_{22}}{q} m^{\frac{r_{22}-q}{q}} M_{22}(s, \frac{q - r_{22}}{q} m^{\frac{r_{22}}{q}}) y(s) + N_{22}(s, \frac{q - r_{22}}{q} m^{\frac{r_{22}}{q}}).$$

Substituting the last inequalities into (2.50) we get

$$\begin{cases} x(t) \le \Psi_1(t) + \dfrac{r_{11}}{p} m^{\frac{r_{11}-p}{p}} \displaystyle\int_{t_0}^{t} k_{11}(t,s) M_{11}(s, \frac{p - r_{11}}{p} m^{\frac{r_{11}}{p}}) x(s)\Delta s \\ \qquad\qquad + \dfrac{r_{12}}{q} m^{\frac{r_{12}-q}{q}} \displaystyle\int_{t_0}^{t} k_{12}(t,s) y(s)\Delta s, \\ \\ y(t) \le \Psi_2(t) + \dfrac{r_{21}}{p} m^{\frac{r_{21}-p}{p}} \displaystyle\int_{t_0}^{t} k_{21}(t,s) x(s)\Delta s \\ \qquad + \dfrac{r_{22}}{q} m^{\frac{r_{22}-q}{q}} \displaystyle\int_{t_0}^{t} k_{22}(t,s) M_{22}(s, \frac{q - r_{22}}{q} m^{\frac{r_{22}}{q}}) y(s)\Delta s \end{cases} \qquad (2.52)$$

for $t \in \mathrm{T}_0$, where $\Psi_1(t)$ and $\Psi_2(t)$ are defined in (2.39) and (2.40), respectively.

Taking the similar way in the proof of Theorem 2.4 to (2.52), we can obtain that

$$y(t) \le e_{\widetilde{K}_{22}}(t, t_0) \left(A_2(t) + \int_{t_0}^{t} \bar{k}_{21}(t,s) e_{\widetilde{K}_{11}}(s, t_0) B(s) e_{D_2}(s, t_0)\Delta s \right), \qquad (2.53)$$

and

$$x(t) \le e_{\widetilde{K}_{11}}(t, t_0) \left(\tilde{A}_2(t) + \int_{t_0}^{t} \bar{k}_{12}(t,s) e_{\widetilde{K}_{22}}(s, t_0) \tilde{B}_2(s) e_{\tilde{D}_2}(s, t_0)\Delta s \right), \qquad (2.54)$$

where $\widetilde{K}_{22}(t), B_2(t)$ and $D_2(t)$ are given in (2.44), (2.41) and (2.43), respectively; $\widetilde{K}_{11}(t), \tilde{B}_2(t)$ and $\tilde{D}_2(t)$ are defined in (2.44), (2.41) and (2.42), respectively.

Combining (2.53) and (2.54) with (2.51)and (2.48), it follows the desired (2.36). □

An application

In this section, we present an application of our results.

Consider the two dimensional delay dynamic system

$$\begin{cases} (x^p(t))^\Delta = G[t, x(\tau(t)), y(\tau(t))] \\ \\ (y^q(t))^\Delta = H[t, x(\tau(t)), y(\tau(t))] \end{cases} \qquad t \in \mathrm{T}_0 \tag{3.1}$$

with the initial condition

$$\begin{cases} x(t) = \varphi(t), \quad y(t) = \psi(t) \qquad \text{for } t \in [\alpha, t_0] \cap \mathrm{T}_0 \ \text{ with} \\ \\ \varphi(\tau(t)) \le |C_1|^{1/p}, \ \psi(\tau(t)) \le |C_2|^{1/q} \ \text{ for every } t \in \mathrm{T}_0 \ \text{ with } \tau(t) \le t_0 \end{cases} \tag{I′}$$

where $G, H : \mathrm{T}_0 \times \mathrm{R}^2 \to \mathrm{R}$ are continuous functions, $C_1 = x^p(t_0), C_2 = y^q(t_0)$, p and q are constants with $p, q \ge 1$, α and τ are as defined in the initial conditions (I), and $\varphi(t), \psi(t) \in C_{rd}([\alpha, t_0] \cap \mathrm{T}, \mathrm{R})$.

Theorem 3.1 *Assume that* (H)

$$|G[t, x(\tau(t)), y(\tau(t))]| \le l_{11}(t)|x(\tau(t))| + l_{12}(t)|y(\tau(t))|$$

and

$$|H[t, x(\tau(t)), y(\tau(t))]| \le l_{21}(t)|x(\tau(t))| + l_{22}(t)|y(\tau(t))|,$$

where $l_{ij}(t) \in C_{rd}(\mathrm{T}_0, \mathrm{R}_+), \mathrm{i}, \mathrm{j} = 1, 2$. *If* $(x(t), y(t))$ *is solution of the system (3.1) satisfies the initial condition* (I′). *Then*

$$\begin{cases} |x(t)| \le e_{L_{11}}^{\frac{1}{p}}(t, t_0) \left(\tilde{A}^*(t) + \displaystyle\int_{t_0}^t L_{12}(s) e_{L_{22}}(s, t_0) \tilde{B}^*(s) e_{\tilde{D}^*}(s, t_0) \Delta s \right)^{\frac{1}{p}}, \\ \\ |y(t)| \le e_{L_{22}}^{\frac{1}{q}}(t, t_0) \left(A^*(t) + \displaystyle\int_{t_0}^t L_{21}(s) e_{L_{11}}(s, t_0) B^*(s) e_{D^*}(s, t_0) \Delta s \right)^{\frac{1}{q}}, \end{cases} \tag{3.2}$$

for any $m > 0, t \in \mathrm{T}_0$, *where*

$$\tilde{A}^*(t) = \Psi_1(t) + \int_{t_0}^t L_{12}(s) e_{L_{22}}(s, t_0) \Psi_2(s) \Delta s, \tag{3.3}$$

$$A^*(t) = \Psi_2(t) + \int_{t_0}^t L_{21}(s) e_{L_{11}}(s, t_0) \Psi_1(s) \Delta s, \tag{3.4}$$

$$\Psi_1(t) = |C_1|^p + \frac{p-1}{p} m^{\frac{1}{p}} \int_{t_0}^t l_{11}(s) \Delta s + \frac{q-1}{q} m^{\frac{1}{q}} \int_{t_0}^t l_{12}(s) \Delta s, \tag{3.5}$$

$$\Psi_2(t) = |C_2|^q + \frac{p-1}{p} m^{\frac{1}{p}} \int_{t_0}^t l_{21}(s) \Delta s + \frac{q-1}{q} m^{\frac{1}{q}} \int_{t_0}^t l_{22}(s) \Delta s, \tag{3.6}$$

$$\tilde{B}^*(t) = \int_{t_0}^t L_{21}(s) e_{L_{11}}(s, t_0) \tilde{A}^*(s) \Delta s, \quad B^*(t) = \int_{t_0}^t L_{12}(s) e_{L_{22}}(s, t_0) A^*(s) \Delta s \tag{3.7}$$

$$\tilde{D}^*(t) = L_{21}(t)e_{L_{11}}(t,t_0)\int_{t_0}^{t} L_{12}(s)e_{L_{22}}(s,t_0)\Delta s \tag{3.8}$$

$$D^*(t) = L_{12}(t)e_{L_{22}}(t,t_0)\int_{t_0}^{t} L_{21}(s)e_{L_{11}}(s,t_0)\Delta s \tag{3.9}$$

and

$$L_{i1}(t) = \frac{1}{p}m^{\frac{1-p}{p}}l_{i1}(t), \quad L_{i2}(t) = \frac{1}{q}m^{\frac{1-q}{q}}l_{i2}(t), \quad i = 1,2. \tag{3.10}$$

Proof. Obviously, the solutions $(x(t), y(t))$ of the system (3.1) with the initial condition (I′) satisfy the equivalent nonlinear delay system on time scales

$$\begin{cases} x^p(t) = C_1 + \displaystyle\int_{t_0}^{t} G[s, x(\tau(s)), y(\tau(s))]\Delta s, \\ \\ y^q(t) = C_2 + \displaystyle\int_{t_0}^{t} H[s, x(\tau(s)), y(\tau(s))]\Delta s, \end{cases} \tag{3.11}$$

with the initial condition (I′). Using (H) in (3.11), we have

$$\begin{cases} |x(t)|^p \le |C_1| + \displaystyle\int_{t_0}^{t} l_{11}(s)|x(\tau(s))| + l_{12}(s)|y(\tau(s))|\Delta s, \\ \\ |y(t)|^q \le |C_2| + \displaystyle\int_{t_0}^{t} l_{21}(s)|x(\tau(s))| + l_{22}(s)|y(\tau(s))|\Delta s. \end{cases} \quad t \in \mathrm{T}_0 \tag{3.12}$$

A suitable applications of Theorem 2.4 to (3.12) easily yields (3.2), which gives the bound of the solution of system (3.1). The proof of Theorem 3.1 is complete. □

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Some relationship between q-Bernstein polynomials and extended q-Euler numbers

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Abstract Recently several authors are studying in the area of (h, q)-Euler numbers and polynomials(see [4]). In this paper we study some identities on the (h, q)-Euler numbers and polynomials related to q-Bernstein polynomials by using fermionic q-integral on $\mathbb{Z}_p$.

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1. Introduction

Let p be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number fields and the completion of algebraic closure of $\mathbb{Q}_p$, respactively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$, and if $q \in \mathbb{C}_p$, then one normally assumes $|q-1|_p < 1$. In this paper, the q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}.$$

Thus, we note that $\lim_{q\to 1}[x]_q = x$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. Let $k, n \in \mathbb{Z}_+$ and $f \in C(\mathbb{Z}_p)$. Then the p-adic q-Bernstein linear operator of order n for f is defined by

$$\begin{aligned}\mathbb{B}_{n,q}(f \mid x) &= \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) [x]_q^k [1-x]_{q^{-1}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q), \text{ see [7, 11].}\end{aligned} \tag{1}$$

Here $B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k}$ are called the q-Bernstein polynomials of degree n.

In [4], the fermionic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [4] .} \tag{2}$$

From (2), we can derive the following integral equation(see [1-15]).

$$\int_{\mathbb{Z}_p} f(x+n) d\mu_{-q}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \text{ for } n \in \mathbb{N}. \tag{3}$$

For $h \in \mathbb{Z}$, the (h,q)-Euler polynomials are defined by

$$\mathcal{E}_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n q^{(h-1)y} d\mu_{-q}(y), \text{ for } n \in \mathbb{Z}_+. \tag{4}$$

In the special case, $x = 0$, $\mathcal{E}_{n,q}^{(h)}(0) = \mathcal{E}_{n,q}^{(h)}$ are called the n-th (h,q)-Euler numbers(see [4]).

From (4), we note that

$$\begin{aligned}\mathcal{E}_{n,q}^{(h)}(x) &= \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1+q^{h+l}}\right) \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^{hm} [x+m]_q^n, \text{ see [4].}\end{aligned} \tag{5}$$

By the definition of (h,q)-Euler numbers, we easily get

$$\begin{aligned}\mathcal{E}_{n,q}^{(h)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \mathcal{E}_{l,q}^{(h)} \\ &= ([x]_q + q^x \mathcal{E}_q^{(h)})^n, n \in \mathbb{Z}_+,\end{aligned} \tag{6}$$

with usual convention about replacing $(\mathcal{E}_q^{(h)})^l$ by $\mathcal{E}_{l,q}^{(h)}$(see [4]).

Let $F_q^{(h)}(t,x)$ be the generating function of (h,q)-Euler polynomials as followings:

$$F_q^{(h)}(t,x) = e^{\mathcal{E}_q^{(h)}(x)t} = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(h)}(x) \frac{t^n}{n!},$$

with usual convention about replacing $(\mathcal{E}_q^{(h)}(x))^l$ by $\mathcal{E}_{l,q}^{(h)}(x)$. Thus, we have

$$F_q^{(h)}(t,x) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{[x+m]_q t}. \tag{7}$$

Let $F_q^{(h)}(t,x) = F_q^{(h)}(t)$. Then we get

$$q^h F_q^{(h)}(t,1) + F_q^{(h)}(t) = [2]_q. \tag{8}$$

From (6), (7) and (8), we obtain the following recurrence formula:

$$\mathcal{E}_{0,q}^{(h)} = \frac{1+q}{1+q^h}, \text{ and } q^h(q\mathcal{E}_q^{(h)}+1)^n + \mathcal{E}_{n,q}^{(h)} = 0 \text{ if } n > 0, \tag{9}$$

with usual convention about replacing $(\mathcal{E}_q^{(h)})^n$ by $\mathcal{E}_{n,q}^{(h)}$.

The purpose of this paper is to give some identities on the (h,q)-Euler numbers concerning q-Bernstein polynomials by using the fermionic q-integral on $\mathbb{Z}_p$.

2. Some identities concerning (h,q)-Euler numbers and polynomials

From (7), we obtain the following functional equation:

$$\frac{1}{q} F_{q^{-1}}^{(h)}(t, 1-x) = q^h F_q^{(h)}(-qt, x). \tag{10}$$

By (10), we get

$$\mathcal{E}_{n,q^{-1}}^{(h)}(1-x) = (-1)^n q^{n+h-1} \mathcal{E}_{n,q}^{(h)}(x), n \in \mathbb{Z}_+. \tag{11}$$

From (9), we can derive the following equation:

$$\begin{aligned}
&q^{2h}\mathcal{E}_{n,q}^{(h)}(2) - \frac{1+q}{1+q^h}q^{2h} - \frac{1+q}{1+q^h}q^h \\
&= q^{2h}\sum_{l=1}^{n}\binom{n}{l} q^l (q\mathcal{E}_q^{(h)}+1)^l - \frac{1+q}{1+q^h}q^h \\
&= -q^h \sum_{l=1}^{n}\binom{n}{l} q^l \mathcal{E}_{l,q}^{(h)} - \frac{1+q}{1+q^h}q^h \\
&= -q^h \sum_{l=0}^{n}\binom{n}{l} q^l \mathcal{E}_{l,q}^{(h)} \\
&= -q^h \mathcal{E}_{n,q}^{(h)}(1) = \mathcal{E}_{n,q}^{(h)} \text{ if } n > 0.
\end{aligned} \tag{12}$$

Therefore, we obtain the following theorem.

Theorem 1. For $n > 0$, we have

$$q^{2h}\mathcal{E}_{n,q}^{(h)}(2) - \frac{1+q}{1+q^h}q^{2h} - \frac{1+q}{1+q^h}q^h = \mathcal{E}_{n,q}^{(h)}.$$

By (2), we get

$$q^{h-1}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n q^{(h-1)x}d\mu_{-q}(x) = (-1)^n q^{n+h-1}\int_{\mathbb{Z}_p}[x-1]_q^n q^{(h-1)x}d\mu_{-q}(x) \\ = (-1)^n q^{n+h-1}\mathcal{E}_{n,q}^{(h)}(-1). \tag{13}$$

Therefore, by (13), we obtain the following theorem.

Theorem 2. For $n \in \mathbb{Z}_+$, we have

$$q^{h-1}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n q^{(h-1)x}d\mu_{-q}(x) = \int_{\mathbb{Z}_p}[x+2]_{q^{-1}}^n q^{-(h-1)x}d\mu_{-q^{-1}}(x).$$

From Theorem 1, we note that

$$\mathcal{E}_{n,q}^{(h)}(2) = \frac{1}{q^{2h}}\mathcal{E}_{n,q}^{(h)} + \frac{[2]_q}{[2]_{q^h}}\frac{1}{q^h} + \frac{[2]_q}{[2]_{q^h}} \text{ if } n > 0. \tag{14}$$

For $n \in \mathbb{N}$, by (12) and (14), we have

$$q^{h-1}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n q^{(h-1)x}d\mu_{-q}(x) = \mathcal{E}_{n,q^{-1}}^{(h)}(2) \\ = q^{2h}\mathcal{E}_{n,q^{-1}}^{(h)} + \frac{[2]_q}{[2]_{q^h}}q^{2h-1} + q^{h-1}\frac{[2]_q}{[2]_{q^h}}. \tag{15}$$

Thus, by (15), we get

$$\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n q^{(h-1)x}d\mu_{-q}(x) = q^{h+1}\mathcal{E}_{n,q^{-1}}^{(h)} + q^h\frac{[2]_q}{[2]_{q^h}} + \frac{[2]_q}{[2]_{q^h}} \text{ if } n \in \mathbb{N}. \tag{16}$$

For $x \in \mathbb{Z}_p$, the p-adic analogues of q-Bernstein polynomials are given by

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k}, \text{ where } n,k \in \mathbb{Z}_+. \tag{17}$$

By (17), we get the symmetry of q-Bernstein polynomials as follows:

$$B_{k,n}(x,q) = B_{n-k,n}(1-x,q^{-1}), \text{ see [7].} \tag{18}$$

Thus, by (17) and (18), we see that

$$\int_{\mathbb{Z}_p}B_{k,n}(x,q)q^{(h-1)x}d\mu_{-q}(x) = \int_{\mathbb{Z}_p}B_{n-k,n}(1-x,q^{-1})q^{(h-1)x}d\mu_{-q}(x) \\ = \binom{n}{k}\sum_{l=0}^{k}\binom{n}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n-l}q^{(h-1)x}d\mu_{-q}(x).$$

For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x,q) q^{(h-1)x} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(q^{h+1} \mathcal{E}^{(h)}_{n-l,q^{-1}} + q^h \frac{[2]_q}{[2]_{q^h}} + \frac{[2]_q}{[2]_{q^h}} \right) \\
&= \begin{cases} q^{h+1} \mathcal{E}^{(h)}_{n,q^{-1}} + q^h \dfrac{[2]_q}{[2]_{q^h}} + \dfrac{[2]_q}{[2]_{q^h}}, & \text{if } k = 0, \\ q^{h+1} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \mathcal{E}^{(h)}_{n-l,q^{-1}}, & \text{if } k \neq 0. \end{cases}
\end{aligned} \tag{19}$$

Let us take the fermionic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials of degree n as follows:

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x,q) q^{(h-1)x} d\mu_{-q}(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} [x]_q^k [1-x]_{q^{-1}}^{n-k} q^{(h-1)x} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \mathcal{E}^{(h)}_{l+k,q}.
\end{aligned} \tag{20}$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 3. Let $n, k \in \mathbb{Z}_+$ with $n > k$. Then we have

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q) q^{(h-1)x} d\mu_{-q}(x) = \begin{cases} q^{h+1} \mathcal{E}^{(h)}_{n,q^{-1}} + q^h \dfrac{[2]_q}{[2]_{q^h}} + \dfrac{[2]_q}{[2]_{q^h}}, & \text{if } k = 0, \\ q^{h+1} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \mathcal{E}^{(h)}_{n-l,q^{-1}}, & \text{if } k \neq 0. \end{cases}$$

Moreover,

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \mathcal{E}^{(h)}_{l+k,q} = \begin{cases} q^{h+1} \mathcal{E}^{(h)}_{n,q^{-1}} + q^h \dfrac{[2]_q}{[2]_{q^h}} + \dfrac{[2]_q}{[2]_{q^h}}, & \text{if } k = 0, \\ q^{h+1} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \mathcal{E}^{(h)}_{n-l,q^{-1}}, & \text{if } k \neq 0. \end{cases}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)B_{k,n_2}(x,q)q^{(h-1)x}d\mu_{-q}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n_1+n_2-l}q^{(h-1)x}d\mu_{-q}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(q^{h+1}\mathcal{E}_{n_1+n_2-l,q^{-1}}^{(h)} + q^h\frac{[2]_q}{[2]_{q^h}} + \frac{[2]_q}{[2]_{q^h}}\right) \qquad (21) \\
&= \begin{cases} q^{h+1}\mathcal{E}_{n_1+n_2,q^{-1}}^{(h)} + q^h\dfrac{[2]_q}{[2]_{q^h}} + \dfrac{[2]_q}{[2]_{q^h}}, & \text{if } k=0, \\ q^{h+1}\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k+l}\mathcal{E}_{n_1+n_2-l,q^{-1}}^{(h)}, & \text{if } k\neq 0. \end{cases}
\end{aligned}$$

Therefore, by (21), we obtain the following theorem.

Theorem 4. For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)B_{k,n_2}(x,q)q^{(h-1)x}d\mu_{-q}(x) \\
&= \begin{cases} q^{h+1}\mathcal{E}_{n_1+n_2,q^{-1}}^{(h)} + q^h\dfrac{[2]_q}{[2]_{q^h}} + \dfrac{[2]_q}{[2]_{q^h}}, & \text{if } k=0, \\ q^{h+1}\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k+l}\mathcal{E}_{n_1+n_2-l,q^{-1}}^{(h)}, & \text{if } k>0. \end{cases}
\end{aligned}$$

From the binomial theorem, we can derive the following equation.

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)B_{k,n_2}(x,q)q^{(h-1)x}d\mu_{-q}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\int_{\mathbb{Z}_p}[x]_q^{2k+l}q^{(h-1)x}d\mu_{-q}(x) \qquad (22) \\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}E_{2k+l,q}^{(h)}.
\end{aligned}$$

Thus, we obtain the following corollary.

Corollary 5. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have

$$\begin{aligned}
&\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}E_{2k+l,q}^{(h)} \\
&= \begin{cases} q^{h+1}\mathcal{E}_{n_1+n_2,q^{-1}}^{(h)} + [2]_q, & \text{if } k=0, \\ q^{h+1}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k+l}\mathcal{E}_{n_1+n_2-l,q^{-1}}^{(h)}, & \text{if } k>0. \end{cases}
\end{aligned}$$

Let $s \in \mathbb{N}$ with $s \geq 2$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q)}_{s-\text{times}} q^{(h-1)x} d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\int_{\mathbb{Z}_p}[x]_q^{sk}[1-x]_{q^{-1}}^{n_1+\cdots+n_s-sk}q^{(h-1)x}d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n_1+\cdots+n_s-l}q^{(h-1)x}d\mu_{-q}(x) \qquad (23)\\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\left(q^{h+1}\mathcal{E}^{(h)}_{n_1+\cdots+n_s-l,q^{-1}}+[2]_q\right) \\
&= \begin{cases} q^{h+1}\mathcal{E}^{(h)}_{n_1+\cdots+n_s,q^{-1}}+[2]_q, & \text{if } k=0, \\ q^{h+1}\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\mathcal{E}^{(h)}_{n_1+\cdots+n_s-l,q^{-1}}, & \text{if } k>0. \end{cases}
\end{aligned}$$

Therefore, by (23), we obtain the following theorem.

Theorem 6. For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q)}_{s-\text{times}} q^{(h-1)x} d\mu_{-q}(x) \\
&= \begin{cases} q^{h+1}\mathcal{E}^{(h)}_{n_1+\cdots+n_s,q^{-1}}+[2]_q, & \text{if } k=0, \\ q^{h+1}\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\mathcal{E}^{(h)}_{n_1+\cdots+n_s-l,q^{-1}}, & \text{if } k>0. \end{cases}
\end{aligned}$$

By the definition of q-Bernstein polynomials and the binomial theorem, we easily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q)}_{s-\text{times}} q^{(h-1)x} d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+\cdots+n_s-sk}(-1)^l\binom{n_1+\cdots+n_s-sk}{l}\int_{\mathbb{Z}_p}[x]_q^{sk+l}q^{(h-1)x}d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+\cdots+n_s-sk}(-1)^l\binom{n_1+\cdots+n_s-sk}{l}\mathcal{E}^{(h)}_{sk+l,q}.
\end{aligned} \qquad (24)$$

Therefore, by (24) and Theorem 6, we have the following corollary.

Corollary 7. Let $s \in \mathbb{N}$ with $s \geq 2$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$, we have

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} \mathcal{E}_{sk+l,q}^{(h)}$$
$$= \begin{cases} q^{h+1}\mathcal{E}_{n_1+\cdots+n_s,q^{-1}}^{(h)} + [2]_q, & \text{if } k = 0, \\ q^{h+1}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\mathcal{E}_{n_1+\cdots+n_s-l,q^{-1}}^{(h)}, & \text{if } k > 0. \end{cases}$$

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MODIFIED NEWTON'S METHODS OF CUBIC CONVERGENCE FOR MULTIPLE ROOTS

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Abstract. From Newton's method of solving nonlinear equations numerically, various modifications concerning accelerated order, or multiple roots had been actively developed. In this paper, we focus on those concerning multiple roots. Based on existing modifications for simple roots, new methods for multiple roots, which do not require knowledge on multiplicity of the desired root to find, are derived. They are proved to be of third order, and numerical experiments show their quality, one of them competitive and the other superior to existing methods of same order.

1. Introduction

Newton's method is one of the most basic applications of differential calculus, developed by Newton, and is widely used to obtain roots of nonlinear equations numerically, for the method has such a simple iterative algorithm and relatively high accuracy, with a quadratic order of convergence. The well-known iterative method is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n \geq 0. \tag{1}$$

Owing to its high usefulness, there had been numerous efforts to improve the method to produce more complicated, and possibly, more efficient methods, two of which are of special interest in this paper: the trapezoidal method [10] and the midpoint method [4].

Weerakoon suggested in [10] to use trapezoidal rule rather than left-side rectangular rule as in the original Newton's method, in approximating a definite integral of f'. Also, it utilized an idea of approximating the root by the original Newton's method, and making use of both an initial value and the approximation to evaluate the next-step iteration. The trapezoidal Newton's method is defined by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})}, \tag{2}$$

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where $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Inspired by [10], Frontini and Sormani suggested in [4] to use the midpoint rule in approximation of the definite integral. They also used the idea of two-step iterations. The midpoint Newton's method is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n + x^*_{n+1}}{2})},$$

where $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Or, written differently,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x^*_{n+1})}, \tag{3}$$

with $x^*_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}$.

Both the trapezoidal and the midpoint methods are of cubic order, while the original Newton's method was of quadratic order. Many other methods of cubic or higher order were developed, see [1]. However, the original Newton's method fails to keep being efficient when it comes to handle an equation with multiple roots, see Remark 3 in [4], because of $f'(x_n)$ in its denominator, which approaches to zero as an approximation reaches near the multiple root. Many ideas to overcome this idiosyncrasy have been developed, one of which defining a new function $u(x) = f(x)/f'(x)$ where f has one or more multiple roots, and finding roots of u instead of f, see [9, p.126] or [2, p.80].

Approximating a multiple root of f, can be done by approximating a single root of u by well-known Newton's method, as mentioned above. Applying the well-known Newton's method on $u(x) = f(x)/f'(x)$, it follows that

$$x_{n+1} = x_n - \frac{u(x_n)}{u'(x_n)},$$

or, in terms of f,

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)},$$

see [2, p.81].

This is a modified Newton's method for multiple roots. Many other methods for finding multiple roots are developed, and some examples of them are organized well in [3]. However, most of existing methods for multiple roots require multiplicity of the desired root, since the methods vary with the multiplicity. Practically in most cases, it may seem unlikely to know the multiplicity of a root before even finding it. Although it is possible to find the multiplicity before finding the root, it seems a method without any requirements on the multiplicity may be found useful in many cases.

In this paper, we present two new modifications of Newton's method for multiple roots, that do not require any prior knowledge on the multiplicity. The modifications are obtained by applying the trapezoidal Newton's method (2) and the midpoint Newton's method (3) on $u(x) = f(x)/f'(x)$, respectively. We derive the modifications with some necessary definitions and a theorem in Section 2, and prove the modifications to be of third order in

Section 3. Numerical comparisons with several existing third-ordered methods for multiple roots are presented in Section 4.

2. Modified Newton's methods for multiple roots

2.1. **Preliminaries.** Before starting, it is necessary to define certain terms and make one thing clear.

Definition 1. *(See* [10]*) With α a real number, and n a non-negative integer, if a real sequence $\{x_n\}$ converges to α and for n large enough there exist constants $c \geq 0$ and $p \geq 0$ that satisfy*

$$| x_{n+1} - \alpha | \leq c \mid x_n - \alpha |^p,$$

then the maximum of p is said to be an order of convergence of $\{x_n\}$ to α.

Definition 2. *(See* [2, p.79]*) A root α of an equation $f(x) = 0$ is said to have the multiplicity m if and only if $f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \ldots, f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$. In this case, f can be written as*

$$f(x) = (x - \alpha)^m g(x),$$

with $g(\alpha) \neq 0$.

Theorem 1. *(See* [2, p.81]*) If α is a root of an equation $f(x) = 0$ with the multiplicity m, a function u defined by*

$$u(x) = \frac{f(x)}{f'(x)},$$

has α as a simple root.

Proof. From Definition 2,

$$u(x) = \frac{(x-\alpha)^m g(x)}{m(x-\alpha)^{m-1} g(x) + (x-\alpha)^m g'(x)} = (x-\alpha)\frac{g(x)}{mg(x) + (x-\alpha)g'(x)}.$$

Since $g(\alpha) \neq 0$, $u(x)$ has α as its simple root. □

2.2. **Development of the methods.** When finding a multiple root of f by finding a simple root of u, it seems very likely that using modified and improved Newton's method on u will give us more accurate results. Applying the trapezoidal Newton's method (2) rather than the original Newton's method (1) on u, we obtain a new method as follows:

$$x_{n+1} = x_n - \frac{2u(x_n)}{u'(x_n) + u'(x_n - \frac{u(x_n)}{u'(x_n)})}.$$

Substitution of the definition $u = f/f'$ and tedious differential operations give us the formula in terms of f, that is,

$$x_{n+1} = x_n - \frac{2f(x_n)/f'(x_n)}{2 - \frac{f(x_n)f''(x_n)}{f'(x_n)^2} - \frac{f(x^*_{n+1})f''(x^*_{n+1})}{f'(x^*_{n+1})^2}}, \tag{4}$$

where $x^*_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)}$. We will refer to (4) *the trapezoidal Newton's method for multiple roots*, briefly TNM.

Also, by applying the midpoint Newton's method (3) on u, we obtain

$$x_{n+1} = x_n - \frac{u(x_n)}{u'(x_n - \frac{u(x_n)}{2u'(x_n)})}.$$

Doing the same thing as in (4), we obtain

$$x_{n+1} = x_n - \frac{f(x_n)/f'(x_n)}{1 - \frac{f(x^*_{n+1})f''(x^*_{n+1})}{f'(x^*_{n+1})^2}}, \tag{5}$$

where $x^*_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{2(f'(x_n)^2 - f(x_n)f''(x_n))}$. We will refer to (5) *the midpoint Newton's method for multiple roots*, simply MNM.

Unlike many existing methods, two new methods do not require any information about the multiplicity of the root. Since we often do not know the multiplicity beforehand in most cases of solving equations, this characteristic may be found very useful in many cases. Obviously, these methods require the evaluation of first and second derivatives of f.

3. Order of convergence

Order of convergence is the most common and basic standard of how accurate an iterative method is. That means, the higher the order is, the faster the approximation by an iterative method converges to the exact root. Finding an order of a method is rather a tedious calculus. Among many ways, using Taylor's expansion is one of the most popular ways.

Notice that for Theorem 2 and Theorem 3, dots were used to indicate terms that have high orders of e_n, thus can be neglected in derivation of the error equation.

Theorem 2. *Let α be an exact root of f and its multiplicity m. Let n be an integer with $n \geq 0$, x_n an approximation after n iterations, $e_n = x_n - \alpha$ an error between the exact root and the nth iteration, and*

$$c_n = \frac{1}{(m+n)!}\frac{f^{(m+n)}(\alpha)}{f^{(m)}(\alpha)}. \tag{6}$$

Then the trapezoidal Newton's method for multiple roots(TNM) defined by (4) is of third order with an error equation

$$e_{n+1} = \frac{(5m+17)c_1^2 - 10mc_0c_2}{2m^2c_0^2}e_n^3 + \cdots. \tag{7}$$

Proof. By the Taylor's expansion for f about α, it follows that

$$f(x_n) = f^{(m)}(\alpha)(c_0e_n^m + c_1e_n^{m+1} + c_2e_n^{m+2} + c_3e_n^{m+3} + \cdots),$$
$$f'(x_n) = f^{(m)}(\alpha)\{mc_0e_n^{m-1} + (m+1)c_1e_n^m + (m+2)c_2e_n^{m+1} + (m+3)c_3e_n^{m+2} + \cdots\},$$

and

$$f''(x_n) = f^{(m)}(\alpha)\{m(m-1)c_0e_n^{m-2} + (m+1)mc_1e_n^{m-1} + (m+2)(m+1)c_2e_n^m + (m+3)(m+2)c_3e_n^{m+1} + \cdots\}.$$

Substituting these in terms of e_n for f and its derivatives in (4) will give us the proof. We have the following equations which are some key procedures:

$$f(x_n)f''(x_n) = \{f^{(m)}(\alpha)\}^2\{m(m-1)c_0^2e_n^{2m-2} + 2m^2c_0c_1e_n^{2m-1} + \big(2(m^2+m+1)c_0c_2 + (m^2+m)c_1^2\big)e_n^{2m} + \cdots\}, \tag{8}$$

$$\{f'(x_n)\}^2 = \{f^{(m)}(\alpha)\}^2\{m^2c_0^2e_n^{2m-2} + 2(m^2+m)c_0c_1e_n^{2m-1} + \big(2(m^2+2m)c_0c_2 + (m^2+2m+1)c_1^2\big)e_n^{2m} + \cdots\}, \tag{9}$$

$$f(x_n)f'(x_n) = \{f^{(m)}(\alpha)\}^2\{mc_0^2e_n^{2m-1} + (2m+1)c_0c_1e_n^{2m} + \big((2m+2)c_0c_2 + (m+1)c_1^2\big)e_n^{2m+1} + \cdots\}, \tag{10}$$

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m}e_n - \frac{c_1}{m^2c_0}e_n^2 + \{\frac{(m+1)c_1^2}{m^3c_0^2} - \frac{2c_2}{m^2c_0}\}e_n^3 + \cdots, \tag{11}$$

and

$$\frac{f(x_n)f''(x_n)}{f'(x_n)^2} = \frac{m-1}{m} + \frac{2c_1}{m^2c_0}e_n + 3\frac{2mc_0c_2 - (m+1)c_1^2}{m^3c_0^2}e_n^2 + \cdots. \tag{12}$$

By the definition $x_{n+1}^* = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)}$ and from (8), (9) and (10), we obtain

$$x_{n+1}^* = \alpha - \{\frac{c_1}{mc_0}e_n^2 + \frac{4c_0c_2 - 2c_1^2}{mc_0^2}e_n^3 + \frac{(2-9m)c_0c_1c_2 + 9mc_0^2c_3 + (3m-1)c_1^3}{m^2c_0^3}e_n^4 + \cdots\}.$$

The Taylor's expansions for f and its derivatives about x_{n+1}^* are

$$f(x_{n+1}^*) = f^{(m)}(\alpha)\{c_0(x_{n+1}^* - \alpha)^m + c_1(x_{n+1}^* - \alpha)^{m+1} + c_2(x_{n+1}^* - \alpha)^{m+2} + \cdots\}, \tag{13}$$

$$f'(x_{n+1}^*) = f^{(m)}(\alpha)\big(mc_0(x_{n+1}^*-\alpha)^{m-1} + (m+1)c_1(x_{n+1}^*-\alpha)^m + (m+2)c_2(x_{n+1}^*-\alpha)^{m+1} + \cdots\big), \tag{14}$$

and

$$f''(x_{n+1}^*) = f^{(m)}(\alpha)\{m(m-1)c_0(x_{n+1}^* - \alpha)^{m-2} + (m+1)mc_1(x_{n+1}^* - \alpha)^{m-1} + (m+2)(m+1)c_2(x_{n+1}^* - \alpha)^m + \cdots\}. \tag{15}$$

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From (13), (14), (15) and the definition of $e_n = x_n - \alpha$, we have

$$
\begin{aligned}
&f(x^*_{n+1})f''(x^*_{n+1}) \\
&= \{f^{(m)}(\alpha)\}^2\{\frac{(m-1)c_1^{2m-2}}{m^{2m-3}c_0^{2m-4}}e_n^{4m-4} + \frac{8(m-1)^2c_0c_1^{2m-3}c_2 - 4(m-1)^2c_1^{2m-1}}{m^{2m-3}c_0^{2m-3}}e_n^{4m-3} \\
&+ \frac{1}{m^{2m-2}c_0^{2m-2}}\big(16m(m-1)^2(2m-3)c_0^2c_1^{2m-4}c_2^2 + 18m(m-1)^2c_0^2c_1^{2m-3}c_3 \\
&- (m-1)(32m^3 - 62m^2 + 26m + 4)c_0c_1^{2m-2}c_2 + (8m^4 - 22m^3 + 18m^2 - 4m - 2)c_1^{2m}\big)e_n^{4m-2} \\
&+ \cdots\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\{f'(x^*_{n+1})\}^2 \\
&= \{f^{(m)}(\alpha)\}^2\{\frac{c_1^{2m-2}}{m^{2m-4}c_0^{2m-4}}e_n^{4m-4} + \frac{8(m-1)c_0c_1^{2m-3}c_2 - 4(m-1)c_1^{2m-1}}{m^{2m-4}c_0^{2m-3}}e_n^{4m-3} \\
&+ \frac{1}{m^{2m-2}c_0^{2m-2}}\Big(16m^2(m-1)(2m-3)c_0^2c_1^{2m-4}c_2^2 + 18m^2(m-1)c_0^2c_1^{2m-3}c_3 \\
&- m(32m^3 - 62m^2 + 26m + 4)c_0c_1^{2m-2}c_2 + (8m^4 - 14m^3 + 4m^2 - 2)c_1^{2m}\Big)e_n^{4m-2} + \cdots\}.
\end{aligned}
$$

Therefore,

$$
\frac{f(x^*_{n+1})f''(x^*_{n+1})}{f'(x^*_{n+1})^2} = \frac{m-1}{m} - 2\frac{c_1^2}{m^3c_0^2}e_n^2 + \cdots . \tag{16}
$$

From (11), (12) and (16), we obtain

$$
x_{n+1} = x_n - 2\frac{\frac{1}{m}e_n - \frac{c_1}{m^2c_0}e_n^2 + \frac{(m+1)c_1^2 - 2mc_0c_2}{m^3c_0^2}e_n^3 + \cdots}{\frac{2}{m} - \frac{2c_1}{m^2c_0}e_n - \frac{3(m+5)c_1^2 - 6mc_0c_2}{m^3c_0^2}e_n^2 + \cdots}.
$$

Definition of $e_n = x_n - \alpha$ gives us the error equation (7). □

Theorem 3. *Let α be an exact root of f and its multiplicity m. Let n be an integer with $n \geq 0$, x_n an approximation after n iterations, $e_n = x_n - \alpha$ an error between the exact root and the nth iteration, and c_n defined as in (6). Then, the midpoint Newton's method for multiple roots(MNM) defined by (5) is of third order with an error equation*

$$
e_{n+1} = \frac{(-16m^3 + 32m^2 - 15m - 3)c_1^2 + m(32m^2 - 64m + 30)c_0c_2}{4m^2c_0^2}e_n^3 + \cdots . \tag{17}
$$

Proof. Following the same procedures as in Theorem 2 with Taylor's expansions, (8) through (12) still hold. From (5), we have a slightly different definition of x^*_{n+1} from that in Theorem 2 as follows:

$$
x^*_{n+1} = \alpha - \big(-\frac{1}{2}e_n + \frac{c_1}{2mc_0}e_n^2 + \frac{2c_0c_2 - c_1^2}{mc_0^2}e_n^3 + \cdots\big).
$$

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Using the same principles as in (13) through (15) and multiplying (13) by (15) give us

$$f(x^*_{n+1})f''(x^*_{n+1}) = \{f^{(m)}(\alpha)\}^2\{\frac{m(m-1)c_0^2}{2^{2m-2}}e_n^{2m-2} - \frac{(2m^2-8m+4)c_0c_1}{2^{2m-1}}e_n^{2m-1} + \frac{\frac{17m^3-55m^2+48m-12}{m}c_1^2 - (30m^2-66m+30)c_0c_2}{2^{2m}}e^{2m} + \cdots\},$$

$$f'(x^*_{n+1})^2 = \{f^{(m)}(\alpha)\}^2\{\frac{m^2c_0^2}{2^{2m-2}}e_n^{2m} - \frac{(2m^2-6m)c_0c_1}{2^{2m-1}}e_n^{2m-1} + \frac{(33m^2-54m+17)c_1^2 - (62m^2-68m)c_0c_2}{2^{2m}}e_n^{2m} + \cdots\}.$$

Therefore, we have

$$\frac{f(x^*_{n+1})f''(x^*_{n+1})}{f'(x^*_{n+1})^2} = \frac{m-1}{m} + \frac{c_1}{m^2c_0}e_n + \frac{(-16m^3+32m^2-19m-7)c_1^2 + m(32m^2-64m+38)c_0c_2}{4m^3c_0^2}e_n^2 + \cdots. \tag{18}$$

Thus, from (11) and (18), we have

$$x_{n+1} = x_n - \frac{\frac{1}{m}e_n - \frac{c_1}{m^2c_0}e_n^2 + \{\frac{(m+1)c_1^2}{m^3c_0^2} - \frac{2c_2}{m^2c_0}\}e_n^3 + \cdots}{\frac{1}{m} - \frac{c_1}{m^2c_0}e_n + \frac{(16m^3-32m^2+19m+7)c_1^2 - m(32m^2-64m+38)c_0c_2}{4m^3c_0^2}e_n^2 + \cdots}.$$

Putting $e_n = x_n - \alpha$ gives us the error equation (17). □

4. Numerical comparisons

In this section, numerical experiments and comparisons between existing methods and our new methods for finding multiple roots are presented. These methods are all of cubic order.

Most of the existing methods require the multiplicity m of the root. We consider the following methods among them.

Halley's method [5] (HM):

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}.$$

Osada's method [7] (OM):

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)}.$$

Euler-Chebyshev method [9], [6] (ECM):

$$x_{n+1} = x_n + \frac{m(m-3)f(x_n)}{2f'(x_n)}\left(1 - \frac{m}{m-3}\frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2}\right).$$

Chun and Neta's method [3] (CNM):

$$x_{n+1} = x_n - \frac{2m^2\{f(x_n)\}^2 f''(x_n)}{m(3-m)f(x_n)f'(x_n)f''(x_n) + (m-1)^2\{f'(x_n)\}^3}.$$

Also we consider the *Petkovič and Džunič's method* [8] (PDM) constructed as follows:

$$x_{n+1} = x_n - \frac{2f(x_n)\{f'(x_n)\}^2 - 2\{f(x_n)\}^2 f''(x_n)}{2\{f'(x_n)\}^3 - 3f(x_n)f'(x_n)f''(x_n) + \{f(x_n)\}^2 f'''(x_n)}.$$

This method is a modified Newton's method which does not require the multiplicity of the desired root, and is the only one method of cubic order without knowing the multiplicity as we know.

The trapezoidal Newton's method for multiple roots (TNM) and the midpoint Newton's method for multiple roots (MNM) were presented in this paper by (4) and (5), respectively.

Table 1 shows the test functions used in the root-finding, approximations of the desired root and their multiplicity. We chose f_5 through f_8 from [3].

Table 1 Test functions, approximate roots and their multiplicity

test function	approximate root	multiplicity
$f_1(x) = x^2(x-1)^6(x-2)$	1	6
$f_2(x) = x^2(x-1)^{12}(x-2)$	1	12
$f_3(x) = (\log x)^5$	1	5
$f_4(x) = (\arctan x)^5$	0	5
$f_5(x) = (x^3 + 4x^2 - 10)^3$	1.37	3
$f_6(x) = (\sin^2 x - x^2 + 1)^2$	1.40	2
$f_7(x) = (\cos x - x)^3$	0.74	3
$f_8(x) = (\sin x - \frac{x}{2})^2$	1.90	2

Table 2 is displaying numbers of iterations needed to reach $\mid f(x_n) \mid < 10^{-1024}$ (denoted by N) for each method and each test function. The values in the parentheses are the absolute value of $f(x_n)$(denoted by FN) after such iterations. Note that absolute value of $f(x_n)$ indicates how near x_n is to the exact root for a same function. All computations were done using Mathematica, inserting inputs with significant figures large enough. Here * denotes where the approximation does not converge into the exact root.

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Table 2 Numbers of iterations N(absolute values of f, FN) of numerical methods

function	method						
initial pt.	HM	OM	ECM	CNM	PDM	TNM	MNM
f_1							
$x_0 = 0.7$	5(2e-1055)	6(4e-2701)	6(7e-2827)	6(2e-3025)	6(2e-2466)	6(2e-3024)	5(1e-1841)
$x_0 = 1.3$	5(6e-1186)	5(6e-1168)	5(6e-1173)	5(1e-1180)	6(2e-2781)	5(6e-1193)	5(2e-1442)
f_2							
$x_0 = 0.7$	5(3e-2597)	5(2e-2457)	5(7e-2478)	5(1e-2574)	5(4e-2084)	5(5e-2542)	4(1e-1145)
$x_0 = 1.3$	5(4e-2807)	5(7e-2791)	5(2e-2793)	5(1e-2804)	5(8e-2294)	5(3e-2814)	4(7e-1091)
f_3							
$x_0 = 0.8$	5(9e-1444)	5(3e-1153)	5(4e-1292)	5(8e-1657)	6(6e-2922)	5(5e-1222)	5(8e-1076)
$x_0 = 1.5$	5(1e-1125)	6(4e-2195)	6(7e-2686)	5(3e-1259)	6(1e-2365)	6(1e-3046)	6(6e-2609)
f_4							
$x_0 = 0.5$	6(1e-2148)	6(5e-1672)	6(5e-1821)	6(6e-1977)	6(4e-2267)	6(3e-2080)	6(6e-3042)
$x_0 = -1$	6(2e-1411)	*	7(6e-1295)	7(3e-2702)	6(2e-1238)	6(2e-1185)	6(3e-1517)
f_5							
$x_0 = 1$	6(2e-1649)	7(6e-2351)	6(1e-1128)	6(1e-1128)	6(7e-1875)	6(2e-1318)	6(1e-1715)
$x_0 = 2$	6(8e-1425)	7(7e-2922)	6(4e-1120)	6(4e-1120)	6(6e-1573)	7(5e-2796)	6(8e-1561)
f_6							
$x_0 = 2.3$	7(5e-1549)	7(6e-1029)	7(3e-1282)	7(4e-1131)	7(1e-1311)	7(3e-1252)	7(4e-1968)
$x_0 = 2$	7(5e-2084)	7(1e-1373)	7(9e-1716)	7(2e-1510)	7(5e-1828)	7(6e-1481)	7(2e-3025)
f_7							
$x_0 = 1$	6(7e-2343)	6(4e-2146)	6(2e-2231)	6(2e-2231)	6(6e-2085)	6(2e-2208)	5(7e-1032)
$x_0 = 1.7$	6(9e-1190)	6(2e-1095)	6(3e-1136)	6(3e-1136)	7(2e-2637)	6(9e-1217)	6(1e-1879)
f_8							
$x_0 = 1.7$	6(4e-1252)	7(2e-2037)	7(7e-2994)	7(5e-2479)	6(4e-1229)	6(1e-1077)	6(1e-1487)
$x_0 = 2$	6(2e-1756)	6(8e-1409)	6(1e-1586)	6(1e-1482)	6(3e-1695)	6(9e-1560)	6(5e-2201)

5. Conclusion

In this paper, new modifications of Newton's method for multiple roots were made by the trapezoidal rule and the midpoint rule, respectively, and the definition of $u = f/f'$. These methods were proved to be of third order, and the numerical comparisons with various methods of third order prove that the trapezoidal method(TNM) is highly competitive to other methods, while the midpoint method(MNM) is overwhelmingly powerful in most functions. The methods are somewhat complicated in the formula, however, that they produce relatively accurate results without requiring any information on multiplicity is an important feature that can be made extremely useful in many cases.

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ORTHOGONAL STABILITY OF A GENERALIZED QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN SPACES

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Abstract. In this paper, by using the fixed point method and the direct method, we prove the Hyers-Ulam stability of an orthogonally generalized quadratic functional equation in non-Archimedean normed spaces.

1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad \langle x, y\rangle = 0.$$

By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

Pinsker [41] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [51] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [19]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [48] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [49] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of Rätz (see [48]).

Suppose that X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

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(O_1) Totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;

(O_2) Independence: if $x, y \in X - \{0\}, x \perp y$, then x and y are linearly independent;

(O_3) Homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O_4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are as follows:

(1) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent;

(2) The ordinary orthogonality on an inner product space $(X, \langle \cdot, \cdot \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$;

(3) The Birkhoff-James orthogonality on a normed space $(X, \| \cdot \|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly, examples (1) and (2) are symmetric but example (3) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[3], [8, 15, 24, 25, 37]).

The stability problem of functional equations was originated from the following question of Ulam [53]:

Under what condition does there exist an additive mapping near an approximately additive mapping?

In 1941, Hyers [21] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [43] extended the theorem of Hyers by considering the unbounded Cauchy difference

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \quad \forall \varepsilon > 0, \, p \in [0, 1).$$

The result of Th.M. Rassias has provided a lot of influence in the development of what we now call *generalized Hyers-Ulam stability* or *Hyers-Ulam stability* of functional equations.

During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [11, 12, 22, 26, 47] and references therein for detailed information on stability of functional equations.

Ger and Sikorska [18] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$

for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was Skof [50] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. Cholewa [9] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by Czerwik [10] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [40], [44]–[46]).

The orthogonally quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x \perp y$$

was first investigated by Vajzović [54] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljević [16], Fochi [17], Moslehian [32, 33] and Szabó [52] generalized this result.

In [4], Baak et al. proved the Hyers-Ulam stability of the following generalized quadratic functional equation

$$(2_{d-2}C_{l-1} -_{d-2} C_l -_{d-2} C_{l-2})r^2 f\left(\frac{\sum_{j=1}^d x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^d \iota(j)=l}} r^2 f\left(\frac{\sum_{j=1}^d (-1)^{\iota(j)} x_j}{r}\right)$$

$$= (_{d-1}C_l +_{d-1} C_{l-1} + 2_{d-2}C_{l-1} -_{d-2} C_l -_{d-2} C_{l-2}) \sum_{j=1}^d f(x_j) \tag{1.1}$$

in Banach spaces.

In 1897, Hensel [20] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [13, 28, 29, 36]).

Definition 1.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

(a) $|r| = 0$ if and only if $r = 0$;
(b) $|rs| = |r||s|$;
(c) $|r + s| \leq \max\{|r|, |s|\}$.

Definition 1.2. ([35]) Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot|| : X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(1) $||x|| = 0$ if and only if $x = 0$;
(2) $||rx|| = |r|||x||$ for all $r \in \mathbb{K}$ and $x \in X$);

(3) The strong triangle inequality (ultrametric); namely,

$$||x+y|| \leq \max\{||x||, ||y||\}, \quad \forall x, y \in X.$$

Then $(X, ||\cdot||)$ is called *a non-Archimedean space.*

Note that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\}, \quad \forall n, m \geq 1 \ (n > m).$$

Definition 1.3. A sequence $\{x_n\}$ is a *Cauchy sequence* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a *complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

Let X be a set. A function $m : X \times X \to [0, \infty]$ is called a *generalized metric* on X if m satisfies the following conditions:

(a) $m(x, y) = 0$ if and only if $x = y$;
(b) $m(x, y) = m(y, x)$ for all $x, y \in X$;
(c) $m(x, z) \leq m(x, y) + m(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.4. ([5, 14]) *Let (X, m) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then, for each given element $x \in X$, either*

$$m(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$m(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid m(J^{n_0} x, y) < \infty\}$;*
(4) *$m(y, y^*) \leq \frac{1}{1-\alpha} m(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Th.M. Rassias [23] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 27, 31, 38, 39, 42]).

In this paper, we prove the Hyers-Ulam stability of the orthogonally generalized quadratic functional equation (1.1) in non-Archimedean normed spaces by using the fixed point method.

2. Hyers-Ulam stability of the orthogonally generalized quadratic functional equation (1.1) in non-Archimedean Banach spaces

Throughout this paper, assume that $(X, \perp)$ is an orthogonality space and $(Y, \|\cdot\|_Y)$ is a non-Archimedean Banach space. Assume that d and l are integers with $1 < l < d$ and r is a positive real number with $|r| \neq 1$.

For any mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x_1, \cdots, x_d) := \; & (2_{d-2}C_{l-1} -_{d-2} C_l -_{d-2} C_{l-2}) r^2 f\left(\frac{\sum_{j=1}^d x_j}{r}\right) \\ & + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^d \iota(j)=l}} r^2 f\left(\frac{\sum_{j=1}^d (-1)^{\iota(j)} x_j}{r}\right) \\ & -(_{d-1}C_l +_{d-1} C_{l-1} + 2_{d-2}C_{l-1} -_{d-2} C_l -_{d-2} C_{l-2}) \sum_{j=1}^d f(x_j) \end{aligned}$$

for all $x_1, \cdots, x_d \in X$ with $x_i \perp x_j$ $(i \neq j)$, where $\perp$ is the orthogonality in the sense of Rätz.

Using the fixed point method and applying some ideas from [18, 22], we prove the Hyers-Ulam stability of the orthogonally generalized quadratic functional equation $Df(x_1, \cdots, x_d) = 0$ in non-Archimedean normed spaces.

Theorem 2.1. *Let $\varphi : X^d \to [0, \infty)$ be a function such that there exists $\alpha < 1$ with*

$$\varphi(x_1, \cdots, x_d) \leq |r|^2 \alpha \varphi\left(\frac{x_1}{r}, \cdots, \frac{x_d}{r}\right) \tag{2.1}$$

for all $x_1, \cdots, x_d \in X$ with $x_i \perp x_j$ $(i \neq j)$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\|Df(x_1, \cdots, x_d)\|_Y \leq \varphi(x_1, \cdots, x_d) \tag{2.2}$$

for all $x_1, \cdots, x_d \in X$ with $x_i \perp x_j$ $(i \neq j)$. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{\alpha}{|4_{d-2}C_{l-1}|(1-\alpha)} \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}) \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x_1 = x$ and $x_2 = \cdots = x_d = 0$ in (2.2), we get

$$\begin{aligned} & \left\| (_{d-1}C_l +_{d-1} C_{l-1} + 2_{d-2}C_{l-1} -_{d-2} C_l -_{d-2} C_{l-2}) \left(r^2 f\left(\frac{x}{r}\right) - f(x)\right) \right\|_Y \\ \leq \; & \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}) \end{aligned}$$

for all $x \in X$ since $x \perp 0$. Since $_{d-1}C_l +_{d-1} C_{l-1} + 2_{d-2}C_{l-1} -_{d-2} C_l -_{d-2} C_{l-2} = 4_{d-2}C_{l-1}$,

$$\left\| 4_{d-2}C_{l-1} \left(r^2 f\left(\frac{x}{r}\right) - f(x)\right) \right\|_Y \leq \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}})$$

for all $x \in X$. So we have

$$\left\| f(x) - r^2 f\left(\frac{x}{r}\right) \right\|_Y \leq \frac{1}{|4_{d-2}C_{l-1}|} \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}) \tag{2.4}$$

for all $x \in X$. Thus it follows that

$$\begin{aligned} \left\| f(x) - \frac{1}{r^2} f(rx) \right\|_Y &\leq \frac{1}{|4_{d-2} C_{l-1}||r|^2} \varphi(rx, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}) \\ &\leq \frac{|r|^2 \alpha}{|4_{d-2} C_{l-1}||r|^2} \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}) \end{aligned} \tag{2.5}$$

for all $x \in X$. Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$m(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq \mu \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, m) is complete (see [30, Lemma 2.1]).

Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{r^2} g(rx)$$

for all $x \in X$. Let $g, h \in S$ be given such that $m(g,h) = \varepsilon$. Then we have

$$\|g(x) - h(x)\|_Y \leq \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}})$$

for all $x \in X$. Hence it follows that

$$\|Jg(x) - Jh(x)\|_Y = \left\| \frac{1}{r^2} g(rx) - \frac{1}{r^2} h(rx) \right\|_Y \leq \alpha \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}})$$

for all $x \in X$. So $m(g,h) = \varepsilon$ implies that $m(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$m(Jg, Jh) \leq \alpha m(g,h)$$

for all $g, h \in S$. It follows from (2.5) that $m(f, Jf) \leq \frac{\alpha}{|4_{d-2} C_{l-1}|}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(rx) = r^2 Q(x) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : m(h,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\|_Y \leq \mu \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}})$$

for all $x \in X$;

(2) $m(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty}\frac{1}{r^{2n}}f\left(r^n x\right)=Q(x)$$

for all $x\in X$;

(3) $m(f,Q)\leq\frac{1}{1-\alpha}m(f,Jf)$, which implies the inequality

$$m(f,Q)\leq\frac{\alpha}{|4_{d-2}C_{l-1}|(1-\alpha)}.$$

This implies that the inequality (2.3) holds. It follows from (2.1) and (2.2) that

$$\begin{aligned}\|DQ(x_1,\cdots,x_d)\|_Y &= \lim_{n\to\infty}\frac{1}{r^{2n}}\|Df(r^n x_1,\cdots,r^n x_d)\|_Y\\ &\leq \lim_{n\to\infty}\frac{1}{r^{2n}}\varphi(r^n x_1,\cdots,r^n x_d)\\ &\leq \lim_{n\to\infty}\frac{|r|^{2n}\alpha^n}{|r|^{2n}}\varphi(x_1,\cdots,x_d)\\ &= 0\end{aligned}$$

for all $x_1,\cdots,x_d\in X$ with $x_i\perp x_j$ $(i\neq j)$ and so

$$DQ(x_1,\cdots,x_d)=0 \tag{2.7}$$

for all $x_1,\cdots,x_d\in X$ with $x_i\perp x_j$ $(i\neq j)$. Hence $Q:X\to Y$ is an orthogonally generalized quadratic mapping. Letting $x_1=x$, $x_2=y$ with $x\perp y$ and $x_3=\cdots=x_d=0$ in (2.7), we get

$$\begin{aligned}&(2_{d-2}C_{l-1}-_{d-2}C_l-_{d-2}C_{l-2})r^2Q\left(\frac{x+y}{r}\right)\\ &+(_{d-2}C_l+_{d-2}C_{l-2})r^2Q\left(\frac{x+y}{r}\right)+2_{d-2}C_{l-1}r^2Q\left(\frac{x-y}{r}\right)\\ =\ &(_{d-1}C_l+_{d-1}C_{l-1}+2_{d-2}C_{l-1}-_{d-2}C_l-_{d-2}C_{l-2})(Q(x)+Q(y))\end{aligned}$$

for all $x,y\in X$ with $x\perp y$. So we have

$$\begin{aligned}&2_{d-2}C_{l-1}r^2\left(Q\left(\frac{x+y}{r}\right)+Q\left(\frac{x-y}{r}\right)\right)\\ =\ &(_{d-1}C_l+_{d-1}C_{l-1}+2_{d-2}C_{l-1}-_{d-2}C_l-_{d-2}C_{l-2})(Q(x)+Q(y))\end{aligned}$$

for all $x,y\in X$ with $x\perp y$. Since $_{d-1}C_l+_{d-1}C_{l-1}+2_{d-2}C_{l-1}-_{d-2}C_l-_{d-2}C_{l-2}=4_{d-2}C_{l-1}$,

$$r^2Q\left(\frac{x+y}{r}\right)+r^2Q\left(\frac{x-y}{r}\right)=2Q(x)+2Q(y) \tag{2.8}$$

for all $x,y\in X$ with $x\perp y$. Letting $y=0$ in (2.8), we get

$$2r^2Q\left(\frac{x}{r}\right)=2Q(x)$$

for all $x\in X$, since $x\perp 0$. So $Q\left(\frac{x}{r}\right)=\frac{1}{r^2}Q(x)$ for all $x\in X$. Thus it follows that

$$Q(x+y)+Q(x-y)=r^2Q\left(\frac{x+y}{r}\right)+r^2Q\left(\frac{x-y}{r}\right)=2Q(x)+2Q(y)$$

for all $x, y \in X$ with $x \perp y$. Hence the mapping $Q : X \to Y$ is orthogonally quadratic. this completes the proof. □

Corollary 2.2. *Assume that $(X, \perp)$ is an orthogonality non-Archimedean normed space. Let θ be a positive real number and p a real number with $p > 2$ when $r > 1$ and with $0 < p < 2$ when $0 < r < 1$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\|Df(x_1, \cdots, x_d)\|_Y \leq \theta \sum_{j=1}^{d} \|x_j\|^p$$

for all $x_1, \cdots, x_d \in X$ with $x_i \perp x_j$ $(i \neq j)$. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{|r|^p \theta}{|4|(|r|^2 - |r|^p)|_{d-2} C_{l-1}|} ||x||^p$$

for all $x \in X$.

Proof. Taking $\varphi(x_1, \cdots, x_d) = \theta \sum_{j=1}^{d} \|x_j\|^p$ for all $x_1, \cdots, x_d \in X$ with $x_i \perp x_j$ $(i \neq j)$ and choosing $\alpha = |r|^{p-2}$ in Theorem 2.1, we get the desired result. □

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SOME IDENTITIES ON THE EXTENDED CARLITZ'S q-BERNOULLI NUMBERS AND POLYNOMIALS

SEOG-HOON RIM, TAE-KYUN KIM, AND BYUNG-JE LEE

ABSTRACT. In [5], Kim introduced the p-adic q-integral on $\mathbb{Z}_p$ (= q-Volkenborn integral) and gave some Witt's formula for the Carlitz's q-Bernoulli numbers and polynomials. In this paper we investigate some properties of the extended Carlitz's q-Bernoulli numbers and polynomials by using p-adic q-integral on $\mathbb{Z}_p$. From these properties, we derive some interesting identities on the extended Carlitz's q-Bernoulli numbers and polynomials.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-v_p(p)} = 1/p$ (see [1-19]).

Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. We use the notation of q-number as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. In [2], Carlitz defined a set of numbers $\eta_k = \eta_k(q)$ inductively by

$$\eta_0 = 1, \quad q(q\eta+1)^k - \eta_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing ξ^k by ξ_k.

These numbers are q-analogues of ordinary Bernoulli number B_k, but they do not remain finite for $q = 1$. So he modified the definition as follows :

$$\beta_{0,q} = 1, \text{ and } \quad q(q\beta+1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing β^k by $\beta_{k,q}$ (see [2]). The Carlitz's q-Bernoulli polynomials are also defined by

$$\beta_{k,q}(x) = (q^x\beta + [x]_q)^k = \sum_{l=0}^{k} \binom{k}{l} \beta_{l,q} q^{lx} [x]_q^{k-l}. \tag{1}$$

As $q \to 1$, we have $\beta_{k,q} \to B_k$, where B_k and $B_k(x)$ are the ordinary Bernoulli numbers and polynomials, respectively (see [2,5]).

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral is constructed by Kim as follows :

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ (see [5,6])}. \tag{2}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then, by (2), we have

(3)

$$qI_q(f_n) = q\int_{\mathbb{Z}_p} f(x+n)d\mu_q(x) = I_q(f) + (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l),$$

$$\text{where } f'(l) = \left.\frac{df(x)}{dx}\right|_{x=l} \text{ , (see [5,6]).}$$

In [1], Carlitz have considered the extension of his q-Bernoulli numbers as follows : $h \in \mathbb{Z}$,

$$\beta_{0,q}^{(h)} = \frac{h}{[h]_q}, \text{ and } \quad q^h(q\beta^{(h)}+1)^n = \begin{cases} 1 & \text{if } n=1, \\ 0 & \text{if } n>1, \end{cases} \tag{4}$$

with the usual convention of replacing $(\beta^{(h)})^n$ by $\beta_{n,q}^{(h)}$. In this paper, $\beta_{n,q}^{(h)}$ are called the extended n-th Carlitz's q-Bernoulli numbers.

The Witt's formula for the Carlitz's q-Bernoulli are given by Kim (see [5]) as follows :

$$\int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \beta_{n,q} = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l+1}{[l+1]_q}. \tag{5}$$

Recently, Kim considered q-Bernstein operator of order n for $f \in C([0,1])$ as follows :

$$\begin{aligned} B_{n,q}(f|x) &= \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k} \\ &= \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x,q), \text{ where } n,k \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}. \end{aligned}$$

Here $B_{k,n}(x,q)$ are called the q-Bernstein polynomials of degree n (see [9]). In [9], the p-adic extension of $B_{k,n}(x,q)$ are given by

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k}, \text{ where } x \in \mathbb{Z}_p, \text{ and } n,k \in \mathbb{Z}_+. \tag{6}$$

In this paper, we investigate some properties of the extended Carlitz's q-Bernoulli numbers and polynomials by using p-adic q-integral on $\mathbb{Z}_p$. From these properties we derive some interesting identities on the extended Carlitz's q-Bernoulli numbers and polynomials.

2. Extended Carlitz's q-Bernoulli Numbers

Let $h \in \mathbb{Z}_+$. Then we consider the following (h,q)-Bernoulli polynomials by using p-adic q-integral :

$$\beta_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n q^{(h-1)y} d\mu_q(y), \ n \in \mathbb{Z}_+. \tag{7}$$

In the special case, $x = 0$, $\beta_{n,q}^{(h)}(0) = \beta_{n,q}^{(h)}$ are called the n-th (h,q)-Bernoulli numbers.

From (7), we can derive the following equation.

(8)

$$\begin{aligned}\beta_{n,q}^{(h)}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n q^{(h-1)y} d\mu_q(y) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+h}{[l+h]_q} \\ &= \frac{-n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{(l+1)x} \frac{1}{1-q^{l+1+h}} + (1-q)h \sum_{m=0}^{\infty} q^{hm} [x+m]_q^n \\ &= -n \sum_{m=0}^{\infty} q^{hm+x+m} [x+m]_q^{n-1} + (1-q)h \sum_{m=0}^{\infty} q^{hm} [x+m]_q^n \\ &= -n \sum_{m=0}^{\infty} q^{hm} [x+m]_q^{n-1} + (1-q)(n+h) \sum_{m=0}^{\infty} q^{hm} [x+m]_q^n.\end{aligned}$$

Therefore, by (8), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}\beta_{n,q}^{(h)}(x) &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+h}{[l+h]_q} \\ &= -n \sum_{m=0}^{\infty} q^{hm} [x+m]_q^{n-1} + (1-q)(n+h) \sum_{m=0}^{\infty} q^{hm} [x+m]_q^n.\end{aligned}$$

Let $F_q^{(h)}(t,x)$ be the generating function of the (h,q)-Bernoulli polynomials as follows :

$$F_q^{(h)}(t,x) = e^{\beta^{(h)}(x)t} = \sum_{n=0}^{\infty} \beta_{n,q}^{(h)}(x) \frac{t^n}{n!}, \tag{9}$$

with the usual convention about replacing $(\beta^{(h)}(x))^n$ by $\beta_{n,q}^{(h)}(x)$.

By (9) and Theorem 2.1, we easily get

$$F_q^{(h)}(t,x) = -t \sum_{m=0}^{\infty} q^{hm+x+m} e^{[x+m]_q t} + (1-q)h \sum_{m=0}^{\infty} q^{hm} e^{[x+m]_q t}. \tag{10}$$

Therefore by (10), we obtain the following theorem.

Theorem 2.2. *Let $F_q^{(h)}(t,x) = \sum_{n=0}^{\infty} \beta_{n,q}^{(h)}(x) \frac{t^n}{n!}$. Then we have*

$$F_q^{(h)}(t,x) = -t \sum_{m=0}^{\infty} q^{hm+x+m} e^{[x+m]_q t} + (1-q)h \sum_{m=0}^{\infty} q^{hm} e^{[x+m]_q t}.$$

Let $F_q^{(h)}(t,0) = F_q^{(h)}(t)$. Then we get

$$q^h F_q^{(h)}(t,1) - F_q^{(h)}(t) = t + (q-1)h. \tag{11}$$

By (11), we obtain the following functional equation :

Corollary 2.3.

$$q^h F_q^{(h)}(t,1) - F_q^{(h)}(t) = t + (q-1)h.$$

By Corollary 2.3, we get

$$\sum_{n=0}^{\infty}\{q^h\beta_{n,q}^{(h)}(1)-\beta_{n,q}^{(h)}\}\frac{t^n}{n!}=t+(q-1)h.$$

Thus we have

$$q^h\beta_{n,q}^{(h)}(1)-\beta_{n,q}^{(h)}=\begin{cases}(q-1)h & \text{if } n=0,\\ 1 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases} \tag{12}$$

Therefore, by (12), we obtain the following theorem.

Theorem 2.4. *For $n\in\mathbb{Z}_+$, we have*

$$\beta_{0,q}^{(h)}=\frac{h}{[h]_q},\ \textit{and}\quad q^h\beta_{n,q}^{(h)}(1)-\beta_{n,q}^{(h)}=\begin{cases}1 & \textit{if } n=1,\\ 0 & \textit{if } n>1.\end{cases}$$

From (7), (8) and (10), we note that

$$\begin{aligned}\beta_{n,q}^{(h)}(x)&=\sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\beta_{l,q}^{(h)}\\ &=(q^x\beta^{(h)}+[x]_q)^n,\end{aligned} \tag{13}$$

with the usual convention about replacing $(\beta^{(h)})^n$ by $\beta_{n,q}^{(h)}$.

By (7), we get

$$\begin{aligned}\beta_{n,q^{-1}}^{(h)}(1-x)&=\int_{\mathbb{Z}_p}[1-x+y]_{q^{-1}}^n q^{-(h-1)y}d\mu_{q^{-1}}(y)\\ &=\frac{1}{(1-q^{-1})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{-l+lx}\int_{\mathbb{Z}_p}q^{-(h+l-1)y}d\mu_{q^{-1}}(y)\\ &=(-1)^n\frac{q^{n+h-1}}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{h+l}{[h+l]_q}\\ &=(-1)^n q^{n+h-1}\beta_{n,q}^{(h)}(x).\end{aligned} \tag{14}$$

Therefore, by (14), we obtain the following theorem.

Theorem 2.5. *For $n\in\mathbb{Z}_+$, we have*

$$\beta_{n,q^{-1}}^{(h)}(1-x)=(-1)^n q^{n+h-1}\beta_{n,q}^{(h)}(x).$$

From (9), (10) and Theorem 2.5, we note that

$$F_{q^{-1}}^{(h)}(t,1-x)=q^{h-1}F_q^{(h)}(-qt,x). \tag{15}$$

Thus by (15), we obtain the following corollary.

Corollary 2.6.

$$F_{q^{-1}}^{(h)}(t,1-x)=q^{h-1}F_q^{(h)}(-qt,x).$$

By (13) and Theorem 2.4, we also get the following corollary.

Corollary 2.7.

$$\beta_{0,q}^{(h)} = \frac{h}{[h]_q}, \text{ and } \quad q^h(q\beta^{(h)}+1)^n - \beta_{n,q}^{(h)} = \begin{cases} 1 & \text{if } n=1, \\ 0 & \text{if } n>1, \end{cases}$$

From Corollary 2.7, we can derive the following equation (16).

$$\begin{aligned} & q^{2h}\beta_{n,q}^{(h)}(2) - \frac{h}{[h]_q}q^{2h} - nq^{h+1} + q^h\frac{h}{[h]_q} \\ &= q^{2h}\sum_{l=0}^{n}\binom{n}{l}q^l(1+q\beta^{(h)})^l - \frac{h}{[h]_q}q^{2h} - nq^{h+1} + q^h\frac{h}{[h]_q} \\ &= q^{2h}\sum_{l=1}^{n}\binom{n}{l}q^l(1+q\beta^{(h)})^l - nq^{h+1} + q^h\frac{h}{[h]_q} \\ &= q^{2h}\sum_{l=2}^{n}\binom{n}{l}q^l(1+q\beta^{(h)})^l + \binom{n}{1}q^{h+1}(1+\beta_{1,q}^{(h)}) - nq^{h+1} + q^h\frac{h}{[h]_q} \\ &= q^h\sum_{l=2}^{n}\binom{n}{l}q^l\beta_{l,q}^{(h)} + q^h\binom{n}{1}\beta_{1,q}^{(h)} + q^h\frac{h}{[h]_q} \\ &= q^h\sum_{l=0}^{n}\binom{n}{l}q^l\beta_{l,q}^{(h)} = q^h\beta_{n,q}^{(h)}(1) = \beta_{n,q}^{(h)} \quad \text{if } n>1. \end{aligned} \tag{16}$$

Therefore, by (16), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{Z}_+$ with $n>1$, we have*

$$\beta_{n,q}^{(h)}(2) = \frac{1}{q^{2h}}\beta_{n,q}^{(h)} + n\frac{1}{q^{h-1}} - \frac{1}{q^h}\frac{h}{[h]_q} + \frac{h}{[h]_q}.$$

By (2) and (3), we get

$$\begin{aligned} q^{h-1}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n q^{(h-1)x}d\mu_q(x) &= (-1)^n q^{n+h-1}\int_{\mathbb{Z}_p}[x-1]_q^n q^{(h-1)x}d\mu_q(x) \\ &= (-1)^n q^{n+h-1}\beta_{n,q}^{(h)}(-1) = \beta_{n,q^{-1}}^{(h)}(2). \end{aligned} \tag{17}$$

Thus, by (17), we obtain the following proposition.

Proposition 2.9. *For $n \in \mathbb{Z}_+$, we have*

$$q^{h-1}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n q^{(h-1)x}d\mu_q(x) = \beta_{n,q^{-1}}^{(h)}(2).$$

Let $n \in \mathbb{N}$ with $n>1$. Then by Theorem 2.8 and Proposition 2.9, we get

$$\begin{aligned} q^{h-1}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n q^{(h-1)x}d\mu_q(x) &= q^{2h}\beta_{n,q^{-1}}^{(h)} + nq^{h-1} - q^{2h-1}\frac{h}{[h]_q} + q^{h-1}\frac{h}{[h]_q} \\ &= q^{2h}\beta_{n,q^{-1}}^{(h)} + nq^{h-1} + q^{h-1}h(1-q). \end{aligned} \tag{18}$$

Therefore, by (18), we obtain the following theorem.

Theorem 2.10. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^{n} q^{(h-1)x} d\mu_q(x) = q^{h+1}\beta_{n,q^{-1}}^{(h)} + n + h(1-q).$$

For $x \in \mathbb{Z}$, and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \cdots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk + 1$. Then we take p-adic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials as follows :

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q) q^{(h-1)x} d\mu_q(x) \\ &= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n_1+\cdots+n_s-l} q^{(h-1)x} d\mu_q(x). \end{aligned} \tag{19}$$

From Theorem 2.10 and (19), we note that

(20)

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q) q^{(h-1)x} d\mu_q(x) \\ &= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\{q^{h+1}\beta_{n_1+\cdots+n_s-l,q^{-1}} + n_1 + \cdots + n_s - l + h(1-q)\} \\ &= \begin{cases} q^{h+1}\beta_{n_1+\cdots+n_s,q^{-1}} + n_1 + \cdots + n_s + h(1-q) & \text{if } k = 0, \\ q^{h+1}\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\beta_{n_1+\cdots+n_s-l,q^{-1}} & \text{if } k > 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q) q^{(h-1)x} d\mu_q(x) \\ &= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+\cdots+n_s-sk}\binom{n_1+\cdots+n_s-sk}{l}(-1)^l\beta_{sk+l,q}^{(h)}. \end{aligned} \tag{21}$$

Therefore, by (20) and (21), we obtain the following theorem.

Theorem 2.11. *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, \cdots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk + 1$. Then we have*

$$\begin{aligned} &\sum_{l=0}^{n_1+\cdots+n_s-sk}\binom{n_1+\cdots+n_s-sk}{l}(-1)^l\beta_{sk+l,q}^{(h)} \\ &= \begin{cases} q^{h+1}\beta_{n_1+\cdots+n_s,q^{-1}} + n_1 + \cdots + n_s + h(1-q) & \text{if } k = 0, \\ q^{h+1}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\beta_{n_1+\cdots+n_s-l,q^{-1}} & \text{if } k > 0. \end{cases} \end{aligned}$$

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Solving an inverse problem from bioluminescence tomography by minimizing an energy-like functional*

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Abstract

A new method is proposed for the reconstruction of the source function in bioluminescence tomography. In contrast with several existing methods [5, 13, 15], the main difference of our method is that the functional used for measuring the data fitting is a certain quantity defined over the whole domain under consideration, so that the finite element approximation is quite convenient to implement in practical applications. The theoretical analysis is developed for the new method, whose computational performance is shown by several numerical tests.

Keywords. Bioluminescence tomography; Inverse source problems; Finite element approximations

1 Introduction

In the last decade, people have witnessed rapid progress in the field of bioluminescence tomography. Briefly speaking, bioluminescence is the process of light emission in living organisms; bioluminescence tomography (BLT) is a technique for noninvasive study of ongoing biological processes in small laboratory animals, by use of native light emission from living organisms which bioluminesce. We refer to [6, 12, 17, 19, 21–26] and references therein for details about the biomedical background of BLT.

If the light emission from living organisms is described by a certain diffusion equation [18], then the BLT corresponds to an inverse source problem of this equation [1,5,8,9,13–15]. To be more precise, we are going to look for a light source function p in the following diffusion equation:

$$-\nabla \cdot (D\nabla u) + \mu u = p\chi_{\Omega_0} \quad \text{in } \Omega, \tag{1.1}$$

equipped with the boundary conditions:

$$D\frac{\partial u}{\partial \nu} = -g \quad \text{on } \Gamma, \tag{1.2}$$

$$2D\frac{\partial u}{\partial \nu} + u = g^- \quad \text{on } \Gamma. \tag{1.3}$$

Here $D := [3(\mu + \mu')]^{-1}$, μ and μ' are prescribed absorption and reduced scattering coefficients, the incoming flux g^- is the imposed data, and the function g is the measured data; $\frac{\partial}{\partial \nu}$ indicates the directional derivative along the unit outward normal to Γ, the boundary of Ω. Moreover, Ω_0 is a given measurable subset of Ω or Ω itself, and χ_{Ω_0} means the characteristic function of Ω_0. In terms of the right side term of (1.1), we actually assume the light source only exists inside the set Ω_0. The determination of a reasonable subset Ω_0 relies on

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a good understanding of the practical problem to be studied. If there is not enough *a priori* information for the problem, it's better to choose $\Omega_0 = \Omega$.

The above problems are a kind of typical ill-posed problems [10, 16], and one requires to use the regularization method to reconstruct the source function p feasibly [10, 20]. To the best of our knowledge, several effective methods were devised for solving such problems [5, 9, 13, 15]. The key point of these methods is to reformulate the inverse source problem under consideration as some minimization problems, with the functionals given by some functionals measuring the data fitting over the boundary Γ plus the $L^2(\Omega_0)$-norm regularization term.

Following [2], we propose in this paper a new method for the reconstruction of the source function in BLT. In contrast with the methods just mentioned, the main difference of our method is that the functional used for measuring the data fitting is a certain quantity related to $L^2(\Omega)$-norm, so that the finite element approximation is quite convenient to implement in practical applications. We establish theoretical analysis for our method, whose computational performance is illustrated by several numerical tests.

2 The new method in the continuous version

Let $\mathbf{\Omega} \subset \mathbb{R}^d$ $(d \leq 3)$ be a bounded domain with a Lipschitz boundary Γ. For the given data used in (1.1)-(1.3), suppose that $D \in L^\infty(\Omega), D \geq D_0$ a.e. in Ω for some constant $D_0 > 0$, and suppose that $\mu \in L^\infty(\Omega), \mu \geq 0$ a.e. in Ω, and μ is positive over a subset of Ω with positive measure. Also assume both the functions g and g^- belong to $L^2(\Gamma)$. To formulate our new method for solving the inverse source problem related to (1.1)-(1.3), we need to introduce the following two auxiliary problems:

$$\begin{cases} -\nabla \cdot (D\nabla u_1) + \mu u_1 = p\chi_{\Omega_0} & \text{in } \Omega, \\ D\frac{\partial u_1}{\partial \nu} = -g & \text{on } \Gamma. \end{cases}$$

$$\begin{cases} -\nabla \cdot (D\nabla u_2) + \mu u_2 = p\chi_{\Omega_0} & \text{in } \Omega, \\ 2D\frac{\partial u_2}{\partial \nu} + u_2 = g^- & \text{on } \Gamma. \end{cases}$$

For a function $q \in L^2(\Omega_0)$, the variational formulations of the above two problems with p replaced by q can be written respectively as follows:

Find $u_1 = u_1(q) \in H^1(\Omega)$ and $u_2 = u_2(q) \in H^1(\Omega)$ such that

$$\int_\Omega (D\nabla u_1 \cdot \nabla v + \mu u_1 v)dx = \int_{\Omega_0} qvdx - \int_\Gamma gvds \quad \forall\, v \in H^1(\Omega), \tag{2.1}$$

$$\int_\Omega (D\nabla u_2 \cdot \nabla v + \mu u_2 v)dx + \frac{1}{2}\int_\Gamma u_2 vds = \int_{\Omega_0} qvdx + \frac{1}{2}\int_\Gamma g^- vds \quad \forall\, v \in H^1(\Omega). \tag{2.2}$$

The unique existence of $u_1(q)$ and $u_2(q)$ is ensured by the well-known Lax-Milgram lemma [3, 11].

The problem (1.1)-(1.3), when p is given, is over-determined, so the solution is not always existent for general prescribed data. On the other hand, if the problem really has a solution for a certain function $p \in L^2(\Omega_0)$, we easily find out that $u_1(p) = u_2(p)$, in view of the unique solvability of the elliptic equations satisfying coercive conditions [11]. Based on this observation and following [2], we are tempted to reformulate the previous inverse source problem related to BLT as the following minimization problem:

Problem 2.1 *Find $p_\varepsilon \in Q_{ad}$ such that*

$$J_\varepsilon(p_\varepsilon) = \inf_{q\in Q_{ad}} J_\varepsilon(q),$$

where

$$J_\varepsilon(q) := \|u_1(q) - u_2(q)\|^2_{L^2(\Omega)} + \varepsilon \|q\|^2_{L^2(\Omega_0)}, \qquad \varepsilon \geq 0,$$

with $u_1(q)$ and $u_2(q)$ given respectively by (2.1) and (2.2), and Q_{ad} is a closed convex subset of the space $L^2(\Omega_0)$.

Remark 2.1 *As in [13], examples for Q_{ad} include $Q_{ad} = L^2(\Omega_0)$, or the subset of $L^2(\Omega_0)$ of non-negatively valued functions, or a finite-dimensional subspace or subset of a linear combinations of specified functions such as the characteristic functions of certain subsets of Ω.*

Remark 2.2 *In the definition of the objective functional $J_\varepsilon(q)$, we can also use the energy norm to measure the deviation between $u_1(q)$ and $u_2(q)$, instead of the L^2-norm used above. And all the derivations given below can be applied to this situation in the similar manners. However, the error bound for the numerical solution can not be improved any more via the new approach. Therefore, we here only consider the L^2-norm case intensively.*

For the theoretical study of the problem 2.1, we first produce a formula evaluating the Gâteaux derivative of the objective functional $J_\varepsilon(\cdot)$. In fact, by use of the principle of superposition, we know that $u_1(q)$ and $u_2(q)$ are two affine operators in q. Hence, for $p,\ q \in L^2(\Omega_0)$ and $t \in \mathbb{R}$,

$$u_i(p + tq) - u_i(p) = t(u_i(p + q) - u_i(p)),\ i = 1, 2; \tag{2.3}$$

and for $p_1,\ p_2,\ q \in L^2(\Omega_0)$,

$$u_i(p_1 + q) - u_i(p_2) = u_i(p_1) - u_i(p_2 - q),\ i = 1, 2. \tag{2.4}$$

Let

$$J_1(p) = \|u_1(p) - u_2(p)\|^2_{L^2(\Omega)}, \qquad J_2(p) = \|p\|^2_{L^2(\Omega_0)}.$$

Then

$$J_\varepsilon(p) = J_1(p) + \varepsilon J_2(p).$$

By the definition of the Gâteaux derivative and the product rule for Gâteaux derives (cf. [3, pp. 225–227]), it follows from the identity (2.3) that

$$\begin{aligned} J_1'(p)q =& \frac{d}{dt}\Big|_{t=0} [\|u_1(p + tq) - u_2(p + tq)\|^2_{L^2(\Omega)}] = 2\langle u_1(p) - u_2(p), u_1'(p)q - u_1'(p)q\rangle_{L^2(\Omega)} \\ =& 2\langle u_1(p) - u_2(p), (u_1(p + q) - u_1(p)) - (u_2(p + q) - u_2(p))\rangle_{L^2(\Omega)}. \end{aligned}$$

It is easy to get

$$J_2'(p)q = \frac{d}{dt}\Big|_{t=0} [\|p + tq\|^2_{L^2(\Omega_0)}] = 2\langle p, q\rangle_{L^2(\Omega_0)}.$$

Hence,

$$\begin{aligned} J_\varepsilon'(p)q =& J_1'(p)q + \varepsilon J_2'(p)q \\ =& 2\left\langle u_1(p) - u_2(p), (u_1(p + q) - u_1(p)) - (u_2(p + q) - u_2(p))\right\rangle_{L^2(\Omega)} \\ &+ 2\varepsilon \left\langle p, q\right\rangle_{L^2(\Omega_0)}. \end{aligned} \tag{2.5}$$

For ease of exposition, we simply write $\widetilde{u}(q) = u_1(q) - u_2(q)$ in what follows. Our first result is concerned with the unique solvability of problem 2.1 and the characterization of the solution.

Theorem 2.1 *For any $\varepsilon > 0$, problem 1 has a unique solution $p_\varepsilon \in Q_{ad}$. Moreover, the solution $p_\varepsilon \in Q_{ad}$ is characterized by a variational inequality*

$$\langle \widetilde{u}(p_\varepsilon), \widetilde{u}(q) - \widetilde{u}(p_\varepsilon) \rangle_{L^2(\Omega)} + \varepsilon \langle p_\varepsilon, q - p_\varepsilon \rangle_{L^2(\Omega_0)} \geq 0 \quad \forall\, q \in Q_{ad}. \tag{2.6}$$

When $Q_{ad} \subset L^2(\Omega_0)$ is a subspace, the inequality is reduced to a variational equation

$$\langle \widetilde{u}(p_\varepsilon), \widetilde{u}(q) - \widetilde{u}(0) \rangle_{L^2(\Omega)} + \varepsilon \langle p_\varepsilon, q \rangle_{L^2(\Omega_0)} = 0 \quad \forall\, q \in Q_{ad}. \tag{2.7}$$

Proof. Note that $Q_{ad} \subset L^2(\Omega_0)$ is convex and closed and $J_\varepsilon : Q_{ad} \to \mathbb{R}$ is strictly convex, continuous, and coercive, i.e., $J_\varepsilon(q) \to \infty$ as $\|q\|_{L^2(\Omega_0)} \to \infty$. Hence, by the standard theory on convex minimization (cf. [3, pp. 135–137]), there is a unique solution $p_\varepsilon \in Q_{ad}$ to the problem 2.1, and the solution is characterized by the relation (cf. [3, p. 233])

$$J_\varepsilon^{'}(p_\varepsilon)(q - p_\varepsilon) \geq 0 \quad \forall\, q \in Q_{ad},$$

which, in conjunction with (2.5), leads to the variational inequality (2.6).

Now assume $Q_{ad} \subset L^2(\Omega_0)$ is a subspace. We easily have $J_\varepsilon^{'}(p_\varepsilon)(q) \geq 0$ and $J_\varepsilon^{'}(p_\varepsilon)(-q) \geq 0$ for all $q \in Q_{ad}$. So

$$J_\varepsilon^{'}(p_\varepsilon)q = 0 \quad \forall\, q \in Q_{ad}.$$

Hence, the relation (2.7) is a direct consequence of the last equation and the identity (2.4). ■

Denote by $S_0 \in Q_{ad}$ the solution set of the problem 2.1 in the case $\varepsilon = 0$. Arguing as in the proof of Proposition 3.5 and Corollary 3.6 in [13],we readily have the following result:

Proposition 2.1 *Assume S_0 is non-empty. Then S_0 is closed and convex. Moreover,*

$$p_\varepsilon \to p_0 \ \text{ in } \ L^2(\Omega_0) \text{ as } \ \varepsilon \to 0^+,$$

where $p_0 \in S_0$ is the solution of problem 2.1 for $\varepsilon = 0$ with minimal $L^2(\Omega_0)$ norm:

$$p_0 = \inf_{q \in S_0} \|q\|_{L^2(\Omega_0)}.$$

Furthermore, if $S_0 = \{p\}$ is a singleton, then

$$p_\varepsilon \to p_0 \in \ L^2(\Omega_0) \qquad \text{as } \varepsilon \to 0.$$

3 Finite element approximations

Now, we turn to the numerical solution of problem 2.1. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of finite element partitions of $\bar{\Omega}$, with h being the mesh size of the partition $\mathcal{T}_h$. Assume that each element at the boundary Γ has at most one non-straight face (for a three-dimensional domain) or side (for a two-dimensional domain). With each partition $\mathcal{T}_h = \{K\}$, we associate the usual conforming linear element space $V^h \subset H^1(\Omega)$. For any $q \in L^2(\Omega_0)$, let

$u_1^h = u_1^h(q) \in V^h$ and $u_2^h = u_2^h(q) \in V^h$ be the solutions of the following two problems, respectively:

$$\int_\Omega (D\nabla u_1^h \cdot \nabla v^h + \mu u_1^h v^h)dx = \int_{\Omega_0} q v^h dx - \int_\Gamma g v^h ds \quad \forall\, v^h \in V^h,$$

$$\int_\Omega (D\nabla u_2^h \cdot \nabla v^h + \mu u_2^h v^h)dx + \frac{1}{2}\int_\Gamma u_2^h v^h ds = \int_{\Omega_0} q v^h dx + \frac{1}{2}\int_\Gamma g^- v^h ds \quad \forall\, v^h \in V^h.$$

The unique existence of $u_1^h(q)$ and $u_2^h(q)$ are ensured by the Lax-Milgram lemma. Let

$$J_\varepsilon^h(q) := \|u_1^h(q) - u_2^h(q)\|^2_{L^2(\Omega)} + \varepsilon\|q\|^2_{L^2(\Omega_0)}, \qquad \varepsilon \geq 0.$$

The admissible source function space Q_{ad} may or may not need to be discretized. In general, let $Q_{ad,1} \subset Q_{ad}$ be non-empty, closed and convex. Later in the section, we will consider two possible choices of $Q_{ad,1}$. We then introduce the following discretization of problem 2.1:

Problem 3.1 *Find $p_\varepsilon^h \in Q_{ad,1}$ such that*

$$J_\varepsilon^h(p_\varepsilon^h) = \inf_{q\in Q_{ad,1}} J_\varepsilon^h(q).$$

In the rest of the paper, we denote $\widetilde{u}^h(p) = u_1^h(p) - u_2^h(p)$ to simplify the exposition. Moreover, for any two quantities a and b, $a \lesssim b$ abbreviates $a \leq C\, b$, with $C > 0$ a generic constant independent of ε and the finite element mesh size h, which may take different values at different appearances. Then, applying the similar arguments for deriving Theorem 2.1 to the above problem, we have

Theorem 3.1 *For any $\varepsilon > 0$, problem 3.1 has a unique solution $p_\varepsilon^h \in Q_{ad,1}$, which is characterized by the discrete variational inequality*

$$\langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}^h(q) - \widetilde{u}^h(p_\varepsilon^h)\rangle_{L^2(\Omega)} + \varepsilon\langle p_\varepsilon^h, q - p_\varepsilon^h\rangle_{L^2(\Omega_0)} \geq 0 \quad \forall\, q \in Q_{ad,1}. \tag{3.1}$$

When $Q_{ad,1} \subset L^2(\Omega_0)$ is a subspace, the inequality is reduced to a variational equation

$$\langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}^h(q) - \widetilde{u}^h(0)\rangle_{L^2(\Omega)} + \varepsilon\langle p_\varepsilon^h, q\rangle_{L^2(\Omega_0)} = 0 \quad \forall\, q \in Q_{ad,1}.$$

We can also get a discrete analogue of Proposition 2.1, given below.

Proposition 3.1 *Suppose problem 3.1 for $\varepsilon = 0$ has a solution, and denote S_0^h by the solution set. Then $S_0^h \subset Q_{ad,1}$ is closed and convex, and $p_\varepsilon^h \to p_0^h$ in $L^2(\Omega_0)$ as $\varepsilon \to 0$, where $p_0^h \in S_0^h$ satisfies: $\|p_0^h\|_{L^2(\Omega_0)} = \inf_{q\in S_0^h} \|q\|_{L^2(\Omega_0)}$.*

We comment that, if $Q_{ad,1}$ is a bounded set, then S_0^h is non-empty by the arguments in [13, pp. 1666–1667]. In concrete situations, it is possible to show the non-empty of the solution set S_0^h directly.

Next, let us consider error estimation and convergence analysis of the finite element approximation (3.1). For this propose, we additionally assume

$$\Gamma \in C^{1,1}, \quad D \in C^1(\bar{\Omega}), \quad g,\ g^- \in H^{1/2}(\Gamma).$$

We note that the condition $\Gamma \in C^{1,1}$ can also be replaced by the condition that the domain Ω is a polyhedral domain. Then we have the solution regularity bound

$$\|u_1(q)\|_{H^2(\Omega)} \lesssim \|q\|_{L^2(\Omega_0)} + \|g\|_{H^{1/2}(\Gamma)}\,, \tag{3.2}$$

$$\|u_2(q)\|_{H^2(\Omega)} \lesssim \|q\|_{L^2(\Omega_0)} + \|g^-\|_{H^{1/2}(\Gamma)}. \tag{3.3}$$

We will make use of the following finite element interpolation error estimate (cf. [4,7,13]):

$$\|u - \Pi_{V^h} u\|_{L^2(\Omega)} + h\,\|u - \Pi_{V^h} u\|_{H^1(\Omega)} \lesssim h^2\,\|u\|_{H^2(\Omega)} \qquad \forall\, u \in H^2(\Omega), \tag{3.4}$$

where $\Pi_{V^h} u \in V^h$ is the piecewise linear interpolant of u.

The following lemma collects some standard estimates for the conforming finite element method (cf. [4, 7]):

Lemma 3.1 *For any $q \in L^2(\Omega_0)$, we have the error bound*

$$\|u_1(q) - u_1^h(q)\|_{L^2(\Omega)} \lesssim h^2\big(\|q\|_{L^2(\Omega_0)} + \|g\|_{H^{1/2}(\Gamma)}\big), \tag{3.5}$$

$$\|u_2(q) - u_2^h(q)\|_{L^2(\Omega)} \lesssim h^2\big(\|q\|_{L^2(\Omega_0)} + \|g^-\|_{H^{1/2}(\Gamma)}\big). \tag{3.6}$$

To proceed further, we distinguish two cases for $Q_{ad,1}$. In the first case, we take $Q_{ad,1} = Q_{ad}$. This is the natural choice when Q_{ad} is a finite-dimensional subspace or subset of linear combinations of specified functions such as the characteristic functions of certain subsets of Ω. Thus, we can establish error estimates of problem 3.1 based on (2.6) and (3.1), given below.

Theorem 3.2 *With the choice $Q_{ad,1} = Q_{ad}$, it holds that*

$$\big\|\widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\big\|^2_{L^2(\Omega)} + \varepsilon\big\|p_\varepsilon - p_\varepsilon^h\big\|^2_{L^2(\Omega_0)} \lesssim h^2 A_\varepsilon^h A_\varepsilon,$$

where

$$A_\varepsilon^h := \|p_\varepsilon^h\|_{L^2(\Omega_0)} + \|g\|_{H^{1/2}(\Gamma)} + \|g^-\|_{H^{1/2}(\Gamma)},$$
$$A_\varepsilon := \|p_\varepsilon\|_{L^2(\Omega_0)} + \|g\|_{H^{1/2}(\Gamma)} + \|g^-\|_{H^{1/2}(\Gamma)}.$$

Proof. Choosing $q = p_\varepsilon$ in (3.1) and $q = p_\varepsilon^h$ in (2.6), we obtain

$$\langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}^h(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h))\rangle_{L^2(\Omega)} + \varepsilon\langle p_\varepsilon^h, p_\varepsilon - p_\varepsilon^h\rangle_{L^2(\Omega_0)} \geq 0,$$
$$\langle \widetilde{u}(p_\varepsilon), \widetilde{u}(p_\varepsilon^h) - \widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)} + \varepsilon\langle p_\varepsilon, p_\varepsilon^h - p_\varepsilon\rangle_{L^2(\Omega_0)} \geq 0.$$

The summation of the last two inequalities implies

$$\varepsilon\|p_\varepsilon - p_\varepsilon^h\|^2_{L^2(\Omega_0)} \leq \langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}^h(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\rangle_{L^2(\Omega)} + \langle \widetilde{u}(p_\varepsilon), \widetilde{u}(p_\varepsilon^h) - \widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)}.$$

Then, adding the term $\big\|\widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\big\|^2_{L^2(\Omega)}$ to both sides of the above inequality and reformulating terms, we get

$$\begin{aligned}
&\big\|\widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\big\|^2_{L^2(\Omega)} + \varepsilon\big\|p_\varepsilon - p_\varepsilon^h\big\|^2_{L^2(\Omega_0)} \\
&\leqslant \big\|\widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\big\|^2_{L^2(\Omega)} + \langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}^h(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\rangle_{L^2(\Omega)} \\
&\quad + \langle \widetilde{u}(p_\varepsilon), \widetilde{u}(p_\varepsilon^h) - \widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)} \\
&= \langle \widetilde{u}(p_\varepsilon), \widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\rangle_{L^2(\Omega)} - \langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\rangle_{L^2(\Omega)} \\
&\quad + \langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}^h(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\rangle_{L^2(\Omega)} + \langle \widetilde{u}(p_\varepsilon), \widetilde{u}(p_\varepsilon^h) - \widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)} \\
&= \langle \widetilde{u}(p_\varepsilon), \widetilde{u}(p_\varepsilon^h) - \widetilde{u}^h(p_\varepsilon^h)\rangle_{L^2(\Omega)} + \langle \widetilde{u}^h(p_\varepsilon^h), \widetilde{u}^h(p_\varepsilon) - \widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)}.
\end{aligned}$$

Hence, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\left\|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^h)\right\|^2_{L^2(\Omega)}+\varepsilon\left\|p_\varepsilon-p_\varepsilon^h\right\|^2_{L^2(\Omega_0)} &\leqslant \|\widetilde{u}(p_\varepsilon)\|_{L^2(\Omega)}\left\|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^h)\right\|_{L^2(\Omega)}\\ &\quad+\left\|\widetilde{u}^h(p_\varepsilon^h)\right\|_{L^2(\Omega)}\left\|\widetilde{u}^h(p_\varepsilon)-\widetilde{u}(p_\varepsilon)\right\|_{L^2(\Omega)}\\ &=: A_1B_1+A_2B_2, \end{aligned}\tag{3.7}$$

where

$$\begin{aligned} A_1 &:= \|\widetilde{u}(p_\varepsilon)\|_{L^2(\Omega)}, \quad B_1 := \left\|\widetilde{u}(p_\varepsilon^h)-\widetilde{u}^h(p_\varepsilon^h)\right\|_{L^2(\Omega)},\\ A_2 &= \left\|\widetilde{u}^h(p_\varepsilon^h)\right\|_{L^2(\Omega)}, \quad B_2 := \left\|\widetilde{u}(p_\varepsilon)-\widetilde{u}(p_\varepsilon)\right\|_{L^2(\Omega)}. \end{aligned}$$

On the other hand, we have by (3.2), (3.3), and (3.5) that

$$A_1 \leqslant \|u_1(p_\varepsilon)\|_{L^2(\Omega)}+\|u_2(p_\varepsilon)\|_{L^2(\Omega)} \lesssim A_\varepsilon,$$

$$\begin{aligned}\|u_1^h(q)\|_{L^2(\Omega)} &\leqslant \|u_1^h(q)-u_1(q)\|_{L^2(\Omega)}+\|u_1(q)\|_{L^2(\Omega)}\\ &\lesssim \|q\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)}+h^2(\|q\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)})\\ &\lesssim \|q\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)}. \end{aligned}\tag{3.8}$$

Similarly, by (3.3) and (3.6),

$$\|u_2^h(q)\|_{L^2(\Omega)} \lesssim \|q\|_{L^2(\Omega_0)}+\|g^-\|_{H^{1/2}(\Gamma)}.\tag{3.9}$$

Therefore, it follows from (3.8) and (3.9) that

$$A_2 \leqslant \|u_1^h(p_\varepsilon^h)\|_{L^2(\Omega)}+\|u_2^h(p_\varepsilon^h)\|_{L^2(\Omega)} \lesssim A_\varepsilon^h.$$

By use of (3.5) and (3.6), we have

$$\begin{aligned} B_1 &\leqslant \|u_1(p_\varepsilon^h)-u_1^h(p_\varepsilon^h)\|_{L^2(\Omega)}+\|u_2(p_\varepsilon^h)-u_2^h(p_\varepsilon^h)\|_{L^2(\Omega)}\\ &\lesssim h^2(\|p_\varepsilon^h\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)})+h^2(\|p_\varepsilon^h\|_{L^2(\Omega_0)}+\|g^-\|_{H^{1/2}(\Gamma)})\\ &\lesssim h^2(\|p_\varepsilon^h\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)}+\|g^-\|_{H^{1/2}(\Gamma)})=h^2A_\varepsilon^h, \end{aligned}$$

$$\begin{aligned} B_2 &\leqslant \|u_1(p_\varepsilon)-u_1^h(p_\varepsilon)\|_{L^2(\Omega)}+\|u_2(p_\varepsilon)-u_2^h(p_\varepsilon)\|_{L^2(\Omega)}\\ &\lesssim h^2(\|p_\varepsilon\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)})+h^2(\|p_\varepsilon\|_{L^2(\Omega_0)}+\|g^-\|_{H^{1/2}(\Gamma)})\\ &\lesssim h^2(\|p_\varepsilon\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)}+\|g^-\|_{H^{1/2}(\Gamma)})=h^2A_\varepsilon. \end{aligned}$$

Inserting the bound of the quantities A_1, A_2, B_1, and B_2 into the inequality (3.7), we obtain

$$\left\|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^h)\right\|^2_{L^2(\Omega)}+\varepsilon\left\|p_\varepsilon-p_\varepsilon^h\right\|^2_{L^2(\Omega_0)} \leqslant A_1B_1+A_2B_2 \lesssim h^2A_\varepsilon^hA_\varepsilon,$$

as required. ■

Corollary 3.1 *Assume Q_{ad} is a bounded set in $L^2(\Omega_0)$. Then with the choice $Q_{ad,1} = Q_{ad}$, it holds that*

$$\left\|\widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^h)\right\|_{L^2(\Omega)} + \varepsilon^{1/2}\left\|p_\varepsilon - p_\varepsilon^h\right\|_{L^2(\Omega_0)} \lesssim h.$$

Proof. Since Q_{ad} is a bounded set in $L^2(\Omega)$, $\|p_\varepsilon\|_{L^2(\Omega_0)}$ and $\|p_\varepsilon^h\|_{L^2(\Omega_0)}$ are uniformly bounded with respect to ε and h. Then this corollary follows directly from Theorem 3.2. ∎

Next, we consider another choice of $Q_{ad,1}$, which is a general closed convex subset of $L^2(\Omega_0)$ and requires further discretization in actual computation. Due to some technical reason, we only consider the case

$$Q_{ad} := \left\{q \in L^2(\Omega_0);\ \alpha_1 \le q(x) \le \alpha_2 \quad \text{a.e. } x \in \Omega\right\}, \tag{3.10}$$

where α_1 and α_2 are two given real constants. We note that if $\alpha_1 = -\infty$ (resp. $\alpha_2 = +\infty$), the lower (resp. upper) bound in the definition (3.10) is dropped out naturally. This choice is frequently encountered when we want to reconstruct a physical parameter through measurement data. In addition to the regular family of finite-element partitions $\{\mathcal{T}_h\}$ of $\bar{\Omega}$, let $\{\mathcal{T}_{0,H}\}$ be a regular family of finite-element partitions of $\bar{\Omega}_0$ such that each element at the boundary $\partial\Omega_0$ has at most one non-straight face (for a three-dimensional domain) or side (for a two-dimensional domain). The partitions $\mathcal{T}_h$ and $\mathcal{T}_{0,H}$ do not need to be related; however, $\mathcal{T}_h$ is allowed to be constructed on the basis of $\mathcal{T}_{0,H}$. Let $Q^H \subset L^2(\Omega_0)$ be the piecewise constant space. Let $Q_{ad,1} = Q_{ad}^H := Q^H \cap Q_{ad}$. In this case, denote the solution of problem 3.1 by $p_\varepsilon^{h,H}$.

Denote by Π^H the orthogonal projection operator from $L^2(\Omega_0)$ onto Q^H, i.e., for $q \in L^2(\Omega_0)$,

$$\Pi^H q \in Q^H, \qquad \left(\Pi^H q, q^H\right)_{L^2(\Omega_0)} = \left(q, q^H\right)_{L^2(\Omega_0)} \quad \forall\, q^H \in Q^H. \tag{3.11}$$

For this operator, it is well-known that (cf. [13])

$$\left\|q - \Pi^H q\right\|_{L^2(\Omega_0)} \lesssim H\,|q|_{H^1(\Omega_0)} \qquad \forall\, q \in H^1(\Omega_0). \tag{3.12}$$

And it is also easy to verify the elementwise formula:

$$\Pi^H q|_K = \frac{1}{|K|}\int_K q dx \qquad \forall\, K \in \mathcal{T}_{0,H}.$$

Consequently, we can easily prove that $\Pi^H : Q_{ad} \to Q_{ad}^H$, i.e., for $q \in Q_{ad}$, its piecewise constant orthogonal projection $\Pi^H q \in Q_{ad}^H$.

Some norm estimates, given below, are useful in the error analysis of problem 3.1 related to the choice of (3.10).

Lemma 3.2 *For any $q \in L^2(\Omega_0)$, it holds that*

$$\left\|u_1^h(q) - u_1^h(\Pi^H q)\right\|_{H^1(\Omega)} \lesssim H\left\|q - \Pi^H q\right\|_{L^2(\Omega_0)}, \tag{3.13}$$

$$\left\|u_2^h(q) - u_2^h(\Pi^H q)\right\|_{H^1(\Omega)} \lesssim H\left\|q - \Pi^H q\right\|_{L^2(\Omega_0)}, \tag{3.14}$$

$$\left\|u_1(q) - u_1^h(\Pi^H q)\right\|_{L^2(\Omega)} \lesssim h^2\left(\|q\|_{L^2(\Omega_0)} + \|g\|_{H^{1/2}(\Gamma)}\right) + (h^2 + H)\left\|q - \Pi^H q\right\|_{L^2(\Omega_0)}, \tag{3.15}$$

$$\left\|u_2(q) - u_2^h(\Pi^H q)\right\|_{L^2(\Omega)} \lesssim h^2\left(\|q\|_{L^2(\Omega_0)} + \|g^-\|_{H^{1/2}(\Gamma)}\right) + (h^2 + H)\left\|q - \Pi^H q\right\|_{L^2(\Omega_0)}. \tag{3.16}$$

Proof. Using Lemma 4.7 in [13] and its proof, we can easily get the first two inequalities. For the derivation of inequality (3.15), let η be a function in $H^1(\Omega)$ satisfying that

$$a_1(v,\eta) := \int_\Omega (D\nabla v \cdot \nabla \eta + \mu v \eta) dx = \int_\Omega \big(u_1^h(q) - u_1^h(\Pi^H q)\big) v dx \quad \forall\, v \in H^1(\Omega).$$

The unique existence of the function η is a direct consequence of the Lax-Milgram lemma, and moreover it holds the estimate

$$\|\eta\|_{H^2(\Omega)} \lesssim \big\|u_1^h(q) - u_1^h(\Pi^H q)\big\|_{L^2(\Omega)}. \tag{3.17}$$

Choosing $v = u_1^h(q) - u_1^h(\Pi^H q) \in V^h \subset H^1(\Omega)$ in the last equation, we get

$$\big\|u_1^h(q) - u_1^h(\Pi^H q)\big\|^2_{L^2(\Omega)} = a_1\big(u_1^h(q) - u_1^h(\Pi^H q), \eta\big). \tag{3.18}$$

we have by the definition of $u_1^h(q)$ that

$$a_1\big(u_1^h(q), v^h\big) = \int_{\Omega_0} q v^h dx - \int_\Gamma g v^h ds \quad \forall\, v^h \in V^h,$$

$$a_1(u_1^h(\Pi^H q), v^h) = \int_{\Omega_0} \Pi^H q \cdot v^h dx - \int_\Gamma g v^h ds \quad \forall\, v^h \in V^h,$$

which implies

$$a_1\big(u_1^h(q) - u_1^h(\Pi^H q), v^h\big) = \int_{\Omega_0} (q - \Pi^H q) v^h dx \quad \forall\, v^h \in V^h.$$

Choosing $v^h = \Pi_{V^h}\eta \in V^h$ in the last equation and noting (3.11) allows

$$\begin{aligned}
a_1\big(u_1^h(q) - u_1^h(\Pi^H q), \Pi_{V^h}\eta\big) &= \int_{\Omega_0} (q - \Pi^H q)\Pi_{V^h}\eta dx \\
&= \int_{\Omega_0} (q - \Pi^H q)(\Pi_{V^h}\eta - \eta) dx + \int_{\Omega_0} (q - \Pi^H q)\eta dx \\
&= \int_{\Omega_0} (q - \Pi^H q)(\Pi_{V^h}\eta - \eta) dx + \int_{\Omega_0} (q - \Pi^H q)(\eta - \Pi^H \eta) dx
\end{aligned}$$

Therefore, using the Cauchy-Schwarz inequality, together with the estimates (3.4), (3.12), and (3.17), we arrive at

$$\begin{aligned}
&a_1\big(u_1^h(q) - u_1^h(\Pi^H q), \Pi_{V^h}\eta\big) \\
&\leqslant \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} \big\|\Pi_{V^h}\eta - \eta\big\|_{L^2(\Omega)} + \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} \big\|\eta - \Pi^H \eta\big\|_{L^2(\Omega_0)} \\
&\lesssim h^2 \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} \|\eta\|_{H^2(\Omega)} + H \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} |\eta|_{H^1(\Omega_0)} \\
&\lesssim (h^2 + H) \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} \|\eta\|_{H^2(\Omega)} \\
&\lesssim (h^2 + H) \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} \big\|u_1^h(q) - u_1^h(\Pi^H q)\big\|_{L^2(\Omega)}.
\end{aligned} \tag{3.19}$$

On the other hand, the combination of (3.4), (3.13), and (3.17) implies

$$\begin{aligned}
a_1\big(u_1^h(q) - u_1^h(\Pi^H q), \eta - \Pi_{V^h}\eta\big) &\lesssim \big\|u_1^h(q) - u_1^h(\Pi^H q)\big\|_{H^1(\Omega)} \|\eta - \Pi_{V^h}\eta\|_{H^1(\Omega)} \\
&\lesssim Hh \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} \|\eta\|_{H^2(\Omega)} \\
&\lesssim Hh \big\|q - \Pi^H q\big\|_{L^2(\Omega_0)} \big\|u_1^h(q) - u_1^h(\Pi^H q)\big\|_{L^2(\Omega)},
\end{aligned} \tag{3.20}$$

Hence, it follows from (3.18)–(3.20) that

$$\begin{aligned}\big\|u_1^h(q)-u_1^h(\Pi^H q)\big\|^2_{L^2(\Omega)} &= a_1\big(\eta, u_1^h(q)-u_1^h(\Pi^H q)\big)\\ &= a_1\big(u_1^h(q)-u_1^h(\Pi^H q), \eta-\Pi_{V^h}\eta\big)+a_1\big(u_1^h(q)-u_1^h(\Pi^H q), \Pi_{V^h}\eta\big)\\ &\lesssim hH\big\|q-\Pi^H q\big\|_{L^2(\Omega_0)}\big\|u_1^h(q)-u_1^h(\Pi^H q)\big\|_{L^2(\Omega)}\\ &\quad +(h^2+H)\big\|q-\Pi^H q\big\|_{L^2(\Omega_0)}\big\|u_1^h(q)-u_1^h(\Pi^H q)\big\|_{L^2(\Omega)}\\ &\lesssim (h^2+hH+H)\big\|q-\Pi^H q\big\|_{L^2(\Omega_0)}\big\|u_1^h(q)-u_1^h(\Pi^H q)\big\|_{L^2(\Omega)},\end{aligned}$$

which leads to

$$\big\|u_1^h(q)-u_1^h(\Pi^H q)\big\|_{L^2(\Omega)}\lesssim (h^2+H)\big\|q-\Pi^H q\big\|_{L^2(\Omega_0)}. \tag{3.21}$$

Moreover, by (3.5) and (3.21),

$$\begin{aligned}\big\|u_1(q)-u_1^h(\Pi^H q)\big\|_{L^2(\Omega)} &\leqslant \|u_1(q)-u_1^h(q)\|_{L^2(\Omega)}+\big\|u_1^h(q)-u_1^h(\Pi^H q)\big\|_{L^2(\Omega)}\\ &\lesssim h^2(\|q\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)})+(h^2+H)\big\|q-\Pi^H q\big\|_{L^2(\Omega_0)},\end{aligned}$$

yielding (3.15). Similarly, we can get the inequality (3.16). ■

Denote

$$E^H(p_\varepsilon):=\big\|p_\varepsilon-\Pi^H p_\varepsilon\big\|_{L^2(\Omega_0)}$$

Note that $E^H(p_\varepsilon)$ is generally expected to be small.

Now, we are ready to present error estimates for our method.

Theorem 3.3 *With the choice $Q_{ad,1}=Q_{ad}^H$ as a subset of piecewise constant functions, it holds that*

$$\big\|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\big\|^2_{L^2(\Omega)}+\varepsilon\big\|p_\varepsilon-p_\varepsilon^{h,H}\big\|^2_{L^2(\Omega_0)}\lesssim h^2A_\varepsilon A_\varepsilon^{h,H}+(h^2+H)E^H(p_\varepsilon)A_\varepsilon^{h,H},$$

where A_ε is defined as in Theorem 3.2, and

$$A_\varepsilon^{h,H}:=\|p_\varepsilon^{h,H}\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)}+\|g^-\|_{H^{1/2}(\Gamma)}.$$

Proof. Choose $q=\Pi^H p_\varepsilon$ in (3.1) to get

$$\big\langle\widetilde{u}^h(p_\varepsilon^{h,H}),\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\big\rangle_{L^2(\Omega)}+\varepsilon\big\langle p_\varepsilon^{h,H},\Pi^H p_\varepsilon-p_\varepsilon^{h,H}\big\rangle_{L^2(\Omega_0)}\geq 0.$$

This combined with (3.11) implies

$$\big\langle\widetilde{u}^h(p_\varepsilon^{h,H}),\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\big\rangle_{L^2(\Omega)}+\varepsilon\big\langle p_\varepsilon^{h,H},p_\varepsilon-p_\varepsilon^{h,H}\big\rangle_{L^2(\Omega_0)}\geq 0. \tag{3.22}$$

Choose $q=p_\varepsilon^{h,H}$ in (2.6) to get

$$\big\langle\widetilde{u}(p_\varepsilon),\widetilde{u}(p_\varepsilon^{h,H})-\widetilde{u}(p_\varepsilon)\big\rangle_{L^2(\Omega)}+\varepsilon\big\langle p_\varepsilon,p_\varepsilon^{h,H}-p_\varepsilon\big\rangle_{L^2(\Omega_0)}\geq 0. \tag{3.23}$$

The summation of the inequalities (3.22) and (3.23) gives

$$\begin{aligned}\varepsilon\big\|p_\varepsilon^{h,H}-p_\varepsilon\big\|^2_{L^2(\Omega_0)}\leq&\big\langle\widetilde{u}^h(p_\varepsilon^{h,H}),\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\big\rangle_{L^2(\Omega)}\\ &+\big\langle\widetilde{u}(p_\varepsilon),\widetilde{u}(p_\varepsilon^{h,H})-\widetilde{u}(p_\varepsilon)\big\rangle_{L^2(\Omega)}.\end{aligned}$$

Thus, adding the term $\|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\|^2_{L^2(\Omega)}$ to both sides of the above inequality and reformulating terms, we get

$$\begin{aligned}
&\|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\|^2_{L^2(\Omega)}+\varepsilon\|p_\varepsilon^{h,H}-p_\varepsilon\|^2_{L^2(\Omega_0)}\\
&\leqslant \|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\|^2_{L^2(\Omega)}+\langle\widetilde{u}^h(p_\varepsilon^{h,H}),\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\rangle_{L^2(\Omega)}\\
&\quad+\langle\widetilde{u}(p_\varepsilon),\widetilde{u}(p_\varepsilon^{h,H})-\widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)}\\
&=\langle\widetilde{u}(p_\varepsilon),\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\rangle_{L^2(\Omega)}-\langle\widetilde{u}^h(p_\varepsilon^{h,H}),\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\rangle_{L^2(\Omega)}\\
&\quad+\langle\widetilde{u}^h(p_\varepsilon^{h,H}),\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\rangle_{L^2(\Omega)}+\langle\widetilde{u}(p_\varepsilon),\widetilde{u}(p_\varepsilon^{h,H})-\widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)}\\
&=\langle\widetilde{u}(p_\varepsilon),\widetilde{u}(p_\varepsilon^{h,H})-\widetilde{u}^h(p_\varepsilon^{h,H})\rangle_{L^2(\Omega)}+\langle\widetilde{u}^h(p_\varepsilon^{h,H}),\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}(p_\varepsilon)\rangle_{L^2(\Omega)}.
\end{aligned}$$

Hence, in view of the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}
&\|\widetilde{u}(p_\varepsilon)-\widetilde{u}^h(p_\varepsilon^{h,H})\|^2_{L^2(\Omega)}+\varepsilon\|p_\varepsilon^{h,H}-p_\varepsilon\|^2_{L^2(\Omega_0)}\\
&\leqslant \|\widetilde{u}(p_\varepsilon)\|_{L^2(\Omega)}\|\widetilde{u}(p_\varepsilon^{h,H})-\widetilde{u}^h(p_\varepsilon^{h,H})\|_{L^2(\Omega)}+\|\widetilde{u}^h(p_\varepsilon^{h,H})\|_{L^2(\Omega)}\|\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}(p_\varepsilon)\|_{L^2(\Omega)}\\
&=: C_1D_1+C_2D_2, \qquad (3.24)
\end{aligned}$$

where

$$\begin{aligned}
C_1 &=: \|\widetilde{u}(p_\varepsilon)\|_{L^2(\Omega)}, \quad D_1 =: \|(\widetilde{u}(p_\varepsilon^{h,H})-\widetilde{u}^h(p_\varepsilon^{h,H}))\|_{L^2(\Omega)},\\
C_2 &=: \|\widetilde{u}^h(p_\varepsilon^{h,H})\|_{L^2(\Omega)}, \quad D_2 =: \|\widetilde{u}^h(\Pi^H p_\varepsilon)-\widetilde{u}(p_\varepsilon)\|_{L^2(\Omega)}.
\end{aligned}$$

It follows from (3.2) and (3.3) that

$$C_1 \leqslant \|u_1(p_\varepsilon)\|_{L^2(\Omega)}+\|u_2(p_\varepsilon)\|_{L^2(\Omega)} \lesssim A_\varepsilon,$$

and from (3.8) and (3.9),

$$C_2 \leqslant \|u_1^h(p_\varepsilon^{h,H})\|_{L^2(\Omega)}+\|u_2^h(p_\varepsilon^{h,H})\|_{L^2(\Omega)} \lesssim A_\varepsilon^{h,H}.$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}
D_1 &\leqslant \|u_1(p_\varepsilon^{h,H})-u_1^h(p_\varepsilon^{h,H})\|_{L^2(\Omega)}+\|u_2(p_\varepsilon^{h,H})-u_2^h(p_\varepsilon^{h,H})\|_{L^2(\Omega)}\\
&\lesssim h^2\big(\|p_\varepsilon^{h,H}\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)}\big)+h^2\big(\|p_\varepsilon^{h,H}\|_{L^2(\Omega_0)}+\|g^-\|_{H^{1/2}(\Gamma)}\big)\\
&\lesssim h^2 A_\varepsilon^{h,H},
\end{aligned}$$

and from (3.15) and (3.16),

$$\begin{aligned}
D_2 &\leqslant \|u_1^h(\Pi^H p_\varepsilon)-u_1(p_\varepsilon)\|_{L^2(\Omega)}+\|u_2^h(\Pi^H p_\varepsilon)-u_2(p_\varepsilon)\|_{L^2(\Omega)}\\
&\lesssim h^2\big(\,\|p_\varepsilon\|_{L^2(\Omega_0)}+\|g\|_{H^{1/2}(\Gamma)}\,\big)+(h^2+H)\|p_\varepsilon-\Pi^H p_\varepsilon\|_{L^2(\Omega_0)}\\
&\quad+h^2\big(\|p_\varepsilon\|_{L^2(\Omega_0)}+\|g^-\|_{H^{1/2}(\Gamma)}\big)+(h^2+H)\|p_\varepsilon-\Pi^H p_\varepsilon\|_{L^2(\Omega_0)}\\
&\lesssim h^2 A_\varepsilon+(h^2+H)\|p_\varepsilon-\Pi^H p_\varepsilon\|_{L^2(\Omega_0)}.
\end{aligned}$$

Consequently, the desired result follows from (3.24) combined with the above bound for C_1, C_2, D_1, and D_2. ■

Corollary 3.2 *With the choice $Q_{ad,1} = Q_{ad}^H$ as a subset of piecewise constant functions, it holds that*

$$\left\|\widetilde{u}(p_\varepsilon) - \widetilde{u}^h(p_\varepsilon^{h,H})\right\|_{L^2(\Omega)}^2 + \varepsilon\left\|p_\varepsilon - p_\varepsilon^{h,H}\right\|_{L^2(\Omega_0)}^2 \lesssim h^2 + (h^2 + H)E^H(p_\varepsilon).$$

Proof. Since $Q_{ad,1}$ and Q_{ad} is a bounded set in $L^2(\Omega_0)$, $\|p_\varepsilon\|_{L^2(\Omega_0)}$ and $\|p_\varepsilon^h\|_{L^2(\Omega_0)}$ are uniformly bounded with respect to ε, h, and H. Then the corollary is a direct consequence of Theorem 3.3. ■

Error bounds of Corollary 3.2 involve the best approximation error term $E^H(p_\varepsilon)$. A crude bound for this term is

$$E^H(p_\varepsilon) \leqslant \|p_\varepsilon\|_{L^2(\Omega_0)} + \left\|\Pi^H p_\varepsilon\right\|_{L^2(\Omega_0)} \leqslant 2\,\|p_\varepsilon\|_{L^2(\Omega_0)} \lesssim 1.$$

However, usually we can expect this term to be small as given in Proposition 4.11 of the paper [13].

4 Numerical results

In this section, we perform several numerical results on a model problem to illustrate the computational performance of our method. We also compare the numerical performance of our method and the method devised in [13]. In our numerical tests, we always take $\Omega = (0,1)\times(0,1)$ as the problem domain and $\Omega_0 = (0.5, 0.75)\times(0.5, 0.75)$ as the permissible region. Moreover, we choose the absorption coefficient $\mu = 0.04$ and the reduced scattering coefficient $\mu^{'} = 1.5$ in the whole domain Ω, so that the diffusion model (1.1)–(1.3) is appropriate for the BLT application.

Without loss of generality, choose $g = -D\frac{\partial u}{\partial \nu} = -D = -1/3(\mu + \mu^{'})$ on Γ. We remark that our program is able to reconstruct the source function numerically for any prescribed function g. When considering the effect of the noise, the measurement data g^δ of g is obtained by the formula:

$$g^\delta(x) = g(x)[1 + \delta * \mathrm{rand}], \quad x \in \Gamma,$$

where δ indicates the noise level of the measurement and the function rand generates a random number with the uniform distribution over $[-1, 1]$.

The admissible set is taken to be $Q_{ad} = \{q \in L^2(\Omega_0);\ q \geq 0 \text{ a.e. in } \Omega_0\}$. We use uniform triangle partitions of the regions $\bar{\Omega}$ and $\bar{\Omega_0}$ with mesh sizes $h = H$ (see Fig. 1), where h and H denote the maximal diameters of the elements in the partitions $\mathcal{T}_h$ and $\mathcal{T}_{0,H}$, respectively. Corresponding to the partitions $\mathcal{T}_h$ and $\mathcal{T}_{0,H}$, we use continuous piecewise linear functions for finite element spaces V^h, and piecewise constant functions for finite element spaces Q_{ad}^H.

For a source function p and the function g given above, we use the finite element method corresponding to V^h with $h = 1/64$ to solve the forward problem (1.1)-(1.2), to get the incoming flux g^- from (1.3).

To make our numerical investigation more reliable, we compute the approximate source functions $p_\varepsilon^{h,h}$ with $h = 1/32$ for different choices of ε and δ, with the exact source function p being non-oscillatory or oscillatory.

First of all, we take $p = (1 + x^3)(1 + y^4)(\mathrm{pW})$ in Ω_0, a non-oscillatory function. The numerical results for our method and the method in [13] are given respectively in Tables 1 and 2. In the second example, we take $p = \sin 3x \cos 2y + 1(\mathrm{pW})$ in Ω_0, an oscillatory function. The numerical results are given respectively in Tables 3 and 4. From these data, we may conclude that

- If the noise level is not very large (δ is no more than 1%), both methods are able to reconstruct the source function with desired accuracy. If the noise level increases large enough (δ is greater than or equal to 5%), the computational accuracy of the two methods would be deteriorated comparatively. Totally speaking, both methods enjoy the similar numerical performance.
- Both methods perform better in the case that the source function is non-oscillatory.
- The choice of the regularization parameter ε is important in reconstructing the source function feasibly. For both methods, ε may be taken as 10^{-4} or 10^{-5}.

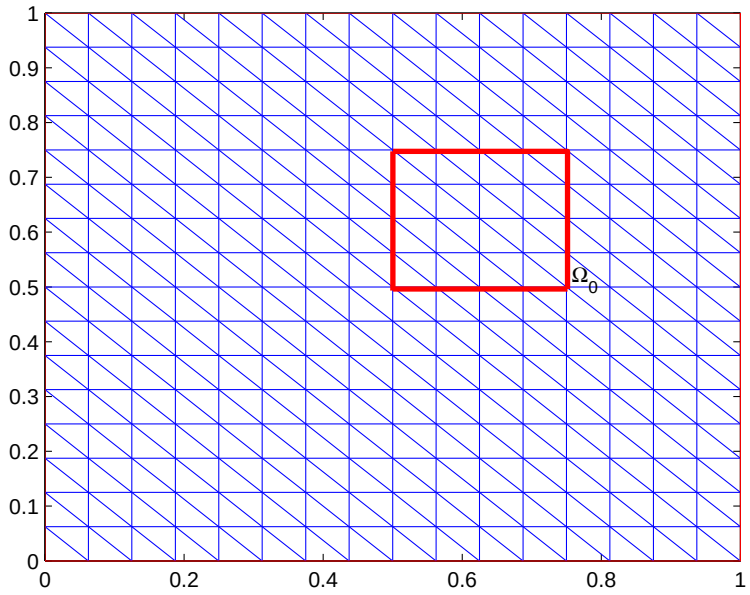


Fig. 1. Schema of uniform triangle partitions of $\bar{\Omega}$ and $\bar{\Omega_0}$.

From our numerical tests, we also find that the choice of permissible regions Ω_0 significantly affect the accuracy of reconstructed source functions. If the region Ω_0 is chosen more close to the support of the exact source function, one can get more accurate reconstructed function. In the above numerical tests, the supports of the exact source functions are $\Omega_0^{'} = (0.5, 0.75) \times (0.5, 0.75)$, and we choose Ω_0 as $\Omega_0^{'}$ exactly. If we expand Ω_0 a little larger, say $\Omega_0 = (0.5 - 2h, 0.75 + 2h) \times (0.5 - 2h, 0.75 + 2h)$ with $h = 1/32$, and still choose $p = (1+x^3)(1+y^4)\chi_{\Omega_0^{'}}$ and $\delta = 0$, then the L^2 errors corresponding to $\varepsilon = 10^{-3} \sim 10^{-7}$ become 0.27122, 0.073970, 0.13659, 0.33737, and 0.56856, respectively, which are much worse than the results in Table 1. In this case, the result for $\varepsilon = 10^{-4}$ is the best one, with the relatively desired accuracy. We mention in passing that the similar numerical phenomenon is also observed for the method in [13]. Intuitively, this phenomenon is very reasonable, which shows the importance of locating the position of light source.

Table 1 $\|p - p_\varepsilon^{h,h}\|_{L^2(\Omega_0)}$ for our method.

ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
δ=0	2.7207e-002	1.3350e-002	6.7328e-003	2.2118e-002	3.9269e-002
δ=1%	3.1296e-002	1.1051e-002	1.3106e-002	3.7565e-002	6.4535e-002
δ=2%	3.5994e-002	1.5093e-002	2.7642e-002	5.6769e-002	9.2449e-002
δ=5%	5.2016e-002	3.8928e-002	7.3048e-002	1.1830e-001	1.7958e-001

Table 2 $\|p - p_\varepsilon^{h,h}\|_{L^2(\Omega_0)}$ for the method in [13].

ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
δ=0	1.2771e-002	7.4124e-003	2.8505e-002	5.0651e-002	1.0253e-001
δ=1%	1.2234e-002	1.3779e-002	4.5165e-002	7.9306e-002	1.5966e-001
δ=2%	1.6934e-002	2.7851e-002	6.5122e-002	1.1018e-001	2.2152e-001
δ=5%	4.0012e-002	7.2380e-002	1.2917e-001	2.0603e-001	4.1402e-001

Table 3 $\|p - p_\varepsilon^{h,h}\|_{L^2(\Omega_0)}$ for our method.

ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
δ=0	4.3434e-002	2.5745e-002	1.0938e-002	2.5070e-002	3.9703e-002
δ=1%	4.6931e-002	3.2205e-002	2.1622e-002	4.0489e-002	6.4902e-002
δ=2%	5.0945e-002	3.9664e-002	3.5795e-002	5.9496e-002	9.2777e-002
δ=5%	6.5084e-002	6.4438e-002	8.0731e-002	1.2074e-001	1.7986e-001

Table 4 $\|p - p_\varepsilon^{h,h}\|_{L^2(\Omega_0)}$ for the method in [13].

ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
δ=0	2.8051e-002	1.2823e-002	3.1738e-002	5.0900e-002	1.0115e-001
δ=1%	3.4421e-002	2.3246e-002	4.8433e-002	7.9580e-002	1.5897e-001
δ=2%	4.1633e-002	3.6976e-002	6.8248e-002	1.1046e-001	2.2116e-001
δ=5%	6.5435e-002	8.0953e-002	1.3204e-001	2.0631e-001	4.1405e-001

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Local Hermite Interpolation by Bivariate C^1 Cubic Splines on Checkerboard Triangulations

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Abstract

Given a so-call checkerboard quadrangulation $\bar{\diamondsuit}$, a checkerboard triangulation $\bar{\Diamond}$ can be obtained by adding two diagonals of all quadrilaterals in $\bar{\diamondsuit}$. In this paper, we develop a local Hermite interpolation method for bivariate C^1 cubic splines on $\bar{\Diamond}$. By enforcing some additional smoothness conditions across the interior edges of $\bar{\Diamond}$, a C^1 piecewise cubic polynomial function based on $\bar{\Diamond}$ is constructed by interpolating only the function values and derivatives of first order at the vertices of $\bar{\diamondsuit}$ and none of the normal derivatives at the midpoints of edges in $\bar{\diamondsuit}$ is needed. It is shown that the new interpolation method produces optimal order approximation of smooth functions.

Keywords: Bivariate cubic spline; Hermite interpolation; Bernstein-Bézier form; Checkerboard triangulation.

AMS(2000) subject classifications. 41A05. 41A15. 41A25

1 Introduction

Let $S_d^r(\triangle)$ be the bivariate spline space of degree at most d and smoothness r on a triangulation $\triangle$ of a polygonal domain Ω in R^2. Bivariate splines are very flexible for approximating known

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or unknown functions or any given data sets, they have been playing an important role in surface fitting and computer aided geometric design (CAGD).

A fundamental problem of bivariate spline space is to construct local interpolation methods which have full approximation order. de Boor and Höllig [1] and Lai and Schumaker [2] have proved that the space $S_d^r(\triangle)$ has optimal approximation order on arbitrary triangulations $\triangle$ when $d \geq 3r+2$, but for $d < 3r+2$, we can not even determine the dimension of the space $S_d^r(\triangle)$ on arbitrary triangulations $\triangle$. However, because of the simplicity and efficiency in calculation, the spline functions with lower degrees are more important and favorable in practical applications, for example, bivariate C^1 cubic spline functions, which are just what we concern in this paper.

Up to now, there has been much typical literature about Hermite interpolations by bivariate C^1 cubic spline functions. For example, in 1974, Ciarlet [3] showed that he can uniquely construct a C^1 cubic spline function on $\triangle_{CT}$ to interpolate the function values and the gradient values at the vertices of $\triangle$ and the normal derivatives at all the edges of $\triangle$, where $\triangle_{CT}$ denotes the Clough-Tocher refinement of $\triangle$, this method has full approximation order. In 1983, de Boor and Höllig [4] showed that the space $S_3^1(\triangle_1)$ has no full approximation order on regular type-I triangulations, but Lai [5] in 1994 proved that the space $S_3^1(\triangle_2)$ has full approximation order on regular type-II triangulations. In 1996, Lai and Shumaker [6] showed that the space $S_3^1(\diamondsuit)$ also has full approximation order, where $\diamondsuit$ denotes a triangulated quadrangulation obtained by drawing in both diagonals of quadrilateral in a general quadrangulation $\diamondsuit$. They pointed out that their interpolation method is more efficient than those methods mentioned above. Following this method, a C^1 cubic spline function on $\diamondsuit$ can be uniquely constructed to interpolate the function values and the gradient values at the vertices of $\diamondsuit$ and the normal derivatives at some points in all the edges of $\diamondsuit$. However, as we know, the values of the normal derivatives at the edges of $\diamondsuit$ is generally unknown in applications.

The aim of this paper is to develop a local Hermite interpolation method for bivariate C^1 cubic splines on a so-called checkerboard triangulation. In our method, a C^1 piecewise cubic polynomial function on $\bar{\diamondsuit}$ is constructed by interpolating the function values and the gradient values at the vertices of $\bar{\diamondsuit}$ without any normal derivative at the edges of $\bar{\diamondsuit}$ being used. This improves the interpolation method presented in [6]. Further, by applying quasi-interpolation techniques, we shall prove that the new method produces optimal order approximation of smooth functions. This can make the interpolation and approximation methods using the splines of subspace of $S_3^1(\bar{\diamondsuit})$ more efficient.

2 Preliminaries

As usual, we use the well-known Bernstein-Bézier form [7] to represent every polynomial $s \in \mathcal{P}_3$ associated with a triangle $T :=< v_1, v_2, v_3 >$ as

$$s := \sum_{i+j+k=3} c_{ijk}^T \mathcal{B}_{ijk}^3, \tag{1}$$

where $\{\mathcal{B}_{ijk}^3\}_{i+j+k=3}$ is the Bernstein basis of degree at most 3 on the triangle T, and c_{ijk}^T are called the B-coefficients of s associated with the domain points $\xi_{ijk}^T := (iv_0 + jv_1 + kv_2)/3$, $i + j + k = 3$. Suppose

$$\mathcal{D}_{3,T} := \{\xi_{ijk}^T,\ i + j + k = 3\}$$

and

$$\mathcal{D}_{3,\triangle} := \bigcup_{T \in \triangle} \mathcal{D}_{3,T}$$

denote the set of domain points on T and the set of domain points on $\triangle$, respectively.

Following the definition in [8], a minimal determining set (MDS) for a spline space $S \subseteq S_3^0(\triangle)$ is a subset $\mathcal{M}$ of $\mathcal{D}_{3,\triangle}$ such that if s is a spline in S whose B-coefficients corresponding to points in $\mathcal{M}$ are set to prescribed values, then all of its remaining B-coefficients associated with domain points of $\triangle$ are uniquely determined. It is well-known that $\dim S = |\mathcal{M}|$, where $|\mathcal{M}|$ denotes the cardinality of $\mathcal{M}$.

According to [9], a MDS $\mathcal{M}$ is said to be stable provided that there exists a constant K depending only on the smallest angle in $\triangle$ such that

$$\max_{\eta \in \mathcal{D}_{3,\triangle}} |c_\eta| \le K \max_{\xi \in \mathcal{M}} |c_\xi|. \tag{2}$$

Let $T :=< v_0, v_1, v_2 >$ and $\tilde{T} :=< v_3, v_2, v_1 >$ be two adjacent triangles, they sharing the common edge $e :=< v_1, v_2 >$. Suppose $s|_T$ and $s|_{\tilde{T}}$ are two polynomials of degree 3 with B-coefficients $c_{ijk} = c_{ijk}^T$ and $\tilde{c}_{ijk} = c_{ijk}^{\tilde{T}}$ relative to T and $\tilde{T}$, respectively. Then it is well known (cf. [10]) that $s|_T$ and $s|_{\tilde{T}}$ join with C^r continuity across the edge e if and only if

$$\tilde{c}_{n,m-n,3-m} = \sum_{i+j+k=n} c_{i,j+3-m,k+m-n} \mathcal{B}_{ijk}^n(v_3) \tag{3}$$

for $n \le m \le 3$, $0 \le n \le r$.

Let $\tau_{m,e}^n$ be the linear functional defined on $S_3^0(\triangle)$ by

$$\tau_{m,e}^n s = \tilde{c}_{n,m-n,3-m} - \sum_{i+j+k=n} c_{i,j+3-m,k+m-n} \mathcal{B}_{ijk}^n(v_3). \tag{4}$$

For $s \in S_3^1(T \cup \tilde{T})$, we have

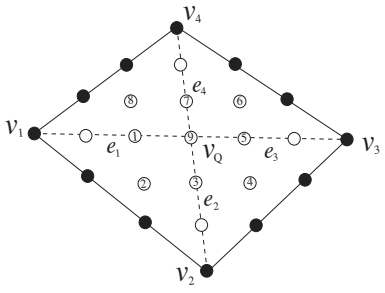


Figure 1: Notation for the domain points in Lemmas 2-5.

Lemma 1([10]). Suppose that all B-coefficients c_{ijk} and $\tilde{c}_{ijk}$ of $s|_T$ and $s|_{\tilde{T}}$ are known except for

$$\begin{aligned} c_\nu &:= c_{\nu,3-m,m-\nu}, \quad \nu = l+1, ..., q, \\ \tilde{c}_\nu &:= c_{\nu,m-\nu,3-m}, \quad \nu = l+1, ..., \tilde{q}, \end{aligned} \tag{5}$$

for some $l, m, q, \tilde{q}$ with $0 \le q, \tilde{q},\ 0 \le l+1 \le q, \tilde{q}$, and $q + \tilde{q} - l \le m \le 3$. Then these B-coefficients are uniquely determined by the smoothness conditions

$$\tau^n_{m,e} s = 0, \ \ l+1 \le n \le q + \tilde{q} - l. \tag{6}$$

3 Several Lemmas

As shown in Figure 1, suppose $\triangle_Q$ is the triangulation which is obtained by inserting both diagonals in a quadrilateral $Q :=< v_1, v_2, v_3, v_4 >$, and suppose $T_i =< v_i, v_{i+1}, v_Q >$, $i = 1, \cdots, 4$, are four triangles and $e_i =< v_Q, v_i >$ are four edges of $\triangle_Q$. Let $\mathcal{P}_Q$ be the set of 12 domain points lying on the edges of Q.

Lemma 2. Let S_1 be the subspace of all splines s in $S^1_3(\triangle_Q)$ satisfying the following additional smoothness conditions:

$$\tau^n_{3,e_2} s = 0, \ \ n = 2, 3 \ \text{ and } \ \tau^2_{2,\mathrm{e_m}} \mathrm{s} = 0, \ \ \mathrm{m} = 1, 3, \tag{7}$$

then $\dim S_1 = 12$, and $\mathcal{P}_Q$ is a stable MDS for S_1.

Proof: Firstly, we show that $\mathcal{M}$ is a MDS for S_1. Suppose s is a spline in S_1 whose B-coefficients corresponding to points in $\mathcal{P}_Q$ are set to prescribed values, then we show that all of its remaining B-coefficients associated with domain points of $\triangle_Q$ are uniquely determined. By using C^1 smoothness condition and the condition (7), the B-coefficients associated with domain points ①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧ and ⑨ are uniquely determined. Then the remaining B-coefficients associated with domain points of $\triangle_Q$ are uniquely determined by C^1 smoothness conditions. We conclude that $\mathcal{M}$ is a MDS for S_1 and $\dim S_1 = 12$. It is well known that the process of computing B-coefficients directly from a smoothness condition is stable [11], the

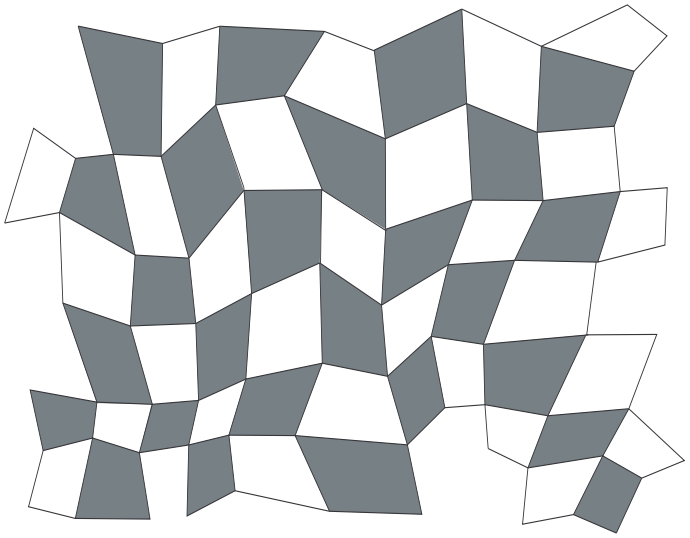


Figure 2: A checkerboard quadrangulation $\bar{\diamondsuit}$.

computation of coefficients by Lemma 1 is also a stable process as the norms of the inverses of the resulting matrices are bounded by a constant depending only on the smallest angle in $\triangle_Q$. This proves that $\mathcal{M}$ is a stable MDS. ■

Similar to the proof of Lemma 2, the following three Lemmas can be obtained immediately.

Lemma 3. Let S_2 be the subspace of all splines s in $S_3^1(\triangle_Q)$ satisfying the following additional smoothness conditions:

$$\tau_{3,e_2}^n s = 0 \text{ and } \tau_{2,\mathrm{e}_1}^2 \mathrm{s} = 0, \quad \mathrm{n} = 2, 3, \tag{8}$$

then $\dim S_2 = 13$, and $\mathcal{P}_Q \cup \{⑥\}$ is a stable MDS for S_2.

Lemma 4. Let S_3 be the subspace of all splines s in $S_3^1(\triangle_Q)$ satisfying the following additional smoothness conditions:

$$\tau_{3,e_2}^n s = 0, \quad n = 2, 3, \tag{9}$$

then $\dim S_3 = 14$, and $\mathcal{P}_Q \cup \{⑥, ⑧\}$ is a stable MDS for S_3.

Lemma 5. Let S_4 be the subspace of all splines s in $S_3^1(\triangle_Q)$ satisfying the following additional smoothness conditions:

$$\tau_{2,e_1}^2 s = 0, \tag{10}$$

then $\dim S_4 = 15$, and $\mathcal{P}_Q \cup \{④, ⑥, ⑧\}$ is a stable MDS for S_4.

4 The Dimension and Local Basis

Definition 1 ([12]). Suppose $\bar{\diamondsuit}$ is a general quadrangulation consisting of quadrilaterals with largest interior angle less than π. Suppose that the quadrilaterals can be colored black and

white in such a way that any two quadrilaterals sharing an edge have the opposite color. Then we call $\bar{\diamondsuit}$ a checkerboard quadrangulation. The triangulation $\bar{\diamondsuit}$ which is obtained by drawing in both diagonals of all quadrilaterals is called a checkerboard triangulation.

A general quadrangulation $\bar{\diamondsuit}$ is said to be a checkerboard quadrangulation provided that the degree of all the interior vertices of $\bar{\diamondsuit}$ are even [12]. In this paper, we concern a checkerboard quadrangulation such that all of interior vertices of $\bar{\diamondsuit}$ are of degree four (See Figure 2). Given a checkerboard quadrangulation $\bar{\diamondsuit}$, let $\mathcal{B}$ and $\mathcal{W}$ be the sets of black and white quadrilaterals of $\bar{\diamondsuit}$, respectively. It is clear that there exists a set of black quadrilaterals, denoted by $\mathcal{G} \subset \mathcal{B}$, such that for every interior vertex v of $\bar{\diamondsuit}$, there is a unique quadrilateral $Q \in \mathcal{G}$ sharing the vertex v. Let $\mathcal{W}_i$ be the set of white quadrilaterals which share i edges with black quadrilaterals, where $i = 1, 2, 3$. In Figure 3, we show the subset $\mathcal{G}$ of $\mathcal{B}$ which is shaded in Figure 2 with black color.

Following Lai [6], we can determine the dimension of space

$$S_3^1(\bar{\diamondsuit}) := \left\{ s \in C^1(\Omega) : s|_T \in \mathcal{P}_3, \forall T \in \bar{\diamondsuit} \right\},$$

where $\mathcal{P}_3$ is the space of bivariate polynomials of degree 3. Namely, $\dim S_3^1(\bar{\diamondsuit}) = 3V + E$, where V and E denote the number of vertices and edges in $\bar{\diamondsuit}$. Just as described in Section 1, in this paper, a new interpolation method by a subspace of $S_3^1(\bar{\diamondsuit})$ is constructed with the dimension of the subspace being reduced to $3V$.

Algorithm 1. Given a checkerboard triangulations $\bar{\diamondsuit}$, let $\mathcal{P}$ be the subset of $\mathcal{D}_{3,\bar{\diamondsuit}}$ obtained from the following steps:

1) For each $Q \in \mathcal{G}$ in $\bar{\diamondsuit}$, choose 12 points as in Lemma 3.

2) For each $v \in \mathcal{V}_{\lfloor}$ $(v \notin \mathcal{G})$, where $\mathcal{V}_{\lfloor}$ denotes the boundary vertices of checkerboard quadrangulation $\bar{\diamondsuit}$, choose 3 points situated on the edges of $\bar{\diamondsuit}$ in $\mathcal{D}_1(v)$.

The set $\mathcal{P}$ constructed by this algorithm is marked by $\bullet$ in Figure 3.

Definition 1. Given a checkerboard quadringulation $\bar{\diamondsuit}$, let $\mathcal{P}$ be as above, and $\mathcal{S}$ be the set of all splines $s \in S_3^1(\bar{\diamondsuit})$ such that

1) For each triangle $T \in \triangle_Q$ in $\mathcal{B}$, s satisfies the additional smoothness conditions described in Lemma 2.

2) For each triangle $T \in \triangle_Q$ in $\mathcal{W}_1$, s satisfies the additional smoothness conditions described in Lemma 5.

3) For each triangle $T \in \triangle_Q$ in $\mathcal{W}_2$, s satisfies the additional smoothness conditions described in Lemma 4.

4) For each triangle $T \in \triangle_Q$ in $\mathcal{W}_3$, s satisfies the additional smoothness conditions described in Lemma 3.

Theorem 1. Let $\mathcal{P}$ and $\mathcal{S}$ be as above, then $\dim \mathcal{S} = 3V = |\mathcal{P}|$.

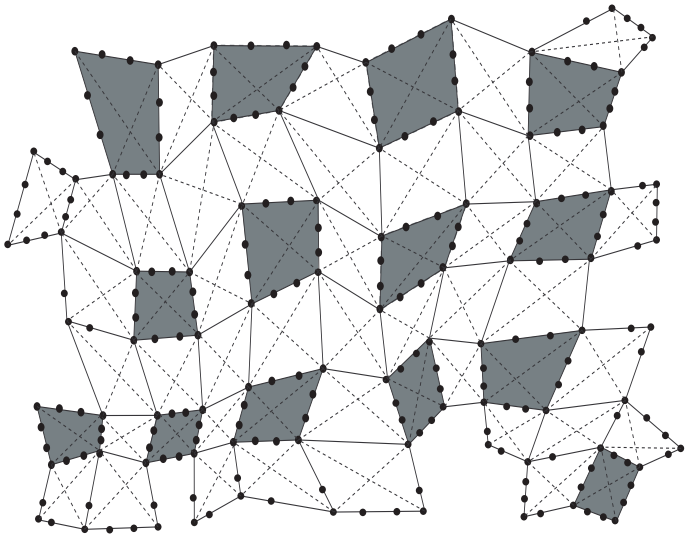


Figure 3: A minimal determining set $\mathcal{P}$ for $\mathcal{S}$.

Proof: We show how to compute the B-coefficients of $s \in \mathcal{S}$. Firstly, compute the B-coefficients of s with respect $T \in \triangle_Q$ in $\mathcal{B}$ following Lemma 2. Secondly, compute the B-coefficients of $T \in \triangle_Q$ in $\mathcal{W}_1$, $T \in \triangle_Q$ in $\mathcal{W}_2$ and $T \in \triangle_Q$ in $\mathcal{W}_3$ following Lemma 5, Lemma 4 and Lemma 3, respectively. Finally, the remaining B-coefficients associated with domain points of $\bar{\diamondsuit}$ are uniquely determined by C^1 smoothness conditions. We conclude that $\mathcal{P}$ is a MDS for $\mathcal{S}$ and $\dim S = |\mathcal{P}| = 3V$. ■

Theorem 2. Let $\mathcal{P}$ and $\mathcal{S}$ be as above. Then $\mathcal{P}$ is a stable local MDS for $\mathcal{S}$.

Proof: Given $\xi \in \mathcal{P}$, let B_ξ be the spline in $\mathcal{S}$ such that $B_\xi(\xi) = 1$ and $B_\xi(\eta) = 0$ for all other points $\eta \in \mathcal{P}$. It is the fundamental spline associated with ξ. It is easy to see that B_ξ has a local support. If ξ is a point as in item 1) of Algorithm 1, then $supp(B_\xi) \subset star^2(Q_\xi)$. If ξ is a point as in item 2) of Algorithm 1, then $supp(B_\xi) \subset star^1(Q_\xi)$. Here $star(Q) := Q$ and $star^i(Q)$ is the union of the set of all quadrilaterals which touch a quadrilateral in $star^{i-1}(Q)$. The proofs of Lemmas 2-5 and Theorem 1 show that $\{B_\xi\}_{\xi\in\mathcal{P}}$ form a stable local basis for $\mathcal{S}$. It is easy to prove that $\mathcal{P}$ is a stable local MDS for $\mathcal{S}$. ■

5 Hermite Interpolations of Scattered Data

In terms of nodal parameters we get the following result.

Theorem 3. Let $\mathcal{S}$ be defined as in above section. Then any spline $s \in \mathcal{S}$ is uniquely determined by the following set of data

$$\{D_x^\alpha D_y^\beta s(v_i)\}_{0\le\alpha+\beta\le 1}, \quad i = 1, ..., V, \tag{11}$$

where v_i are the vertices in the checkerboard quadrangulation $\bar{\diamondsuit}$.

Proof : It is well-known [13] that setting this nodal data is equivalent to setting the B-coefficients of $\mathcal{P}$ listed in Theorem 1. ■

In what following, we consider the Hermite interpolation problem.

Theorem 4. For any function f which is sufficiently smooth so that the needed derivatives exist, there is a unique spline $s \in \mathcal{S}$ such that

$$D_x^\alpha D_y^\beta s(x_i, y_i) = D_x^\alpha D_y^\beta f(x_i, y_i), \quad 0 \le \alpha + \beta \le 1, \quad i = 1, ..., V. \tag{12}$$

Proof : For each quadrilateral Q of $\bar{\diamondsuit}$, the interpolant s can be constructed locally since the given data uniquely determine the nodal data of Theorem 3. ■

6 Bounds on the Error of Interpolation

Let $\mathcal{P}$ and $\mathcal{S}$ as described in the previous section. Let $\{B_\xi\}_{\xi\in\mathcal{P}}$ be the dual basis of s. Given any function f defined on Ω, let

$$\mathcal{I}f := \sum_{\xi\in\mathcal{P}} f(\xi) B_\xi. \tag{13}$$

It is clear that the cubic spline $\mathcal{I}f$ interpolates f at all the points of $\mathcal{P}$. We now give $\mathcal{I}f(\xi) = f(\xi)$, for all $\xi \in \mathcal{P}$. We now give an error bound for $f - \mathcal{I}f$ and its derivatives in the infinity norm. Similar results hold for general $p-$norms.

Theorem 5. There exists a constant C depending only on the smallest angle in $\bar{\diamondsuit}$ such that if f belongs to the Sobolev space $W_\infty^{m+1}(\Omega)$ with $0 \le m \le 3$,

$$\| D_x^\alpha D_y^\beta (f - \mathcal{I}f) \|_{\infty,\Omega} \le C|\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\infty,\Omega} \tag{14}$$

for all $0 \le \alpha + \beta \le m$, where $|\triangle|$ is the maximum of the diameters of the triangles in $\bar{\diamondsuit}$ and

$$|f|_{m+1,\infty,\Omega} = \sum_{\nu+\mu=m+1} \| D_x^\nu D_y^\mu f \|_{\infty,\Omega} .$$

Proof: Let T be a triangle of a $\triangle_Q$ in $\bar{\diamondsuit}$, and $\Omega_T := \text{star}^2(Q) \subset \text{star}^4(v_i)$, where v_i $(i = 1, 2, 3, 4)$ are four vertices of Q. It follows from Lemma 4.6 in [2] that there exists a polynomial $p := p_f \in \mathcal{P}_m$ such that

$$\| D_x^\alpha D_y^\beta (f - p) \|_{\infty,\Omega_T} \le C_0 |\Omega_T|^{m+1-\alpha-\beta} |f|_{m+1,\infty,\Omega_T},$$

for $0 \le \alpha + \beta \le m$, where C_0 is an absolute constant. Since $|\Omega_T| \le 8|\triangle|$ (cf. the proof of Lemma 3.2 in [2]), we have

$$\| D_x^\alpha D_y^\beta (f - p) \|_{\infty,\Omega_T} \le C_1 |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\infty,\Omega_T},$$

where $C_1 := 8^{m+1-\alpha-\beta}C_0$. Since $\mathcal{I}p = p$, we have

$$\| D_x^\alpha D_y^\beta (f - \mathcal{I}f) \|_{\infty,T} \leq \| D_x^\alpha D_y^\beta (f-p) \|_{\infty,T} + \| D_x^\alpha D_y^\beta \mathcal{I}(f-p) \|_{\infty,T} .$$

By the Markov inequality [14], we have

$$\| D_x^\alpha D_y^\beta \mathcal{I}(f-p) \|_{\infty,T} \leq C_2 |\triangle|^{-(\alpha+\beta)} \| \mathcal{I}(f-p) \|_{\infty,T},$$

for some C_2 depending on the smallest angle in T. Let $\mathcal{I}(f-p) \mid_T = \sum_{\xi \in \mathcal{D}_T} c_\xi B_\xi^3$, then, we have

$$\| \mathcal{I}(f-p) \|_{\infty,T} \leq \max_{\xi \in \mathcal{D}_T} |c_\xi|.$$

But by Theorem 2, we have

$$|c_\xi| \leq C \max_{\eta \in \mathcal{P} \cap \Omega_T} |(f-p)(\eta)| \leq C \| f - p \|_{\infty,\Omega_T}, \quad \xi \in \mathcal{D}_T.$$

Combining the above inequalities leads immediately to (14). ■

7 Conclusion

Local Lagrange interpolations by bivariate C^1 cubic splines on checkerboard triangulations $\bar{\diamondsuit}$ and triangulated quadrangulation $\diamondsuit$ have been developed in [12] and [15]. Comparing with Lagrange interpolation schemes in [12], our method needs fewer locally supported basis functions because of a fewer dimension of the spline space. Following [15], domain point set $\mathcal{P}$ constructed in this paper is also a Lagrange minimal determining set for $\mathcal{S}$, namely, $\mathcal{P} := \{\eta_i\}_{i=1}^{3V}$ and $\mathcal{S}$ is a Lagrange interpolation pair. In addition, Lemmas 3-5 in this paper are also used to construct Local Lagrange interpolation methods for C^1 cubic splines on reduced checkerboard triangulations [12]. Finally, it is clear that our results can be applied to type-II triangulation, and we can also reduce these results to more general classes of quadrangulations.

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APPROXIMATIONS RELATED TO GAMMA FUNCTION

CRISTINEL MORTICI

ABSTRACT. The aim of this paper is to introduce a new method for establishing new, increasingly accurate formulas for approximating the gamma function, having a simple form. Inequalities related to that formulas are given.

1. INTRODUCTION

Special functions, in particular the Euler's gamma function Γ, have been studied for centuries. One of the most used formula for approximating the gamma function is the following [1, Theorem 1.4.1]

$$\Gamma(x+1)=\left(\frac{x}{e}\right)^x\sqrt{2\pi x}\left(1+O\left(\frac{1}{x}\right)\right),\quad \text{as}\ \ x\to\infty, \tag{1.1}$$

now generally known as Stirling's formula. See also [4]-[9] for recent studies.

Slightly better results as $x\to\infty$ can be obtained using Burnside's formula [2]

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}}\left(1+O\left(\frac{1}{x}\right)\right), \tag{1.2}$$

or Gosper's formula [3]

$$\Gamma(x+1)=\left(\frac{x}{e}\right)^x\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(1+O\left(\frac{1}{x^2}\right)\right). \tag{1.3}$$

In fact, (1.1) is the first approximation of the following asymptotic series

$$\begin{aligned}\Gamma(x+1)\ \sim\ &\left(\frac{x}{e}\right)^x\sqrt{2\pi x}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}\right.\\&\left.+\frac{163879}{209018880x^5}+\frac{5246819}{75246796800x^6}-\frac{534703531}{902961561600x^7}-\cdots\right),\end{aligned} \tag{1.4}$$

also called the Stirling series. It is true that by truncation of only a few terms, such series provide approximations of any desired accuracy, but a sacrifice of simplicity.

Nemes [9, Rel. 12] remarked the surpringsly fact that the function

$$\left(1+\frac{1}{15x^2}\right)^{\frac{5}{4}x}=1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\frac{16997}{149299200x^5}+\cdots \tag{1.5}$$

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has a very similar series expansion with (1.4) and proposed the following accurate approximation formula

$$\Gamma(x+1) \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5}{4}x}. \tag{1.6}$$

The great advantage of this formula consists also in its simple form requiring a minimum of computation. To be more precise, notice that in order to obtain comparable results with (1.6), we should consider at least five terms of the asymptotic series (1.4).

Formula (1.6) is also more accurate than the following approximation in "The lost notebook and other unpublished papers" of Srinivasa Ramanujan:

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}. \tag{1.7}$$

Moreover, for all $x \geq 1$, it holds

$$\begin{aligned}\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{800x^3}} &< \Gamma(x+1) \\ &< \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}.\end{aligned} \tag{1.8}$$

For proofs and other details, see [10].

Related to (1.6), we prove in this paper the following

Theorem 1. *For every $x \geq 2$, we have*

$$\frac{19}{28350x^5}-\frac{167}{283500x^7} < \ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5}{4}x}} < \frac{19}{28350x^5}.$$

A direct consequence is the following asymptotic formula

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5}{4}x}\left(1+O\left(\frac{1}{x^5}\right)\right), \quad \text{as} \quad x\to\infty.$$

First let us introduce the following function depending on a real parameter a,

$$\begin{aligned}\left(1+\frac{1}{15x^2}\right)^{\frac{5}{4}x+\frac{a}{x^3}} &= 1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51\,840x^3}-\frac{571}{2488320x^4} \\ &\quad +\left(\frac{1}{15}a+\frac{16\,997}{149\,299\,200}\right)\frac{1}{x^5}+O\left(\frac{1}{x^6}\right)\end{aligned}$$

By equating the coefficients of x^{-5} here and in (1.4) we obtain the value $a=\frac{19}{1890}$. In consequence we derive the formula

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5}{4}x+\frac{19}{1890x^3}}\left(1+O\left(\frac{1}{x^7}\right)\right), \quad \text{as} \quad x\to\infty \tag{1.9}$$

and we prove the following

Theorem 2. *For every $x \geq 4$, we have*

$$\frac{-241}{425250x^7} < \ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5}{4}x+\frac{19}{1890x^3}}} < \frac{-241}{425250\left(x+\frac{884\,546}{4175325x}\right)^7}.$$

Going further, notice that

$$\left(1+\frac{1}{15x^2}+\frac{b}{x^8}\right)^{\frac{5}{4}x+\frac{19}{1890x^3}} = 1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4} + \frac{163879}{209018880x^5}+\frac{5246\,819}{75246796800x^6} + \left(\frac{5}{4}b-\frac{114\,861\,143}{4514\,807\,808\,000}\right)\frac{1}{x^7}+O\left(\frac{1}{x^8}\right)$$

and by equating the coefficients of x^{-7} here and in (1.4) we obtain the value $b=-\frac{482}{1063125}$. The corresponding approximation formula as $x\to\infty$ is the following

$$\Gamma(x+1)=\left(\frac{x}{e}\right)^x\sqrt{2\pi x}\left(1+\frac{1}{15x^2}-\frac{482}{1063125x^8}\right)^{\frac{5}{4}x+\frac{19}{1890x^3}}\left(1+O\left(\frac{1}{x^9}\right)\right). \tag{1.10}$$

By using a similar method, estimates as in Theorem 1-2 can be established for (1.10), but the computations become difficult. We omit the details for sake of simplicity.

Numerical computations show us that for obtaining comparable results with (1.9) and (1.10) using series (1.4), at least the first eight, respective ten terms of (1.4) should be considered.

2. Proofs of the results and conclusion

There is the following record in [1, p. 29] for all $x>0$ and $n\geq 1$

$$\ln\Gamma(x+1) = \left(x+\frac{1}{2}\right)\ln x-x+\frac{1}{2}\ln 2\pi+\sum_{j=1}^{n}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} + (-1)^n\frac{2}{x^{n-1}}\int_0^\infty\left(\int_0^t\frac{z^{2n}}{x^2+z^2}dz\right)\frac{dt}{e^{2\pi t}-1}, \tag{2.1}$$

where B_s are the Bernoulli numbers ($B_{2j+1}=0$, for all $j\geq 1$, and the first few Bernoulli numbers are $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$). See e.g. [1, p. 12]. Formula (2.1) provides the well known asymptotic expansion of the gamma function

$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x\sqrt{2\pi x}\exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5} -\frac{1}{1680x^7}+\frac{1}{1188x^9}-\frac{691}{360\,360x^{11}}+\cdots\right) \tag{2.2}$$

as $x\to\infty$. As a direct consequence of (2.1), we have for all $x>0$ and $k\geq 1$

$$\left(x+\frac{1}{2}\right)\ln x-x+\frac{1}{2}\ln 2\pi+\sum_{j=1}^{2k}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} < \ln\Gamma(x+1) < \left(x+\frac{1}{2}\right)\ln x-x+\frac{1}{2}\ln 2\pi+\sum_{j=1}^{2k-1}\frac{B_{2j}}{2j(2j-1)x^{2j-1}},$$

or

$$a_{2k}(x)<\ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}<a_{2k-1}(x), \text{ where } a_s(x)=\sum_{j=1}^{s}\frac{B_{2j}}{2j(2j-1)x^{2j-1}}. \tag{2.3}$$

Proof of Theorem 1. The required inequality can be written as

$$\begin{aligned}&\frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right)+\frac{19}{28\,350x^5}-\frac{167}{283\,500x^7}\\ <\;&\ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}<\frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right)+\frac{19}{28350x^5}.\end{aligned}$$

Using (2.3), it suffices to prove that $f>0$ and $g<0$ on $[2,\infty)$, where

$$f(x)=a_6(x)-\frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right)-\frac{19}{28\,350x^5}+\frac{167}{283\,500x^7}$$

and

$$g(x)=a_5(x)-\frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right)-\frac{19}{28\,350x^5}.$$

Using a computer software such as Maple, we get

$$f''(x)=\frac{P(x)}{20270250x^{13}\left(15x^2+1\right)^2},\qquad g''(x)=-\frac{Q(x)}{2970x^{11}\left(15x^2+1\right)^2},$$

where

$$\begin{aligned}P(x)\;=\;&3756240073+30236631356(x-2)+56138769807(x-2)^2\\&+46394157344(x-2)^3+19614489268(x-2)^4\\&+4144565880(x-2)^5+345380490(x-2)^6\end{aligned}$$

and

$$\begin{aligned}Q(x)\;=\;&621331+2680532(x-2)+4139925(x-2)^2+3145712(x-2)^3\\&+1274974(x-2)^4+264528(x-2)^5+22044(x-2)^6.\end{aligned}$$

Now f is strictly convex, g is strictly concave on $[2,\infty)$, with $f(\infty)=g(\infty)=0$. Thus $f(x)>0$ and $g(x)<0$, for all $x\geq 2$ and the theorem is proved. □

Proof of Theorem 2. The required inequality can be written as

$$\begin{aligned}&\left(\frac{5x}{4}+\frac{19}{1890x^3}\right)\ln\left(1+\frac{1}{15x^2}\right)-\frac{241}{425250x^7}\\ <\;&\ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}}<\left(\frac{5x}{4}+\frac{19}{1890x^3}\right)\ln\left(1+\frac{1}{15x^2}\right)-\frac{241}{425250\left(x+\frac{884\,546}{4175325x}\right)^7}.\end{aligned}$$

By multiplying by x^3 and using (2.3), it suffices to prove that $u>0$ and $v<0$ on $[4,\infty)$, where

$$u(x)=x^3a_6(x)-\left(\frac{5x^4}{4}+\frac{19}{1890}\right)\ln\left(1+\frac{1}{15x^2}\right)+\frac{241}{425250x^4}$$

and

$$v(x)=x^3a_7(x)-\left(\frac{5x^4}{4}+\frac{19}{1890}\right)\ln\left(1+\frac{1}{15x^2}\right)+\frac{241x^3}{425250\left(x+\frac{884\,546}{4175325x}\right)^7}.$$

Using Maple software, we get

$$u^{(V)}(x)=-\frac{4R(x)}{2027025x^{13}\left(15x^2+1\right)^5},$$

$$v^{(V)}(x)=\frac{4S(x)}{45045x^{15}\left(15x^2+1\right)^5\left(4175\,325x^2+884\,546\right)^{12}},$$

where R, S are polynomials and the polynomials $R(x+4)$ and $S(x+4)$ have all coefficients positive.

Since $u^{(V)}<0$ and $v^{(V)}>0$ on $[4,\infty)$ and $u^{(j)}(\infty)=v^{(j)}(\infty)=0$, for every $0\leq j\leq 4$, it results that $u(x)>0$ and $v(x)<0$, for all $x\geq 4$ and the theorem is proved. □

Remark that our new aproximation formulas have great superiority over classical results (1.1)-(1.3), (1.6)-(1.7) and the inequalities stated in Theorems 1-2 improve much Ramanujan's inequality (1.8).

As a final example, mention that by searching a new factor of the form $\left(1+\frac{1}{px^2}\right)^{qx+r}$ for Burnside result (1.2), the following asymptotic formula can be obtained

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}}\left(1-\frac{23}{60x^2}\right)^{\frac{5}{46}x-\frac{5}{92}}\left(1+O\left(\frac{1}{x^4}\right)\right),\quad \text{as}\quad x\to\infty.$$

By replacing x by $x-\frac{1}{2}$, the following interesting half-argument formula can be stated

$$\Gamma\left(x+\frac{1}{2}\right)\approx\sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(1-\frac{23}{15\left(2x-1\right)^2}\right)^{\frac{5}{46}(x-1)}.$$

Moreover, it can be proved that for all $x\geq 10$,

$$\begin{aligned}-\frac{7}{2880x^4}-\frac{2311}{907200x^5}-\frac{73}{64800x^6} &< \ln\frac{\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(1-\frac{23}{15(2x-1)^2}\right)^{\frac{5}{46}(x-1)}}\\ &< -\frac{7}{2880x^4}-\frac{2311}{907200x^5}.\end{aligned}$$

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Numerical Solution of Klein-Gordon-Zakharov Equations using Chebyshev Cardinal Functions

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Abstract

In this article, a numerical method is used for solving Klein-Gordon-Zakharov equations. The method consists of expanding the required approximate solutions as the basis of Chebyshev cardinal functions. The partial differential equations are reduced to nonlinear algebraic equations by using operational matrices of derivatives. The computational results are compared with those obtained in previous work and it was found that the method is accurate and efficient.

Keywords: Klein-Gordon-Zakharov equations, Chebyshev cardinal functions, operational matrices of derivatives.

1 Introduction

The Klein-Gordon-Zakharov (KGZ) equations are given by the following partial differential equations [1]:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \phi + w\phi + (|\phi|^2)\phi = 0, \qquad x \in \Omega,\ t \in (0,T], \tag{1.1}$$

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2}{\partial x^2}(|\phi|^2) = 0, \qquad x \in \Omega,\ t \in (0,T], \tag{1.2}$$

where $\Omega = (-L, L)$, $\phi(x,t)$ is an unknown complex-valued function, $w(x,t)$ is an unknown real-valued function. The exact solution of the system (1.1)-(1.2), which has been derived in [18, 19] is given by:

$$\phi(x,t) = \frac{\sqrt{10}-\sqrt{2}}{2}\text{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right).\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\right)\right], \tag{1.3}$$

$$w(x,t) = -2\text{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right). \tag{1.4}$$

The system (1.1)-(1.2) is known to satisfy the following conservation laws [20]:

$$\int_{-L}^{L}\left[|\phi_t|^2 + |\phi_x|^2 + |\phi|^2 + w|\phi|^2 + \frac{1}{2}|\xi|^2 + \frac{1}{2}|w|^2 + \frac{1}{2}|\phi|^4\right]dx = \text{const}, \tag{1.5}$$

where $\xi = \theta_x$ and $\theta_{xx} = w_t$. The KGZ equations describes the interaction of the Langmuir wave and the ion acoustic wave in high frequency plasma [1, 2]. The KGZ system can be derived from the two fluid Euler-Maxwell system, and further details can be found in [3, 4, 5, 6] and references therein.

Many exact and numerical methods have been introduced in the literature for the KGZ equations and equations similar to the KGZ equations, Guo and Yuan [7] found the global smooth solution for KGZ equations and Guo [8] developed a global smooth solution in non-homogenous medium for the KGZ equations. Gan [10, 9] investigated the orbital instability of standing waves for the KGZ equations.

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Exact traveling waves for the KGZ equations have been developed by Li in [11]. In the last decade, much attention has been given in the literature to the design, development, analysis and implementation of methods for KGZ equations, see for example [12, 13, 14, 15, 16, 17].

This paper presents a numerical method based on Chebyshev cardinal functions for solving KGZ equations. In this work we reduce the partial differential equations to nonlinear algebraic equations by using operational matrices of derivatives. The computational results are compared with those obtained in previous work [20]. The paper is organized as follows: In Section 2 we propose the Chebyshev cardinal functions. Numerical method for KGZ equations is described in Section 3. Numerical results that illustrate the efficiency of the proposed method are reported in Section 4. Section 5 contains concluding remarks.

2 Chebyshev Cardinal Functions

J.P. Boyd defined the cardinal functions of order N over the interval $[-1, 1]$ in his book [23]:

$$C_j(\zeta) = \frac{T_{N+1}(\zeta)}{T'_{N+1}(\zeta_j)(\zeta - \zeta_j)}, \qquad j = 1, \ldots, N+1, \tag{2.1}$$

where prime denotes the differentiation with respect to ζ, while $T_{N+1}(\zeta)$ is the second kind Chebyshev function of order $N+1$ in $[-1, 1]$ defined by

$$T_{N+1}(\zeta) = \cos((N+1)\arccos(\zeta)). \tag{2.2}$$

We change the variable $x = L\zeta$ to use these functions on $[-L, L]$. Now any function $\eta(x)$ on $[-L, L]$ can be approximated as

$$\eta(x) \approx \sum_{j=1}^{N+1} \eta(x_j) C_j(x) = \Psi^T \Phi_N(x), \tag{2.3}$$

where

$$x_j = L\left(\frac{2(j-1)}{N} - 1\right), \quad j = 1, 2, \cdots, N+1,$$

$$-L = x_1 < x_2 < x_3, \cdots, < x_{N+1} = L,$$

$$\Psi = [\eta(x_1), \eta(x_2), \ldots, \eta(x_{N+1})]^T, \tag{2.4}$$

and

$$\Phi_N(x) = [C_1(x), C_2(x), \ldots, C_{N+1}(x)]^T. \tag{2.5}$$

2.1 The operational matrix of derivative

The differentiation of function Φ_N in (2.5) can be expressed as [22]:

$$\Phi'_N = (D_N)\Phi_N \tag{2.6}$$

where D_N is $(N+1) \times (N+1)$ operational matrix of derivative for cardinal Chebyshev functions. The matrix D_N can be obtained by the following process. Let

$$\Phi'_N = [C'_1(x), C'_2(x), \ldots, C'_{N+1}(x)]^T. \tag{2.7}$$

Using (2.3) any function $C'_j(x)$ can be approximated as

$$C'_j(x) = \sum_{k=1}^{N+1} C'_j(x_k) C_k(x). \tag{2.8}$$

Comparing (2.6) and (2.8), we get

$$D_N = \begin{bmatrix} C_1'(x_1) & \cdots & C_1'(x_{N+1}) \\ \vdots & & \vdots \\ C_{N+1}'(x_1) & \cdots & C_{N+1}'(x_{N+1}) \end{bmatrix}. \tag{2.9}$$

3 Description of Numerical Method

Suppose $\Phi_N(x)$ is the vector of Chebshev cardinal functions on $[-L, L]$ defined in (2.5) and $\Phi_N(t)$ is the vector of cardinal functions on $[0, T]$. Let $\phi = u + iv$, then (1.1)-(1.2) can be written as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u + wu + (u^2 + v^2)u = 0, \tag{3.1}$$

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v + wv + (u^2 + v^2)u = 0, \tag{3.2}$$

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2}{\partial x^2}(u^2 + v^2) = 0, \tag{3.3}$$

with initial conditions

$$u(x,0) = \mathrm{Re}(\phi_0(x)), \qquad v(x,0) = \mathrm{Im}(\phi_0(x)), \tag{3.4}$$

$$\frac{\partial u}{\partial t}(x,0) = \mathrm{Re}(\phi_1(x)), \qquad \frac{\partial v}{\partial t}(x,0) = \mathrm{Im}(\phi_1(x)), \tag{3.5}$$

$$w(x,0) = w_0(x), \qquad \frac{\partial w}{\partial t}(x,0) = w_1(x), \tag{3.6}$$

and the boundary conditions

$$u(-L,0) = u(L,0) = 0, \quad v(-L,0) = v(L,0) = 0, \quad w(-L,0) = w(L,0) = 0. \tag{3.7}$$

By applying the definitions of [22, 23] we can derive our method as follows:
The unknown functions $u(x,t)$, $v(x,t)$ and $w(x,t)$ can be approximated as

$$u(x,t) \simeq \sum_{i=1}^{N+1}\sum_{j=1}^{M+1} U_{ij} C_i(x) C_j(t) = \Phi_N^T(x) U \Phi_M(t), \tag{3.8}$$

$$v(x,t) \simeq \sum_{i=1}^{N+1}\sum_{j=1}^{M+1} V_{ij} C_i(x) C_j(t) = \Phi_N^T(x) V \Phi_M(t), \tag{3.9}$$

$$w(x,t) \simeq \sum_{i=1}^{N+1}\sum_{j=1}^{M+1} W_{ij} C_i(x) C_j(t) = \Phi_N^T(x) W \Phi_M(t), \tag{3.10}$$

where

$$U = \begin{bmatrix} U_{1,1} & \cdots & U_{1,M+1} \\ \vdots & & \vdots \\ U_{N+1,1} & \cdots & U_{N+1,M+1} \end{bmatrix}, \qquad V = \begin{bmatrix} V_{1,1} & \cdots & V_{1,M+1} \\ \vdots & & \vdots \\ V_{N+1,1} & \cdots & V_{N+1,M+1} \end{bmatrix}, \tag{3.11}$$

and

$$W = \begin{bmatrix} W_{1,1} & \cdots & W_{1,M+1} \\ \vdots & & \vdots \\ W_{N+1,1} & \cdots & W_{N+1,M+1} \end{bmatrix}, \tag{3.12}$$

Now we can approximate the second order derivatives with respect to t of unknown functions $u(x,t)$, $v(x,t)$ and $w(x,t)$ as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial t^2}\Phi_N^T(x)U\Phi_M(t) = \Phi_N^T(x)U\Phi_M''(t) = \Phi_N^T(x)U(D_M^2)\Phi_M(t), \tag{3.13}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2}{\partial t^2}\Phi_N^T(x)V\Phi_M(t) = \Phi_N^T(x)V\Phi_M''(t) = \Phi_N^T(x)V(D_M^2)\Phi_M(t), \tag{3.14}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2}{\partial t^2}\Phi_N^T(x)W\Phi_M(t) = \Phi_N^T(x)W\Phi_M''(t) = \Phi_N^T(x)W(D_M^2)\Phi_M(t). \tag{3.15}$$

The second order derivatives with respect to x of unknown functions $u(x,t)$, $v(x,t)$ and $w(x,t)$ can be approximated as

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\Phi_N^T(x)U\Phi_M(t) = (\Phi'')_N^T(x)U\Phi_M(t) = \Phi_N^T(x)(D_N^2)^T U\Phi_M(t), \tag{3.16}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2}\Phi_N^T(x)V\Phi_M(t) = (\Phi'')_N^T(x)V\Phi_M(t) = \Phi_N^T(x)(D_N^2)^T V\Phi_M(t), \tag{3.17}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2}{\partial x^2}\Phi_N^T(x)W\Phi_M(t) = (\Phi'')_N^T(x)W\Phi_M(t) = \Phi_N^T(x)(D_N^2)^T W\Phi_M(t). \tag{3.18}$$

By applying (3.8)-(3.10), we can write the initial conditions as

$$u(x,0) = \Phi_N^T(x)U\Phi_M(0) = Re(\phi_0(x)), \tag{3.19}$$

$$v(x,0) = \Phi_N^T(x)V\Phi_M(0) = Im(\phi_0(x)), \tag{3.20}$$

$$w(x,0) = \Phi_N^T(x)W\Phi_M(0) = w_0(x), \tag{3.21}$$

and

$$\frac{\partial u}{\partial t}(x,0) = \Phi_N^T(x)U\Phi_M'(0) = \Phi_N^T(x)U(D_M)\Phi_M(0) = Re(\phi_1(x)), \tag{3.22}$$

$$\frac{\partial v}{\partial t}(x,0) = \Phi_N^T(x)V\Phi_M'(0) = \Phi_N^T(x)V(D_M)\Phi_M(0) = Im(\phi_1(x)), \tag{3.23}$$

$$\frac{\partial w}{\partial t}(x,0) = \Phi_N^T(x)W\Phi_M'(0) = \Phi_N^T(x)W(D_M)\Phi_M(0) = w_1(x). \tag{3.24}$$

Substituting (3.13)-(3.18) into (3.1)-(3.3), and collecting the equations at the points at $x = x_i$, $i = 1,2,\ldots,N+1$, $t = t_j$, $j = 1,2,\ldots,M+1$, we obtained

$$\begin{cases} \Phi_N^T(x_i)U(D_M^2)\Phi_M(t_j) - \Phi_N^T(x_i)(D_N^2)U\Phi_M(t_j) + \Phi_N^T(x_i)U\Phi_M(t_j) \\ + \left(\left[\Phi_N^T(x_i)W\Phi_M(t_j)\right] \times \left[\Phi_N^T(x_i)U\Phi_M(t_j)\right]\right) + \Big(\left[\Phi_N^T(x_i)U\Phi_M(t_j)\right]^2 \\ + \left[\Phi_N^T(x_i)V\Phi_M(t_j)\right]^2\Big)\left[\Phi_N^T(x_i)U\Phi_M(t_j)\right] = 0, \end{cases} \tag{3.25}$$

$$\begin{cases} \Phi_N^T(x_i)V(D_M^2)\Phi_M(t_j) - \Phi_N^T(x_i)(D_N^2)V\Phi_M(t_j) + \Phi_N^T(x_i)V\Phi_M(t_j) \\ + \left(\left[\Phi_N^T(x_i)W\Phi_M(t_j)\right] \times \left[\Phi_N^T(x_i)V\Phi_M(t_j)\right]\right) + \Big(\left[\Phi_N^T(x_i)U\Phi_M(t_j)\right]^2 \\ + \left[\Phi_N^T(x_i)V\Phi_M(t_j)\right]^2\Big)\left[\Phi_N^T(x_i)U\Phi_M(t_j)\right] = 0, \end{cases} \tag{3.26}$$

$$\begin{cases} \Phi_N^T(x_i)W(D_M^2)\Phi_M(t_j) - \Phi_N^T(x_i)(D_N^2)W\Phi_M(t_j) - 2\left(\left[\Phi_N^T(x_i)U\Phi_M(t_j)\right]\right. \\ \times[\Phi_N^T(x_i)(D_N^2)U\Phi_M(t_j)] + [\Phi_N^T(x_i)(D_N)U\Phi_M(t_j)]^2\big) + \left(\left[\Phi_N^T(x_i)V\Phi_M(t_j)\right]\right. \\ \left.\times[\Phi_N^T(x_i)(D_N^2)V\Phi_M(t_j)] + [\Phi_N^T(x_i)(D_N)V\Phi_M(t_j)]^2\right) = 0, \end{cases} \tag{3.27}$$

with initial conditions

$$\Phi_N^T(x_i)U\Phi_M(0) = Re(\phi_0(x_i)), \tag{3.28}$$

$$\Phi_N^T(x_i)V\Phi_M(0) = Im(\phi_0(x_i)), \tag{3.29}$$

$$\Phi_N^T(x_i)W\Phi_M(0) = w_0(x_i), \tag{3.30}$$

and

$$\Phi_N^T(x_i)U(D_M)\Phi_M(0) = Re(\phi_1(x_i)), \tag{3.31}$$
$$\Phi_N^T(x_i)V(D_M)\Phi_M(0) = Im(\phi_1(x_i)), \tag{3.32}$$
$$\Phi_N^T(x_i)W(D_M)\Phi_M(0) = w_1(x_i), \tag{3.33}$$

Equations (3.25)-(3.27) together with equations (3.28)-(3.33) give a system of nonlinear equations, which can be solved for U_{ij}, V_{ij} and W_{ij}, for $i = 1, 2, \ldots, N+1$, $j = 1, 2, \ldots, M+1$.

4 Numerical Results

In this section, we present some numerical results of our scheme for the Klein-Gordon-Zakharov equations. All computation were carried out in Matlab 6.5 on a personal computer. We study Langmuir wave and ion acoustic wave. The accuracy of the scheme is measured by using the following error norm:

$$L_\infty = \max_{1\le i\le N+1}\{\|\eta^{\text{exact}}(x_i,t)\| - \|\eta^{\text{approx}}(x_i,t)\|\},$$

where η= u, v and w, while $\eta^{\text{exact}}(x_i,t)$ represents the exact solution and $\eta^{\text{approx}}(x_i,t)$ represents the approximate solution.

4.1 Solution of Conserved Quantities

In this section we solve the conserved quantities by the numerical method based on Chebyshev Cardinal functions. Let

$$I = \int_a^b f(x)dx \tag{4.1}$$

We note that an integral is the special case of ordinary differential equation $u' = f(x,u)$ in which f is independent of u. Equation (4.1) can be written as an initial value problem

$$u' = f(x), \qquad u(a) = 0, \qquad x > a,$$

We use the Chebyshev Cardinal functions for $[a,b]$ on Chebyshev-Gauss-Lobatto points. To impose the boundary condition $u(a) = 0$, we strip off the last row and column of the differentiation matrix D_N. We obtain the linear system of equations

$$\Phi_N^T(x_i)(D_N)U\Phi_M(0) = f, \qquad i = 1, 2, \ldots, N+1$$

where $f = (f(x_1), f(x_1), ..., f(x_{N+1}))$. The system of linear equations can be solved by any standard numerical procedure. For more detail see [21].

4.2 Example

We consider a example of KGZ equations (1.1)-(1.2). The initial condition at $t = 0$ are given by [20]:

$$u(x,0) = \text{Re}\left(\frac{\sqrt{10}-\sqrt{2}}{2}\text{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right]\right), \tag{4.2}$$

$$v(x,0) = \text{Im}\left(\frac{\sqrt{10}-\sqrt{2}}{2}\text{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right]\right), \tag{4.3}$$

$$w(x,0) = w_0(x) = -2\text{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \qquad -20 \le x \le 20. \tag{4.4}$$

$$u_1(x) = \mathrm{Re}\left(\frac{\sqrt{10}-\sqrt{2}}{2}.\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right]\right.$$
$$\left. - \frac{\sqrt{10}-\sqrt{2}}{2}.\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right]\right), \qquad -20 \le x \le 20, \tag{4.5}$$

$$v_1(x) = \mathrm{Im}\left(\frac{\sqrt{10}-\sqrt{2}}{2}.\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right]\right.$$
$$\left. - \frac{\sqrt{10}-\sqrt{2}}{2}.\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right]\right), \qquad -20 \le x \le 20, \tag{4.6}$$

$$w_1(x) = -4\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right).\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \qquad -20 \le x \le 20, \tag{4.7}$$

where α, μ and ν are constants. The analytical solution of the KGZE equation (1.1)-(1.2) is given by equation (1.3)-(1.4). The boundary conditions are given by (3.7).

The error L_∞ is computed and are shown in Table 1 along with the results of the previous method by Wang and Chen [20] for comparison. From the numerical result give in Table 1 to Table 3, it is observed that throughout the simulation, the error norm L_∞ remains less than 0.2×10^{-4}. The numerical results of the present method are better than the the results obtained from the finite difference method of Wang and Chen [20].

The KGZ equation have Langmuir wave and ion acoustic wave with initial conditions (4.3)-(4.7). The Langmuir wave has amplitude 0.8740 given in the exact solution. The peak position of the Langmuir wave is located at $x = 0$ in $t = 0$, while the peak position of Langmuir wave at $t = 7.25$ is located at $x = 5$ (see figure 1). Therefore the Langmuir wave move towards the right with velocity 0.6897, while the ion acoustic wave has the similar behavior as Langmuir wave but the peak position are in opposite directions and moves towards right with velocity 0.6897 (see figure 1.). The space-time graphs of Langmuir wave are given in figure 2, it is clear that the waves move toward right with the increases of time. From Table 4, we can see the present method conserve the conserved quantities by exactly at least six decimal places.

Table 1: Comparison of maximum error at $(x, t) = (-20, 1)$

T.Wang [20]			Present Method		
u	v	w	u	v	w
1.1249E-8	3.7928E-10	2.2268E-13	2.6017E-11	2.2350E-9	1.3279E-14
2.2922E-8	8.9837E-12	6.2031E-14	3.30114E-9	9.8748E-13	7.3140E-15

Table 2: Comparison of maximum error at $(x, t) = (0, 1)$

T.Wang [20]			Present Method		
u	v	w	u	v	w
2.8104E-4	3.3176E-4	2.6012E-4	3.9293E-5	2.2265E-5	3.7123E-5
7.4145E-5	2.2816E-5	6.8993E-5	8.3256E-6	3.3727E-6	5.7872E-6

Table 3: Comparison of maximum error at $(x,t) = (10,1)$

T.Wang [20]			Present Method		
u	v	w	u	v	w
1.9035E-8	1.4914E-8	9.1287E-13	2.8146E-9	2.3025E-9	8.2398E-14
4.8106E-9	3.7165E-9	2.1403E-13	5.7215E-10	4.6274E-10	3.2514E-14

Table 4: Conserved quantities for ϕ and w at $N = 128$

t	ϕ	w
0.0	1.049828	3.664638
7.25	1.049828	3.664648
14.50	1.049828	3.664648
21.75	1.049828	3.664648

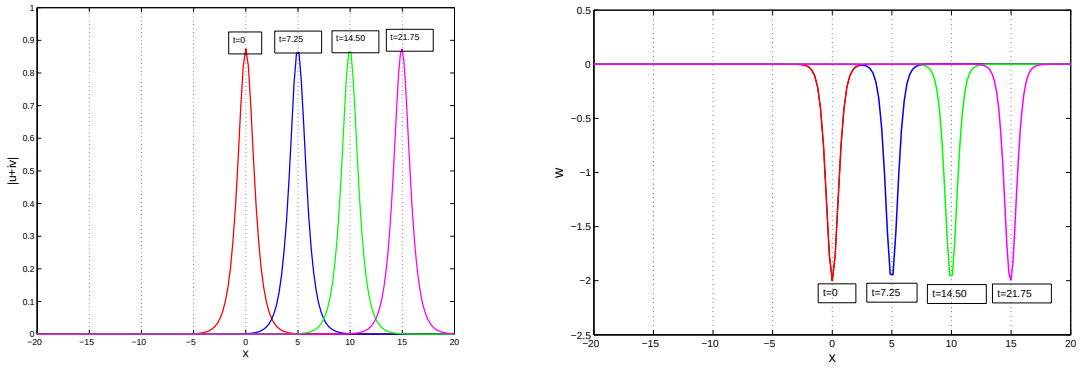


Figure 1: The left figure shows the graph of numerical solution of $|u + iv|$, while the right figure shows the numerical solution of w

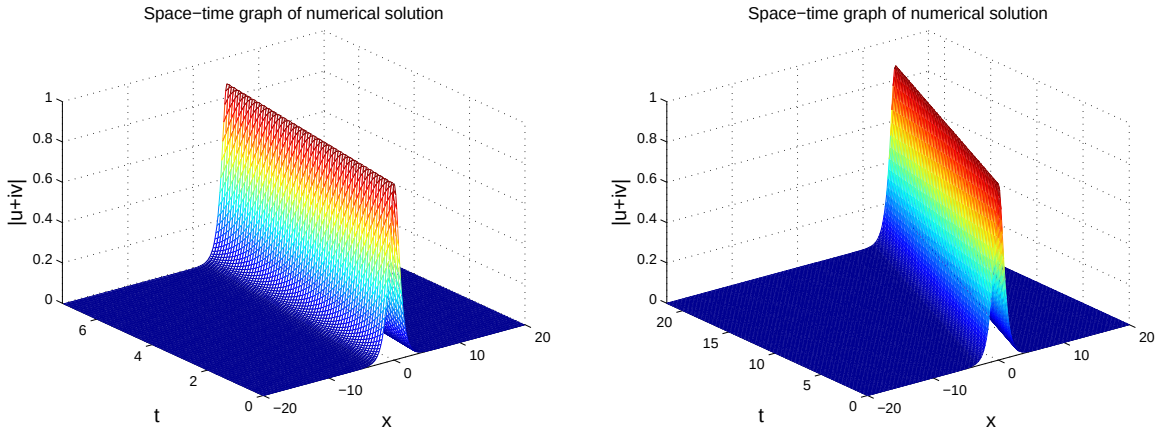


Figure 2: The left figure shows the space-time graph for $0 \leq t \leq 7.25$, while the right figure shows the space-time graph for $0 \leq t \leq 21.75$

5 Conclusion

In this article, we presented a numerical scheme for solving one dimensional Klein-Gordon-Zakharov equations. The shifted Chebyshev cardinal functions [23] on interval $[0, 1]$ are employed to solve the equa-

tions. The partial differential equations are reduce to nonlinear algebraic equations by using operational matrices of derivatives. Numerical results demonstrate the present method is more accurate and efficient than finite difference method.

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New splitting iteration methods for Toeplitz-plus-Hankel systems

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Abstract

In this paper, an efficient splitting iteration method for solving Toeplitz-plus-Hankel linear systems is presented based on the circulant and skew-circulant splitting of the coefficient matrix. We analyze the computational complexity and convergence properties of the new splitting iteration method. Moreover, good estimations of the optimal parameter are discussed for further accelerating. Numerical results are given to illustrate the performance of the new splitting iteration method.

Keywords: Discrete Fourier transform; Toeplitz-plus-Hankel matrix; Circulant and skew-circulant splitting; Iteration method

1 Introduction

An n-by-n matrix T of the form

$$\begin{bmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & & t_{2-n} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{bmatrix} \tag{1}$$

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is called a Toeplitz matrix, i.e. matrices with constant entries along their diagonal. An n-by-n matrix H of the form

$$\begin{bmatrix} h_0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ h_1 & \ddots & \ddots & \ddots & h_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ h_{n-2} & \ddots & \ddots & \ddots & h_{2n-2} \\ h_{n-1} & h_n & \cdots & h_{2n-2} & h_{2n-1} \end{bmatrix} \tag{2}$$

is called a Hankel matrix, i.e. matrices with constant entries along their anti-diagonal. The linear systems with Toeplitz-plus-Hankel coefficient matrices arise from a variety of scientific and engineering applications, such as signal processing, image deblurring; see references [1, 2]. So efficient approaches for solving Toeplitz-plus-Hankel systems have received much attention. Most of the recent works for solving Toeplitz-plus-Hankel systems were focused on fast direct methods [3, 4, 5, 6, 7, 8]. Little attention was paid to iteration methods for solving Toeplitz-plus-Hankel systems.

Bai, Golub and Ng proposed the Hermitian and skew-Hermitian splitting iteration method for the non-Hermitian positive definite systems based on the Hermitian and skew-Hermitian splitting(HSS) of the coefficient matrix [9]. Moreover Ng presented the circulant and skew-circulant splitting iteration method for Toeplitz systems based on the circulant and skew-circulant splitting(CSCS) of the coefficient matrix [10]. Recently, HSS-like iteration methods were developed for positive definite Toeplitz linear systems [11, 12].

It is worth noting that JH is a Toeplitz matrix where H is a Hankel matrix and J is the anti-identity matrix, and Toeplitz matrix T possesses a circulant and skew-circulant splitting (CSCS) [13], i.e.

$$T = C + S,$$

where

$$C = \frac{1}{2}\begin{bmatrix} t_0 & t_{-1}+t_{n-1} & \cdots & t_{-n+2}+t_2 & t_{-n+1}+t_1 \\ t_1+t_{-n+1} & t_0 & t_{-1}+t_{n-1} & & t_{-n+2}+t_2 \\ \vdots & t_1+t_{-n+1} & t_0 & \ddots & \vdots \\ t_{n-2}+t_{-2} & & \ddots & \ddots & t_{-1}+t_{n-1} \\ t_{n-1}+t_{-1} & t_{n-2}+t_{-2} & \cdots & t_1+t_{-n+1} & t_0 \end{bmatrix} \tag{3}$$

and

$$S = \frac{1}{2}\begin{bmatrix} t_0 & t_{-1}-t_{n-1} & \cdots & t_{-n+2}-t_2 & t_{-n+1}-t_1 \\ t_1-t_{-n+1} & t_0 & t_{-1}-t_{n-1} & & t_{-n+2}-t_2 \\ \vdots & t_1-t_{-n+1} & t_0 & \ddots & \vdots \\ t_{n-2}-t_{-2} & & \ddots & \ddots & t_{-1}-t_{n-1} \\ t_{n-1}-t_{-1} & t_{n-2}-t_{-2} & \cdots & t_1-t_{-n+1} & t_0 \end{bmatrix}. \tag{4}$$

Then we propose a new splitting for Toeplitz-plus-Hankel matrix, i.e.

$$T_1 + H_1 = T_1 + JT_2 = (C_1 + JC_2) + (S_1 + JS_2), \tag{5}$$

where T_i, C_i and S_i are Toeplitz matrices, their corresponding circulant matrices and skew-circulant matrices defined in (3) and (4) for i=1, 2. Noting that the circulant matrix C and the skew-circulant matrix S possess following spectral decompositions:

$$C = F_n^* \Lambda_1 F_n \tag{6}$$

and

$$S = \Omega_n^* F_n^* \Lambda_2 F_n \Omega_n, \tag{7}$$

where $\Omega_n = \text{diag}[1, (-1)^{-1/n}, \cdots, (-1)^{-(n-1)/n}]$, F_n is the Fourier matrix, Λ_1 is the diagonal matrix containing the eigenvalues of C and Λ_2 is the diagonal matrix containing the eigenvalues of S. It allows us to multiply $C_1 + JC_2$ or $S_1 + JS_2$ by a vector with just two fast fourier transforms(FFT) after obtaining the eigenvalues of C_i or S_i for $i = 1, 2$. And it will be shown later that the solution of the systems

$$(C_1 + JC_2)z = r \quad \text{or} \quad (S_1 + JS_2)z = r$$

can be easily obtained by just two fast fourier transforms and solving a simple system. Above discussion motivate us to propose an efficient splitting iteration method for Toeplitz-plus-Hankel systems based on the circulant and skew-circulant splitting.

By iterating alternatively between the two splitting parts, we obtain the new splitting iteration method as follows:

Given an initial guess x^0, for $k = 0, 1, 2, \cdots$, until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + C_1 + JC_2)x^{(k+\frac{1}{2})} = (\alpha I - S_1 - JS_2)x^{(k)} + b, \\ (\alpha I + S_1 + JS_2)x^{(k+1)} = (\alpha I - C_1 - JC_2)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{8}$$

where α is a positive constant.

It is obvious that the roles of $C_1 + JC_2$ and $S_1 + JS_2$ are equivalent.

This paper is structured as follows. In Section 2, We analyze the computational complexity and convergence properties of the new splitting iteration method. Moreover, good estimations of the optimal parameter are discussed for further accelerating. Numerical examples are reported to illustrate the effectiveness of the splitting iteration method in Section 3.

2 Convergence analysis of the new splitting iteration

To give a two-step splitting iteration framework, we first recall the following lemma in [9].

Lemma 2.1 *Let $A \in \mathbb{C}^{n\times n}$, $A = M_i - N_i (i = 1, 2)$ be two splittings of the matrix A, and $x^0 \in \mathbb{C}^n$ be a given initial vector. If $\{x^k\}$ is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b, \end{cases}$$

$k = 0, 1, 2, \cdots$, then

$$x^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1}(I + N_2 M_1^{-1})b,\ k = 0, 1, 2, \cdots .$$

Moreover, if the spectral radius $\rho(M_2^{-1} N_2 M_1^{-1} N_1) < 1$, the iteration sequence $\{x^k\}$ converges to the unique solution $x^* \in \mathbb{C}^n$ of the system $Ax = b$ for arbitrary initial vector $x^0 \in \mathbb{C}^n$.

Since $C_1 + JC_2$ and $S_1 + JS_2$ are similar to $F_n^*(\Lambda_1 + F_n J F_n^* \Lambda_2)F_n$ and $\Omega_n^* F_n^*(\Sigma_1 + F_n \Omega_n J \Omega_n^* F_n^* \Sigma_2)F_n \Omega_n$ respectively, the special structure of $F_n J F_n^*$ and $F_n \Omega_n J \Omega_n^* F_n^*$ required to be investigated. The following Lemmas give explicit formulas of these matrices.

Lemma 2.2 The nonzero entries of the matrix $F_n J F_n^*$ are $(1, 1)$-entry 1, (i, j)-entry ω^{-i+1} along anti-diagonal $i + j = n + 2$.

Proof. It can be easily verified that the (i, j)-entry

$$\frac{1}{n}\left(\omega^{(n-1)(i-1)} + \omega^{(n-2)(i-1)}\overline{\omega} j - 1 + \cdots + \omega^{(i-1)}\overline{\omega}^{(n-2)(j-1)} + \overline{\omega}^{(n-1)(j-1)}\right)$$

of the matrix $F_n J F_n^*$ is the sum of the first n terms of a geometric sequence with a common ratio of $-i - j + 2$. From the fact ω^n=1 and the formula for the sum of the first n terms of a geometric sequence, it can be known that the nonzero entries of the matrix are (1,1)-entry 1, (i, j)-entry ω^{-i+1} along anti-diagonal $i + j = n + 2$. □

Remark 2.1 The matrix $F_n J F_n^*$ is a Hermitian and sparse matrix with only n nonzero entries. Then the solution of the system $(C_1 + JC_2)z = r$ can be easily obtained by just two fast fourier transforms and solving a simple system $(\alpha I + \Lambda_1 + F_n J F_n^* \Lambda_2)r = z$ with nonzero entries along the principal diagonal and anti-diagonal $i + j = n + 2$.

Lemma 2.3 The nonzero entries of the matrix $F_n \Omega_n J \Omega_n^* F_n^*$ are (i, j)-entry $(-1)^{\frac{-n+2i-1}{n}}$ along anti-diagonal $i + j = n + 1$.

Proof. Let θ be $(-1)^{-\frac{1}{n}}$. It can be easily verified that the (i, j)-entry

$$\frac{1}{n}\left(\theta^{n-1}\omega^{(n-1)(i-1)} + \theta^{n-2}\overline{\theta}^{1}\omega^{(n-2)(i-1)}\overline{\omega}^{j-1} + \cdots + \theta^{1}\overline{\theta}^{n-2}\omega^{(i-1)}\overline{\omega}^{(n-2)(j-1)} + \overline{\theta}^{n-1}\overline{\omega}^{(n-1)(j-1)}\right)$$

of the matrix $F_n \Omega_n J \Omega_n^* F_n^*$ is the sum of the first n terms of a geometric sequence with a common ratio of $-i - j + 1$. From the fact $\omega^n = 1$ and the formula for the sum of the first n terms of a geometric sequence, we observed that the nonzero entries of the matrix are (i, j)-entry $(-1)^{\frac{-n+2i-1}{n}}$ along anti-diagonal $i + j = n + 1$. □

Remark 2.2 The matrix $F_n\Omega_n J\Omega_n^* F_n^*$ is a Hermitian and sparse matrix with only n nonzero entries. Then the solution of the system $(S_1 + JS_2)z = r$ can be easily obtained by just two fast fourier transforms and solving a simple system $(\alpha I + \Sigma_1 + F_n\Omega_n J\Omega_n^* F_n^*\Sigma_2)r = z$ with nonzero entries along the principal diagonal and anti-diagonal $i + j = n + 1$.

For the convergence property of the new splitting iteration method, we recall the following lemma in [14].

Lemma 2.4 Let

$$V(\alpha) = (\alpha I - P)(\alpha I + P)^{-1}.$$

If $P \in \mathbb{C}^{n\times n}$ is a positive definite matrix(i.e., $\frac{1}{2}(P^* + P)$, the Hermitian part of P, is a Hermitian positive definite matrix), then it holds that

$$\|V(\alpha)\|_2 < 1 \quad \forall \alpha > 0.$$

Theorem 2.1 Let α be a positive constant. If $C_1 + JC_2$ and $S_1 + JS_2$ defined in (5) are positive definite matrices, then the iteration matrix $M(\alpha)$ of the new splitting iteration is given by

$$M(\alpha) = (\alpha I + S_1 + JS_2)^{-1}(\alpha I - C_1 - JC_2)(\alpha I + C_1 + JC_2)^{-1}(\alpha I - S_1 - JS_2) \quad (9)$$

and its spectral radius $\rho(M(\alpha))$ is bounded by

$$\|(\alpha I - C_1 - JC_2)(\alpha I + C_1 + JC_2)\|_2 \, \|(\alpha I - S_1 - JS_2)(\alpha I + S_1 + JS_2)\|_2.$$

Therefore, it holds that

$$\rho(M(\alpha)) \le \|(\alpha I - C_1 - JC_2)(\alpha I + C_1 + JC_2)\|_2 \, \|(\alpha I - S_1 - JS_2)(\alpha I + S_1 + JS_2)\|_2 < 1 \quad for \ \forall \alpha > 0; \quad (10)$$

i.e. the new splitting iteration converges to the exact solution $x^* \in \mathbb{C}^n$ of the system $Ax = b$.

Proof. Let $\tilde{M}(\alpha) = (\alpha I - C_1 - JC_2)(\alpha I + C_1 + JC_2)(\alpha I - S_1 - JS_2)(\alpha I + S_1 + JS_2)$, then

$$\begin{aligned}\rho(M(\alpha)) = \rho(\tilde{M}(\alpha)) &\le \|(\alpha I - C_1 - JC_2)(\alpha I + C_1 + JC_2)(\alpha I - S_1 - JS_2)(\alpha I + S_1 + JS_2)\|_2 \\ &\le \|(\alpha I - C_1 - JC_2)(\alpha I + C_1 + JC_2)\|_2 \, \|(\alpha I - S_1 - JS_2)(\alpha I + S_1 + JS_2)\|_2 .\end{aligned}$$

Since $C_1 + JC_2$ and $S_1 + JS_2$ are positive definite matrices, from the Lemma 2.4 we can show the inequality (10) holds. □

Remark 2.3 The cost for solving two spares linear systems $(\Lambda_1 + F_n J F_n^* \Lambda_2)r = z$ and $(\Sigma_1 + F_n\Omega_n J\Omega_n^* F_n^*\Sigma_2)r = z$ is trivial. Thus in each iteration the leading computational cost are five fast Fourier transforms.

To further improve the efficiency of the new splitting iteration method, it is desirable to find good estimations of the optimal parameter α^* which minimizes the spectral

radius. If $C_1 + JC_2$ and $S_1 + JS_2$ defined in (5) are normal matrices, we can calculate the estimation

$$\hat{\alpha}^* \equiv \min_{\alpha>0} \max_{\gamma+i\eta\in\Omega} \frac{(\alpha-\gamma)^2+\eta^2}{(\alpha+\gamma)^2+\eta^2}$$

by the formula in Theorem 2 of [10].

Remark 2.4 The problems of computing eigenvalues of matrices $\Lambda_1 + F_n J F_n^* \Lambda_2$ and $\Sigma_1 + F_n \Omega_n J \Omega_n^* F_n^* \Sigma_2$ can be transformed into the problems of solving the following equation

$$\begin{vmatrix} \lambda - a_{11} & & & & & -a_{1n} \\ & \ddots & & & \iddots & \\ & & \lambda - a_{\frac{n}{2},\frac{n}{2}} & -a_{\frac{n}{2},\frac{n}{2}+1} & & \\ & & -a_{\frac{n}{2}+1,\frac{n}{2}} & \lambda - a_{\frac{n}{2}+1,\frac{n+1}{2}} & & \\ & \iddots & & & \ddots & \\ -a_{n1} & & & & & \lambda - a_{nn} \end{vmatrix} = 0. \tag{11}$$

Then the eigenvalues of matrices $\Lambda_1 + F_n J F_n^* \Lambda_2$ and $\Sigma_1 + F_n \Omega_n J \Omega_n^* F_n^* \Sigma_2$ can be computed efficiently by the following recurrence

$$D_n = [\lambda^2 - (a_{11} + a_{nn})\lambda + a_{11}a_{nn} - a_{n1}a_{1n}]D_{n-2},$$

where D_n is the n-th order determinant in (11).

3 Numerical examples

In this section, the effectiveness of the proposed splitting iteration method for solving Toeplitz-plus-Hankel systems is illustrated. Let

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta, \qquad k = 0, \pm1, \pm2, \cdots$$

be the Fourier coefficients of f. The function f is called the generating function of the Toeplitz matrix. Similarly a Hankel matrix H can be generated by the generating function of the corresponding Toeplitz matrix JH.

We test the performance of the new splitting method for two Toeplitz-plus-Hankel systems. The Toeplitz matrices T and JH of the first test are generated by the generating functions $f(x) = x^4 + 1$ and $f(x) = 0.1x^4 + 0.1$ respectively. The Toeplitz matrices T and JH of the second test are generated by the generating functions $f(x) = x^4 + 1$ and $f(x) = (|x|^3 + 0.01)$ respectively. All the generating functions are from [15]. The right-hand side b is a vector with all entries one and the zero vector is our initial guess. And the stopping criterion used is $\|r_q\|_2/\|r_0\|_2 < 10^{-7}$, where r_q is the residual vector at q

Table 1: Number of iterations for two test Toeplitz-plus-Hankel systems

n	16	32	64	128	256	512	1024
test 1	36	40	42	42	43	44	44
test 2	41	46	48	49	50	51	52

iterations. Table 1 demonstrates the performance of the new splitting iteration method in terms of number of iterations for solving the two test Toeplitz-plus-Hankel systems. It shows that the iteration number grows very slowly as the size of linear systems goes from 64 to 1024. We can safely conclude that the new splitting iteration method provides a good alternative for the fast direct methods with $(nlog^2n)$ complexity.

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SOME IDENTITIES OF THE TWISTED (h,q)-EULER NUMBERS AND POLYNOMIALS ASSOCIATED WITH BERNSTEIN POLYNOMIALS

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Abstract : Recently, we introduced some interesting relations between twisted q-Euler number and Bernstein polynomials(see [10]). In this paper, we give some interesting identities on the twisted (h,q)-Euler polynomials and Bernstein polynomials.

Key words : Euler numbers and polynomials, twisted (h,q)-Euler numbers and polynomials, Bernstein polynomials

1. Introduction

Throughout this paper, let p be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the p-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^r \frac{t}{s}$ with $(t,p) = (s,p) = (t,s) = 1$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ see [1, 2, 3, 4] .} \tag{1.1}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$ be translation. As well known equation, by (1.1), we have

$$I_{-1}(f_n) = \int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.2}$$

The Euler polynomials are defined by the generating function as follows:

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ see [1-10].} \tag{1.3}$$

In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers(see [1-10]).

Let $C_{p^n} = \{w | w^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$T_p = \lim_{n\to\infty} C_{p^n} = \cup_{n\geq 1} C_{p^n}, \text{ see [8-10].}$$

For $w \in T_p$ and $h \in \mathbb{Z}$, we defined the twisted (h,q)-Euler polynomials as follows:

$$\frac{2}{wq^h e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{1.4}$$

In the special case, $x = 0$, $E_{n,q,w}^{(h)}(0) = E_{n,q,w}^{(h)}$ are called the n-th twisted (h,q)-Euler numbers.

From (1.4), we note that

$$E_{n,q,w}^{(h)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q,w}^{(h)} x^{n-l}. \tag{1.5}$$

From (1.2) and (1.4), for $n = 1$, we have

$$\int_{\mathbb{Z}_p} q^{hy} w^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{wq^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{1.6}$$

By (1.6), we obtain

$$\int_{\mathbb{Z}_p} q^{hy} w^y (x+y)^n d\mu_{-1}(y) = E_{n,q,w}^{(h)}(x), \text{ for } n \in \mathbb{Z}_+. \tag{1.7}$$

As well known definition, Bernstein polynomials of degree n are given by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } x \in [0,1], n, k \in \mathbb{Z}_+. \tag{1.8}$$

In [1], Kim introduced p-adic extension of Bernstein polynomials as follows:

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } x \in \mathbb{Z}_p \text{ and } n, k \in \mathbb{Z}_+. \tag{1.9}$$

In this paper, we investigate some properties for the twisted (h,q)-Euler numbers and polynomials. By using these properties, we give some interesting identities on the twisted (h,q)-Euler polynomials and Bernstein polynomials.

2. Some identities on the Bernstein and twisted (h,q)-Euler polynomials

From (1.6), we can derive the following recurrence formula for the twisted (h,q)-Euler numbers:

$$E_{0,q,w}^{(h)} = \frac{2}{1+wq^h}, \text{ and } wq^h (E_{q,w}^{(h)} + 1)^n + E_{n,q,w}^{(h)} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{2.1}$$

with usual convention about replacing $(E_{q,w}^{(h)})^n$ by $E_{n,q,w}^{(h)}$.

By (1.4), we easily get

$$\sum_{n=0}^{\infty} E_{n,q^{-1},w^{-1}}^{(h)} (1-x)(-1)^n \frac{t^n}{n!} = \frac{2wq^h}{wq^h e^t + 1} e^{xt} = wq^h \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{2.2}$$

By (2.2), we obtain the following theorem.

Theorem 1. Let $n \in \mathbb{Z}_+$. For $w \in T_p$, we have

$$E_{n,q,w}^{(h)}(x) = (-1)^n w^{-1} q^{-h} E_{n,q^{-1},w^{-1}}^{(h)}(1-x).$$

From (1.7), we note that

$$\int_{\mathbb{Z}_p} q^{hx} w^x x^n d\mu_{-1}(x) = E_{n,q,w}^{(h)}, \text{ for } n \in \mathbb{Z}_+. \tag{2.3}$$

By (2.1), for $n \in \mathbb{N}$, we get

$$\begin{aligned} E_{n,q,w}^{(h)}(2) &= (E_{q,w}^{(h)} + 1 + 1)^n \\ &= \sum_{l=0}^{n} \binom{n}{l} E_{l,q,w}(1) \\ &= E_{0,q,w}^{(h)} + \frac{1}{wq^h} \sum_{l=1}^{n} \binom{n}{l} wq^h E_{l,q,w}^{(h)}(1) \\ &= \frac{2}{1+wq^h} - \frac{1}{wq^h} \sum_{l=1}^{n} \binom{n}{l} E_{l,q,w}^{(h)} \\ &= \frac{2}{wq^h} - \frac{1}{(wq^h)^2} wq^h E_{n,q,w}(1) \\ &= \frac{2}{wq^h} + \frac{1}{(q^h w)^2} E_{n,q,w}^{(h)}. \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2. For $n \in \mathbb{N}$, we have

$$wq^h E_{n,q,w}^{(h)}(2) = 2 + \frac{1}{wq^h} E_{n,q,w}^{(h)}.$$

By (2.3) and Theorem 2, we have the following corollary.

Corollary 3. For $n \in \mathbb{N}$, we have

$$\frac{1}{wq^h} \int_{\mathbb{Z}_p} q^{-hx} w^{-x} (x+2)^n d\mu_{-1}(x) = 2 + wq^h E_{n,q^{-1},w^{-1}}^{(h)}.$$

By (1.7), (2.3) and Corollary 3, we know that

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{hx} w^x (1-x)^n d\mu_{-1}(x) &= (-1)^n \int_{\mathbb{Z}_p} q^{hx} w^x (x-1)^n d\mu_{-1}(x) \\ &= (-1)^n E_{n,q,w}^{(h)}(-1) \\ &= \frac{1}{wq^h} E_{n,q^{-1},w^{-1}}^{(h)}(2) \\ &= \frac{1}{wq^h} \int_{\mathbb{Z}_p} q^{-hx} w^{-x} (x+2)^n d\mu_{-1}(x) \\ &= 2 + wq^h E_{n,q^{-1},w^{-1}}^{(h)} \\ &= 2 + wq^h \int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^n d\mu_{-1}(x). \end{aligned}$$

Therefore, we have the following theorem.

Theorem 4. For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} q^{hx} w^x (1-x)^n d\mu_{-1}(x) = 2 + wq^h \int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^n d\mu_{-1}(x).$$

In (1.9), we take the twisted fermionic p-adic invariant integral on $\mathbb{Z}_p$ for one Bernstein polynomials as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{hx} w^x B_{k,n}(x) d\mu_{-1}(x) &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \int_{\mathbb{Z}_p} q^{hx} w^x x^{n-l} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} E^{(h)}_{n-l,q,w} \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} E^{(h)}_{k+l,q,w}, \text{ where } n,k \in \mathbb{Z}_+.\end{aligned} \tag{2.5}$$

From the reflection symmetric properties of Bernstein polynomials, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \text{ where } n,k \in \mathbb{Z}_+ \text{ and } x \in \mathbb{Z}_p. \tag{2.6}$$

For $n,k \in \mathbb{Z}_+$ with $n > k$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{hx} w^x B_{k,n}(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} q^{hx} w^x B_{n-k,n}(1-x) d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} q^{hx} w^x (1-x)^{n-l} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + wq^h \int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^{n-l} d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we have the following theorem.

Theorem 5. For $n,k \in \mathbb{Z}_+$ with $n > k$, we have

$$\int_{\mathbb{Z}_p} q^{hx} w^x B_{k,n}(x) d\mu_{-1}(x) = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + wq^h E^{(h)}_{n-l,q^{-1},w^{-1}}\right).$$

By (2.5) and Theorem 5, we have the following theorem.

Theorem 6. Let $n,k \in \mathbb{Z}_+$ with $n > k$. Then we have

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l E^{(h)}_{k+l,q,w} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + wq^h E^{(h)}_{n-l,q^{-1},w^{-1}}\right).$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) q^{hx} w^x d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} q^{hx} w^x (1-x)^{n_1+n_2-l} d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \frac{1}{wq^h} \int_{\mathbb{Z}_p} q^{-hx} w^{-x} (x+2)^{n_1+n_2-l} d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(2 + wq^h \int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^{n_1+n_2-l} d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 7. For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)q^{hx}w^x d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}(2 + wq^h E^{(h)}_{n_1+n_2-l,q^{-1},w^{-1}}).$$

By simple calculation, we easily see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)q^{hx}w^x d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\int_{\mathbb{Z}_p} q^{hx}w^x x^{l+2k}d\mu_{-1}(x) \qquad (2.7)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}E^{(h)}_{l+2k,q,w}, \text{ where } n_1, n_2, k \in \mathbb{Z}_+.\end{aligned}$$

Therefore, by (2.7) and Theorem 7, we obtain the following theorem.

Theorem 8. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have

$$\begin{aligned}&\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2 + wq^h E^{(h)}_{n_1+n_2-l,q^{-1},w^{-1}}\right)\\ &= \sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}E^{(h)}_{l+2k,q,w}.\end{aligned}$$

For $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k$, by the symmetry of Bernstein polynomials, we see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}w^x d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\int_{\mathbb{Z}_p} q^{hx}w^x(1-x)^{n_1+n_2+n_3-l}d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\frac{1}{wq^h}\int_{\mathbb{Z}_p} q^{-hx}w^{-x}(x+2)^{n_1+n_2+n_3-l}d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2 + wq^h\int_{\mathbb{Z}_p} q^{-hx}w^{-x}x^{n_1+n_2+n_3-l}d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we have the following theorem.

Theorem 9. For $n_1, n_2, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k$, we have

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}w^x d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}(2 + wq^h E^{(h)}_{n_1+n_2+n_3-l,q^{-1},w^{-1}}).$$

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}w^x d\mu_{-1}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}\int_{\mathbb{Z}_p} q^{hx}w^x x^{l+3k}d\mu_{-1}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}E^{(h)}_{l+3k,q,w},
\end{aligned}$$

where $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k$.

Therefore, by Theorem 9, we obtain the following theorem.

Theorem 10. Let $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k$. Then we have

$$\begin{aligned}
&\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2+wq^hE^{(h)}_{n_1+n_2+n_3-l,q^{-1},w^{-1}}\right)\\
&= \sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}E^{(h)}_{l+3k,q,w}.
\end{aligned}$$

Using the above theorem and mathematical induction, we obtain the following theorem.

Theorem 11. Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1+\cdots+n_m > mk$, the multiplication of the sequence of Bernstein polynomials $B_{k,n_1}(x), \ldots, B_{k,n_m}(x)$ with different degrees under fermionic p-adic invariant integral on $\mathbb{Z}_p$ can be given as

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left(\prod_{i=1}^m B_{k,n_i}(x)\right)q^{hx}w^x d\mu_{-1}(x)\\
&= \left(\prod_{i=1}^m\binom{n_i}{k}\right)\sum_{l=0}^{mk}(-1)^{l+mk}(2+wq^hE^{(h)}_{n_1+\cdots+n_m-l,q^{-1},w^{-1}}).
\end{aligned}$$

We also easily see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left(\prod_{i=1}^m B_{k,n_i}(x)\right)q^{hx}w^x d\mu_{-1}(x)\\
&= \left(\prod_{i=1}^m\binom{n_i}{k}\right)\sum_{l=0}^{n_1+\cdots+n_m-mk}\binom{n_1+\cdots+n_m-mk}{l}(-1)^lE^{(h)}_{l+mk,q,w}.
\end{aligned}\tag{2.8}$$

By Theorem 11 and (2.8), we have the following corollary.

Corollary 12. Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1+\cdots+n_m > mk$, we have

$$\begin{aligned}
&\sum_{l=0}^{mk}\binom{mk}{l}(-1)^{l+mk}\left(2+wq^hE^{(h)}_{n_1+\cdots+n_m-l,q^{-1},w^{-1}}\right)\\
&= \sum_{l=0}^{n_1+\cdots+n_m-mk}\binom{n_1+\cdots+n_m-mk}{l}(-1)^lE^{(h)}_{l+mk,q,w}.
\end{aligned}$$

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Numerical solution of fractional ordinary and partial differential equations by an operational technique

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Abstract

The purpose of this paper is to present a numerical technique for solving the fractional ordinary differential equations (FODEs) and fractional partial differential equations (FPDEs). The proposed technique is based on operational Tau method (OTM). The main idea behind the OTM is to convert the desired problem to some operational matrices. The development of the technique for FODEs and FPDEs is considered. Some linear and nonlinear experiments are applied to solve 1D and 2D fractional differential equations (FDEs) and a comparison is made between the proposed technique and the other methods. The results demonstrate the accuracy and the validity of the new technique.

Key-words: Spectral methods; Operational approach of the Tau method; Improved algebraic formulation; Fractional ordinary differential equations; Fractional partial differential equations.
2010 Mathematics Subject Classification: 76M22; 34A99; 08A99; 26A33; 35A25.

1. Introduction

The Tau method has been applied to a wide class of ordinary differential equations (ODEs); partial differential equations (PDEs); integral equations (IEs) and integro-differential equations (IDEs). Tau method as a branch of spectral methods have become increasingly popular in recent years, especially since the development of fast transform methods, with applications in numerical simulation of turbulent flows and other problems where high accuracy is desired for complicated solutions. In 1981, Ortiz and Samara [1] presented an operational technique for the numerical solution of nonlinear ODEs with some supplementary conditions based on the standard Tau method [2]. The theoretical analysis and numerical applications of this method

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has been described in a series of papers [3-7] for linear ordinary differential eigenvalue problems. The method was developed for numerical solution of PDEs and their related eigenvalue problems, iterated solutions of linear operator equations [8-13] and for IDEs, too [14]. The object of this work is the development of the operational Tau method (OTM) for the numerical solution of FODEs and FPDEs.

FODEs and FPDEs have been extensively investigated and applied for many real problems which are modeled in various areas. For instance; in mathematical physics [15], in fluid and continuum mechanics [16], viscoplastic and viscoelastic flows [17], biology, chemistry, acoustics and psychology [18]. Also, FODEs and FPDEs have been the focus of numerous studies. In addition, considerable attention has been given to the solutions of FODEs and FPDEs of physical interest [15,19-21].

There are several approximation and numerical methods for the solution of FDEs. The most commonly used ones are; homotopy perturbation method (HPM) [22, 23], Adomian decomposition method (ADM) [24-26], variational iteration method (VIM) [22, 27], fractional iteration method (FIM) [28] and fractional difference method (FDM) [27-29].

Out of the aforesaid mentioned methods, we are desired to present the operational approach of the Tau method to solve FODEs and FPDEs. Because in the Tau method, we are dealing with a system of equations wherein the matrix of unknown coefficients is sparse and can be easily invertible. Also, the integral part appearing in the equation is replaced by its operational Tau representation and then we obtain a system of algebraic equations wherein its solution is easy.

The paper has been organized as follows: Section 2 gives notations and basic definitions of the fractional calculus. Section 3 is devoted to introduce the OTM and its application to the FODEs. In Section 4, we develop the OTM to solve FPDEs. In Section 5, the Chebyshev polynomials as base functions are considered. Some illustrative numerical experiments are given in Section 6 followed by the discussion and conclusion is presented in Section 7.

2. Basic definitions of the fractional calculus

In this Section, we state some preliminaries and definitions of fractional calculus [15, 30].

Definition 1. A real function $u(x),\ x > 0$ is said to be in the space $C_\mu,\ \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $u(x) = x^p v(x)$, where $v(x) \in C[0, \infty)$ and it is said to be in the space C_μ^m iff $u^{(m)}(x) \in C_\mu,\ m \in \mathbb{N}$.

Definition 2. The fractional derivative of $u(x)$ in the Caputo's sense is defined as:

$$D^\alpha u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} u^{(m)}(t) dt, \tag{1}$$

when $m - 1 < \alpha \leq m,\ m \in \mathbb{N},\ x > 0,\ u(x) \in C_{-1}^m$.

Definition 3. For m to be the smallest integer that exceeds α, the Caputo's space–fractional derivative operator of order $\alpha > 0$ is defined as:

$$D_x^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x (x-\sigma)^{m-\alpha-1} \frac{\partial^m u(\sigma,t)}{\partial \sigma^m} d\sigma, & \text{if } \ m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial x^m}, & \text{if } \ \alpha = m \in \mathbb{N}, \end{cases} \tag{2}$$

and the time-fractional derivative operator of order $\alpha > 0$ is defined as:

$$D_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & \text{if } \ m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{if } \ \alpha = m \in \mathbb{N}. \end{cases} \tag{3}$$

3. Implementation of the OTM on FODEs

For clarifying the concepts of applied operational approach of the Tau method on FODEs, we need assert some preliminaries and some notations.

Let us consider the matrices $\mu = [\mu_{i,j}]_{i,j=0}^n$, $\mathbf{M} = [m_{i,j}]_{i,j=0}^n$ and $\eta = \mu^T \mathbf{M}$, where $\mu_{i,j} = \delta_{i+1,j}$ and $\mathbf{M}_{i,j} = (i+1)\delta_{i,j}$; $\delta_{i,j}$ is the Kronecker symbol.
We are seeking a polynomial solution to approximate $u(x) \in L^2[a,b]$. Let $\underline{\phi} = \{\phi_i(x)\}_{i=0}^\infty = \Phi \mathbf{x}_x$ are a set of arbitrary orthogonal polynomial bases defined by a lower triangular matrix Φ and $\mathbf{x}_x = \left[1, x, x^2, ..., x^n, ...\right]^T$. In continuation, we consider the following lemma.

Lemma 1. Suppose that $u(x)$ is a polynomial as $u(x) = \displaystyle\sum_{i=0}^\infty u_i x^i = \mathbf{u}\mathbf{x}_x$, then we have:

$$\frac{d^r}{dx^r} u(x) = \mathbf{u}\eta^r \mathbf{x}_x, \qquad r = 0, 1, 2, \ldots, \tag{4}$$

$$x^s u(x) = \mathbf{u}\mu^s \mathbf{x}_x, \qquad s = 0, 1, 2, \ldots, \tag{5}$$

and

$$\int_a^x u(t)dt = \mathbf{u}\zeta \mathbf{x}_x - \mathbf{u}\zeta \mathbf{x}_a, \tag{6}$$

where $\mathbf{u} = [u_0, u_1, \ldots, u_n, \ldots]$, $\mathbf{x}_a = \left[1, a, a^2, \ldots, a^n, \ldots\right]^T, a \in \mathbb{R}$ and ζ is an infinite matrix with only nonzero elements $\zeta_{i,i+1} = \frac{1}{i+1}$, $i = 0, 1, 2, \ldots$.
Proof: See ref. [31]. □

From the above assumptions concern to orthogonal basis, we consider orthogonal series expansion of the exact solution $u(x) \in L^2[a,b]$ as:

$$u(x) = \sum_{i=0}^\infty u_i \phi_i(x) = \mathbf{u}\Phi \mathbf{x}_x, \tag{7}$$

where $\mathbf{u} = \{u_i\}_{i=0}^{\infty}$ and any distinct function $f(x)$ can also be considered as:

$$f(x) = \sum_{i=0}^{\infty} f_i \phi_i(x) = \mathbf{f}\Phi\mathbf{x}_x, \qquad \mathbf{f} = [f_0, f_1, \ldots]. \tag{8}$$

Now, the idea is approximating $D^{\alpha}u(x)$. Simple computations reveal that:

$$\int_0^x (x-t)^{m-\alpha-1} t^p dt = \frac{\Gamma(m-\alpha)\Gamma(p+1)}{\Gamma(m-\alpha+p+1)} x^{m-\alpha+p}, \quad p = 0, 1, 2, \ldots. \tag{9}$$

Substituting equation (7) in equation (1) and using equations (4) and (9), we have:

$$\begin{aligned} D^{\alpha}u(x) &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{\partial^m}{\partial t^m}(\mathbf{u}\Phi\mathbf{x}_t)dt = \\ &\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} (\mathbf{u}\Phi\eta^m\mathbf{x}_t)dt = \\ &\mathbf{u}\Phi\eta^m \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1}\mathbf{x}_t dt = \mathbf{u}\Phi\eta^m\Gamma\Psi, \end{aligned} \tag{10}$$

where Γ is an infinite diagonal matrix as $[\Gamma]_{i,j} = \dfrac{\Gamma(p+1)}{\Gamma(m-\alpha+p+1)}\delta_{i,j}$ and

$$\Psi = \left[x^{m-\alpha}, x^{m-\alpha+1}, x^{m-\alpha+2}, \ldots, x^{m-\alpha+p}, \ldots\right]^T.$$

By approximating $x^{m-\alpha+p}, p = 0, 1, 2, \ldots$, as follows:

$$x^{m-\alpha+p} = \sum_{i=0}^{\infty} a_{p,i}\phi_i(x) = \mathbf{a}_p\Phi\mathbf{x}_x, \qquad \mathbf{a}_p = [a_{p,0}, a_{p,1}, a_{p,2}, \ldots],$$

we have:

$$\Psi = [\mathbf{a}_0\Phi\mathbf{x}_x, \mathbf{a}_1\Phi\mathbf{x}_x, \mathbf{a}_2\Phi\mathbf{x}_x, \ldots, \mathbf{a}_p\Phi\mathbf{x}_x \ldots]^T = [\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p, \ldots]^T \Phi\mathbf{x}_x.$$

Let us suppose that $\Pi = [\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p, \ldots]^T$, be the coefficient matrix of the last relation. Thus, we have:

$$D^{\alpha}u(x) = \mathbf{u}\Phi\eta^m\Gamma\Psi = \mathbf{u}\Phi\eta^m\Gamma\Pi\Phi\mathbf{x}_x = \mathbf{v}\Phi\mathbf{x}_x, \quad \text{where} \quad \mathbf{v} = \mathbf{u}\Phi\eta^m\Gamma\Pi. \tag{11}$$

Lemma 2. Let $\mathbf{x}_x = \left[1, x, x^2, \ldots\right]^T$ and $\mathbf{u} = [u_0, u_1, u_2, \ldots]$ are infinite vectors and $\Phi = [\phi_0|\phi_1|\phi_2|\ldots]$ is infinite matrix (ϕ_j's are columns of Φ). Then, we have:

$$\mathbf{x}_x\mathbf{u}\Phi\mathbf{x}_x = \mathbf{U}\mathbf{x}_x, \tag{12}$$

where $\mathbf{U}$ is an upper triangular matrix as:

$$\mathbf{U}_{i,j} = \begin{cases} \sum_{k=0}^{\infty} u_k\phi_{k,j-i}, & i = 1, 2, \ldots, \quad j \geq i, \\ 0, & j < i, \end{cases}$$

and if suppose that $u(x) = \mathbf{u}\Phi\mathbf{x}_x$ represents a polynomial, then for any positive integer s, the relation

$$u^s(x) = \mathbf{u}\Phi\mathbf{U}^{s-1}\mathbf{x}_x, \tag{13}$$

is valid.

Proof:

$$\mathbf{x}_x\mathbf{u}\Phi = \left[1, x, x^2, ...\right]^T \left[\mathbf{u}\phi_0|\mathbf{u}\phi_1|\mathbf{u}\phi_2|\ldots\right] = \begin{bmatrix} \mathbf{u}\phi_0 & \mathbf{u}\phi_1 & \mathbf{u}\phi_2 & \cdots \\ \mathbf{u}\phi_0 x & \mathbf{u}\phi_1 x & \mathbf{u}\phi_2 x & \cdots \\ \mathbf{u}\phi_0 x^2 & \mathbf{u}\phi_1 x^2 & \mathbf{u}\phi_2 x^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

therefore

$$\mathbf{x}_x\mathbf{u}\Phi\mathbf{x}_x = \begin{bmatrix} \mathbf{u}\phi_0 & \mathbf{u}\phi_1 & \mathbf{u}\phi_2 & \cdots \\ \mathbf{u}\phi_0 x & \mathbf{u}\phi_1 x & \mathbf{u}\phi_2 x & \cdots \\ \mathbf{u}\phi_0 x^2 & \mathbf{u}\phi_1 x^2 & \mathbf{u}\phi_2 x^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \left[1, x, x^2, ...\right]^T = \begin{bmatrix} \sum_{i=0}^{\infty} \mathbf{u}\phi_i x^{i+0} \\ \sum_{i=0}^{\infty} \mathbf{u}\phi_i x^{i+1} \\ \sum_{i=0}^{\infty} \mathbf{u}\phi_i x^{i+2} \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{u}\phi_0 & \mathbf{u}\phi_1 & \mathbf{u}\phi_2 & \cdots \\ 0 & \mathbf{u}\phi_0 & \mathbf{u}\phi_1 & \cdots \\ 0 & 0 & \mathbf{u}\phi_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix},$$

if we call the last upper triangular coefficient matrix as $\mathbf{U}$, then we have:

$$\mathbf{U}_{i,j} = \begin{cases} \mathbf{u}\phi_{j-i}, & i = 1, 2, \ldots, \quad j \geq i, \\ 0, & j < i, \end{cases} = \begin{cases} \sum_{k=0}^{\infty} u_k \phi_{k,j-i}, & i = 1, 2, \ldots, \quad j \geq i, \\ 0, & j < i. \end{cases}$$

Now, in order to prove equation (13), we start by induction. For $s = 1$, it is obvious that $u(x) = \mathbf{u}\Phi\mathbf{x}_x$. For $s = 2$, we rewrite $u^2(x) = \mathbf{u}\Phi\mathbf{x}_x\mathbf{u}\Phi\mathbf{x}_x$ and using equation (12), we have:

$$u^2(x) = \mathbf{u}\Phi\mathbf{U}\mathbf{x}_x,$$

therefore equation (13) is hold for $s = 2$, too. Now, suppose that equation (13) is hold for $s = k$, then we must prove that for $s = k + 1$ the relation is valid, too. Thus,

$$u^{k+1}(x) = u^k(x)u(x) = (\mathbf{u}\Phi\mathbf{U}^{k-1}\mathbf{x}_x)(\mathbf{u}\Phi\mathbf{x}_x) = \mathbf{u}\Phi\mathbf{U}^{k-1}(\mathbf{x}_x\mathbf{u}\Phi\mathbf{x}_x) = \mathbf{u}\Phi\mathbf{U}^k\mathbf{x}_x,$$

hence, equation (13) is proved. □

We have the following corollary to compute $(D^\alpha u(x))^r$.

Corollary 1. With assumption in lemma 2, any positive power of equation (11) can also be written as:

$$(D^\alpha u(x))^r = \mathbf{v}\Phi\ \mathbf{V}^{r-1}\mathbf{x}_x, \tag{14}$$

where

$$\mathbf{V}_{i,j} = \begin{cases} \sum_{k=0}^{\infty} v_k \phi_{k,j-i}, & i = 1, 2, \ldots, \quad j \geq i, \\ 0, & j < i. \end{cases}$$

Lemma 2 and corollary 1 have effective rules for numerical solution of nonlinear problems. By using these advantages of Tau method, most of nonlinear problems can be solved and expressed as polynomials, easily. Also, for any positive integers r and s, we can write $u^s(x)$ and $(D^\alpha u(x))^r$ with respect to any arbitrary polynomial basis as follows:

$$u^s(x) = \mathbf{u}(\Phi \mathbf{U}^{s-1} \Phi^{-1}) \Phi \mathbf{x}_x, \qquad \text{and} \qquad (D^\alpha u(x))^r = \mathbf{v}(\Phi \mathbf{V}^{r-1} \Phi^{-1}) \Phi \mathbf{x}_x, \tag{15}$$

where, $\Phi \mathbf{x}_x$ is a polynomial basis. By attention to equations (4) and (7), it is obvious that in each FODEs any boundary condition of the problem can be replaced by an algebraic expression based on polynomial basis. Finally, if we choose finite algebraic equations gained in the problem and imposing boundary conditions, then we can obtain a square system wherein its solution gives an unknown vector as $\mathbf{u}_n = \{u_k\}_{k=0}^n$.

4. Implementation of the OTM on FPDEs

In this Section, we state the required preliminaries of the Tau method to apply on PDEs of fractional order. In this case, we suppose that the function $u(x,t)$ is defined on $[a,b] \times [c,d]$ and is infinitely differentiable. Let us suppose that

$$\mathbf{x}_{x,t} = \left[\{1, t, t^2, \ldots\}, \{1, t, t^2, \ldots\}x, \{1, t, t^2, \ldots\}x^2, \ldots\right]^T,$$

be the standard base for polynomials in $\mathbb{R}^2$. We seek a polynomial in $\mathbb{R}^2$ to approximate $u(x,t)$ such that

$$u(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} \phi_i(x) \psi_j(t),$$

where $\phi_x = \{\phi_i(x)\}_{i=0}^{\infty}$ and $\psi_t = \{\psi_j(t)\}_{j=0}^{\infty}$ are the sets of arbitrary orthogonal polynomial bases defined on 1D space. Since, we use finite series in the Tau method, thus we suppose that the approximation is as:

$$u(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} u_{i(n+1)+j} \phi_i(x) \psi_j(t). \tag{16}$$

By multiplying $\phi_i(x)$ and $\psi_j(t)$ in the above summation and extracting its matrix of coefficients as Φ with respect to base $\mathbf{x}_{x,t}$, we can assume that the approximate solution of $u(x,t)$ is as below.

Lemma 3. Suppose that $\phi_x = \mathbf{V}_\phi \mathbf{x}_x$ and $\psi_t = \mathbf{V}_\psi \mathbf{x}_t$ where $\mathbf{V}_\phi$ and $\mathbf{V}_\psi$ are coefficient matrices with respect to $\mathbf{x}_x$, $\mathbf{x}_t$ and $\mathbf{u} = \left[u_0, u_1, u_2, \ldots, u_{(n+1)^2}\right]$. Then equation (16) can be written as:

$$u(x,t) = \mathbf{u} \Phi \mathbf{x}_{x,t}, \tag{17}$$

where Φ is blocky matrix as:

$$\Phi = \begin{bmatrix} (V_\phi)_{0,0}V_\psi & (V_\phi)_{0,1}V_\psi & (V_\phi)_{0,2}V_\psi & \cdots \\ (V_\phi)_{1,0}V_\psi & (V_\phi)_{1,1}V_\psi & (V_\phi)_{1,3}V_\psi & \cdots \\ (V_\phi)_{2,0}V_\psi & (V_\phi)_{2,2}V_\psi & (V_\phi)_{2,3}V_\psi & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{(n+1)^2\times(n+1)^2}.$$

Proof: Expanding equation (16) yields

$$u(x,t) = \mathbf{u}\ [\phi_0(x)\psi_0(t), \phi_0(x)\psi_1(t), \ldots, \phi_0(x)\psi_n(t), \phi_1(x)\psi_0(t), \phi_1(x)\psi_1(t), \\ \ldots, \phi_1(x)\psi_n(t), \ldots, \phi_n(x)\psi_0(t), \phi_n(x)\psi_1(t), \ldots, \phi_n(x)\psi_n(t)]^T. \tag{18}$$

Each $\phi_r(x)$ and $\psi_s(t)$ can be written as:

$$\phi_r(x) = [\phi_{r,0}, \phi_{r,1}, \ldots \phi_{r,n}]\mathbf{x}_x, \qquad \text{and} \qquad \psi_s(t) = [\psi_{s,0}, \psi_{s,1}, \ldots \psi_{s,n}]\mathbf{x}_t.$$

To evaluate equation (18), we must compute $\phi_r(x)\psi_s(t), r, s = 0, 1, \ldots, n$. Thus,

$$\phi_r(x)\psi_s(t) = [\phi_{r,0}, \phi_{r,1}, \ldots \phi_{r,n}]\mathbf{x}_x[\psi_{s,0}, \psi_{s,1}, \ldots \psi_{s,n}]\mathbf{x}_t = \sum_{i=0}^{n}\sum_{j=0}^{n}\psi_{s,i}\phi_{r,j}x^j t^i = \\ [\phi_{r,0}(\psi_{s,0}, \psi_{s,1}, \ldots, \psi_{s,n}), \phi_{r,1}(\psi_{s,0}, \psi_{s,1}, \ldots \psi_{s,n}), \ldots, \phi_{r,n}(\psi_{s,0}, \psi_{s,1}, \ldots \psi_{s,n})]\mathbf{x}_{x,t}.$$

Therefore we can construct a matrix such that:

$$u(x,t) = \mathbf{u}\begin{bmatrix} (V_\phi)_{0,0}V_\psi & (V_\phi)_{0,1}V_\psi & (V_\phi)_{0,2}V_\psi & \cdots \\ (V_\phi)_{1,0}V_\psi & (V_\phi)_{1,1}V_\psi & (V_\phi)_{1,3}V_\psi & \cdots \\ (V_\phi)_{2,0}V_\psi & (V_\phi)_{2,2}V_\psi & (V_\phi)_{2,3}V_\psi & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\mathbf{x}_{x,t}. \tag{19}$$

If we call the above coefficient matrix as Φ, then this lemma is valid. □

Lemma 4. Suppose that $\mathbf{u} = [u_0, u_1, u_2, \ldots, u_{(n+1)^2}, 0, 0\ldots]$, $\Phi = [\phi_0|\phi_1|\phi_2|\ldots]$, $\mathbf{x}_{x,t} = \left[\{1,t,t^2,\ldots\},\{1,t,t^2,\ldots\}x,\{1,t,t^2,\ldots\}x^2,\ldots\right]^T$ and $u(x,t) = \mathbf{u}\Phi\mathbf{x}_{x,t}$ is a polynomial. Then, we have:

i. $$\frac{d^r}{dx^r}u(x,t) = \mathbf{u}\Phi\eta_x^r\mathbf{x}_{x,t}, \quad r = 1,2,\ldots, \tag{20}$$

where

$$[\eta_x]_{i,j} = \begin{cases} k, & k(n+1)+1 \le i = j \le (k+1)(n+1), \quad k = 0,1,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$

ii. $$\frac{d^r}{dt^r}u(x,t) = \mathbf{u}\Phi\eta_t^r\mathbf{x}_{x,t}, \quad r = 1,2,\ldots, \tag{21}$$

where

$$[\eta_t]_{i,j} = \begin{cases} \mathbf{A}_0, & i = j, \quad i = 1,2,\ldots n, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad [\mathbf{A}_0]_{p,q} = \begin{cases} q, & p = q+1, \quad q = 1,2,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$

iii. $t^s u(x,t) = \mathbf{u}\Phi\mu_t^s\mathbf{x}_{x,t}, \quad s = 1,2,\ldots$ (22)

where

$$[\mu_t]_{i,j} = \begin{cases} \mathbf{B}_0, & i = j, \quad i = 1,2,\ldots,n, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad [\mathbf{B}_0]_{p,q} = \begin{cases} 1, & q = p+1, \quad p = 1,2,\ldots,n, \\ 0, & \text{otherwise}. \end{cases}$$

iv. $x^s u(x,t) = \mathbf{u}\Phi\mu_x^s\mathbf{x}_{x,t}, \quad s = 1,2,\ldots$ (23)

where

$$[\mu_x]_{i,j} = \begin{cases} 1, & j = i+n+1, \quad i = 1,2,\ldots,n(n+1), \\ 0, & \text{otherwise}. \end{cases}$$

Proof: The proof of these properties can be obtained by simple computations, which we neglect here due to the increased volume of the paper. □

Lemma 5. By all assumptions in lemma 4, we have:

$$\mathbf{x}_{x,t}\mathbf{u}\Phi\mathbf{x}_{x,t} = \mathbf{U}\mathbf{x}_{x,t}, \tag{24}$$

where $\mathbf{U}$ is an upper triangular matrix as:

$$\mathbf{U}_{i,j} = \begin{cases} \sum_{k=0}^{\infty} u_k\phi_{k,j-i}, & i = 1,2,\ldots, \quad j \geq i, \\ 0, & j < i, \end{cases}$$

and for any positive integer s, the relation

$$u^s(x,t) = \mathbf{u}\Phi\mathbf{U}^{s-1}\mathbf{x}_{x,t}, \tag{25}$$

is valid.

Proof: We have:

$$\mathbf{x}_{x,t}\mathbf{u} = \left[\{1,t,t^2,\ldots\},\{1,t,t^2,\ldots\}x,\{1,t,t^2,\ldots\}x^2,\ldots\right]^T[u_0,u_1,u_2,\ldots] = \left[\{1,t,t^2,\ldots\}\mathbf{u},\{1,t,t^2,\ldots\}x\mathbf{u},\{1,t,t^2,\ldots\}x^2\mathbf{u},\ldots\right]^T,$$

$$\mathbf{x}_{x,t}\ \mathbf{u}\Phi = \left[\{1,t,t^2,\ldots\}\mathbf{u},\{1,t,t^2,\ldots\}x\mathbf{u},\{1,t,t^2,\ldots\}x^2\mathbf{u},\ldots\right]^T[\phi_0|\phi_1|\phi_2|\ldots] = \left[\mathbf{u}\phi_0\mathbf{x}_{x,t}|\mathbf{u}\phi_1\mathbf{x}_{x,t}|\mathbf{u}\phi_2\mathbf{x}_{x,t}|\ldots\right]^T,$$

so

$$\mathbf{x}_{x,t}\mathbf{u}\Phi\mathbf{x}_{x,t} = [\mathbf{u}\phi_0\mathbf{x}_{x,t}|\mathbf{u}\phi_1\mathbf{x}_{x,t}|\mathbf{u}\phi_2\mathbf{x}_{x,t}|\ldots]^T[\{1,t,t^2,\ldots\},\{1,t,t^2,\ldots\}x,\{1,t,t^2,\ldots\}x^2,\ldots]^T$$
$$= \begin{bmatrix} \mathbf{u}\phi_0 & \mathbf{u}\phi_1 & \mathbf{u}\phi_2 & \mathbf{u}\phi_3 & \cdots \\ 0 & \mathbf{u}\phi_0 & \mathbf{u}\phi_1 & \mathbf{u}\phi_2 & \cdots \\ 0 & 0 & \mathbf{u}\phi_0 & \mathbf{u}\phi_1 & \cdots \\ 0 & 0 & 0 & \mathbf{u}\phi_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}[\{1,t,t^2,\ldots\},\{1,t,t^2,\ldots\}x,\{1,t,t^2,\ldots\}x^2,\ldots]^T.$$

If we let $\mathbf{U}$ to be the last coefficient matrix, then we have:

$$\mathbf{x}_{x,t}\ \mathbf{u}\Phi\mathbf{x}_{x,t} = \mathbf{U}\mathbf{x}_{x,t},$$

where

$$\mathbf{U}_{i,j} = \begin{cases} \mathbf{u}\phi_{j-i}, & i = 1,2,\ldots, \quad j \geq i, \\ 0, & j < i, \end{cases} = \begin{cases} \sum_{k=0}^{\infty} u_k \phi_{k,j-i}, & i = 1,2,\ldots, \quad j \geq i, \\ 0, & j < i. \end{cases}$$

To prove the second part of this lemma, applying the induction as the same way in lemma 2, gains the desired result, easily. □

Now, we want to compute $D_x^\alpha u(x,t)$ and $D_t^\alpha u(x,t)$ in operational forms. For computing $D_x^\alpha u(x,t)$, substituting equation (17) in equation (2), yields:

$$D_x^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x (x-\sigma)^{m-\alpha-1} \frac{\partial^m}{\partial \sigma^m}(\mathbf{u}\Phi\mathbf{x}_{\sigma,t})d\sigma, & \text{if } \ m-1 < \alpha < m, \\ \dfrac{\partial^m}{\partial x^m}(\mathbf{u}\Phi\mathbf{x}_{x,t}), & \text{if } \ \alpha = m \in \mathbb{N}. \end{cases}$$

When $\alpha = m \in \mathbb{N}$, lemma 4 implies that

$$D_x^\alpha u(x,t) = \frac{\partial^m}{\partial x^m}(\mathbf{u}\Phi\mathbf{x}_{x,t}) = \mathbf{u}\Phi\eta_x^m\mathbf{x}_{x,t}, \tag{26}$$

but when $m-1 < \alpha < m$, using lemma 4 again, we have:

$$D_x^\alpha u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\sigma)^{m-\alpha-1} \frac{\partial^m}{\partial \sigma^m}(\mathbf{u}\Phi\mathbf{x}_{\sigma,t})d\sigma =$$

$$\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\sigma)^{m-\alpha-1} \mathbf{u}\Phi\eta_x^m\mathbf{x}_{\sigma,t}d\sigma = \mathbf{u}\Phi\eta_x^m \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\sigma)^{m-\alpha-1}\mathbf{x}_{\sigma,t}d\sigma.$$

Since each element of the base $\mathbf{x}_{\sigma,t}$, is as $\sigma^p t^q,\ p,q = 0,1,\ldots$; from equation (9), we have:

$$\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\sigma)^{m-\alpha-1}\sigma^p t^q d\sigma = t^q \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\sigma)^{m-\alpha-1}\sigma^p d\sigma =$$
$$\frac{\Gamma(p+1)}{\Gamma(m-\alpha+p+1)} x^{m-\alpha+p}t^q = [\Gamma\Psi]t^q = \Gamma\Pi\Phi t^q\mathbf{x}_x, \tag{27}$$

where Γ and Ψ have the same structure in equation (10).

For computing $D_t^\alpha u(x,t)$, by substituting equation (17) in equation (3), we have:

$$D_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m}{\partial \tau^m}(\mathbf{u}\Phi\mathbf{x}_{x,\tau})d\tau, & \text{if } \ m-1 < \alpha < m, \\ \dfrac{\partial^m}{\partial t^m}(\mathbf{u}\Phi\mathbf{x}_{x,t}), & \text{if } \ \alpha = m \in \mathbb{N}. \end{cases}$$

When $\alpha = m \in \mathbb{N}$, using lemma 4 yields

$$D_t^{\alpha} u(x,t) = \frac{\partial^m}{\partial t^m}(\mathbf{u}\Phi\mathbf{x}_{x,t}) = \mathbf{u}\Phi\eta_t^m\mathbf{x}_{x,t}, \tag{28}$$

but when $m-1 < \alpha < m$, by using lemma 4 again, we have:

$$D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m}{\partial \tau^m}(\mathbf{u}\Phi\mathbf{x}_{x,\tau})d\tau =$$

$$\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\mathbf{u}\Phi\eta_t^m\mathbf{x}_{x,\tau}d\tau = \mathbf{u}\Phi\eta_t^m\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\mathbf{x}_{x,\tau}d\tau.$$

Since each element of the base $\mathbf{x}_{x,\tau}$, is as $x^p\tau^q,\ p,q = 0,1,\ldots$; from equation (9), we have:

$$\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}x^p\tau^q d\tau = x^p\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\tau^q d\tau =$$

$$\frac{\Gamma(q+1)}{\Gamma(m-\alpha+q+1)}t^{m-\alpha+q}x^p = [\Gamma\Psi]x^p = \Gamma\Pi\Phi x^p\mathbf{x}_t, \tag{29}$$

where Γ and Ψ have the same structure in equation (10), too.

Lemma 6. Suppose that $\mathbf{x}_{x,t} = \left[\{1,t,t^2,\ldots\},\{1,t,t^2,\ldots\}x,\{1,t,t^2,\ldots\}x^2,\ldots\right]^T$, $\mathbf{x}_x = \left[1,x,x^2,\ldots,x^n\right]^T$ and $\mathbf{x}_t = \left[1,t,t^2,\ldots,t^n\right]^T$. Then, we have:

$$x^p\mathbf{x}_t = \mathbf{W}\mathbf{x}_{x,t},\ p = 0,1,\ldots; \quad \text{where} \quad \mathbf{W}_{i,j} = \begin{cases} 1, & j = p(n+1)+i \le (n+1)^2, \\ 0, & \text{otherwise}, \end{cases} \tag{30}$$

and

$$t^q\mathbf{x}_x = \mathbf{V}\mathbf{x}_{x,t},\ q = 0,1,\ldots; \quad \text{where} \quad \mathbf{V}_{i,j} = \begin{cases} 1, & j = (i-1)n+q+i \le (n+1)^2, \\ 0, & \text{otherwise}. \end{cases} \tag{31}$$

Proof: The proof of these properties can be obtained by simple computations, which we neglect here due to the increased volume of the paper. □

Using the above lemma for equations (27) and (29), for $m-1 < \alpha < m$, we have:

$$D_x^{\alpha} u(x,t) = \mathbf{u}\Phi\eta_x^m\Gamma\ \Pi_p\Phi\mathbf{V}\mathbf{x}_{x,t} = \mathbf{u}\Upsilon\mathbf{x}_{x,t}, \qquad \text{where} \qquad \Upsilon = \Phi\eta_x^m\Gamma\Pi\Phi\mathbf{V}, \tag{32}$$

and

$$D_t^{\alpha} u(x,t) = \mathbf{u}\Phi\eta_t^m\Gamma\ \Pi_p\Phi\mathbf{W}\mathbf{x}_{x,t} = \mathbf{u}\Omega\mathbf{x}_{x,t}, \qquad \text{where} \qquad \Omega = \Phi\eta_t^m\Gamma\Pi\Phi\mathbf{W}. \tag{33}$$

Also, these relations can be expressed in arbitrary polynomial bases as $D_x^{\alpha} u(x,t) = \mathbf{u}\Upsilon\Phi^{-1}\Phi\mathbf{x}_{x,t}$ and $D_t^{\alpha} u(x,t) = \mathbf{u}\Omega\Phi^{-1}\Phi\mathbf{x}_{x,t}$, where $\Phi\mathbf{x}_{x,t}$ is the polynomial base.

Lemma 7. With assumption in lemma 6, for any positive integers s and v, the following relations are valid.

$$(D_x^\alpha u(x,t))^s = \mathbf{u}\Upsilon\mathbf{K}^{s-1}\mathbf{x}_{x,t}, \qquad \text{where} \qquad \mathbf{K}_{i,j} = \begin{cases} \sum_{k=0}^{\infty} u_k \Upsilon_{k,j-i}, & i = 1,2,\ldots, \quad j \geq i, \\ 0, & j < i, \end{cases} \tag{34}$$

and

$$(D_t^\alpha u(x,t))^v = \mathbf{u}\Omega\mathbf{P}^{v-1}\mathbf{x}_{x,t}, \qquad \text{where} \qquad \mathbf{P}_{i,j} = \begin{cases} \sum_{k=0}^{\infty} u_k \Omega_{k,j-i}, & i = 1,2,\ldots, \quad j \geq i, \\ 0, & j < i. \end{cases} \tag{35}$$

Proof: The proof is the same as lemma 5. □

5. Converted Chebyshev polynomials

In Sections 3 and 4, we have considered OTM based on arbitrary orthogonal polynomials. Since converted Chebyshev polynomials are more applicable, then we can use them for a wide range of problems. In consequence, we consider operational Tau approximation of the FODEs and FPDEs with converted Chebyshev polynomials $\underline{\mathbf{T}}^* = \{T_i^*(x)\}_{i=0}^{\infty} = \mathbf{T}^*\mathbf{x}_x$. It is well known that the coefficient matrix of them have the same structure of general Chebyshev polynomials. They are orthogonal functions and they can be used to obtain a good approximation for transcendental functions. In the sequel of this Section, we introduce a convertor matrix for converting Chebyshev polynomials. The Chebyshev polynomials are defined on $[-1,1]$ as

$$\begin{cases} T_0(x) = 1, T_1(x) = x, \\ T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x), \quad i = 1,2,3,\ldots, \end{cases} \tag{36}$$

and shifted Chebyshev polynomials are defined as

$$\begin{cases} T_0^*(x) = 1, T_1^*(x) = \frac{2x-(b+a)}{b-a}, \quad x \in [a,b], \\ T_{i+1}^*(x) = 2\left(\frac{2x-(b+a)}{b-a}\right) T_i^*(x) - T_{i-1}^*(x), \quad i = 1,2,3.\ldots. \end{cases} \tag{37}$$

Now, we consider the following lemma.

Lemma 8. Suppose that $\mathbf{T}$ and $\mathbf{T}^*$ are coefficient matrices of Chebyshev polynomials $\{T_i(x) |\ x \in [-1,1],\ i = 0,1,2,\ldots\}$ and shifted Chebyshev polynomials $\{T_i^*(x) |\ x \in [a,b],\ i = 0,1,2,\ldots\}$ respectively. Then we have:

$$\mathbf{T}^* = \mathbf{T}\mathbf{Q},$$

where

$$\mathbf{Q}_{i,j} = \begin{cases} \binom{i}{j} v^{i-j} w^j, & i \geq j, \quad i,j = 0,1,2,\ldots, \\ 0, & i < j, \end{cases}$$

with $v = \frac{2}{b-a}$ and $w = \frac{a+b}{a-b}$.

Proof : Definition of $\mathbf{T}$ states that:

$$\begin{bmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ \vdots \end{bmatrix} = \mathbf{T} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix} \Rightarrow \begin{bmatrix} T_0^*(x) \\ T_1^*(x) \\ T_2^*(x) \\ \vdots \end{bmatrix} = \mathbf{T} \begin{bmatrix} 1 \\ \frac{2x-(b+a)}{b-a} \\ (\frac{2x-(b+a)}{b-a})^2 \\ \vdots \end{bmatrix} = \mathbf{T} \begin{bmatrix} 1 \\ vx+w \\ (vx+w)^2 \\ \vdots \end{bmatrix}.$$

We know that $(vx+w)^n = \sum_{i=0}^{n} \binom{n}{i} v^i w^{n-i} x^i$, thus

$$\begin{bmatrix} 1 \\ vx+w \\ (vx+w)^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ v & w & 0 & 0 & \cdots \\ v^2 & 2vw & w^2 & 0 & \cdots \\ v^3 & 3vw^2 & 3v^2w & w^3 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix}.$$

If we let $\mathbf{Q}$ be the last coefficient matrix, then

$$[T_0^*(x), T_1^*(x), T_2^*(x)\ldots]^T = \mathbf{TQx}_x, \quad \text{where} \quad \mathbf{Q}_{i,j} = \begin{cases} \binom{i}{j} v^{i-j} w^j, & i \geq j, \quad i,j = 0,1,2,\ldots, \\ 0, & i < j, \end{cases}$$

so

$$\mathbf{T}^* = \mathbf{TQ}.$$

Therefore, the statement of the lemma is valid. □

Note 1. It is noticeable that if the solution i.e. $u(x,t)$ has no singular point in $[-1,1]\times[-1,1]$ then the expansion of $u(x,t)$ based on Chebyshev polynomial is convergent. If $u(x,t)$ has singularity at some points $(x_i,t_j), i = 0,1,\ldots,n, j = 0,1,\ldots,m$, and let us suppose that $\beta = \min|x_i|, i = 0,1,\ldots,n$, then Chebyshev expansion of $u(x,t)$ is convergent for points (x,t) such that $x < \beta$ [32]. This is the main condition which guarantees the approximate solution to be convergent.

6. Illustrative numerical experiments

In this section, some experiments of linear and nonlinear FDEs on 1D and 2D are given to illustrate our results. In all experiments, we consider the shifted Chebyshev polynomials as basis function. In this work, the orthogonality of the basis functions is important, because it reduces the run time of the source programs and computations time. The computations associated with the experiments discussed above were performed in ***Maple 14*** on a PC with a CPU of 2.4 GHz.

Experiment 6.1 Consider the following FODE,

$$D^\alpha u(x) + u''(x) + u(x) = 8, \qquad x > 0, \quad 0 < \alpha \leq 2,$$

with the initial conditions: $u(0) = u'(0) = 0$. We have solved this experiment for $n = 30$, different α and have compared it with the closed form series solution of the exact solution [27] as $\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{j}{k}\frac{(-1)^k x^{2k-j\alpha+2}}{\Gamma(2k-j\alpha+3)}$. The results given in [27, 33], state that the OTM is the best in accuracy among them. The comparison is shown in Table 1.

Table 1: Comparison of the solutions of FDM, ADM, VIM and OTM with the exact solution for different α and x of experiment 6.1

x	u_{FDM}	u_{ADM}	u_{VIM}	u_{OTM}	u_{Exact}
$\alpha = .5$					
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.039473	0.039874	0.039874	0.039765	0.039750
0.2	0.157703	0.158512	0.158512	0.157070	0.157036
0.3	0.352402	0.353625	0.353625	0.347422	0.347370
0.4	0.620435	0.622083	0.622083	0.604764	0.604695
0.5	0.957963	0.960047	0.960047	0.921851	0.921768
0.6	1.360551	1.363093	1.363093	1.290552	1.290457
0.7	1.823267	1.826257	1.826257	1.702114	1.702008
0.8	2.340749	2.344224	2.344224	2.147401	2.147287
0.9	2.907324	2.911278	2.911278	2.617120	2.617001
1.0	3.517013	3.521462	3.521462	3.102013	3.101906
$\alpha = 1$					
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.039325	0.038925	0.038925	0.386673	0.386673
0.2	0.154541	0.153742	0.153742	0.149353	0.149353
0.3	0.341385	0.340182	0.340182	0.324153	0.324153
0.4	0.595453	0.593846	0.593846	0.555303	0.555303
0.5	0.912227	0.910214	0.910214	0.835243	0.835243
0.6	1.267112	1.284685	1.284685	1.156668	1.156668
0.7	1.715433	1.712597	1.712597	1.512574	1.512574
0.8	2.192488	2.189258	2.189258	1.896296	1.896296
0.9	2.713612	2.709964	2.709964	2.301537	2.301537
1.0	3.273081	3.270029	3.270029	2.722398	2.722398
$\alpha = 1.5$					
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.036111	0.036478	0.036478	0.032852	0.033507
0.2	0.139904	0.140640	0.140640	0.123882	0.125221
0.3	0.306402	0.307485	0.307485	0.265664	0.267609
0.4	0.531856	0.533284	0.533284	0.452944	0.455435
0.5	0.812989	0.814757	0.814757	0.681350	0.684335
0.6	1.146733	1.148840	1.148840	0.946961	0.950393
0.7	1.530132	1.532571	1.532571	1.246125	1.249959
0.8	1.960252	1.963033	1.963033	1.575365	1.579557
0.9	2.434223	2.437331	2.437331	1.931357	1.935832
1.0	2.949144	2.952567	2.952567	2.311672	2.315526

In the above experiment, we compared the OTM with the FDM, ADM and VIM for solving a FODE. We observe that higher-order accuracy can be achieved by OTM than using the same terms in the other methods. These results, demonstrate another capability of the proposed method. Also, when α is an integer, the OTM is so much accurate than the other methods.

Here, we desire to show that the proposed method is convergent. We test this case by considering the behavior of the solutions when n increases. In Table 2, the results demonstrate that by increasing n, the round-off error of the method decreases.

Table 2: Comparison of the OTM for different n and α with the exact solution of experiment 6.1

x	$n=5$	$n=15$	$n=25$	$n=35$	u_{Exact}
$\alpha=0.5$					
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.040224	0.039818	0.039773	0.039761	0.039750
0.2	0.158542	0.157207	0.157089	0.157060	0.157036
0.3	0.350021	0.347642	0.347451	0.347406	0.347370
0.4	0.608462	0.605059	0.604802	0.604742	0.604695
0.5	0.926525	0.922214	0.921898	0.921825	0.921768
0.6	1.296005	1.290973	1.290606	1.290522	1.290457
0.7	1.708089	1.702584	1.702174	1.702080	1.702008
0.8	2.153604	2.147913	2.147466	2.147365	2.147287
0.9	2.623269	2.617689	2.617190	2.617083	2.617001
1.0	3.107950	3.102761	3.102128	3.101999	3.101906
$\alpha=1.0$					
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.038667	0.038667	0.038667	0.038667	0.038667
0.2	0.149354	0.149353	0.149353	0.149353	0.149353
0.3	0.324162	0.324153	0.324153	0.324153	0.324153
0.4	0.555349	0.555303	0.555303	0.555303	0.555303
0.5	0.835416	0.835243	0.835243	0.835243	0.835243
0.6	1.157184	1.156668	1.156668	1.156668	1.156668
0.7	1.513871	1.512574	1.512574	1.512574	1.512574
0.8	1.899178	1.896296	1.896296	1.896296	1.896296
0.9	2.307366	2.301537	2.301537	2.301537	2.301537
1.0	2.733333	2.722398	2.722398	2.722398	2.722398
$\alpha=1.5$					
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.029606	0.032038	0.032687	0.033169	0.033507
0.2	0.114693	0.121909	0.123507	0.124709	0.125221
0.3	0.250115	0.262625	0.265101	0.267005	0.267609
0.4	0.431165	0.448945	0.452213	0.454765	0.455435
0.5	0.653355	0.676483	0.680467	0.683607	0.684335
0.6	1.202977	0.941308	0.945940	0.949616	0.950393
0.7	1.530132	1.239750	1.244981	1.249143	1.249959
0.8	1.520555	1.568232	1.574104	1.578719	1.579557
0.9	1.859128	1.922864	2.929872	1.934997	1.935832
1.0	2.212022	2.297889	2.307849	2.314162	2.315526

In this experiment, we observe that higher-order accuracy can be achieved by increasing n. This fact ensures us that the method is convergent for higher polynomial degree. These results, demonstrate another capability of the proposed method. Furthermore, when α is an integer, the OTM is so much accurate.

Experiment 6.2 Consider the following nonlinear FODE,

$$D^{\alpha}u(x) + e^{u(x)} = 0, \qquad 0 < \alpha \leq 1,$$

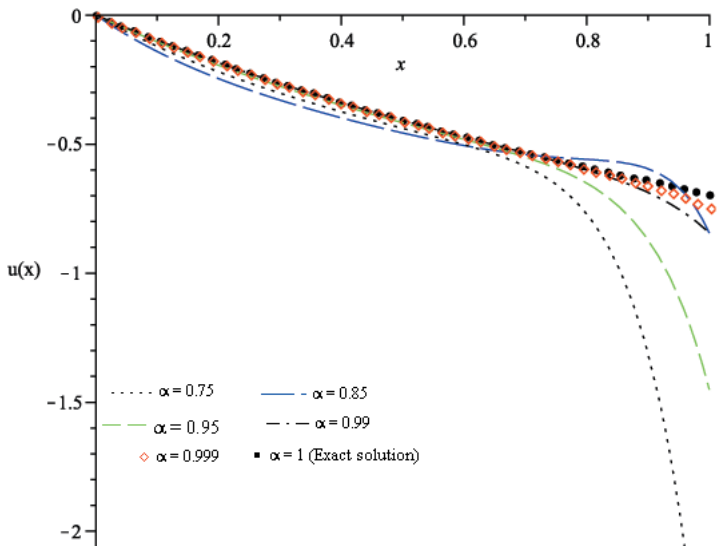
$$u(0) = 0.$$

The exact solution of this initial value problem for $\alpha = 1$, the ODE case, is $u(x) = -ln(x+1)$. In this problem, we use from the following expansion to linearize $e^{u(x)}$ as:

$$e^{u(x)} \simeq \sum_{i=0}^{n} \frac{u^i(x)}{i!}.$$

We have solved this problem for $n = 11$ and $\alpha = 0.75, 0.85, 0.95, 0.99, 0.999$. Figure 1, shows the numerical results. In the Figure, there is a good agreement with the same results obtained by using the VIM and ADM in [28]. Also, the results show that when $\alpha \to 1$, then approximate solution tends to exact solution, rapidly.

Figure 1. The graphs of the experiment 6.2 for $n = 11$ and different α



Experiment 6.3 Consider the following nonlinear FODE [34],

$$D^{\alpha}u(x) = \frac{9}{4}\sqrt{u(x)} + u(x), \quad 1 < \alpha \leq 2, \quad x > 0,$$

$$u(0) = 1, \quad u'(0) = 2.$$

The exact solution of this initial value problem for $\alpha = 2$, the ODE case, is

$$u(x) = \frac{9}{4}\left(\frac{3}{2}e^{0.5x} + \frac{1}{6}e^{-0.5x} - 1\right)^2.$$

The numerical results of the solution for $n = 10$ and $\alpha = 1.5, 1.75, 2$, compared with the HPM, are given in Table 3. Also, the nonlinear term in the problem is expanded as the following series:

$$\sqrt{u} \simeq 1 + \frac{1}{2}(u-1) - \frac{1}{8}(u-1)^2 + \frac{1}{16}(u-1)^3 - \frac{5}{128}(u-1)^4 + \cdots .$$

Table 3: Comparison of the solutions of HPM and OTM with the exact solution for different α and x of experiment 6.3

	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2$		
x	u_{HPM}	u_{OTM}	u_{HPM}	u_{OTM}	u_{HPM}	u_{OTM}	u_{Exact}
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.28245453	1.25127654	1.23780724	1.22934593	1.21697814	1.21697814	1.21697814
0.2	1.65028248	1.59134824	1.53420555	1.51587669	1.47098992	1.47099037	1.47099037
0.3	2.09444288	2.01182972	1.88844768	1.86064977	1.76704394	1.76704923	1.76704923
0.4	2.61939160	2.51346262	2.30514239	2.26766722	2.11071067	2.11074085	2.11074085
0.5	3.23340806	3.09921740	2.79109495	2.74266841	2.50817057	2.50828744	2.50828744
0.6	3.94743003	3.76140179	3.35490799	3.29186910	2.96626225	2.96661653	2.96661653
0.7	4.77490977	4.44407982	4.00703217	3.91920598	3.49253054	3.49343764	3.49343764
0.8	5.73195406	4.94458363	4.75997836	4.61955254	4.09527467	4.09732725	4.09732726
0.9	6.83756778	4.69087327	5.62861104	5.36359893	4.78359634	4.78782294	4.78782298
1.0	8.11394079	2.29381291	6.63049227	6.06756297	5.56744792	5.57552751	5.57552764

Since the exact solution for $\alpha = 1.5$ and $\alpha = 1.75$ does not exist, we are in doubt which of the solution of HPM or OTM is more accurate, but this doubt removes soon by attention to the obtained results for $\alpha = 2$. Here, you can easily see the OTM is more accurate and has well agreement to the exact solution. Thus we prefer using OTM instead of HPM to solve nonlinear FODEs.

Experiment 6.4 Consider the following linear time-fractional convective-diffusion equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2(1 + t + x^2), \quad t > 0, x \in \mathbb{R},\ 0 < \alpha \leq 1,$$

with the initial condition: $u(x, 0) = x^2$.
We have solved this problem for $n = 8$ and $\alpha = 0.75, 0.85, 0.95$ and compared it with HPM [22] to show the efficiency of the OTM. The results are given in Table 4.

Table 4: Comparison of the HPM and OTM for different t and x of experiment 6.4

t/x	u_{HPM}			u_{OTM}		
	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$
t=0.1						
0.00	0.017325	0.014486	0.011452	0.025199	0.017920	0.012299
0.25	0.079825	0.076986	0.073952	0.087699	0.080420	0.074799
0.50	0.267326	0.264486	0.261452	0.275199	0.267920	0.026229
0.75	0.579826	0.576986	0.573952	0.587699	0.580420	0.574799
1.00	1.017326	1.014486	1.011453	1.025199	1.017920	1.012299
t=0.2						
0.00	0.067432	0.056444	0.045243	0.077557	0.060178	0.046034
0.25	0.129932	0.118944	0.107743	0.140057	0.122678	0.108534
0.50	0.317432	0.306444	0.295243	0.327557	0.310178	0.296034
0.75	0.629932	0.618944	0.607743	0.640057	0.622678	0.608534
1.00	1.067432	1.056444	1.045243	1.077558	1.060179	1.046035
t=0.3						
0.00	0.142801	0.121285	0.099954	0.154136	0.125161	0.100719
0.25	0.205301	0.183785	0.162454	0.216636	0.187661	0.163219
0.50	0.392801	0.371285	0.349954	0.404136	0.375161	0.350719
0.75	0.705301	0.683785	0.662454	0.716636	0.687661	0.663219
1.00	1.142801	1.121285	1.099955	1.154136	1.125162	1.100719
t=0.4						
0.00	0.241393	0.207880	0.175242	0.252852	0.211689	0.175989
0.25	0.303893	0.270380	0.237742	0.315352	0.274189	0.238489
0.50	0.491393	0.457880	0.425242	0.502852	0.461689	0.425989
0.75	0.803893	0.770380	0.737742	0.815352	0.774189	0.738489
1.00	1.241394	1.207880	1.175243	1.252852	1.211688	1.175989
t=0.5						
0.00	0.361591	0.315341	0.270835	0.372193	0.318879	0.271561
0.25	0.424091	0.377841	0.333335	0.434693	0.381379	0.334061
0.50	0.611591	0.565341	0.520835	0.622193	0.568879	0.521561
0.75	0.924091	0.877841	0.833335	0.934693	0.881379	0.834061
1.00	1.361591	1.315341	1.270835	1.372194	1.318879	1.271561

The obtained results in Table 4, demonstrate the approximate solution obtained using the OTM is in good agreement with the approximate solution obtained using the HPM for all values of t and x.

Experiment 6.5 Consider the following nonlinear space-fractional Fisher's equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^{1.5} u}{\partial x^{1.5}} - u(x,t)(1-u(x,t)) = x^2, \quad x > 0,$$

with the initial condition: $u(x,0) = x$.
We have obtained the approximate solution for $n = 7$ with different t and x. Comparison of the OTM with VIM [35] and generalized differential transform method (GDTM) [36] are given in Table 5.

Table 5: Comparison of the solutions of GDTM, VIM and OTM for different t and x of experiment 6.5

x	t=0.1			t=0.2		
	u_{GDTM}	u_{VIM}	u_{OTM}	u_{GDTM}	u_{VIM}	u_{OTM}
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.110197	0.110123	0.110123	0.119817	0.119386	0.119302
0.2	0.220333	0.220243	0.220239	0.240073	0.239548	0.239453
0.3	0.330252	0.330158	0.330153	0.359392	0.588663	0.358809
0.4	0.439954	0.439870	0.439869	0.477771	0.477364	0.477393
0.5	0.549439	0.549383	0.549389	0.595204	0.595069	0.595229
0.6	0.658706	0.658701	0.658716	0.711688	0.712008	0.712343
0.7	0.767755	0.767825	0.767854	0.827220	0.828207	0.828759
0.8	0.876585	0.876761	0.876806	0.941795	0.943694	0.944504
0.9	0.985196	0.985512	0.985576	1.055411	1.058498	1.059601
1.0	1.093587	1.094081	1.094166	1.168066	1.172648	1.174077
x	t=0.3			t=0.4		
	u_{GDTM}	u_{VIM}	u_{OTM}	u_{GDTM}	u_{VIM}	u_{OTM}
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.127035	0.126129	0.125536	0.129536	0.128692	0.126286
0.2	0.256895	0.255777	0.255291	0.267899	0.266790	0.265133
0.3	0.384480	0.383473	0.383307	0.401893	0.401330	0.401004
0.4	0.509772	0.509305	0.509667	0.531477	0.532516	0.534090
0.5	0.632755	0.633359	0.634452	0.656611	0.660556	0.664574
0.6	0.753415	0.755727	0.757743	0.777260	0.785664	0.792632
0.7	0.871740	0.876498	0.879621	0.893392	0.908056	0.918431
0.8	0.987717	0.995769	1.000160	1.004976	1.027956	1.042129
0.9	1.101336	1.113633	1.119438	1.111987	1.145590	1.163878
1.0	1.212587	1.230189	1.237524	1.214400	1.261189	1.283822

From the above comparison, a good agreement of OTM with the other methods is observable.

7. Conclusion

Nowadays, most of the real physical world problems can be best modeled by the FODEs and FPDEs. Besides modeling, the solution techniques and their use are most important to obtain high accurate solutions. The main idea of the proposed method (OTM) is to convert the problem including linear and nonlinear terms to an algebraic system to simplify the computations. Several experiments of FODEs and FPDEs were numerically solved and compared with the most powerful methods such as HPM, VIM, FDM, GDTM and ADM. From the results, it can be easily seen that OTM obtains results as accurate as possible. So, OTM is a powerful tool which enables us to handle even nonlinear problems. It is seen that, for the case where the exact solution is known, the numerical results demonstrate the high accuracy of the scheme. These results supported the confidence in applying this method to FODEs and FPDEs in which the theoretical solution is not known.

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Generalized induced linguistic harmonic mean operators based approach to multiple attribute group decision making

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Abstract

Two generalized induced linguistic aggregation operator called the generalized induced linguistic ordered weighted harmonic mean (GILOWHM) operator and generalized induced uncertain linguistic ordered weighted harmonic mean (GIULOWHM) operator is defined. Each object processed by these operators consists of three components, where the first component represents the importance degree or character of the second component, and the second component is used to induce an ordering, through the first component, over the third components which are linguistic variables (or uncertain linguistic variables) and then aggregated. Based on the GILOWHM and GIULOWHM operators respectively, we develop two procedures to solve the multiple attribute group decision making problems where all attribute values are expressed in linguistic variables or uncertain linguistic variables. Finally, an example is used to illustrate the developed procedures.

Keywords: Group decision making, linguistic variable, generalized induced linguistic ordered weighted harmonic mean (GILOWHM) operator, generalized induced uncertain linguistic ordered weighted harmonic mean (GIULOWHM) operator, operational laws.

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1 Introduction

Information aggregation is essential process of gathering relevant information from multiple sources. Many techniques have been developed to aggregate data

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information [12,13,20-23,27]. Yager and Filev [26] introduced an induced aggregation operator called the induced ordered weighted averaging (IOWA) operator, which takes as its argument pairs, called OWA pairs, in which one component is used to induce an ordering over the second components which are exact numerical values and then aggregated. Later, some new induced aggregation operators have been developed, including the induced ordered weighted geometric (IOWG) operator [21], induced fuzzy integral aggregation (IFIA) operator [24] and induced Choquet ordered averaging (ICOA) operator [25]. Xu and Da [21] introduced two more general aggregation techniques called generalized IOWA (GIOWA) and generalized IOWG (GIOWG) operators, and proved that the OWA and IOWA operators are the special cases of the GIOWA operator, and that the OWG and IOWG operators are the special cases of the GIOWG operator. All this operators have been used in situations in which the input arguments are numerical values. In some situations, however, the input arguments take the form of linguistic variables or uncertain linguistic variables rather than numerical ones because of time pressure, lack of knowledge, and the decision maker's limit attention and information processing capabilities [1-10,14-19,28-30]. Recently, Xu [19] developed various generalized induced linguistic aggregation operators, such as the generalized induced linguistic ordered weighted averaging (GILOWA) and generalized induced linguistic ordered weighted geometric (GILOWG) operator, both of which can be used to deal with the linguistic information, and generalized induced uncertain linguistic ordered weighted averaging (GIULOWA) operator and generalized induced uncertain linguistic ordered weighted geometric (GIULOWG) operator, both of which can be used to deal with the uncertain linguistic information. In this paper, we shall develop two new aggregation operators called generalized induced linguistic ordered weighted harmonic mean (GILOWHM) operator and generalized induced uncertain linguistic ordered weighted harmonic mean (GIULOWHM) operator, which can be used to deal with linguistic information or uncertain linguistic information, and study some of their desirable properties. Each object processed by these operators consists of three components, where the first component represents the importance degree or character of the second component, and the second component is used to induce an ordering, through the first component, over the third components which are linguistic variables or uncertain linguistic variables and then aggregated. It is shown that the induced linguistic ordered weighted harmonic mean (ILOWHM) [11] operator and linguistic ordered weighted harmonic mean (LOWHM) [11] operator are the special cases of the GILOWHM operator and that the induced uncertain linguistic ordered weighted harmonic mean (IULOWHM) operator and uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator are the special cases of the GIULOWHM operator. Two procedures based on the GILOWHM and GIULOWHM operators respectively, are developed to solve the multiple attribute decision making (MADM) problems where all decision information about attribute values take the forms of linguistic variables or uncertain linguistic variables. Finally, an illustrative example is pointed out.

2 Preliminaries

The linguistic approach is approximate technique, which represents qualitative aspects as linguistic values by means of linguistic variables [28-30].

Let $S = \{s_i : i = 1, 2, \ldots, t\}$ be a finite and totally ordered discrete label set, whose cardinality value is odd. Any label, s_i, represents a possible value for a linguistic variable, and it must have the following characteristics [6]: (1) the set is ordered: $s_i \geq s_j$ if $i \geq j$, and (2) there is the negation operator: $\text{neg}(s_i) = s_{t-i+1}$. We call this linguistic label set S the additive linguistic scale. For example, S can be defined as:

$$\begin{aligned} S = \{&s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, \\ &s_4 = \text{slightly poor}, s_5 = \text{fair}, s_6 = \text{slightly good}, \\ &s_7 = \text{good}, s_8 = \text{very good}, s_9 = \text{extremely good}\}. \end{aligned}$$

To preserve all the given information, we extend the discrete linguistic label set S to a continuous linguistic label set $\bar{S} = \{s_\alpha : \alpha \in [1, q]\}$, where q $(q > t)$ is a sufficiently large positive integer. If $s_\alpha \in S$, then Xu [16] called s_α the original linguistic label, otherwise, Xu [16] called s_α the virtual linguistic label. The decision maker, in general, uses the original linguistic labels to evaluate alternatives, and the virtual linguistic labels can only appear in operations [16].

Let $\tilde{s} = [s_\alpha, s_\beta]$, where $s_\alpha, s_\beta \in \bar{S}$, s_α and s_β are the lower and upper limits, respectively, then Xu [15] called $\tilde{s}$ the uncertain linguistic variables. For convenience, we let $\tilde{S}$ be the set of all the uncertain linguistic variables.

Consider any three linguistic variables $s_\lambda, s_{\lambda_1}, s_{\lambda_2} \in \bar{S}$, and any three uncertain linguistic variables $\tilde{s} = [s_\alpha, s_\beta]$, $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$, and let $\mu \in [0, 1]$, then we define their operations as follows:

(1) $s_{\lambda_1} \oplus s_{\lambda_2} = s_{\lambda_1 + \lambda_2}$;
(2) $\mu s_\lambda = s_{\mu\lambda}$;
(3) $\frac{1}{s_\lambda} = s_{\frac{1}{\lambda}}$;
(4) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}]$;
(5) $\lambda\tilde{s} = \lambda[s_\alpha, s_\beta] = [\lambda s_\alpha, \lambda s_\beta]$;
(6) $\frac{1}{\tilde{s}} = \frac{1}{[s_\alpha, s_\beta]} = [\frac{1}{s_\beta}, \frac{1}{s_\alpha}]$.

In order to compare uncertain linguistic variables, Xu [17] provided the following definition:

Definition 2.1 Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain linguistic variables, and let $\text{len}(\tilde{s}_1) = \beta_1 - \alpha_1$ and $\text{len}(\tilde{s}_2) = \beta_2 - \alpha_2$, then the degree of possibility of $\tilde{s}_1 \geq \tilde{s}_2$ is defined as

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \frac{\max\{0, \text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2) - \max(\beta_2 - \alpha_1, 0)\}}{\text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2)}. \tag{1}$$

From Definition 2.1, we can easily get the following results:

(1) $0 \leq p(\tilde{s}_1 \geq \tilde{s}_2) \leq 1$, $0 \leq p(\tilde{s}_2 \geq \tilde{s}_1) \leq 1$;
(2) $p(\tilde{s}_1 \geq \tilde{s}_2) + p(\tilde{s}_2 \geq \tilde{s}_1) = 1$. Especially, $p(\tilde{s}_1 \geq \tilde{s}_1) = p(\tilde{s}_2 \geq \tilde{s}_2) = \frac{1}{2}$.

3 Generalized induced linguistic aggregation operators

3.1 The GILOWHM and GIULOWHM operators

Definition 3.1 [11] Let LWHM : $\bar{S}^n \to \bar{S}$, if

$$\mathrm{LWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \frac{1}{\frac{w_1}{s_{\alpha_1}} \oplus \frac{w_2}{s_{\alpha_2}} \oplus \cdots \oplus \frac{w_n}{s_{\alpha_n}}}, \tag{2}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of the s_{α_i} with $w_i \in [0,1]$, $\sum_{i=1}^n w_i = 1$, $s_{\alpha_i} \in \bar{S}$, then LWHM is called the linguistic weighted harmonic mean (LWHM) operator. Especially, if $w_i = 1$ and $w_j = 0$, $j \neq i$, then $\mathrm{LWHM}(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = s_{\alpha_i}$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then LWHM operator is called the linguistic harmonic mean (LHM) operator, i.e.,

$$\mathrm{LHM}(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \frac{n}{s_{\sum_{i=1}^n \frac{1}{\alpha_i}}}. \tag{3}$$

The fundamental aspect of the LWHM operator is that it compute an aggregated value taking into the importance of the sources of information.

Definition 3.2 [11] A LOWHM operator of dimension n is a mapping LOWHM : $\bar{S}^n \to \bar{S}$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, such that

$$\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}}, \tag{4}$$

where s_{β_i} is the ith largest of the s_{α_i}.

The weighted vector $w = (w_1, w_2, \ldots, w_n)^T$ can be determined by using some weight determining methods like the normal distribution based method.

Definition 3.3 An ULOWHM operator of dimension n is a mapping ULOWHM : $\tilde{S}^n \to \tilde{S}$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, such that

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}}, \tag{5}$$

where $\tilde{s}_{\beta_i}$ is the ith largest of the $\tilde{s}_i$.

The fundamental aspect of the LOWHM and ULOWHM operators is reordering of the arguments to be aggregated, based on their values. When using the ULOWHM operator, we need to rank the uncertain linguistic arguments $\tilde{s}_i$ $(i = 1, 2, \ldots, n)$. To do so, we first compare each argument $\tilde{s}_i$ with all arguments $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$ by using (1), and let $p_{ij} = p(\tilde{s}_i \geq \tilde{s}_j)$. Then we can contract a complementary matrix $\mathbf{P} = (p_{ij})_{n \times n}$ where:

$$p_{ij} \geq 0, p_{ij} + p_{ji} = 1, p_{ii} = \frac{1}{2},\ i, j = 1, 2, \ldots, n.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$. Then, in accordance with the values of p_i $(i = 1, 2, \ldots, n)$, we can rank the arguments $\tilde{s}_i$ $(i = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_i $(i = 1, 2, \ldots, n)$.

Definition 3.4 [11] An induced LOWHM (ILOWHM) operator is defined as follows:

$$\text{ILOWHM}_w(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \tag{6}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $w_i \in [0, 1]$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} w_i = 1$, s_{β_i} is the s_{α_i} value of the LOWHM pair $\langle u_i, s_{\alpha_i}\rangle$ having the ith largest u_i, and u_i in $\langle u_i, s_{\alpha_i}\rangle$ is referred to as the order inducing variable and s_{α_i} as the linguistic argument variable. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then ILOWHM is reduced to the LHM operator; if $u_i = s_{\alpha_i}$, for all i, then ILOWHM is reduced to the LOWHM operator; if $u_i =$No. i, for all i, where No. i is the ordered position of the a_i, then ILOWHM is the LHM operator.

However, if there is a tie between $\langle u_i, s_{\alpha_i}\rangle$, $\langle u_j, s_{\alpha_j}\rangle$ with respect to order-inducing variables, in this case, we can follow the policy presented by Yager and Filov [26] - to replace the arguments of the tied objects by the mean of the arguments of the tied objects (i.e., we replace the argument component of each of $\langle u_i, s_{\alpha_i}\rangle$ and $\langle u_j, s_{\alpha_j}\rangle$ by their average $(s_{\alpha_i} \oplus s_{\alpha_j})/2$). If k items are tied, we replace these by k replicas of their average.

In the following, we shall give example to specify the special cases with respect to the inducing variables.

Example 3.5 Consider the following collection of LOWHM pairs:

$$\langle s_4, s_3\rangle, \langle s_6, s_7\rangle, \langle s_3, s_1\rangle, \langle s_5, s_4\rangle.$$

Performing the ordering the LOWHM pairs with respect to the first component, we have

$$\langle s_6, s_7\rangle, \langle s_5, s_4\rangle, \langle s_4, s_3\rangle, \langle s_3, s_1\rangle.$$

This ordering induces the ordered linguistic arguments

$$s_{\beta_1} = s_7, s_{\beta_2} = s_4, s_{\beta_3} = s_3, s_{\beta_4} = s_1.$$

If the weighting vector $w = (0.3, 0.1, 0.4, 0.2)^T$, then we get an aggregated value:

$$\begin{aligned}&\text{ILOWHM}_w(\langle s_4, s_3\rangle, \langle s_5, s_7\rangle, \langle s_3, s_1\rangle, \langle s_6, s_4\rangle)\\ &\quad = \frac{1}{\frac{0.3}{s_7} \oplus \frac{0.1}{s_4} \oplus \frac{0.4}{s_3} \oplus \frac{0.2}{s_1}} = s_{2.49}.\end{aligned}$$

Definition 3.6 An induced uncertain LOWHM (IULOWHM) operator is defined as follows:

$$\text{IULOWHM}_w(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \ldots, \langle u_n, \tilde{s}_n\rangle) = \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}} \tag{7}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $w_i \in [0,1]$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} w_i = 1$, $\tilde{s}_{\beta_i}$ is the $\tilde{s}_i$ value of the ULOWHM pair $\langle u_i, \tilde{s}_i \rangle$ having the ith largest u_i, and u_i in $\langle u_i, \tilde{s}_i \rangle$ is referred to as the order inducing variable and $\tilde{s}_i$ as the uncertain linguistic argument variable. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then IULOWHM is reduced to the ULHM operator; if $u_i = \tilde{s}_i$, for all i, then IULOWHM is reduced to the ULOWHM operator; if $u_i =$No. i, for all i, where No. i is the ordered position of the a_i, then IULOWHM is the ULHM operator.

However, if there is a tie between $\langle u_i, \tilde{s}_i \rangle$, $\langle u_j, \tilde{s}_j \rangle$ with respect to order-inducing variables. In this case, we can replace the argument component of each of $\langle u_i, \tilde{s}_i \rangle$ and $\langle u_j, \tilde{s}_j \rangle$ by their average $(\tilde{s}_i \oplus \tilde{s}_j)/2)$. If k items are tied, we replace these by k replicas of their average.

Example 3.7 Consider the following collection of ULOWHM pairs:

$$\langle 0.5, [s_3, s_4] \rangle, \langle 0.3, [s_6, s_7] \rangle, \langle 0.7, [s_2, s_3] \rangle, \langle 0.4, [s_2, s_4] \rangle.$$

Performing the ordering the ULOWHM pairs with respect to the first component, we have

$$\langle 0.7, [s_2, s_3] \rangle, \langle 0.5, [s_3, s_4] \rangle, \langle 0.4, [s_2, s_4] \rangle, \langle 0.3, [s_6, s_7] \rangle.$$

This ordering induces the ordered linguistic arguments

$$\tilde{s}_{\beta_1} = [s_2, s_3], \tilde{s}_{\beta_2} = [s_3, s_4], \tilde{s}_{\beta_3} = [s_2, s_4], \tilde{s}_{\beta_4} = [s_6, s_7].$$

If the weighting vector $w = (0.3, 0.1, 0.4, 0.2)^T$, then we get an aggregated value:

$$\begin{aligned} &\text{ILOWHM}_w(\langle 0.5, [s_3, s_4] \rangle, \langle 0.3, [s_6, s_7] \rangle, \langle 0.7, [s_2, s_3] \rangle, \langle 0.4, [s_2, s_4] \rangle) \\ &\quad = [s_{2.40}, s_{3.94}]. \end{aligned}$$

An important feature of the ILOWHM operator is that the argument ordering process is guided by a variable called the order inducing value. This operator essentially aggregate objects, which are pairs, and provide a very general family of aggregations operators. In some situations, however, when we need to provide more information about the objects, i.e. each object may consist of three components, a direct locator, an indirect locator and a prescribed value, it is unsuitable to use this induced aggregation operator as an aggregation tool. In following we shall present some more general linguistic aggregation technique.

Definition 3.8 A generalized induced LOWHM (GILOWHM) operator is given by

$$\begin{aligned} &\text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1} \rangle, \langle v_2, u_2, s_{\alpha_2} \rangle, \ldots, \langle v_n, u_n, s_{\alpha_n} \rangle) \\ &\quad = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \end{aligned} \tag{8}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weighting vector with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, the object $\langle v_i, u_i, s_{\alpha_i} \rangle$ consists of three components, where the first component v_i represents the importance degree or character of second

component u_i, and the second component u_i is used to induce an ordering, through the first component v_i, over the third component s_{α_i} which are then aggregated. Here, s_{β_j} is the s_{α_i} value of the object having the jth largest v_i. In discussing the object $\langle v_i, u_i, s_{\alpha_i}\rangle$, because of its role we shall refer to the v_i as the direct order inducing variable, the u_i as the indirect inducing variable, and s_{α_i} as the linguistic argument variable.

Especially, if $v_i = u_i$, for all i, then the GILOWHM operator is reduced to the ILOWHM operator; if $v_i = s_{\alpha_i}$, for all i, then the GILOWHM operator is reduced to the LOWHM operator; if $v_i =$ No. i, for all i, where No. i is the ordered position of the s_{α_i}, then the GILOWHM operator is reduced to the LWHM operator; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the GILOWHM operator is reduced to the LHM operator.

Example 3.9 Consider the collection of the objects

$$\langle \text{No. } 3, \text{Kim}, s_1\rangle, \langle \text{No. } 1, \text{Park}, s_7\rangle, \langle \text{No. } 2, \text{Lee}, s_2\rangle, \langle \text{No. } 4, \text{Jung}, s_5\rangle.$$

By the first component, we get the ordered objects

$$\langle \text{No. } 1, \text{Park}, s_7\rangle, \langle \text{No. } 2, \text{Lee}, s_2\rangle, \langle \text{No. } 3, \text{Kim}, s_1\rangle, \langle \text{No. } 4, \text{Jung}, s_5\rangle.$$

The ordering induces the ordered arguments $s_{\beta_1} = s_7$, $s_{\beta_2} = s_2$, $s_{\beta_3} = s_1$, $s_{\beta_4} = s_5$. If the weighting vector for this aggregation is $w = (0.3, 0.1, 0.2, 0.4)^T$, then we get

$$\begin{aligned}&\text{GILOWHM}_w(\langle \text{No.3}, \text{Kim}, s_1\rangle, \langle \text{No.1}, \text{Park}, s_7\rangle, \langle \text{No.2}, \text{Lee}, s_2\rangle, \langle \text{No.4}, \text{Jung}, s_5\rangle)\\ &\quad = \frac{1}{\frac{0.3}{s_7} \oplus \frac{0.1}{s_2} \oplus \frac{0.2}{s_1} \oplus \frac{0.4}{s_5}} = s_{2.70}.\end{aligned}$$

However, if we replace the objects in Example 3.9 with

$$\langle \text{No. } 3, \text{Kim}, s_1\rangle, \langle \text{No. } 1, \text{Park}, s_7\rangle, \langle \text{No. } 2, \text{Lee}, s_2\rangle, \langle \text{No. } 3, \text{Jung}, s_5\rangle,$$

then there is a tie between $\langle \text{No. } 3, \text{Kim}, s_1\rangle$ and $\langle \text{No. } 3, \text{Jung}, s_5\rangle$ with respect to order direct inducing variable, in this case, we can follow the policy: we replace the linguistic argument component of each of $\langle \text{No. } 3, \text{Kim}, s_1\rangle$ and $\langle \text{No. } 3, \text{Jung}, s_5\rangle$ by their average $(s_1 \oplus s_5)/2 = s_3$. This substitution gives us ordered arguments $s_{\beta_1} = s_7$, $s_{\beta_2} = s_2$, $s_{\beta_3} = s_3$, $s_{\beta_4} = s_3$. Thus

$$\begin{aligned}&\text{GILOWHM}_w(\langle \text{No.3}, \text{Kim}, s_1\rangle, \langle \text{No.1}, \text{Park}, s_7\rangle, \langle \text{No.2}, \text{Lee}, s_2\rangle, \langle \text{No.3}, \text{Jung}, s_5\rangle)\\ &\quad = \frac{1}{\frac{0.3}{s_7} \oplus \frac{0.1}{s_2} \oplus \frac{0.2}{s_3} \oplus \frac{0.4}{s_3}} = s_{3.44}.\end{aligned}$$

If we replace (8) with

$$\begin{aligned}&\text{GIULOWHM}_w(\langle v_1, u_1, \tilde{s}_1\rangle, \langle v_2, u_2, \tilde{s}_2\rangle, \ldots, \langle v_n, u_n, \tilde{s}_n\rangle) \qquad (9)\\ &\quad = \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}}\end{aligned}$$

then by Definition 3.8, we get a GIULOWHM operator. Especially, if $v_i = u_i$, for all i, then the GIULOWHM operator is reduced to the IULOWGM operator; if $v_i = \tilde{s}_i$, for all i, then the GIULOWHM operator is reduced to the ULOWHM operator; if $v_i =$ No. i, for all i, where No. i is ordered position of the $\tilde{s}_i$, then the GIULOWHM operator is reduced to the ULWHM operator; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the GIULOWHM operator is reduced to the ULHM operator.

Example 3.10 Consider a collection of the objects

$$\langle 0.3, \text{Kim}, [s_1, s_3]\rangle, \langle 0.1, \text{Park}, [s_7, s_8]\rangle, \langle 0.2, \text{Lee}, [s_2, s_3]\rangle.$$

Performing the ordering of the objects with respect to the first component, we get the ordered objects

$$\langle 0.3, \text{Kim}, [s_1, s_3]\rangle, \langle 0.2, \text{Lee}, [s_2, s_3]\rangle, \langle 0.1, \text{Park}, [s_7, s_8]\rangle.$$

The ordering induces the ordered uncertain linguistic arguments $\tilde{s}_{\beta_1} = [s_1, s_3]$, $\tilde{s}_{\beta_2} = [s_2, s_3]$, $s_{\beta_3} = [s_7, s_8]$. If the weighting vector for this aggregation is $w = (0, 2, 0.6, 0.2)^T$, then we have

$$\begin{aligned}\text{GIULOWHM}_w(&\langle 0.3, \text{Kim}, [s_1, s_3]\rangle, \langle 0.1, \text{Park}, [s_7, s_8]\rangle, \langle 0.2, \text{Lee}, [s_2, s_3]\rangle)\\ &= [s_{2.33}, s_{3.42}].\end{aligned}$$

If the direct order inducing variables v_i $(i = 1, 2, \ldots, n)$ take the form of uncertain linguistic variables $\tilde{s}'_i$ $(i = 1, 2, \ldots, n)$, then we shall use, to rank these uncertain linguistic variables, the procedure for ranking uncertain linguistic arguments when using the ULOWHM operator.

Example 3.11 Consider a collection of the objects

$$\langle [s_1, s_2], \text{Kim}, [s_2, s_4]\rangle, \langle [s_4, s_5], \text{Park}, [s_7, s_8]\rangle, \langle [s_3, s_5], \text{Lee}, [s_2, s_3]\rangle.$$

To rank the first components v_i $(i = 1, 2, 3)$ of the objects, we first compare each v_i with all these first components v_i $(i = 1, 2, 3)$ by using (1), and then construct a complementary matrix

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.000 & 0.000 \\ 1.000 & 0.500 & 0.667 \\ 1.000 & 0.333 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_1 = 0.500,\ p_2 = 2.167,\ p_3 = 1.833.$$

Then we rank all the variables v_i $(i = 1, 2, 3)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3)$

$$v_2 = [s_4, s_5],\ v_3 = [s_3, s_5],\ v_1 = [s_1, s_2].$$

Performing the ordering of the objects with respect to the first component, we get the ordered objects

$$\langle [s_4, s_5], \text{Park}, [s_7, s_8]\rangle, \langle [s_3, s_5], \text{Lee}, [s_2, s_3]\rangle, \langle [s_1, s_2], \text{Kim}, [s_2, s_4]\rangle.$$

The ordering induces the ordered uncertain linguistic arguments $\tilde{s}_{\beta_1} = [s_7, s_8]$, $\tilde{s}_{\beta_2} = [s_2, s_3]$, $s_{\beta_3} = [s_2, s_4]$. If the weighting vector for this aggregation is $w = (0, 2, 0.6, 0.2)^T$, then we have

$$\mathrm{GIULOWHM}_w\Big(\langle[s_1, s_2], \mathrm{Kim}, [s_2, s_4]\rangle, \langle[s_4, s_5], \mathrm{Park}, [s_7, s_8]\rangle, \langle[s_3, s_5], \mathrm{Lee}, [s_2, s_3]\rangle\Big) = [s_{2.33}, s_{3.64}].$$

3.2 Some properties of the GILOWHM operator

In the following we shall make an investigation on some desirable properties of the GILOWHM operator.

Theorem 3.12 (Commutativity) If $(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_1, u'_1, s'_{\alpha_1}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle)$ is any permutation of $(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_1, u_1, s_{\alpha_1}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)$, then

$$\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) = \mathrm{GILOWHM}_w(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle).$$

Proof Let

$$\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}}$$

$$\mathrm{GILOWHM}_w(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s'_{\beta_1}} \oplus \frac{w_2}{s'_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s'_{\beta_n}}}.$$

Since $(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_1, u'_1, s'_{\alpha_1}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle)$ is a permutation of $(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_1, u_1, s_{\alpha_1}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)$, we have $s_{\beta_j} = s'_{\beta_j}$ $(j = 1, 2, \ldots, n)$, then

$$\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) = \mathrm{GILOWHM}_w(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle).$$

Theorem 3.13 (Idempotency) If $s_{\alpha_i} = s_\alpha$, for all i, then

$$\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) = s_\alpha.$$

Proof Since $s_{\alpha_i} = s_\alpha$, for all i, we have

$$\begin{aligned}\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) &= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} = \frac{1}{\frac{w_1}{s_\alpha} \oplus \frac{w_2}{s_\alpha} \oplus \cdots \oplus \frac{w_n}{s_\alpha}} \\ &= s_\alpha.\end{aligned}$$

Theorem 3.14 (Monotonicity) If $s_{\alpha_i} \le s'_{\alpha_i}$, for all i, then

$$\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) \le \mathrm{GILOWHM}_w(\langle v_1, u_1, s'_{\alpha_1}\rangle, \langle v_2, u_2, s'_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s'_{\alpha_n}\rangle).$$

Proof Let

$$\mathrm{GILOWHM}_w(\langle v_1,u_1,s_{\alpha_1}\rangle,\langle v_2,u_2,s_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s_{\alpha_n}\rangle)=\frac{1}{\frac{w_1}{s_{\beta_1}}\oplus\frac{w_2}{s_{\beta_2}}\oplus\cdots\oplus\frac{w_n}{s_{\beta_n}}}$$

$$\mathrm{GILOWHM}_w(\langle v_1,u_1,s'_{\alpha_1}\rangle,\langle v_2,u_2,s'_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s'_{\alpha_n}\rangle)=\frac{1}{\frac{w_1}{s'_{\beta_1}}\oplus\frac{w_2}{s'_{\beta_2}}\oplus\cdots\oplus\frac{w_n}{s'_{\beta_n}}}.$$

Since $s_{\alpha_i}\le s'_{\alpha_i}$, for all i, it follows that $s_{\beta_i}\le s'_{\beta_i}$, then

$$\begin{aligned}&\mathrm{GILOWHM}_w(\langle v_1,u_1,s_{\alpha_1}\rangle,\langle v_2,u_2,s_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s_{\alpha_n}\rangle)\\&\qquad\le\mathrm{GILOWHM}_w(\langle v_1,u_1,s'_{\alpha_1}\rangle,\langle v_2,u_2,s'_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s'_{\alpha_n}\rangle).\end{aligned}$$

Theorem 3.15 (Boundedness)

$$\min_i(s_{\alpha_i})\le\mathrm{GILOWHM}_w(\langle v_1,u_1,s_{\alpha_1}\rangle,\langle v_2,u_2,s_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s_{\alpha_n}\rangle)\le\max_i(s_{\alpha_i}).$$

Proof Let $\max_i(s_{\alpha_i})=s_\beta$ and $\min_i(s_{\alpha_i})=s_\alpha$, then

$$\begin{aligned}&\mathrm{GILOWHM}_w(\langle v_1,u_1,s_{\alpha_1}\rangle,\langle v_2,u_2,s_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s_{\alpha_n}\rangle)\\&\qquad=\frac{1}{\frac{w_1}{s_{\beta_1}}\oplus\frac{w_2}{s_{\beta_2}}\oplus\cdots\oplus\frac{w_n}{s_{\beta_n}}}\le\frac{1}{\frac{w_1}{s_\beta}\oplus\frac{w_2}{s_\beta}\oplus\cdots\oplus\frac{w_n}{s_\beta}}=s_\beta,\\&\mathrm{GILOWHM}_w(\langle v_1,u_1,s_{\alpha_1}\rangle,\langle v_2,u_2,s_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s_{\alpha_n}\rangle)\\&\qquad=\frac{1}{\frac{w_1}{s_{\beta_1}}\oplus\frac{w_2}{s_{\beta_2}}\oplus\cdots\oplus\frac{w_n}{s_{\beta_n}}}\ge\frac{1}{\frac{w_1}{s_\alpha}\oplus\frac{w_2}{s_\alpha}\oplus\cdots\oplus\frac{w_n}{s_\alpha}}=s_\alpha.\end{aligned}$$

Hence we have

$$\min_i(s_{\alpha_i})\le\mathrm{GILOWHM}_w(\langle v_1,u_1,s_{\alpha_1}\rangle,\langle v_2,u_2,s_{\alpha_2}\rangle,\ldots,\langle v_n,u_n,s_{\alpha_n}\rangle)\le\max_i(s_{\alpha_i}).$$

Similarly, we can prove that GIULOWHM operator also has the desirable properties above.

4 An approach to group decision making

For a group decision making with linguistic information, let $X=\{x_1,x_2,\ldots,x_n\}$ be a set of alternatives, and $G=\{G_1,G_2,\ldots,G_m\}$ be the set of attributes, and $\omega=(\omega_1,\omega_2,\ldots,\omega_m)^T$ be the weight vector of attributes, where $\omega_i\ge0$, $i=1,2,\ldots,m$, $\sum_{i=1}^m\omega_i=1$. Let $U=\{u_1,u_2,\ldots,u_l\}$ be a set of decision makers, and $V=\{v_1,v_2,\ldots,v_l\}$ be the set of importance degrees or characters of decision makers u_k $(k=1,2\ldots,l)$. Suppose that $A^{(k)}=(a_{ij}^{(k)})_{m\times n}$ is the linguistic decision matrix, where $a_{ij}^{(k)}\in\bar{S}$ is preference value, which takes the form of linguistic variables, given by the decision maker $u_k\in U$, for the alternative $x_j\in X$ with respect to the attribute $G_i\in G$, for all $k=1,2,\ldots,l$; $i=1,2,\ldots,m$; $j=1,2,\ldots,n$.

In the following, we apply the GILOWHM operator (whose exponential weighting vector $w = (w_1, w_2, \ldots, w_l)^T$, $w_k \geq 0$, $k = 1, 2, \ldots l$, $\sum_{k=1}^{l} w_k = 1$) and the LWHM operator to group decision making with linguistic information:

Procedure I.

Step 1: Utilize the GILOWHM operator

$$\begin{aligned} a_{ij} &= \mathrm{GILOWHM}_w(\langle v_1, u_1, a_{ij}^{(1)}\rangle, \langle v_2, u_2, a_{ij}^{(2)}\rangle, \ldots, \langle v_l, u_l, a_{ij}^{(l)}\rangle), \\ &= \frac{1}{\frac{w_1}{a_{ij}^{(1)}} \oplus \frac{w_2}{a_{ij}^{(2)}} \oplus \cdots \oplus \frac{w_l}{a_{ij}^{(l)}}}, \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n \end{aligned}$$

to aggregate all the decision matrices $A^{(k)}$ $(k = 1, 2, \ldots, l)$ into a collective decision matrix $A = (a_{ij})_{m\times n}$, where v_k $(k = 1, 2, \ldots l)$ are direct order inducing variables and u_k $(k = 1, 2, \ldots l)$ are indirect order inducing variables.

Step 2: Utilize the decision information given in matrix A, and the LWHM operator

$$\begin{aligned} a_j &= \mathrm{LWHM}_\omega(a_{1j}, a_{2j}, \ldots, a_{mj}) \\ &= \frac{1}{\frac{\omega_1}{a_{1j}} \oplus \frac{\omega_2}{a_{2j}} \oplus \cdots \oplus \frac{\omega_m}{a_{mj}}}, \ j = 1, 2, \ldots, n \end{aligned}$$

to derive the collective overall preference values a_j of the alternative x_j, where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weight vector of attributes.

Step 3: Rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ and select the best one(s) in accordance with the collective overall preference values a_j $(j = 1, 2, , \ldots, n)$.

Step 4: End.

Now we consider the group decision making problems under interval uncertainty where all the attribute values are expressed in uncertain linguistic variables. The following notations are used to depict the considered problems:

Let X, G, ω, U and V be presented as above-mentioned, and let $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times m}$ be an uncertain linguistic decision matrix, where $\tilde{a}_{ij}^{(k)} \in \tilde{S}$ is preference value, which takes the form of uncertain linguistic variables, given by the decision maker $u_k \in U$, for the alternative $x_j \in X$ with respect to the attribute $G_i \in G$, for all $k = 1, 2, \ldots, l$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Similar to the Procedure I, a procedure for solving the above problems can be described as follows:

Procedure II.

Step 1: Utilize the GIULOWHM operator

$$\begin{aligned} a_{ij} &= \mathrm{GIULOWHM}_w(\langle v_1, u_1, \tilde{a}_{ij}^{(1)}\rangle, \langle v_2, u_2, \tilde{a}_{ij}^{(2)}\rangle, \ldots, \langle v_l, u_l, \tilde{a}_{ij}^{(l)}\rangle), \\ &= \frac{1}{\frac{w_1}{\tilde{a}_{ij}^{(1)}} \oplus \frac{w_2}{\tilde{a}_{ij}^{(2)}} \oplus \cdots \oplus \frac{w_l}{\tilde{a}_{ij}^{(l)}}}, \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n \end{aligned}$$

to aggregate all the decision matrices $\tilde{A}^{(k)}$ $(k = 1, 2, \ldots, l)$ into a collective decision matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$, where v_k $(k = 1, 2, \ldots l)$ are direct order inducing variables and u_k $(k = 1, 2, \ldots l)$ are indirect order inducing variables.

Step 2: Utilize the decision information given in matrix $\tilde{A}$, and the ULWHM operator

$$\begin{aligned}\tilde{a}_j &= \mathrm{ULWHM}_{\omega}(\tilde{a}_{1j}, \tilde{a}_{2j}, \ldots, \tilde{a}_{mj}) \\ &= \frac{1}{\frac{\omega_1}{\tilde{a}_{1j}} \oplus \frac{\omega_2}{\tilde{a}_{2j}} \oplus \cdots \oplus \frac{\omega_m}{\tilde{a}_{mj}}},\ j = 1, 2, \ldots, n\end{aligned}$$

to derive the collective overall preference values $\tilde{a}_j$ of the alternative x_j, where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weight vector of attributes.

Step 3: To rank these collective attribute values $\tilde{a}_i$ $(i = 1, 2, \ldots, n)$, we first compare each $\tilde{a}_i$ with all $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ by using (1). For simplicity, we let $p_{ij} = p(\tilde{a}_i \geq \tilde{a}_j)$, then we develop a complementary matrix as $\mathbf{P} = (p_{ij})_{n\times n}$, where:

$$p_{ij} \geq 0,\ p_{ij} + p_{ji} = 1,\ p_{ii} = 0.5,\ i, j = 1, 2, \ldots, n.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_i = \sum_{j=1}^{n} p_{ij},\quad i = 1, 2, \ldots, n.$$

Then we rank the $\tilde{a}_i$ $(i = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_i $(i = 1, 2, \ldots, n)$.

Step 4: Rank all the alternatives x_i $(i = 1, 2, \ldots, n)$ and select the best one(s) in accordance with the $\tilde{a}_i$ $(i = 1, 2, \ldots n)$.

Step 5: End.

5 Illustrative example

Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted by Herrera et al. [5]). There is a panel with five possible alternatives in which to invest the money: (1) x_1 is a car industry; (2) x_2 is a food company; (3) x_3 is a computer company; (4) x_4 is an arms company; (5) x_5 is a TV company.

The investment company must make a decision according to the following four attributes (suppose that the weight vector of four attributes is $\omega = (0.3, 0.4, 0.2, 0.1)^T$): (1) G_1 is the risk analysis; (2) G_2 is the growth analysis; (3) G_3 is the social-political impact analysis; (4) G_4 is the environmental impact analysis.

There is three decision makers u_k $(k = 1, 2, 3)$ to evaluate five alternatives as follows: u_1 is Anderson; u_2 is Smith; and u_3 is Brown, where $v_1 =$ No. 3, $v_2 =$ No. 2 and $v_3 =$ No. 1 are order positions of relative importance of decision

makers u_k $(k = 1, 2, 3)$, respectively. The five possible alternatives x_j $(j = 1, 2, 3, 4, 5)$ are evaluated using the linguistic scale:

$$\begin{aligned} S = \{&s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, \\ &s_4 = \text{slightly poor}, s_5 = \text{fair}, s_6 = \text{slightly good}, \\ &s_7 = \text{good}, s_8 = \text{very good}, s_9 = \text{extremely good}\}. \end{aligned}$$

by three decision makers under the above four attributes G_i $(i = 1, 2, 3, 4)$, and construct, respectively, the decision matrices $A^{(k)} = (a_{ij}^{(k)})_{4\times 5}$ $(k = 1, 2, 3)$ as listed in Tables 1-3.

Table 1: Linguistic decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_6	s_9	s_4	s_3	s_6
G_2	s_3	s_7	s_8	s_8	s_4
G_3	s_7	s_4	s_6	s_8	s_7
G_4	s_2	s_4	s_6	s_7	s_8

Table 2: Linguistic decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_6	s_8	s_4	s_7	s_3
G_2	s_3	s_6	s_8	s_8	s_4
G_3	s_7	s_4	s_6	s_7	s_9
G_4	s_2	s_3	s_4	s_6	s_8

Table 3: Linguistic decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_6	s_8	s_4	s_7	s_2
G_2	s_4	s_6	s_8	s_7	s_4
G_3	s_7	s_3	s_7	s_9	s_8
G_4	s_3	s_4	s_4	s_7	s_7

Now we utilize the proposed procedure I to prioritize these alternatives:

Step 1: Utilize the GILOWHM operator (whose weight vector is $w = (0.3, 0.4, 0.3)^T$)

$$\begin{aligned} a_{ij} = \text{GILOWHM}_w(&\langle v_1, u_1, a_{ij}^{(1)}\rangle, \langle v_2, u_2, a_{ij}^{(2)}\rangle, \langle v_3, u_3, a_{ij}^{(3)}\rangle), \\ &i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5 \end{aligned}$$

to aggregate all the decision matrices $A^{(k)}$ $(k = 1, 2, 3)$ into a collective decision matrix $A = (a_{ij})_{4\times 5}$ (Table 4).

Step 2: Utilize the decision information given in matrix A, and the LWHM operator

$$\begin{aligned} a_j &= \text{LWHM}_\omega(a_{1j}, a_{2j}, a_{3j}, a_{4j}) \\ &= \frac{1}{\frac{\omega_1}{a_{1j}} \oplus \frac{\omega_2}{a_{2j}} \oplus \frac{\omega_3}{a_{3j}} \oplus \frac{\omega_4}{a_{4j}}}, \ j = 1, 2, 3, 4, 5 \end{aligned}$$

Table 4: Collective linguistic decision matrix A

	x_1	x_2	x_3	x_4	x_5
G_1	$s_{6.0}$	$s_{8.3}$	$s_{4.0}$	$s_{5.0}$	$s_{3.0}$
G_2	$s_{3.2}$	$s_{6.3}$	$s_{8.0}$	$s_{7.7}$	$s_{4.0}$
G_3	$s_{7.0}$	$s_{3.6}$	$s_{6.3}$	$s_{7.8}$	$s_{8.0}$
G_4	$s_{2.2}$	$s_{3.5}$	$s_{4.4}$	$s_{6.6}$	$s_{7.7}$

to derive the collective overall preference values a_j of the alternative x_j:

$$a_1 = s_{4.02}, a_2 = s_{5.44}, a_3 = s_{5.57}, a_4 = s_{6.55}, a_5 = s_{4.20}.$$

Step 3: Rank all the alternatives x_j $(j = 1, 2, 3, 4, 5)$ and select the best one(s) in accordance with the collective overall preference values a_j $(j = 1, 2, 3, 4, 5)$:

$$x_4 \succ x_3 \succ x_2 \succ x_1 \succ x_5$$

thus the best alternative is x_4.

If three decision makers evaluate the performance of five companies x_j $(j = 1, 2, 3, 4, 5)$ according to attributes G_i $(i = 1, 2, 3, 4)$ by using the uncertain linguistic terms in the set $\tilde{S}$ and constructs, respectively, the uncertain linguistic decision matrices $\tilde{A}^{(k)}$ $(k = 1, 2, 3)$ as listed in Tables 5-7.

In such case, we can utilize the proposed procedure II to prioritize these alternatives as follows.

Step 1: Utilize the GIULOWHM operator (whose weight vector $w = (0.3, 0.4, 0.3)^T$)

$$\tilde{a}_{ij} = \text{GIULOWHM}_w(\langle v_1, u_1, \tilde{a}_{ij}^{(1)}\rangle, \langle v_2, u_2, \tilde{a}_{ij}^{(2)}\rangle, \langle v_3, u_3, \tilde{a}_{ij}^{(3)}\rangle), \\ i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5$$

to aggregate all the uncertain linguistic decision matrices $\tilde{A}^{(k)}$ $(k = 1, 2, 3)$ into a collective uncertain linguistic decision matrix $\tilde{A} = (a_{ij})_{4\times 5}$ (Table 8).

Step 2: Utilize the decision information given in matrix $\tilde{A}$, and the ULWHM operator

$$\begin{aligned} \tilde{a}_j &= \text{ULWHM}_\omega(\tilde{a}_{1j}, \tilde{a}_{2j}, \tilde{a}_{3j}, \tilde{a}_{4j}) \\ &= \frac{1}{\frac{\omega_1}{\tilde{a}_{1j}} \oplus \frac{\omega_2}{\tilde{a}_{2j}} \oplus \frac{\omega_3}{\tilde{a}_{3j}} \oplus \frac{\omega_4}{\tilde{a}_{4j}}}, \ j = 1, 2, 3, 4, 5 \end{aligned}$$

to derive the collective overall preference values $\tilde{a}_j$ of the alternative x_j:

$$\tilde{a}_1 = [s_{2.49}, s_{3.98}], \tilde{a}_2 = [s_{2.67}, s_{4.59}], \tilde{a}_3 = [s_{1.78}, s_{3.48}], \\ \tilde{a}_4 = [s_{2.36}, s_{4.09}], \tilde{a}_5 = [s_{2.77}, s_{4.61}].$$

Step 3: To rank these collective overall preference values $\tilde{a}_j$ $(j = 1, 2, 3, 4, 5)$, we first compare each $\tilde{a}_j$ with all $\tilde{a}_i$ $(i = 1, 2, 3, 4, 5)$ by using (1), and develop

Table 5: Uncertain linguistic decision matrix $\tilde{A}^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_5, s_7]$	$[s_7, s_9]$	$[s_2, s_4]$	$[s_3, s_5]$	$[s_4, s_6]$
G_2	$[s_2, s_3]$	$[s_6, s_7]$	$[s_7, s_9]$	$[s_3, s_5]$	$[s_4, s_6]$
G_3	$[s_2, s_4]$	$[s_5, s_6]$	$[s_1, s_3]$	$[s_6, s_7]$	$[s_4, s_5]$
G_4	$[s_3, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_3, s_4]$

Table 6: Uncertain linguistic decision matrix $\tilde{A}^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_6, s_7]$	$[s_8, s_9]$	$[s_1, s_2]$	$[s_3, s_5]$	$[s_1, s_3]$
G_2	$[s_2, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_4]$	$[s_4, s_5]$
G_3	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$	$[s_2, s_4]$	$[s_5, s_6]$
G_4	$[s_3, s_5]$	$[s_4, s_6]$	$[s_2, s_3]$	$[s_1, s_3]$	$[s_4, s_6]$

Table 7: Uncertain linguistic decision matrix $\tilde{A}^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_6, s_8]$	$[s_6, s_8]$	$[s_1, s_3]$	$[s_2, s_3]$	$[s_4, s_5]$
G_2	$[s_3, s_4]$	$[s_1, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_3, s_4]$
G_3	$[s_1, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$
G_4	$[s_2, s_3]$	$[s_2, s_4]$	$[s_4, s_5]$	$[s_1, s_2]$	$[s_2, s_4]$

Table 8: Collective uncertain linguistic decision matrix $\tilde{A}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_{5.66}, s_{7.27}]$	$[s_{7.00}, s_{8.67}]$	$[s_{1.18}, s_{2.67}]$	$[s_{2.61}, s_{4.17}]$	$[s_{1.82}, s_{4.11}]$
G_2	$[s_{2.22}, s_{3.64}]$	$[s_{1.82}, s_{3.62}]$	$[s_{3.98}, s_{5.77}]$	$[s_{2.50}, s_{4.26}]$	$[s_{3.64}, s_{4.88}]$
G_3	$[s_{1.54}, s_{2.67}]$	$[s_{2.78}, s_{4.11}]$	$[s_{1.18}, s_{2.50}]$	$[s_{3.08}, s_{4.93}]$	$[s_{3.92}, s_{4.96}]$
G_4	$[s_{2.61}, s_{3.92}]$	$[s_{2.50}, s_{4.14}]$	$[s_{2.67}, s_{3.95}]$	$[s_{1.18}, s_{2.61}]$	$[s_{2.86}, s_{4.62}]$

a complementary matrix:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.384 & 0.690 & 0.503 & 0.363 \\ 0.616 & 0.500 & 0.776 & 0.611 & 0.484 \\ 0.310 & 0.224 & 0.500 & 0.327 & 0.201 \\ 0.497 & 0.389 & 0.673 & 0.500 & 0.370 \\ 0.637 & 0.516 & 0.799 & 0.630 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of the matrix $\mathbf{P}$, we have

$$p_1 = 2.440,\ p_2 = 2.987,\ p_3 = 1.562,\ p_4 = 2.429,\ p_5 = 3.082$$

and then we rank $\tilde{a}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_j $(j = 1, 2, 3, 4, 5)$:

$$\tilde{a}_5 > \tilde{a}_2 > \tilde{a}_1 > \tilde{a}_4 > \tilde{a}_3.$$

Step 4. Rank all alternatives x_j $(j = 1, 2, 3, 4, 5)$ by the ranking $\tilde{a}_j$ $(j = 1, 2, 3, 4, 5)$:

$$x_5 \succ x_2 \succ x_1 \succ x_4 \succ x_3$$

and thus the most desirable alternative is x_5.

6 Conclusions

In this paper, we have defined the GILOWHM and GIULOWHM operators, by which each object processed consists of three components, where the first component represents the importance degree or character of the second component, and the second component is used to induce an ordering, through the first component, over the third components which are linguistic variables (or uncertain linguistic variables) and then aggregated. We have also shown that the ILOWHM operator and LOWHM operator are the special cases of the GILOWHM operator, and that the IULOWHM operator and the ULOWHM operator are the special cases of the GIULOWHM operator. In the process of aggregating information, these operators can avoid losing the original linguistic or uncertain linguistic information and thus ensure exactness and rationality of the aggregated results. Moreover, based on the GILOWHM and GIULOWHM operators respectively, we have developed two procedures for solving the MADM problems where all decision information about attribute values take the forms of linguistic variables or uncertain linguistic variables. To verify the effectiveness and practicality of the developed procedures, we have given an illustrative example.

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A GENERALIZED RANDOM NONLINEAR VARIATIONAL INCLUSION FOR MULTI-VALUED RANDOM OPERATORS IN A UNIFORMLY SMOOTH BANACH SPACE*

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Abstract. In this paper, we introduce and study the random nonlinear variational inclusion problem with multi-valued random operators. We define a random iterative algorithm for finding the approximate solutions of class of random variational inclusions and establish the convergence of random iterative sequence generated by proposed algorithm in a uniformly smooth Banach space. Our result in this paper improves and generalizes some know corresponding results in the literature.

1. Introduction

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in the optimization and control, economics and transportation equilibrium, engineering science and physical sciences. For these reasons, many existence result and iterative algorithms for various variational inclusion have been studied extensively many authors (see, for example, [10]). Very recently, Lan [23] first introduced a new concept of (A,η)-monotone operators, which generalizes the (H,η)-monotonicity, A-monotonicity and other existing monotone operators as special cases, and studied some properties of (A,η)-monotone operators and defined resolvent operators associated with (A,η)-monotone operators. Lan et al. [24] introduced a new class of general nonlinear random multi-valued operator equations involving generalized m-accretive mappings in Banach spaces. By using the Chang's lemma and the resolvent operator technique for generalized m-accretive mapping due to Huang et al [18], they also prove the existence theorems of the solution and convergence theorems of the generalized random iterative procedures with errors for this nonlinear random multi-valued operator equations in q-uniformly smooth Banach spaces. In the past years, many existence results and random iterative algorithms for various random variational inequality and random variational inclusion problems have been studied. For more details, see [8, 11, 12, 13, 22, 21, 30] and the references therein.

Motivated and inspired by the recent research works in this fascinating area, the purpose of this paper is to introduce and study a new class of general nonlinear multi-valued random operators with generalized m-accretive operators in Banach spaces, give the notion of proximal random operators associated with the A-monotone random operators. An iterative algorithm is defined to compute the approximate solutions of random variational inclusion problems. The convergence of the iterative sequences generated by the proposed algorithm is also shown.

2. Preliminaries

Throughout this paper, let $(\Omega,\mathcal{A},\mu)$ be a complete σ-finite measure space and E be a separable real Banach space. We denote by $\mathcal{B}(E)$, $(\cdot,\cdot)$ and $\|\cdot\|$ the class of Borel σ-fields in E, the inner product and the norm on E, respectively. In the sequel, we denote 2^E, $CB(E)$ and H by

$2^E=\{A:A\in E\}$, $CB(E)=\{A\subset E:A$ is nonempty, bounded and closed$\}$ and the Hausdorff metric on $CB(E)$ respectively.

Next, we will use the following definitions and lemmas.

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Definition 2.1. A operator $x : \Omega \longrightarrow E$ is said to be *measurable* if, for any $B \in \mathcal{B}(E)$, $\{t \in \Omega : x(t) \in B\} \in \mathcal{A}$.

Definition 2.2. A operator $F : \Omega \times E \longrightarrow E$ is called a *random operator* if for any $x \in E$, $F(t, x) = y(t)$ is measurable. A random operator F is said to be *continuous* (resp. *linear, bounded*) if for any $t \in \Omega$, the operator $F(t, \cdot) : E \longrightarrow E$ is continuous (resp. linear, bounded).

Similarly, we can define a random operator $a : \Omega \times E \times E \longrightarrow E$. We shall write $F_t(x) = F(t, x(t))$ and $a_t(x, y) = a(t, x(t), y(t))$ for all $t \in \Omega$ and $x(t)$, $y(t) \in E$.

It is well known that a measurable operator is necessarily a random operator.

Definition 2.3. A multi-valued operator $G : \Omega \longrightarrow 2^E$ is said to be *measurable* if, for any $B \in \mathcal{B}(E)$, $G^{-1}(B) = \{t \in \Omega : G(t) \cap B \neq \emptyset\} \in \mathcal{A}$.

Definition 2.4. A operator $u : \Omega \longrightarrow E$ is called a *measurable selection* of a multi-valued measurable operator $\Gamma : \Omega \longrightarrow 2^E$ if u is measurable and for any $t \in \Omega$, $u(t) \in \Gamma(t)$.

Lemma 2.5. [4] *Let $M : \Omega \times E \longrightarrow CB(E)$ be a H-continuous random multi-valued operator. Then, for any measurable operator $x : \Omega \longrightarrow E$, the multi-valued operator $M(\cdot, x(\cdot)) : \Omega \longrightarrow CB(E)$ is measurable.*

Lemma 2.6. [4] *Let $M, V : \Omega \times E \longrightarrow CB(E)$ be two measurable multi-valued operators, $\epsilon > 0$ be a constant and $x : \Omega \longrightarrow E$ be a measurable selection of M. Then there exists a measurable selection $y : \Omega \longrightarrow E$ of V such that, for any $t \in \Omega$,*

$$\|x(t) - y(t)\| \leq (1 + \epsilon)H(M(t), V(t)).$$

Definition 2.7. A multi-valued operator $F : \Omega \times E \longrightarrow 2^E$ is called a *random multi-valued operator* if, for any $x \in E$, $F(\cdot, x)$ is measurable. A random multi-valued operator $F : \Omega \times E \longrightarrow CB(E)$ is said to be *H-continuous* if, for any $t \in \Omega$, $F(t, \cdot)$ is continuous in $H(\cdot, \cdot)$, where $H(\cdot, \cdot)$ is *the Hausdorff metric* on $CB(E)$ defined as follows: for any given $A, B \in CB(E)$,

$$H(A, B) = \max\Big\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y)\Big\}.$$

Suppose that $f, g, P, S : \Omega \times E \to E$ and $M : \Omega \times E \times E \to 2^E$ with $Im(g) \cap dom(M(t, \cdot, s)) \neq \emptyset$, $K, T, G : \Omega \times E \to 2^E$ are three multi-valued operators. Let $N : \Omega \times E \times E \times E \longrightarrow E$ be single-valued. Now, we consider the following problem:
Find measurable mappings $x, u, v, w : \Omega \to E$ such that $u(t) \in K(t, x(t))$, $v(t) \in T(t, x(t))$, $w(t) \in G(t, x(t))$ and

$$0 \in M(t, g(t, x(t), w(t)) + N(t, S(t, x(t)), u(t), v(t)) + P(t, u(t)) - \{f(t, v(t)) - g(t, x(t))\}. \tag{2.1}$$

The inequality (2.1) is called *random variational inclusion problem with random multi-valued operators*. The set of measurable mappings (x, u, v, w) is called a random solution of (2.1).
Some special cases of the problem (2.1):
(1) If $P \equiv 0$ and $f(t, v(t)) = g(t, x(t))$ for all $t \in \Omega$, then problem (2.1) is equivalent to finding $x, v, w : \Omega \longrightarrow E$ such that $v(t) \in T(t, x(t))$, $w(t) \in G(t, x(t))$ and

$$0 \in N(t, S(t, x(t)), u(t), v(t)) + M(t, g(t, x(t)), w(t)) \tag{2.2}$$

for all $t \in \Omega$ and $u(t) \in M(t, x(t))$. Problem (2.2) is introduced and studied by Lan et al, [24].
If G is a single-valued operator, $g \equiv I$, the identity mapping and $N(t, x, y, z) = \widetilde{f}(t, z) + \widetilde{g}(t, x, y)$ for all $t \in \Omega$ and $x, y, z \in E$, then problem (2.1) is equivalent to finding $x, v : \Omega \longrightarrow E$ such that $v(t) \in T(t, x(t))$ and

$$0 \in \widetilde{f}(t, v(t)) + \widetilde{g}(t, S(t, x(t)), u(t)) + M(t, x(t), G(t, x(t))), \tag{2.3}$$

for all $t \in \Omega$ and $u \in M(t, x(t))$. The problem (2.3) was considered and studied by Agarwal et al [2]. when $G \equiv I$.

If $M(t, x, s) = M(t, x)$ for all $t \in \Omega$, $x, s \in E$ and, for all $t \in \Omega$, $M(t, \cdot) : E \longrightarrow 2^E$ is a generalized m-accretive mapping, then the problem (2.1) reduces to the following generalized nonlinear random multi-valued operator equation involving generalized m-accretive mapping in Banach spaces:

Find $x, v : \Omega \longrightarrow E$ such that $v(t) \in T(t, x(t))$ and

$$0 \in N(t, S(t, x(t)), u(t), v(t)) + M(t, g(t, x(t))) \tag{2.4}$$

for all $t \in \Omega$ and $u(t) \in M(t, x(t))$.

(2) If $E = H$, $N(t, S(t, x(t)), u(t), v(t)) = 0$ and for all $t \in \Omega$, then the problem (2.1) reduces to the following random variational inclusion problem in Hilbert space was considered and studied by Ahmad and Farajzadeh [1].

The generalized duality mapping $J_q : E \longrightarrow 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\}$$

for all $x \in E$,where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if E^* is strictly convex (see, for example, [31]). If $E = H$ is a Hilbert space, then J_2 becomes the identity mapping of H. In what follows we shall denote the single-valued generalized duality mapping by j_q.

The *modules of smoothness* of E is the function $\rho_E : [0, \infty) \longrightarrow [0, \infty)$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}\|x + y\| + \|x - y\| - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space E is called *uniformly smooth* if $\lim\limits_{t \longrightarrow 0} \frac{\rho_E(t)}{t} = 0$ and E is called *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_{E(t)} \leq ct^q$, where $q > 1$ is a real number.

It is well known that Hilbert spaces, L_p(or l_p) spaces, $1 < p < \infty$ and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q-uniformly smooth.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [33] proved the following result.

Lemma 2.8. *Let $q > 1$ be a given real number and E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that, for all $x, y \in E$ and $j_q(x) \in J_q(x)$, the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

Definition 2.9. A random operator $g : \Omega \times E \longrightarrow E$ is said to be:
(a) *α-strongly accretive* if there exists $j_2(x(t) - y(t)) \in J_2(x(t) - y(t))$ such that

$$\langle g_t(x) - g_t(y), j_2(x(t) - y(t))\rangle \geq \alpha(t)\|x(t) - y(t)\|^2$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$, where $\alpha(t) > 0$ is a real-valued random variable;
(b) *β-Lipschitz continuous* if there exists a real-valued random variable $\beta(t) > 0$ such that

$$\|g_t(x) - g_t(y)\| \leq \beta(t)\|x(t) - y(t)\|$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$.

Definition 2.10. Let $S : \Omega \times E \longrightarrow E$ be a random operator. A operator $N : \Omega \times E \times E \times E \longrightarrow E$ is said to be:
(a) *ϱ-strongly accretive* with respect to S in the first argument if there exists $j_2(x(t) - y(t)) \in J_2(x(t) - y(t))$ such that

$$\langle N_t(S_t(x), \cdot, \cdot) - N_t(S_t(y), \cdot, \cdot), j_2(x(t) - y(t))\rangle \geq \varrho(t)\|x(t) - y(t)\|^2$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$, where $\varrho(t) > 0$ is a real-valued random variable;
(b) *ϵ-Lipschitz continuous* in the first argument if there exists a real-valued random variable $\epsilon(t) > 0$ such that

$$\|N_t(x, \cdot, \cdot) - N_t(y, \cdot, \cdot)\| \leq \epsilon(t)\|x(t) - y(t)\|$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$.

Similarly, we can define the Lipschitz continuity in the second argument and third argument of $N(\cdot, \cdot, \cdot)$.

Definition 2.11. Let $\eta : \Omega \times E \times E \longrightarrow E^*$ be a random operator $A : \Omega \times E \longrightarrow E$ be a random operator and $M : \Omega \times E \longrightarrow 2^E$ be a random multi-valued operator. Then M is said to be:
(a) *η-accretive* if

$$\langle u(t) - v(t), \eta_t(x, y)\rangle \geq 0$$

for all $x(t)$, $y(t) \in E$, $u(t) \in M_t(x)$ and $v(t) \in M_t(y)$ where $M_t(z) = M(t, z(t))$,$\forall t \in \Omega$;
(b) *strictly η-accretive* if

$$\langle u(t) - v(t), \eta_t(x, y)\rangle \geq 0$$

for all $x(t)$, $y(t) \in E$, $u(t) \in M_t(x)$, $v(t) \in M_t(y)$ and $t \in \Omega$ and the equality holds if and only if $u(t) = v(t)$ for all $t \in \Omega$;
(c) *strongly η-accretive* if there exists a real-valued random variable $r(t) > 0$ such that, for any $t \in \Omega$,

$$\langle u(t) - v(t), \eta_t(x, y)\rangle \geq r(t)\|x(t) - y(t)\|^2$$

for all $x(t)$, $y(t) \in E$, $u(t) \in M_t(x)$, $v(t) \in M_t(y)$ and $t \in \Omega$;
(d) *generalized m-accretive* if M is η-accretive and $(A_t + \rho(t)M(t, \cdot))(E) = E$ for all $t \in \Omega$ and $\rho(t) > 0$.

Remark 2.12. If $E = E^* = H$ is a Hilbert space, then (a)-(d) of Definition 2.11 reduce to the definition of η-monotonicity, strict η-monotonicity, strong η-monotonicity and maximal η-monotonicity respectively; if E is uniformly smooth and $\eta(x, y) = j_2(x-y) \in J_2(x-y)$, then (a)-(d) of Definition 2.11 reduce to the definitions of accretive, strictly accretive, strongly accretive and m-accretive operators in uniformly smooth Banach spaces, respectively.

Definition 2.13. The operator $\eta : \Omega \times E \times E \longrightarrow E^*$ is said to be:
(a) *monotone* if

$$\langle x(t) - y(t), \eta_t(x, y)\rangle \geq 0$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$;
(b) *strictly monotone* if

$$\langle x(t) - y(t), \eta_t(x, y)\rangle \geq 0$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$ and the equality holds if and only if $x(t) = y(t)$ for all $t \in \Omega$;
(c) *δ-strongly monotone* if there exists a measurable function $\delta : \Omega \longrightarrow (0, \infty)$ such that

$$\langle x(t) - y(t), \eta_t(x, y)\rangle \geq \delta(t)\|x(t) - y(t)\|^2$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$;
(d) *τ-Lipschitz continuous* if there exists a real-valued random variable $\tau(t) > 0$ such that

$$\|\eta_t(x, y)\| \leq \tau(t)\|x(t) - y(t)\|$$

for all $x(t)$, $y(t) \in E$ and $t \in \Omega$.

Definition 2.14. A multi-valued measurable operator $T : \Omega \times E \longrightarrow CB(E)$ is said to be *γ-H-Lipschitz continuous* if there exists a measurable function $\gamma : \Omega \longrightarrow (0, +\infty)$ such that, for any $t \in \Omega$,

$$H(T_t(x), T_t(y)) \leq \gamma(t)\|x(t) - y(t)\|,$$

for all $x(t)$, $y(t) \in E$.

Definition 2.15. A random operator $A : \Omega \times E \to E$ is said to be
(i) *monotone*, if
$\langle A(t, x_1(t)) - A(t, x_2(t)), x_1(t) - x_2(t)\rangle \geq 0, \ \ \forall x_1(t), x_2(t) \in E, t \in \Omega$,
(ii) *r-strongly monotone*, if there exists a measurable function $r : \Omega \to (0, \infty)$ such that
$\langle A(t, x_1(t)) - A(t, x_2(t)), x_1(t) - x_2(t)\rangle \geq r(t)\|x_1(t) - x_2(t)\|^2 \ \ \forall x_1(t), x_2(t) \in E, t \in \Omega$,

Definition 2.16. Let $M : \Omega \times E \times E \longrightarrow 2^E$ be a generalized m-accretive random operator and $A : \Omega \times E \to E$ be r-strongly monotone random operator. Then the *proximal operator* $J^{\rho(t), A_t}_{M_{t(\cdot, x)}}$ is defined as follows:

$$J^{\rho(t), A_t}_{M_{t(\cdot, x)}}(z) = (A_t + \rho(t)M_t)^{-1}(z)$$

for all $t \in \Omega$ and $z \in E$, where $\rho : \Omega \longrightarrow (0, \infty)$ is a measurable function and $\eta : \Omega \times E \times E \longrightarrow E^*$ is a strictly monotone operator.

Lemma 2.17. [1] *Suppose that E is a reflexive Banach space with the dual space E^*. Let $A : \Omega \times E \to E^*$ be a r-strongly monotone mapping such that for each fixed $t \in \Omega$ and $M : \Omega \times E \to 2^{E^*}$ be a A-monotone mapping, then $J^{\rho(t),A_t}_{M(t)}(x) : \Omega \times E^* \to E$ is Lipschitz continuous with constant $\frac{1}{r(t)-\rho(t)m(t)}$, i.e.,*

$$\|J^{\rho(t),A_t}_{M(t)}(x) - J^{\rho(t),A_t}_{M(t)}(y)\| \leq \frac{1}{r(t)-\rho(t)m(t)}\|x-y\|, \ \ \forall x, y \in E^*,$$

where $\rho(t) \in \left(0, \frac{r(t)}{m(t)}\right)$ is a real valued random variable.

3. Random Iterative Algorithms

In this section, we suggest and analyze a new class of iterative methods and construct some new random iterative algorithms for solving the problems (2.1).

Lemma 3.1. *The set of measurable mapping $x, u, v, w : \Omega \to E$ a random solution of problem (2.1) if and only if for all $t \in \Omega$, and*

$$g_t(x) \quad = \quad J^{\rho(t),A_t}_{M_{t(\cdot,w)}}[A_t(g_t(x)) - \rho(t)\{P_t(u) - (f_t(v) - g_t(x)) + N_t(S_t(x), u, v)\}].$$

Proof. The proof directly follows from the definition of $J^{\rho(t),A_t}_{M_{t(\cdot,w)}}$ as follows:

$$\begin{aligned}
g_t(x) \quad &= \quad J^{\rho(t),A_t}_{M_{t(\cdot,w)}}[A_t(g_t(x)) - \rho(t)\{P_t(u) - (f_t(v) - g_t(x)) + N_t(S_t(x), u, v)\}] \\
&\Leftrightarrow \quad g_t(x) = (A_t + \rho(t)M_t)^{-1}[A_t(g_t(x)) - \rho(t)\{P_t(u) - (f_t(v) - g_t(x)) + N_t(S_t(x), u, v)\}] \\
&\Leftrightarrow \quad (A_t + \rho(t)M_t)g_t(x) = A_t(g_t(x)) - \rho(t)\{P_t(u) - (f_t(v) - g_t(x)) + N_t(S_t(x), u, v)\} \\
&\Leftrightarrow \quad \rho(t)M_t(g_t(x), w) = -\rho(t)\{P_t(u) - (f_t(v) - g_t(x)\} + N_t(S_t(x), u, v)\} \\
&\Leftrightarrow \quad 0 \in M_t(g_t(x), w) + P_t(u) - (f_t(v) - g_t(x)) + N_t(S_t(x), u, v).
\end{aligned}$$

□

Algorithm 3.2. *Let $M : \Omega \times E \times E \longrightarrow 2^E$ be a random multi-valued operator such that for each fixed $t \in \Omega$ and $s \in E$, $M(t, \cdot, s) : E \longrightarrow 2^E$ be a generalized m-accretive mapping and $Range(g) \bigcap domM(t, \cdot, s) \neq \emptyset$. Let $S, g : \Omega \times E \longrightarrow E$, $\eta : \Omega \times E \times E \longrightarrow E$ and $N : \Omega \times E \times E \times E \longrightarrow E$ be single-valued operators, $K, T, G : \Omega \times E \longrightarrow 2^E$ be multi-valued operators and $\lambda : \Omega \longrightarrow (0, 1]$ be a measurable step size function. Then, by Lemma 2.5 and Himmelberg [14], we know that, for given $x_0(\cdot) \in E$, the multi-valued operators $K(\cdot, x_0(\cdot)), T(\cdot, x_0(\cdot))$ and $G(\cdot, x_0(\cdot))$ are measurable and there exist measurable selections $u_0(\cdot) \in K(\cdot, x_0(\cdot)), v_0(\cdot) \in T(\cdot, x_0(\cdot))$ and $w_0(\cdot) \in G(\cdot, x_0(\cdot))$. Set*

$$x_1(t) = x_0(t) - \lambda(t)\Big\{g_t(x_0) - J^{\rho(t),A_t}_{M_{t(\cdot,w_0)}}[A_t(g_t(x_0)) - \rho(t)\{P_t(u_0) - (f_t(v_0) - g_t(x_0)) + N_t(S_t(x_0), u_0, v_0)\}]\Big\}$$

where ρ and A are the same as in Lemma 3.1. Then it is easy to know that $x_1 : \Omega \longrightarrow E$ is measurable. Since $u_0(t) \in K_t(x_0) \in CB(E), v_0(t) \in T_t(x_0) \in CB(E)$ and $w_0(t) \in G_t(x_0) \in CB(E)$, by Lemma 2.6, there exist measurable selections $u_1(t) \in K_t(x_1), v_1(t) \in T_t(x_1)$ and $w_1(t) \in G_t(x_1)$ such that for all $t \in \Omega$,

$$\begin{cases}
\|u_0(t) - u_1(t)\| \leq (1 + \frac{1}{1})H(K_t(x_0), K_t(x_1)), \\
\|v_0(t) - v_1(t)\| \leq (1 + \frac{1}{1})H(T_t(x_0), T_t(x_1)), \\
\|w_0(t) - w_1(t)\| \leq (1 + \frac{1}{1})H(G_t(x_0), G_t(x_1)).
\end{cases}$$

By induction, we can define the sequences $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ inductively satisfying

$$\begin{cases}
x_{n+1}(t) = x_n(t) - \lambda(t)\Big\{g_t(x_n) - J^{\rho(t),A_t}_{M_{t(\cdot,w_n)}}[A_t(g_t(x_n)) - \rho(t)\{P_t(u_n) - (f_t(v_n) - g_t(x_n)) + N_t(S_t(x_n), u_n, v_n)\}]\Big\} \\
u_n(t) \in M_t(x_n), \ \|u_n(t) - u_{n+1}(t)\| \leq (1 + \frac{1}{n+1})H(K_t(x_n), K_t(x_{n+1})), \\
v_n(t) \in T_t(x_n), \ \|v_n(t) - v_{n+1}(t)\| \leq (1 + \frac{1}{n+1})H(T_t(x_n), T_t(x_{n+1})), \\
w_n(t) \in G_t(x_n), \ \|w_n(t) - w_{n+1}(t)\| \leq (1 + \frac{1}{n+1})H(G_t(x_n), G_t(x_{n+1})).
\end{cases}$$

From Algorithm 3.2, we can get the following algorithms.

Algorithm 3.3. *Suppose that E, M, g, η, S, N, K, T, G and λ be the same as in Algorithm 3.2 and setting $P \equiv 0, f_t(v_n) = g_t(x_n)$ and $A \equiv I$ for all $t \in \Omega$. Then, for given measurable $x_0 : \Omega \longrightarrow E$, we have*

$$(3.1)\quad \begin{cases} x_{n+1}(t) = x_n(t) - \lambda(t)\{g_t(x_n) - J^{\rho(t)}_{M_t(\cdot,w_n)}[g_t(x_n) - \rho(t)N_t(S_t(x_n), u_n, v_n)]\} \\ u_n(t) \in K_t(x_n),\ \|u_n(t) - u_{n+1}(t)\| \le (1 + \frac{1}{n+1})H(K_t(x_n), K_t(x_{n+1})), \\ v_n(t) \in T_t(x_n),\ \|v_n(t) - v_{n+1}(t)\| \le (1 + \frac{1}{n+1})H(T_t(x_n), T_t(x_{n+1})), \\ w_n(t) \in G_t(x_n),\ \|w_n(t) - w_{n+1}(t)\| \le (1 + \frac{1}{n+1})H(G_t(x_n), G_t(x_{n+1})). \end{cases}$$

Algorithm 3.4. *Suppose that E, M, η, S, K, T and λ are the same as in Algorithm 3.2 and setting $P \equiv 0, f_t(v_n) = g_t(x_n)$ and $g, A \equiv I$ for all $t \in \Omega$. Let $G : \Omega \times E \longrightarrow E$ be a random single-valued operator and $N(t, x, y, z) = \widetilde{f}(t, z) + \widetilde{g}(t, x, y)$ for all $t \in \Omega$ and $x, y, z \in E$. Then, for given measurable $x_0 : \Omega \longrightarrow E$, we have*

$$\begin{cases} x_{n+1}(t) = (1 - \lambda(t))x_n(t) + \lambda(t)J^{\rho(t)}_{M_t(\cdot,G_t(x_n))}\{x_n(t) - \rho(t)[\widetilde{f}_t(v_n) + \widetilde{g}_t(S_t(x_n), u_n)]\} \\ u_n(t) \in K_t(x_n),\ \|u_n(t) - u_{n+1}(t)\| \le (1 + \frac{1}{n+1})H(K_t(x_n), K_t(x_{n+1})), \\ v_n(t) \in T_t(x_n),\ \|v_n(t) - v_{n+1}(t)\| \le (1 + \frac{1}{n+1})H(T_t(x_n), T_t(x_{n+1})), \end{cases}$$

Algorithm 3.5. *Let $M : \Omega \times E \longrightarrow 2^E$ be a random multi-valued operator such that for each fixed $t \in \Omega$, $M(t, \cdot) : E \longrightarrow 2^E$ is a generalized m-accretive mapping and $Range(g) \bigcap domM(t, \cdot) \neq \emptyset$. If S, g, η, N, K, T and λ are the same as in Algorithm 2.1 and $P \equiv 0, f_t(v_n) = g_t(x_n)$ and $A \equiv I$ for all $t \in \Omega$. Then for given measurable $x_0 : \Omega \longrightarrow E$, we have*

$$\begin{cases} x_{n+1}(t) = x_n(t) - \lambda(t)\{g_t(x_n) - J^{\rho(t)}_{M_t(\cdot)}[g_t(x_n) - \rho(t)N_t(S_t(x_n), u_n, v_n)]\} \\ u_n(t) \in K_t(x_n),\ \|u_n(t) - u_{n+1}(t)\| \le (1 + \frac{1}{n+1})H(K_t(x_n), K_t(x_{n+1})), \\ v_n(t) \in T_t(x_n),\ \|v_n(t) - v_{n+1}(t)\| \le (1 + \frac{1}{n+1})H(T_t(x_n), T_t(x_{n+1})). \end{cases}$$

4. Main Results

Theorem 4.1. *Let E be a q-uniformly smooth and separable Banach space, $M : \Omega \times E \times E \to 2^E$ be a random multi-valued mapping such that for each fixed $t \in \Omega$ and $s \in E$, $M(t, \cdot, s) : E \to 2^E$ is a generalized m-accretive mapping $g : \Omega \times E \to E$ is α-strongly accretive and $\mu_g(t)$-Lipschitz continuous and $Range(g) \cap dom(M(t, \cdot, s)) \neq \emptyset$, for all $t \in \Omega$. Let $\eta : \Omega \times E \times E \to E$ be a δ-strongly monotone and τ-Lipschitz continuous and $S : \Omega \times E \to E$ be a σ-Lipschitz continuous random operator. Let $N : \Omega \times E \times E \times E \to E$ be a ϱ-strongly accretive with respect to S and ϵ-Lipschitz continuous in the first argument, and μ-Lipschitz continuous in the second argument, ν-Lipschitz continuous in the third argument, respectively. Suppose that $A : \Omega \times E \to E$ is r-strongly monotone and $\mu_A(t)$-Lipschitz continuous. Let $f, P : \Omega \times E \to E$ be Lipschitz continuous random mappings with constants $\psi_f(t)$ and $\psi_p(t)$, respectively. Let $K, T, G : \Omega \times E \to CB(E)$ be H-Lipschitz continuous with constants $\mu_K(t), \mu_T(t)$ and $\mu_G(t)$ respectively. If there exist real-valued random variables $\rho(t) > 0$ and $\pi(t) > 0$ such that, for any $t \in \Omega$, $x, y, z \in E$,*

$$(4.1)\qquad \|J^{\rho(t),A_t}_{M_{t(\cdot,x)}}(z) - J^{\rho(t),A_t}_{M_{t(\cdot,y)}}(z)\| \le \pi(t)\|x - y\|$$

and the following conditions hold:

$$(4.2)\begin{cases} k(t) = \left(1 + \frac{\rho(t)}{r(t) - \rho(t)m(t)}\right)(1 - q\alpha(t) + c_q\beta(t)^q)^{\frac{1}{q}} + \pi(t)\mu_{H_G}(t) + \frac{\mu_A\psi_g}{r(t) - \rho(t)m(t)} < 1, \\ \left(\psi_p(t)\mu_K(t) + \psi_f(t)\mu_T(t) + (1 - q\rho(t)\varrho(t) + c_q\epsilon(t)^q\sigma(t)^q)^{\frac{1}{q}}\right) < \frac{(r(t) - \rho(t)m(t))(1 - k(t))}{\rho(t)} \end{cases}$$

where c_q is the same as in Lemma 2.8 for any $t \in \Omega$. If there exist real-valued random variables $\lambda(t)$, $\rho(t) > 0$ then there exist $x^*(t) \in E, u^*(t) \in K_t(x^*), v^*(t) \in T_t(x^*)$ and $w^*(t) \in G_t(x^*)$ such that $\big(x^*(t), u^*(t), v^*(t), w^*(t)\big)$ is solution of the problem (2.1) and

$$x_n(t) \to x^*(t), u_n(t) \to u^*(t), v_n(t) \to v^*(t), w_n(t) \to w^*(t)$$

as $n \to \infty$, where $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ are iterative sequences generated by Algorithm 3.2.

Proof. From Algorithm 3.2, Lemma 2.17 and (4.1), we compute

$$\|x_{n+1}(t) - x_n(t)\| \quad = \quad \|x_n(t) - \lambda(t)\{g_t(x_n) - J^{\rho(t),A_t}_{M_{t(\cdot,w_n)}}[A_t(g_t(x_n)) - \rho(t)\{P_t(u_n) - (f_t(v_n) - g_t(x_n)\}$$

$$
\begin{aligned}
&+N_t(S_t(x_n),u_n,v_n)]\}\\
&-x_{n-1}(t)-\lambda(t)\{g_t(x_{n-1})-J^{\rho(t),A_t}_{M_{t(\cdot,w_{n-1})}}[A_t(g_t(x_{n-1}))-\rho(t)\{P_t(u_{n-1})\\
&-(f_t(v_{n-1})-g_t(x_{n-1})\}+N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})]\}\|\\
\leq\ &\|x_n(t)-x_{n-1}(t)-\lambda(t)(g_t(x_n)-g_t(x_{n-1}))\|\\
&+\lambda(t)\|J^{\rho(t),A_t}_{M_{t(\cdot,w_n)}}[A_t(g_t(x_n))-\rho(t)\{P_t(u_n)-(f_t(v_n)-g_t(x_n)\\
&+N_t(S_t(x_n),u_n,v_n)]\\
&-J^{\rho(t),A_t}_{M_{t(\cdot,w_{n-1})}}[A_t(g_t(x_{n-1}))-\rho(t)\{P_t(u_{n-1})-(f_t(v_{n-1})-g_t(x_{n-1})\}\\
&+N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})]\|\\
\leq\ &\|x_n(t)-x_{n-1}(t)-\lambda(t)(g_t(x_n)-g_t(x_{n-1}))\|\\
&+\lambda(t)\|J^{\rho(t),A_t}_{M_{t(\cdot,w_n)}}[A_t(g_t(x_n))-\rho(t)\{P_t(u_n)-(f_t(v_n)-g_t(x_n)\\
&+N_t(S_t(x_n),u_n,v_n)]\\
&-J^{\rho(t),A_t}_{M_{t(\cdot,w_n)}}[A_t(g_t(x_{n-1}))-\rho(t)\{P_t(u_{n-1})-(f_t(v_{n-1})-g_t(x_{n-1}))\\
&+N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})]\|\\
&+\lambda(t)\|J^{\rho(t),A_t}_{M_{t(\cdot,w_n)}}[A_t(g_t(x_{n-1}))-\rho(t)\{P_t(u_{n-1})-(f_t(v_{n-1})-g_t(x_{n-1})\\
&+N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})]\\
&-J^{\rho(t),A_t}_{M_{t(\cdot,w_{n-1})}}[A_t(g_t(x_{n-1}))-\rho(t)\{P_t(u_{n-1})-(f_t(v_{n-1})-g_t(x_{n-1})\}\\
&+N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})]\|\\
\leq\ &\|x_n(t)-x_{n-1}(t)-\lambda(t)(g_t(x_n)-g_t(x_{n-1}))\|\\
&+\frac{\lambda(t)}{r(t)-\rho(t)m(t)}\|[A_t(g_t(x_n))-\rho(t)\{P_t(u_n)-(f_t(v_n)-g_t(x_n))+N_t(S_t(x_n),u_n,v_n)\}]\\
&-[A_t(g_t(x_{n-1}))-\rho(t)\{P_t(u_{n-1})-(f_t(v_{n-1})-g_t(x_{n-1}))+N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})]\}\|\\
&+\lambda(t)\pi(t)\|w_n-w_{n-1}\|\\
\leq\ &\|x_n(t)-x_{n-1}(t)-\lambda(t)(g_t(x_n)-g_t(x_{n-1}))\|\\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\|(g_t(x_n)-g_t(x_{n-1}))+N_t(S_t(x_n),u_n,v_n)-N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})\|\\
&+\frac{\lambda(t)}{r(t)-\rho(t)m(t)}\|A_t(g_t(x_n))-A_t(g_t(x_{n-1}))\|\\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\|P_t(u_n)-P_t(u_{n-1})\|\\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\|f_t(v_n)-f_t(v_{n-1})\|\\
&+\lambda(t)\pi(t)\|w_n-w_{n-1}\|\\
\leq\ &\|x_n(t)-x_{n-1}(t)-\lambda(t)(g_t(x_n)-g_t(x_{n-1}))\|\\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\Big\{\|x_n(t)-x_{n-1}(t)-(g_t(x_n)-g_t(x_{n-1}))\|\\
&+\|x_n(t)-x_{n-1}(t)+\big(N_t(S_t(x_n),u_n,v_n)-N_t(S_t(x_{n-1}),u_n,v_n)\big)\|\\
&+\|N_t(S_t(x_{n-1}),u_n,v_n)-N_t(S_t(x_{n-1}),u_{n-1},v_n)\|\\
&+\|N_t(S_t(x_{n-1}),u_{n-1},v_n)-N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})\Big\}\\
&+\frac{\lambda(t)}{r(t)-\rho(t)m(t)}\|A_t(g_t(x_n))-A_t(g_t(x_{n-1}))\|\\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\|P_t(u_n)-P_t(u_{n-1}\|\\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\|f_t(v_n)-f_t(v_{n-1})\|
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda(t)\pi(t)\|w_n - w_{n-1}\| \\
\leq\ & (1-\lambda(t))\|x_n(t) - x_{n-1}(t)\| \\
&+\lambda(t)\big(1+\frac{\rho(t)}{r(t)-\rho(t)m(t)}\big)\|x_n(t) - x_{n-1}(t) - (g_t(x_n) - g_t(x_{n-1}))\| \\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\Big\{\|x_n(t) - x_{n-1}(t) + \big(N_t(S_t(x_n),u_n,v_n) - N_t(S_t(x_{n-1}),u_n,v_n)\big)\| \\
&+\|N_t(S_t(x_{n-1}),u_n,v_n) - N_t(S_t(x_{n-1}),u_{n-1},v_n)\| \\
&+\|N_t(S_t(x_{n-1}),u_{n-1},v_n) - N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})\|\Big\} \\
&+\frac{\lambda(t)}{r(t)-\rho(t)m(t)}\|A_t(g_t(x_n)) - A_t(g_t(x_{n-1}))\| \\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\|P_t(u_n) - P_t(u_{n-1})\| \\
&+\frac{\lambda(t)\rho(t)}{r(t)-\rho(t)m(t)}\|f_t(v_n) - f_t(v_{n-1})\| \\
&+\lambda(t)\pi(t)\|w_n - w_{n-1}\|. \qquad (4.3)
\end{aligned}
$$

By using g_t is a strongly accretive and Lipschitz continuous, we have

$$
\begin{aligned}
&\|x_n(t) - x_{n-1}(t) - (g_t(x_n) - g_t(x_{n-1}))\|^q \\
&\leq \|x_n(t) - x_{n-1}(t)\|^q - q\langle(g_t(x_n) - g_t(x_{n-1})), j_q(x_n(t) - x_{n-1}(t))\rangle \\
&\quad +c_q\|g_t(x_n) - g_t(x_{n-1})\|^q \\
&\leq (1 - q\alpha(t) + c_q\beta(t)^q)\|x_n(t) - x_{n-1}(t)\|^q,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\|x_n(t) - x_{n-1}(t) - (g_t(x_n) - g_t(x_{n-1}))\| \\
&\leq (1 - q\alpha(t) + c_q\beta(t)^q)^{\frac{1}{q}}\|x_n(t) - x_{n-1}(t)\|, \qquad (4.4)
\end{aligned}
$$

where c_q is the same as in Lemma 2.8. Also from the strongly accretivity of N with respect to S and the Lipschitz continuity of N in the first argument, we obtain

$$
\begin{aligned}
&\|x_n(t) - x_{n-1}(t) - (N_t(S_t(x_n),u_n,v_n) - N_t(S_t(x_{n-1}),u_n,v_n))\| \\
&\leq (1 - q\varrho(t) + c_q\epsilon(t)^q\sigma(t)^q)^{\frac{1}{q}}\|x_n(t) - x_{n-1}(t)\|. \qquad (4.5)
\end{aligned}
$$

By Lipschitz continuity of N in the second and third argument and since K, T, G are H-Lipschitz continuous and f, g, A and P are Lipschitz continuous, we have

$$
\begin{aligned}
\|N_t(S_t(x_{n-1}),u_n,v_n)) - N_t(S_t(x_{n-1}),u_{n-1},v_n)\| &\leq \mu(t)\|u_n - u_{n-1}\| \\
&\leq \mu(t)(1+\frac{1}{n})H(K_t(x_{n-1}),K_t(x_n)) \\
&\leq (1+\frac{1}{n})\mu(t)\mu_K(t)\|x_n(t) - x_{n-1}(t)\|, \qquad (4.6) \\
\|N_t(S_t(x_{n-1}),u_{n-1},v_n) - N_t(S_t(x_{n-1}),u_{n-1},v_{n-1})\| &\leq \nu(t)\|v_n - v_{n-1}\| \\
&\leq \nu(t)(1+\frac{1}{n})H(T_t(x_{n-1}),T_t(x_n)) \\
&\leq (1+\frac{1}{n})\nu(t)\mu_T(t)\|x_n(t) - x_{n-1}(t)\|, \qquad (4.7) \\
\|w_n - w_{n-1}\| &\leq (1+\frac{1}{n})H(G_t(x_{n-1}),G_t(x_n)) \\
&\leq (1+\frac{1}{n})\mu_G(t)\|x_n(t) - x_{n-1}(t)\|, \qquad (4.8) \\
\|P_t(u_n) - P_t(u_{n-1}(t))\| &\leq \psi_p(t)\|u_n(t) - u_{n-1}(t)\| \\
&\leq \psi_p(t)\mu_K(t)(1+\frac{1}{n})\|x_n(t) - x_{n-1}(t)\|, \qquad (4.9) \\
\|f_t(v_n(t)) - f_t(v_{n-1}(t))\| &\leq \psi_f(t)\|v_n(t) - v_{n-1}(t)\|
\end{aligned}
$$

$$\begin{aligned} &\leq \psi_f(t)\mu_T(t)(1+\frac{1}{n})\|x_n(t)-x_{n-1}(t)\|, \quad (4.10)\\ \|A_t(g_t(x_n))-A_t(g_t(x_{n-1}))\| &\leq \mu_A\|g_t(x_n)-g_t(x_{n-1})\| \\ &\leq \mu_A\psi_g\|x_n(t)-x_{n-1}(t)\|. \quad (4.11)\end{aligned}$$

Using (4.4)-(4.11) in (4.3), we obtain for all $t \in \Omega$,

$$\|x_{n+1}(t)-x_n(t)\| \leq \theta_n(t)\|x_n(t)-x_{n-1}(t)\| \quad (4.12)$$

where

$$\begin{aligned}\theta_n(t) &= 1-\lambda(t)+\lambda(t)\kappa_n(t),\\ \kappa_n(t) &= \Big(1+\frac{\rho(t)}{r(t)-\rho(t)m(t)}\Big)(1-q\alpha(t)+c_q\beta(t)^q)^{\frac{1}{q}}\\ &\quad+\frac{\rho(t)}{r(t)-\rho(t)m(t)}(1-q\varrho(t)+c_q\epsilon(t)^q\sigma(t)^q)^{\frac{1}{q}}+\frac{\mu_A\psi_g}{r(t)-\rho(t)m(t)}\\ &\quad+(1+\frac{1}{n})\frac{\rho(t)}{r(t)-\rho(t)m(t)}\Big(\psi_p(t)\mu_K(t)+\psi_f(t)\mu_T(t)\Big)+(1+\frac{1}{n})\pi(t)\mu_G(t)\end{aligned}$$

Letting

$$\begin{aligned}\kappa(t) &= \Big(1+\frac{\rho(t)}{r(t)-\rho(t)m(t)}\Big)(1-q\alpha(t)+c_q\beta(t)^q)^{\frac{1}{q}}+\pi(t)\mu_{H_G}(t)+\frac{\mu_A\psi_g}{r(t)-\rho(t)m(t)}\\ &\quad+\frac{\rho(t)}{r(t)-\rho(t)m(t)}\Big(\psi_p(t)\mu_K(t)+\psi_f(t)\mu_T(t)+(1-q\varrho(t)+c_q\epsilon(t)^q\sigma(t)^q)^{\frac{1}{q}}\Big),\\ \theta(t) &= 1-\lambda(t)+\lambda(t)\kappa(t).\end{aligned}$$

Thus $\kappa_n(t) \longrightarrow \kappa(t)$, $\theta_n(t) \longrightarrow \theta(t)$ as $n \longrightarrow \infty$. From the condition (4.2), we know that $0 < \theta(t) < 1$, for all $t \in \Omega$. Using the same arguments as those used in the proof in [Lan et al, Theorem 3.1, pp 14, [24]] it follow that $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ are Cauchy sequences. Thus by the completeness of E, there exist $u^*(t), v^*(t), w^*(t) \in E$ such that $u_n(t) \to u^*(t), v_n(t) \to v^*(t), w_n(t) \to w^*(t)$ as $n \to \infty$. Next, we show that $u^*(t) \in K_t(x^*)$, we have

$$\begin{aligned} d(u^*(t), K_t(x^*)) &= \inf\{\|u^*(t)-y\| : y \in K_t(x^*)\}\\ &\leq \|u^*(t)-u_n(t)\|+d(u_n(t), K_t(x^*))\\ &\leq \|u^*(t)-u_n(t)\|+H(K_t(x_n), K_t(x^*))\\ &\leq \|u^*(t)-u_n(t)\|+\mu_K(t)\|x_n(t)-x^*(t)\| \longrightarrow 0.\end{aligned}$$

This implies that $u^*(t) \in K_t(x^*)$. Similarly, we can get $v^*(t) \in T_t(x^*)$ and $w^*(t) \in G_t(x^*)$ for all $t \in \Omega$. Therefore, from algorithm 3.2 and the continuity of $J^{\rho(t)}_{A_t(\cdot,w)}$, P, N and S, we obtain

$$g_t(x) = J^{\rho(t),A_t}_{M_{t(\cdot,w)}}[A_t(g_t(x))-\rho(t)\{P_t(u)-(f_t(v)-g_t(x))+N_t(S_t(x),u,v)\}]$$

By Lemma 3.1, we know that $(x^*(t), u^*(t), v^*(t), w^*(t))$ is a solution of the problem (2.1). This completes the proof. □

If $P \equiv 0$, $f(t, v(t)) = g(t, x(t))$ and $A \equiv I$, the identity mapping, in Theorem 4.1, then we can get the following theorems.

Theorem 4.2. *Let E, M, η, g, N, S, K, T, G and λ be the same as in Theorem 4.1. Assume that $M : \Omega \times E \times E \longrightarrow 2^E$ is a random multi-valued operator such that for each fixed $t \in \Omega$ and $s \in E$, $M(t, \cdot, s) : E \longrightarrow 2^E$ is a generalized m-accretive mapping. If there exist real-valued random variables $\rho(t) > 0$, $\pi(t) > 0$ and $\frac{\tau(t)}{\delta(t)} = \frac{1}{r(t)-\rho(t)m(t)}$ such that (4.1) holds and*

$$k(t) = \Big(1+\tau(t)\delta(t)^{-1}\Big)(1-q\alpha(t)+c_q\beta(t)^q)^{\frac{1}{q}}+\pi(t)\mu_{H_G}(t) < 1,$$

$$\Big(\rho(t)(\psi_p(t)\mu_K(t)+\psi_f(t)\mu_T(t))+(1-q\rho(t)\varrho(t)+c_q\epsilon(t)^q\sigma(t)^q)^{\frac{1}{q}}\Big) < \frac{\delta(t)(1-k(t))}{\tau(t)}, \quad (4.13)$$

where c_q is the same as in Lemma 2.8 for any $t \in \Omega$. the iterative sequences $\{x_n(t)\}$, $\{u_n(t)\}$ and $\{v_n(t)\}$ defined by Algorithm 3.3 converge strongly to the solution $(x^*(t), u^*(t), v^*(t))$ of the problem (2.2).

Theorem 4.3. *Let E, η, M, S, K, T and λ be the same as in Theorem 3.1. Assume that $M : \Omega \times E \times E \longrightarrow 2^E$ is a random multi-valued operator such that, for each fixed $t \in \Omega$ and $s \in E$, $M(t, \cdot, s) : E \longrightarrow 2^E$ is a generalized m-accretive mapping. Let $\widetilde{f} : \Omega \times E \longrightarrow E$ be ν-Lipschitz continuous, $S : \Omega \times E \longrightarrow E$ be a σ-Lipschitz continuous random operator, $G : \Omega \times E \longrightarrow E$ be ζ-Lipschitz continuous and $\widetilde{g} : \Omega \times E \times E \longrightarrow E$ be ϱ-strongly accretive with respect to S and ϵ-Lipschitz continuous in the first argument and μ-Lipschitz continuous in the second argument, respectively. If there exist real-valued random variables $\rho(t) > 0$ and $\pi(t) > 0$ such that (4.1) holds and*

$$\rho(t)(\psi_p(t)\mu_K(t) + \psi_f(t)\mu_T(t)) + (1 - q\rho(t)\varrho(t) + c_q\rho(t)^q\epsilon(t)^q\sigma(t)^q)^{\frac{1}{q}} < \frac{\delta(t)(1 - \pi(t)\zeta(t))}{\tau(t)},$$

for all $t \in \Omega$, where c_q is the same as in Lemma 1.1, then, for any $t \in \Omega$, the iterative sequences $\{x_n(t)\}$, $\{u_n(t)\}$ and $\{v_n(t)\}$ defined by Algorithm 3.5 converge strongly to the solution $(x^*(t), u^*(t), v^*(t))$ of the problem (2.3).

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Univariate sigmoidal neural network approximation

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Abstract

Here we study the univariate quantitative approximation of real and complex valued continuous functions on a compact interval or all the real line by quasi-interpolation sigmoidal neural network operators. This approximation is derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its high order derivative. Our operators are defined by using a density function induced by the logarithmic sigmoidal function. The approximations are pointwise and with respect to the uniform norm. The related feed-forward neural network is with one hidden layer.

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Keywords and Phrases: sigmoidal function, neural network approximation, quasi-interpolation operator, modulus of continuiuty, complex approximation.

1 Introduction

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x,

and σ is the activation function of the network. In many fundamental network models, the activation function is the sigmoidal function of logistic type.

It is well known that FNNs are universal approximators. Theoretically, any continuous function defined on a compact set can be approximated to any desired degree of accuracy by increasing the number of hidden neurons. It was proved by Cybenko [10] and Funahashi [12], that any continuous function can be approximated on a compact set with uniform topology by a network of the form $N_n(x)$, using any continuous, sigmoidal activation function. Hornik et al. in [14], have shown that any measurable function can be approached with such a network. Furthermore, these authors proved in [15], that any function of the Sobolev spaces can be approached with all derivatives. A variety of density results on FNN approximations to multivariate functions were later established by many authors using different methods, for more or less general situations: [17] by Leshno et al., [21] by Mhaskar and Micchelli, [9] by Chui and Li, [7] by Chen and Chen, [13] by Hahm and Hong, etc.

Usually these results only give theorems about the existence of an approximation. A related and important problem is that of complexity: determining the number of neurons required to guarantee that all functions belonging to a space can be approximated to the prescribed degree of accuracy ϵ.

Barron [4] shows that if the function is supposed to satisfy certain conditions expressed in terms of its Fourier transform, and if each of the neurons evaluates a sigmoidal activation function, then at most $O\left(\epsilon^{-2}\right)$ neurons are needed to achieve the order of approximation ϵ. Some other authors have published similar results on the complexity of FNN approximations: Mhaskar and Micchelli [22], Suzuki [23], Maiorov and Meir [19], Makovoz [20], Ferrari and Stengel [11], Xu and Cao [25], Cao et al. [6], etc.

The author in [1] and [2], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treates there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

For this article the author is greatly motivated by the very important article [8] by Z. Chen and F. Cao.

He does related to it work and much more beyond however [8] remains the initial point. So the author here performs univariate sigmoidal neural network approximations to continuous functions over compact intervals of the real line

or over the whole $\mathbb{R}$, then he extends his results to complex valued functions. All convergences here are with rates expressed via the modulus of continuity of the involved function or its high order derivative, and given by very tight Jackson type inequalities.

The author comes up with the correct precisely defined quasi-interpolation neural network operator related to compact intervals, and among others, improves results from [8]. Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish further properties of the basic density function defining our operators.

2 Background and Auxilliary results

We consider here the sigmoidal function of logarithmic type

$$s(x) = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R}.$$

It has the properties $\lim_{x\to+\infty} s(x) = 1$ and $\lim_{x\to-\infty} s(x) = 0$.

This function plays the role of an activation function in the hidden layer of neural networks, also has application in biology, demography, etc ([5, 16]).

As in [8], we consider

$$\Phi(x) := \frac{1}{2}\left(s(x+1) - s(x-1)\right), \quad x \in \mathbb{R}.$$

We notice the following properties:

i) $\Phi(x) > 0, \ \forall\, x \in \mathbb{R}$,

ii) $\sum_{k=-\infty}^{\infty} \Phi(x-k) = 1, \ \forall\, x \in \mathbb{R}$,

iii) $\sum_{k=-\infty}^{\infty} \Phi(nx-k) = 1, \ \forall\, x \in \mathbb{R};\ n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi(x)\,dx = 1$,

v) Φ is a density function,

vi) Φ is even: $\Phi(-x) = \Phi(x), \ x \geq 0$.

We see that ([8])

$$\Phi(x) = \left(\frac{e^2-1}{2e}\right)\frac{e^{-x}}{\left(1+e^{-x-1}\right)\left(1+e^{-x+1}\right)} = \left(\frac{e^2-1}{2e^2}\right)\frac{1}{\left(1+e^{x-1}\right)\left(1+e^{-x-1}\right)},$$

and

$$\Phi'(x) = \left(\frac{e^2-1}{2e^2}\right)\left[-\frac{(e^x-e^{-x})}{e\left(1+e^{x-1}\right)^2\left(1+e^{-x-1}\right)^2}\right] \leq 0, \quad x \geq 0.$$

Hence

vii) Φ is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$.

Let $0 < \alpha < 1$, $n \in \mathbb{N}$. We observe the following

$$\sum_{\substack{k=-\infty \\ :\, |nx-k| > n^{1-\alpha}}}^{\infty} \Phi(nx-k) = \sum_{\substack{k=-\infty \\ :\, |nx-k| > n^{1-\alpha}}}^{\infty} \Phi(|nx-k|) \leq$$

$$\left(\frac{e^2-1}{2e^2}\right)\int_{(n^{1-\alpha}-1)}^{\infty} \frac{1}{\left(1+e^{x-1}\right)\left(1+e^{-x-1}\right)}dx \leq$$

$$\left(\frac{e^2-1}{2e}\right)\int_{(n^{1-\alpha}-1)}^{\infty} e^{-x}dx = \left(\frac{e^2-1}{2e}\right)\left(e^{-\left(n^{1-\alpha}-1\right)}\right)$$

$$= \left(\frac{e^2-1}{2}\right)e^{-n^{(1-\alpha)}} = 3.1992e^{-n^{(1-\alpha)}}.$$

We have found that:

viii) for $n \in \mathbb{N}$, $0 < \alpha < 1$, we get

$$\sum_{\substack{k=-\infty \\ :\, |nx-k| > n^{1-\alpha}}}^{\infty} \Phi(nx-k) < \left(\frac{e^2-1}{2}\right)e^{-n^{(1-\alpha)}} = 3.1992e^{-n^{(1-\alpha)}}.$$

Denote by $\lceil \cdot \rceil$ the ceiling of a number, and by $\lfloor \cdot \rfloor$ the integral part of a number. Consider $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We observe that

$$1 = \sum_{k=-\infty}^{\infty} \Phi(nx-k) > \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(|nx-k|) > \Phi(|nx-k_0|),$$

for any $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$.

Here we can choose $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$ such that $|nx-k_0| < 1$.

Therefore $\Phi(|nx-k_0|) > \Phi(1) = 0.19046485$.

Consequently,

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k) > \Phi(1) = 0.19046485.$$

Therefore we obtain

ix) $\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)} < \frac{1}{\Phi(1)} = 5.250312578, \ \forall\, x \in [a,b]$.

We also notice that

$$1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nb-k) = \sum_{k=-\infty}^{\lceil na\rceil-1} \Phi(nb-k) + \sum_{k=\lfloor nb\rfloor+1}^{\infty} \Phi(nb-k)$$

$$> \Phi(nb - \lfloor nb\rfloor - 1)$$

(call $\varepsilon := nb - \lfloor nb\rfloor$, $0 \le \varepsilon < 1$)

$$= \Phi(\varepsilon - 1) = \Phi(1-\varepsilon) \ge \Phi(1) > 0.$$

Therefore $\lim_{n\to\infty} \left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nb-k)\right) > 0$.

Similarly,

$$1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(na-k) = \sum_{k=-\infty}^{\lceil na\rceil-1} \Phi(na-k) + \sum_{k=\lfloor nb\rfloor+1}^{\infty} \Phi(na-k)$$

$$> \Phi(na - \lceil na\rceil + 1)$$

(call $\eta := \lceil na\rceil - na$, $0 \le \eta < 1$)

$$= \Phi(1-\eta) \ge \Phi(1) > 0.$$

Therefore again $\lim_{n\to\infty} \left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(na-k)\right) > 0$.

Therefore we derive that

x) $\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k) \ne 1$, for at least some $x \in [a,b]$.

Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na\rceil \le \lfloor nb\rfloor$.

We introduce and define the positive linear neural network operator

$$G_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)}, \quad x \in [a.b]. \tag{1}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

We study here the pointwise convergence of $G_n(f,x)$ to $f(x)$ with rates.

For convinience we call

$$G_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k), \tag{2}$$

that is

$$G_n(f,x) := \frac{G_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}. \tag{3}$$

Thus,

$$G_n(f,x) - f(x) = \frac{G_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} - f(x)$$

$$= \frac{G_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}. \tag{4}$$

Consequently we derive

$$|G_n(f,x) - f(x)| \leq \frac{1}{\Phi(1)} \left| G_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right|. \tag{5}$$

That is

$$|G_n(f,x) - f(x)| \leq (5.250312578) \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \Phi(nx-k) \right|. \tag{6}$$

We will estimate the right hand side of (6).

For that we need, for $f \in C([a,b])$ the first modulus of continuity

$$\omega_1(f,h) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq h}} |f(x) - f(y)|, \quad h > 0. \tag{7}$$

Similarly it is defined for $f \in C_B(\mathbb{R})$ (continuous and bounded on $\mathbb{R}$). We have that $\lim_{h \to 0} \omega_1(f,h) = 0$.

When $f \in C_B(\mathbb{R})$ we define, (see also [8])

$$\overline{G}_n(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k), \quad n \in \mathbb{N}, x \in \mathbb{R}, \tag{8}$$

the quasi-interpolation neural network operator.

By [3] we derive the following three theorems on extended Taylor formula.

Theorem 1 *Let $N \in \mathbb{N}$, $0 < \varepsilon < \frac{\pi}{2}$ small, and $f \in C^N\left(\left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]\right)$; $x, y \in \left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]$. Then*

$$f(x) = f(y) + \sum_{k=1}^{N} \frac{\left(f \circ \sin^{-1}\right)^{(k)}(\sin y)}{k!} (\sin x - \sin y)^k + K_N(y, x), \tag{9}$$

where

$$K_N(y, x) = \frac{1}{(N-1)!} \cdot \tag{10}$$

$$\int_y^x (\sin x - \sin s)^{N-1} \left(\left(f \circ \sin^{-1}\right)^{(N)}(\sin s) - \left(f \circ \sin^{-1}\right)^{(N)}(\sin y)\right) \cos s \, ds.$$

Theorem 2 *Let $f \in C^N([\varepsilon, \pi - \varepsilon])$, $N \in \mathbb{N}$, $\varepsilon > 0$ small; $x, y \in [\varepsilon, \pi - \varepsilon]$. Then*

$$f(x) = f(y) + \sum_{k=1}^{N} \frac{\left(f \circ \cos^{-1}\right)^{(k)}(\cos y)}{k!} (\cos x - \cos y)^k + K_N^*(y, x), \tag{11}$$

where

$$K_N^*(y, x) = -\frac{1}{(N-1)!} \cdot \tag{12}$$

$$\int_y^x (\cos x - \cos s)^{N-1} \left[\left(f \circ \cos^{-1}\right)^{(N)}(\cos s) - \left(f \circ \cos^{-1}\right)^{(N)}(\cos y)\right] \sin s \, ds.$$

Theorem 3 *Let $f \in C^N([a, b])$ (or $f \in C^N(\mathbb{R})$), $N \in \mathbb{N}$; $x, y \in [a, b]$ (or $x, y \in \mathbb{R}$). Then*

$$f(x) = f(y) + \sum_{k=1}^{N} \frac{\left(f \circ \ln_{\frac{1}{e}}\right)^{(k)}(e^{-y})}{k!} \left(e^{-x} - e^{-y}\right)^k + \overline{K}_N(y, x), \tag{13}$$

where

$$\overline{K}_N(y, x) = -\frac{1}{(N-1)!} \cdot \tag{14}$$

$$\int_y^x \left(e^{-x} - e^{-s}\right)^{N-1} \left[\left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}\left(e^{-s}\right) - \left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}\left(e^{-y}\right)\right] e^{-s} ds.$$

Remark 1 *Using the mean value theorem we get*

$$\begin{aligned} &|\sin x - \sin y| \leq |x - y|, \\ &|\cos x - \cos y| \leq |x - y|, \quad \forall\ x, y \in \mathbb{R}, \end{aligned} \tag{15}$$

furthermore we have

$$\begin{aligned} &|\sin x - \sin y| \leq 2, \\ &|\cos x - \cos y| \leq 2, \quad \forall\ x, y \in \mathbb{R}. \end{aligned} \tag{16}$$

Similarly we get

$$\left|e^{-x}-e^{-y}\right| \leq e^{-a}\left|x-y\right|, \tag{17}$$

and

$$\left|e^{-x}-e^{-y}\right| \leq e^{-a}-e^{-b}, \quad \forall\ x,y\in[a,b]. \tag{18}$$

Let $g(x)=\ln_{\frac{1}{e}} x$, $\sin^{-1}x$, $\cos^{-1}x$ and assume $f^{(j)}(x_0)=0$, $k=1,...,N$. Then, by [3], we get $\left(f\circ g^{-1}\right)^{(j)}(g(x_0))=0$, $j=1,...,N$.

Remark 2 *It is well known that $e^x > x^m$, $m\in\mathbb{N}$, for large $x>0$.*

Let fixed $\alpha,\beta>0$, then $\left\lceil \frac{\alpha}{\beta}\right\rceil \in\mathbb{N}$, and for large $x>0$ we have

$$e^x > x^{\left\lceil \frac{\alpha}{\beta}\right\rceil} \geq x^{\frac{\alpha}{\beta}}.$$

So for suitable very large $x>0$ we get

$$e^{x^\beta} > \left(x^\beta\right)^{\frac{\alpha}{\beta}} = x^\alpha.$$

We proved for large $x>0$ and $\alpha,\beta>0$ that

$$e^{x^\beta} > x^\alpha. \tag{19}$$

Therefore for large $n\in\mathbb{N}$ and fixed $\alpha,\beta>0$, we have

$$e^{n^\beta} > n^\alpha. \tag{20}$$

That is

$$e^{-n^\beta} < n^{-\alpha},\ \text{for large } n\in\mathbb{N}. \tag{21}$$

So for $0<\alpha<1$ we get

$$e^{-n^{(1-\alpha)}} < n^{-\alpha}. \tag{22}$$

Thus be given fixed $A,B>0$, for the linear combination $\left(An^{-\alpha}+Be^{-n^{(1-\alpha)}}\right)$ the (dominant) rate of convergence to zero is $n^{-\alpha}$.

The closer α is to 1 we get faster and better rate of convergence to zero.

3 Real Neural network Approximations

Here we present a series of neural network approximations to a function given with rates.

We first give.

Theorem 4 *Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $x \in [a,b]$. Then*

i)

$$|G_n(f,x) - f(x)| \leq (5.250312578)\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 6.3984\|f\|_\infty e^{-n^{(1-\alpha)}}\right] =: \lambda, \tag{23}$$

and

ii)

$$\|G_n(f) - f\|_\infty \leq \lambda, \tag{24}$$

where $\|\cdot\|_\infty$ is the supremum norm.

Proof. We see that

$$\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right)\Phi(nx-k)\right| \leq$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|f\left(\frac{k}{n}\right) - f(x)\right|\Phi(nx-k) =$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left|f\left(\frac{k}{n}\right) - f(x)\right|\Phi(nx-k) +$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left|f\left(\frac{k}{n}\right) - f(x)\right|\Phi(nx-k) \leq$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\omega_1\left(f, \left|\frac{k}{n}-x\right|\right)\Phi(nx-k) +$$

$$2\|f\|_\infty \sum_{\substack{k=\lceil na\rceil \\ |k-nx| > n^{1-\alpha}}}^{\lfloor nb\rfloor}\Phi(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right)\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\infty}\Phi(nx-k) +$$

$$2\left\|f\right\|_{\infty}\sum_{\substack{k=-\infty\\ \left|k-nx\right|>n^{1-\alpha}}}^{\infty}\Phi\left(nx-k\right)\underset{\text{(by (viii))}}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+2\left\|f\right\|_{\infty}\left(3.1992\right)e^{-n^{(1-\alpha)}}$$

$$=\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+6.3984\left\|f\right\|_{\infty}e^{-n^{(1-\alpha)}}.$$

That is

$$\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f\left(x\right)\right)\Phi\left(nx-k\right)\right|\leq\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+6.3984\left\|f\right\|_{\infty}e^{-n^{(1-\alpha)}}.$$

Using (6) we prove the claim. ■

Theorem 4 improves a lot the model of neural network approximation of [8], see Theorem 3 there.

Next we present

Theorem 5 *Let $f\in C_B\left(\mathbb{R}\right)$, $0<\alpha<1$, $n\in\mathbb{N}$, $x\in\mathbb{R}$. Then*

i)

$$\left|\overline{G}_n\left(f,x\right)-f\left(x\right)\right|\leq\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+6.3984\left\|f\right\|_{\infty}e^{-n^{(1-\alpha)}}=:\mu,\tag{25}$$

and

ii)

$$\left\|\overline{G}_n\left(f\right)-f\right\|_{\infty}\leq\mu.\tag{26}$$

Proof. We observe that

$$\left|\overline{G}_n\left(f,x\right)-f\left(x\right)\right|=\left|\sum_{k=-\infty}^{\infty}f\left(\frac{k}{n}\right)\Phi\left(nx-k\right)-f\left(x\right)\sum_{k=-\infty}^{\infty}\Phi\left(nx-k\right)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{n}\right)-f\left(x\right)\right)\Phi\left(nx-k\right)\right|\leq\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{n}\right)-f\left(x\right)\right|\Phi\left(nx-k\right)=$$

$$\sum_{\substack{k=-\infty\\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\infty}\left|f\left(\frac{k}{n}\right)-f\left(x\right)\right|\Phi\left(nx-k\right)+$$

$$\sum_{\substack{k=-\infty\\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\infty}\left|f\left(\frac{k}{n}\right)-f\left(x\right)\right|\Phi\left(nx-k\right)\leq$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) \Phi(nx-k) +$$

$$2\|f\|_{\infty} \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}}}^{\infty} \Phi(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \Phi(nx-k) +$$

$$2\|f\|_{\infty} \sum_{\substack{k=-\infty \\ |k-nx| > n^{1-\alpha}}}^{\infty} \Phi(nx-k) \underset{\text{(by (viii))}}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) + 2\|f\|_{\infty} (3.1992)\, e^{-n^{(1-\alpha)}}$$

$$= \omega_1\left(f, \frac{1}{n^{\alpha}}\right) + 6.3984\|f\|_{\infty} e^{-n^{(1-\alpha)}},$$

proving the claim. ■

Theorem 5 improves Theorem 4 of [8].

In the next we discuss high order of approximation by using the smoothness of f.

Theorem 6 *Let $f \in C^N([a,b])$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$. Then*

i)

$$|G_n(f,x) - f(x)| \leq (5.250312578) \cdot \tag{27}$$

$$\left\{ \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \left[\frac{1}{n^{\alpha j}} + (3.1992)(b-a)^j e^{-n^{(1-\alpha)}} \right] + \right.$$

$$\left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!} + \frac{(6.3984)\left\|f^{(N)}\right\|_{\infty} (b-a)^N}{N!} e^{-n^{(1-\alpha)}} \right] \right\},$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$|G_n(f,x_0) - f(x_0)| \leq (5.250312578) \cdot \tag{28}$$

$$\left[\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!} + \frac{(6.3984)\left\|f^{(N)}\right\|_{\infty} (b-a)^N}{N!} e^{-n^{(1-\alpha)}} \right],$$

notice here the extremely high rate of convergence at $n^{-(N+1)\alpha}$,

iii)

$$\left\|G_n\left(f\right)-f\right\|_\infty \leq \left(5.250312578\right)\cdot \tag{29}$$

$$\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_\infty}{j!}\left[\frac{1}{n^{\alpha j}}+\left(3.1992\right)\left(b-a\right)^j e^{-n^{(1-\alpha)}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(6.3984\right)\left\|f^{(N)}\right\|_\infty\left(b-a\right)^N}{N!}e^{-n^{(1-\alpha)}}\right]\right\}.$$

Proof. Next we apply Taylor's formula with integral remainder.

We have (here $\frac{k}{n}, x\in[a,b]$)

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N}\frac{f^{(j)}\left(x\right)}{j!}\left(\frac{k}{n}-x\right)^j+\int_x^{\frac{k}{n}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Then

$$f\left(\frac{k}{n}\right)\Phi\left(nx-k\right)=\sum_{j=0}^{N}\frac{f^{(j)}\left(x\right)}{j!}\Phi\left(nx-k\right)\left(\frac{k}{n}-x\right)^j+$$

$$\Phi\left(nx-k\right)\int_x^{\frac{k}{n}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi\left(nx-k\right)-f\left(x\right)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)=$$

$$\sum_{j=1}^{N}\frac{f^{(j)}\left(x\right)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)\left(\frac{k}{n}-x\right)^j+$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)\int_x^{\frac{k}{n}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Thus

$$G_n^*\left(f,x\right)-f\left(x\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)\right)=\sum_{j=1}^{N}\frac{f^{(j)}\left(x\right)}{j!}G_n^*\left(\left(\cdot-x\right)^j\right)+\Lambda_n\left(x\right),$$

where

$$\Lambda_n\left(x\right):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)\int_x^{\frac{k}{n}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

We assume that $b-a>\frac{1}{n^{\alpha}}$, which is always the case for large enough $n\in\mathbb{N}$, that is when $n>\left\lceil (b-a)^{-\frac{1}{\alpha}}\right\rceil$.

Thus $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$ or $\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}$.

As in [2], pp. 72-73 for

$$\gamma:=\int_{x}^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt,$$

in case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$, we find that

$$|\gamma|\leq\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}$$

(for $x\leq\frac{k}{n}$ or $x\geq\frac{k}{n}$).

Notice also for $x\leq\frac{k}{n}$ that

$$\left|\int_{x}^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\int_{x}^{\frac{k}{n}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\leq$$

$$2\left\|f^{(N)}\right\|_{\infty}\int_{x}^{\frac{k}{n}}\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt=2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\frac{k}{n}-x\right)^{N}}{N!}\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}.$$

Next assume $\frac{k}{n}\leq x$, then

$$\left|\int_{x}^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|=$$

$$\left|\int_{\frac{k}{n}}^{x}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\int_{\frac{k}{n}}^{x}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!}dt\leq$$

$$2\left\|f^{(N)}\right\|_{\infty}\int_{\frac{k}{n}}^{x}\frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!}dt=2\left\|f^{(N)}\right\|_{\infty}\frac{\left(x-\frac{k}{n}\right)^{N}}{N!}\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}.$$

Thus

$$|\gamma|\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!},$$

in all two cases.

Therefore

$$\Lambda_n(x) = \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|\le\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k)\,\gamma + \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k)\,\gamma.$$

Hence

$$|\Lambda_n(x)| \le \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|\le\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k)\left(\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{N!n^{N\alpha}}\right) +$$

$$\left(\sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k)\right) 2\left\|f^{(N)}\right\|_\infty \frac{(b-a)^N}{N!} \le$$

$$\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{N!n^{N\alpha}} + 2\left\|f^{(N)}\right\|_\infty \frac{(b-a)^N}{N!}\,(3.1992)\,e^{-n^{(1-\alpha)}}.$$

Consequently we have

$$|\Lambda_n(x)| \le \omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + (6.3984)\,\frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}e^{-n^{(1-\alpha)}}.$$

We further see that

$$G_n^*\left((\cdot-x)^j\right)(x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)\left(\frac{k}{n}-x\right)^j.$$

Therefore

$$\left|G_n^*\left((\cdot-x)^j\right)(x)\right| \le \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)\left|\frac{k}{n}-x\right|^j =$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|\le\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k)\left|\frac{k}{n}-x\right|^j + \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k)\left|\frac{k}{n}-x\right|^j \le$$

$$\frac{1}{n^{\alpha j}}\sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|\le\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k) + (b-a)^j \sum_{\substack{k=\lceil na\rceil \\ |k-nx|>n^{1-\alpha}}}^{\lfloor nb\rfloor} \Phi(nx-k)$$

$$\le \frac{1}{n^{\alpha j}} + (b-a)^j\,(3.1992)\,e^{-n^{(1-\alpha)}}.$$

Hence

$$\left|G_n^*\left(\left(\cdot-x\right)^j\right)(x)\right| \leq \frac{1}{n^{\alpha j}} + (b-a)^j\,(3.1992)\,e^{-n^{(1-\alpha)}},$$

for $j = 1, ..., N$.

Putting things together we have proved

$$\left|G_n^*(f,x) - \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)\right) f(x)\right| \leq \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{1}{n^{\alpha j}} + (3.1992)\,(b-a)^j\,e^{-n^{(1-\alpha)}}\right] +$$

$$\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + \frac{(6.3984)\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}e^{-n^{(1-\alpha)}}\right],$$

that is establishing theorem. ■

We make

Remark 3 *We notice that*

$$G_n(f,x) - \sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}G_n\left(\left(\cdot-x\right)^j\right)(x) - f(x) =$$

$$\frac{G_n^*(f,x)}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)} - \frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)}\left(\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}G_n^*\left(\left(\cdot-x\right)^j\right)(x)\right)$$

$$-f(x) = \frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)}\cdot$$

$$\left[G_n^*(f,x) - \left(\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}G_n^*\left(\left(\cdot-x\right)^j\right)(x)\right) - \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)f(x)\right].$$

Therefore we get

$$\left|G_n(f,x) - \sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}G_n\left(\left(\cdot-x\right)^j\right)(x) - f(x)\right| \leq (5.250312578)\cdot$$

$$\left|G_n^*(f,x) - \left(\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}G_n^*\left(\left(\cdot-x\right)^j\right)(x)\right) - \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)f(x)\right|, \tag{30}$$

$\forall\ x \in [a,b]$.

In the next three Theorems 7-9 we present more general and flexible upper bounds to our error quantities.

We give

Theorem 7 *Let $f \in C^N([a,b])$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$. Then*

1)

$$\left| G_n(f,x) - \sum_{j=1}^{N} \frac{\left(f \circ \ln_{\frac{1}{e}}\right)^{(j)}(e^{-x})}{j!} G_n\left(\left(e^{-\cdot} - e^{-x}\right)^j, x\right) - f(x) \right| \leq$$

$$(5.250312578)\left[\frac{e^{-aN}}{N! n^{N\alpha}} \omega_1\left(\left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}, \frac{e^{-a}}{n^{\alpha}}\right) + \right.$$

$$\left. \frac{(6.3984)\left(e^{-a} - e^{-b}\right)^N}{N!} \left\| \left(f \circ \ln_{\frac{1}{e}}\right)^{(N)} \right\|_{\infty} e^{-n^{(1-\alpha)}} \right], \tag{31}$$

2)

$$|G_n(f,x) - f(x)| \leq (5.250312578) \cdot \tag{32}$$

$$\left\{ \sum_{j=1}^{N} \frac{\left| \left(f \circ \ln_{\frac{1}{e}}\right)^{(j)}(e^{-x}) \right|}{j!} e^{-aj} \left[\frac{1}{n^{\alpha j}} + (3.1992)(b-a)^j e^{-n^{(1-\alpha)}} \right] + \right.$$

$$\left[\frac{e^{-aN}}{N! n^{\alpha N}} \omega_1\left(\left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}, \frac{e^{-a}}{n^{\alpha}}\right) + \right.$$

$$\left. \left. \frac{(6.3984)\left(e^{-a} - e^{-b}\right)^N}{N!} \left\| \left(f \circ \ln_{\frac{1}{e}}\right)^{(N)} \right\|_{\infty} e^{-n^{(1-\alpha)}} \right] \right\},$$

3) If $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, it holds

$$|G_n(f,x_0) - f(x_0)| \leq (5.250312578) \cdot \tag{33}$$

$$\left[\frac{e^{-aN}}{N! n^{N\alpha}} \omega_1\left(\left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}, \frac{e^{-a}}{n^{\alpha}}\right) + \right.$$

$$\left. \frac{(6.3984)\left(e^{-a} - e^{-b}\right)^N}{N!} \left\| \left(f \circ \ln_{\frac{1}{e}}\right)^{(N)} \right\|_{\infty} e^{-n^{(1-\alpha)}} \right].$$

Observe here the speed of convergence is extremely high at $\frac{1}{n^{(N+1)\alpha}}$.

Proof. Call $F := f \circ \ln_{\frac{1}{e}}$. Let $x, \frac{k}{n} \in [a,b]$. Then

$$f\left(\frac{k}{n}\right) - f(x) = \sum_{j=1}^{N} \frac{F^{(j)}(e^{-x})}{j!} \left(e^{-\frac{k}{n}} - e^{-x}\right)^j + \overline{K}_N\left(x, \frac{k}{n}\right),$$

where

$$\overline{K}_N\left(x, \frac{k}{n}\right) := -\frac{1}{(N-1)!} \cdot$$

$$\int_x^{\frac{k}{n}} \left(e^{-\frac{k}{n}} - e^{-s}\right)^{N-1} \left[F^{(N)}\left(e^{-s}\right) - F^{(N)}\left(e^{-x}\right)\right] e^{-s} ds.$$

Thus

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi\left(nx-k\right) f\left(\frac{k}{n}\right) - f\left(x\right) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi\left(nx-k\right) =$$

$$\sum_{j=1}^{N} \frac{F^{(j)}\left(e^{-x}\right)}{j!} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi\left(nx-k\right) \left(e^{-\frac{k}{n}} - e^{-x}\right)^{j} + \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi\left(nx-k\right) \frac{1}{(N-1)!} \cdot$$

$$\int_x^{\frac{k}{n}} \left(e^{-\frac{k}{n}} - e^{-s}\right)^{N-1} \left[F^{(N)}\left(e^{-s}\right) - F^{(N)}\left(e^{-x}\right)\right] de^{-s}.$$

Therefore

$$G_n^*\left(f,x\right) - f\left(x\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi\left(nx-k\right)\right) =$$

$$\sum_{j=1}^{N} \frac{F^{(j)}\left(e^{-x}\right)}{j!} G_n^*\left(\left(e^{-\cdot} - e^{-x}\right)^{j}, x\right) + U_n\left(x\right),$$

where

$$U_n\left(x\right) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi\left(nx-k\right)\mu,$$

with

$$\mu := \frac{1}{(N-1)!} \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w\right)^{N-1} \left[F^{(N)}\left(w\right) - F^{(N)}\left(e^{-x}\right)\right] dw.$$

Case of $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\alpha}}$.

i) Subcase of $x \geq \frac{k}{n}$. I.e. $e^{-\frac{k}{n}} \geq e^{-x}$.

$$|\mu| \leq \frac{1}{(N-1)!} \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w\right)^{N-1} \left|F^{(N)}\left(w\right) - F^{(N)}\left(e^{-x}\right)\right| dw \leq$$

$$\frac{1}{(N-1)!} \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w\right)^{N-1} \omega_1\left(F^{(N)}, \left|w - e^{-x}\right|\right) dw \leq$$

$$\frac{1}{(N-1)!} \omega_1\left(F^{(N)}, \left|e^{-\frac{k}{n}} - e^{-x}\right|\right) \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w\right)^{N-1} dw \leq$$

$$\frac{\left(e^{-\frac{k}{n}} - e^{-x}\right)^{N}}{N!} \omega_1\left(F^{(N)}, e^{-a}\left|x - \frac{k}{n}\right|\right) \leq$$

$$e^{-aN}\frac{\left|x-\frac{k}{n}\right|^N}{N!}\omega_1\left(F^{(N)},e^{-a}\left|x-\frac{k}{n}\right|\right)\leq$$
$$\frac{e^{-aN}}{N!n^{\alpha N}}\omega_1\left(F^{(N)},\frac{e^{-a}}{n^\alpha}\right).$$

Hence when $x\geq\frac{k}{n}$ we get

$$|\mu|\leq\frac{e^{-aN}}{N!n^{\alpha N}}\omega_1\left(F^{(N)},\frac{e^{-a}}{n^\alpha}\right).$$

ii) Subcase of $\frac{k}{n}\geq x$. Then $e^{-\frac{k}{n}}\leq e^{-x}$ and

$$|\mu|\leq\frac{1}{(N-1)!}\int_{e^{-\frac{k}{n}}}^{e^{-x}}\left(w-e^{-\frac{k}{n}}\right)^{N-1}\left|F^{(N)}(w)-F^{(N)}\left(e^{-x}\right)\right|dw\leq$$
$$\frac{1}{(N-1)!}\int_{e^{-\frac{k}{n}}}^{e^{-x}}\left(w-e^{-\frac{k}{n}}\right)^{N-1}\omega_1\left(F^{(N)},\left|w-e^{-x}\right|\right)dw\leq$$
$$\frac{1}{(N-1)!}\omega_1\left(F^{(N)},e^{-x}-e^{-\frac{k}{n}}\right)\int_{e^{-\frac{k}{n}}}^{e^{-x}}\left(w-e^{-\frac{k}{n}}\right)^{N-1}dw\leq$$
$$\frac{1}{(N-1)!}\omega_1\left(F^{(N)},e^{-a}\left|x-\frac{k}{n}\right|\right)\frac{\left(e^{-x}-e^{-\frac{k}{n}}\right)^N}{N}\leq$$
$$\frac{1}{N!}\omega_1\left(F^{(N)},\frac{e^{-a}}{n^\alpha}\right)e^{-aN}\left|x-\frac{k}{n}\right|^N\leq$$
$$\frac{1}{N!}\omega_1\left(F^{(N)},\frac{e^{-a}}{n^\alpha}\right)\frac{e^{-aN}}{n^{\alpha N}},$$

i.e.

$$|\mu|\leq\frac{e^{-aN}}{N!n^{\alpha N}}\omega_1\left(F^{(N)},\frac{e^{-a}}{n^\alpha}\right),$$

when $\frac{k}{n}\geq x$. So in general when $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}$ we proved that

$$|\mu|\leq\frac{e^{-aN}}{N!n^{\alpha N}}\omega_1\left(F^{(N)},\frac{e^{-a}}{n^\alpha}\right).$$

Also we observe:

i)' When $\frac{k}{n}\leq x$, we get

$$|\mu|\leq\frac{1}{(N-1)!}\left(\int_{e^{-x}}^{e^{-\frac{k}{n}}}\left(e^{-\frac{k}{n}}-w\right)^{N-1}dw\right)2\left\|F^{(N)}\right\|_\infty$$
$$=\frac{2\left\|F^{(N)}\right\|_\infty}{(N-1)!}\frac{\left(e^{-\frac{k}{n}}-e^{-x}\right)^N}{N}\leq\frac{2\left\|F^{(N)}\right\|_\infty}{N!}\left(e^{-a}-e^{-b}\right)^N.$$

ii)' When $\frac{k}{n} \geq x$, we obtain

$$|\mu| \leq \frac{1}{(N-1)!}\left(\int_{e^{-\frac{k}{n}}}^{e^{-x}} \left(w - e^{-\frac{k}{n}}\right)^{N-1} dw\right) 2\left\|F^{(N)}\right\|_{\infty}$$

$$= \frac{2\left\|F^{(N)}\right\|_{\infty}}{N!}\left(e^{-x} - e^{-\frac{k}{n}}\right)^{N} \leq \frac{2\left\|F^{(N)}\right\|_{\infty}}{N!}\left(e^{-a} - e^{-b}\right)^{N}.$$

We proved always true that

$$|\mu| \leq \frac{2\left(e^{-a} - e^{-b}\right)^{N}\left\|F^{(N)}\right\|_{\infty}}{N!}.$$

Consequently we get

$$|U_n(x)| \leq \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} \Phi(nx-k)|\mu| + \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} \Phi(nx-k)|\mu| \leq$$

$$\left(\frac{e^{-aN}}{N!n^{\alpha N}}\omega_1\left(F^{(N)}, \frac{e^{-a}}{n^{\alpha}}\right)\right)\left(\sum_{k=-\infty}^{\infty} \Phi(nx-k)\right) +$$

$$\left(\frac{2\left(e^{-a} - e^{-b}\right)^{N}\left\|F^{(N)}\right\|_{\infty}}{N!}\right)\left(\sum_{\substack{k=-\infty \\ |k-nx| > n^{1-\alpha}}}^{\infty} \Phi(nx-k)\right) \leq$$

$$\frac{e^{-aN}}{N!n^{\alpha N}}\omega_1\left(F^{(N)}, \frac{e^{-a}}{n^{\alpha}}\right) + \left(\frac{2\left(e^{-a} - e^{-b}\right)^{N}\left\|F^{(N)}\right\|_{\infty}}{N!}\right)(3.1992)\, e^{-n^{(1-\alpha)}}.$$

So we have proved that

$$|U_n(x)| \leq \frac{e^{-aN}}{N!n^{\alpha N}}\omega_1\left(F^{(N)}, \frac{e^{-a}}{n^{\alpha}}\right) + \frac{(6.3984)\left(e^{-a} - e^{-b}\right)^{N}\left\|F^{(N)}\right\|_{\infty}}{N!}e^{-n^{(1-\alpha)}}.$$

We also notice that

$$\left|G_n^*\left(\left(e^{-\cdot} - e^{-x}\right)^{j}, x\right)\right| \leq G_n^*\left(\left|e^{-\cdot} - e^{-x}\right|^{j}, x\right) \leq$$

$$e^{-aj}G_n^*\left(|\cdot - x|^{j}, x\right) = e^{-aj}\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)\left|\frac{k}{n} - x\right|^{j}\right) = e^{-aj}.$$

$$\left[\sum_{\substack{k=\lceil na \rceil \\ \left| x-\frac{k}{n} \right| \leq \frac{1}{n^{\alpha}}}}^{\lfloor nb \rfloor} \Phi\left(nx-k\right) \left| \frac{k}{n}-x \right|^{j} + \sum_{\substack{k=\lceil na \rceil \\ \left| x-\frac{k}{n} \right| > \frac{1}{n^{\alpha}}}}^{\lfloor nb \rfloor} \Phi\left(nx-k\right) \left| \frac{k}{n}-x \right|^{j} \right] \leq$$

$$e^{-aj} \left[\frac{1}{n^{\alpha j}} + (b-a)^{j} \sum_{\substack{k=\lceil na \rceil \\ \left| x-\frac{k}{n} \right| > \frac{1}{n^{\alpha}}}}^{\lfloor nb \rfloor} \Phi\left(nx-k\right) \right] \leq$$

$$e^{-aj} \left[\frac{1}{n^{\alpha j}} + (3.1992)\left(b-a\right)^{j} e^{-n^{(1-\alpha)}} \right].$$

Thus we have proved

$$\left| G_n^{*}\left(\left(e^{-\cdot}-e^{-x} \right)^{j}, x \right) \right| \leq e^{-aj} \left[\frac{1}{n^{\alpha j}} + (3.1992)\left(b-a\right)^{j} e^{-n^{(1-\alpha)}} \right],$$

for $j=1,...,N$, and the theorem. ■

We continue with

Theorem 8 *Let $f \in C^{N}\left(\left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]\right)$, $n, N \in \mathbb{N}$, $0<\varepsilon<\frac{\pi}{2}$, ε small, $x \in \left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]$, $0<\alpha<1$. Then*

1)

$$\left| G_n\left(f,x\right) - \sum_{j=1}^{N} \frac{\left(f\circ \sin^{-1}\right)^{(j)}\left(\sin x\right)}{j!} G_n\left(\left(\sin\cdot - \sin x\right)^{j}, x\right) - f\left(x\right) \right| \leq \quad (34)$$

$$(5.250312578)\left[\frac{\omega_1\left(\left(f\circ \sin^{-1}\right)^{(N)}, \frac{1}{n^{\alpha}}\right)}{n^{\alpha N} N!} + \left(\frac{(3.1992)\, 2^{N+1} \left\| \left(f\circ \sin^{-1}\right)^{(N)} \right\|_{\infty}}{N!} \right) e^{-n^{(1-\alpha)}} \right],$$

2)

$$\left| G_n\left(f,x\right) - f\left(x\right) \right| \leq (5.250312578)\cdot$$

$$\left\{ \sum_{j=1}^{N} \frac{\left| \left(f\circ \sin^{-1}\right)^{(j)}\left(\sin x\right) \right|}{j!} \left[\frac{1}{n^{\alpha j}} + (3.1992)\left(\pi-2\varepsilon\right)^{j} e^{-n^{(1-\alpha)}} \right] + \right.$$

$$\left[\frac{\omega_1\left(\left(f\circ\sin^{-1}\right)^{(N)},\frac{1}{n^\alpha}\right)}{N!n^{\alpha N}}+\left(\frac{(3.1992)\,2^{N+1}}{N!}\left\|\left(f\circ\sin^{-1}\right)^{(N)}\right\|_\infty\right)e^{-n^{(1-\alpha)}}\right]\Bigg\}, \tag{35}$$

3) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$ *for some* $x_0\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]$, *it holds*

$$|G_n(f,x_0)-f(x_0)|\le(5.250312578)\cdot \tag{36}$$

$$\left[\frac{\omega_1\left(\left(f\circ\sin^{-1}\right)^{(N)},\frac{1}{n^\alpha}\right)}{n^{\alpha N}N!}+\left(\frac{(3.1992)\,2^{N+1}\left\|\left(f\circ\sin^{-1}\right)^{(N)}\right\|_\infty}{N!}\right)e^{-n^{(1-\alpha)}}\right].$$

Notice in the last the high speed of convergence of order $n^{-\alpha(N+1)}$.

Proof. Call $F:=f\circ\sin^{-1}$ and let $\frac{k}{n},x\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]$. Then

$$f\left(\frac{k}{n}\right)-f(x)=\sum_{j=1}^{N}\frac{F^{(j)}(\sin x)}{j!}\left(\sin\frac{k}{n}-\sin x\right)^j+$$

$$\frac{1}{(N-1)!}\int_x^{\frac{k}{n}}\left(\sin\frac{k}{n}-\sin s\right)^{N-1}\left(F^{(N)}(\sin s)-F^{(N)}(\sin x)\right)d\sin s.$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi(nx-k)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)=$$

$$\sum_{j=1}^{N}\frac{F^{(j)}(\sin x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\left(\sin\frac{k}{n}-\sin x\right)^j+\frac{1}{(N-1)!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\cdot$$

$$\int_x^{\frac{k}{n}}\left(\sin\frac{k}{n}-\sin s\right)^{N-1}\left(F^{(N)}(\sin s)-F^{(N)}(\sin x)\right)d\sin s.$$

Set here $a=-\frac{\pi}{2}+\varepsilon$, $b=\frac{\pi}{2}-\varepsilon$. Thus

$$G_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)=$$

$$\sum_{j=1}^{N}\frac{F^{(j)}(\sin x)}{j!}G_n^*\left((\sin\cdot-\sin x)^j,x\right)+M_n(x),$$

where

$$M_n(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\,\rho,$$

with

$$\rho := \frac{1}{(N-1)!}\int_x^{\frac{k}{n}} \left(\sin\frac{k}{n} - \sin s\right)^{N-1} \left(F^{(N)}\left(\sin s\right) - F^{(N)}\left(\sin x\right)\right) d\sin s.$$

Case of $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\alpha}}$.

i) Subcase of $\frac{k}{n} \geq x$. The function sin is increasing on $[a, b]$, i.e. $\sin\frac{k}{n} \geq \sin x$. Then

$$|\rho| \leq \frac{1}{(N-1)!}\int_x^{\frac{k}{n}} \left(\sin\frac{k}{n} - \sin s\right)^{N-1} \left|F^{(N)}\left(\sin s\right) - F^{(N)}\left(\sin x\right)\right| d\sin s =$$

$$\frac{1}{(N-1)!}\int_{\sin x}^{\sin\frac{k}{n}} \left(\sin\frac{k}{n} - w\right)^{N-1} \left(F^{(N)}\left(w\right) - F^{(N)}\left(\sin x\right)\right) dw \leq$$

$$\frac{1}{(N-1)!}\int_{\sin x}^{\sin\frac{k}{n}} \left(\sin\frac{k}{n} - w\right)^{N-1} \omega_1\left(F^{(N)}, |w - \sin x|\right) dw \leq$$

$$\omega_1\left(F^{(N)}, \left|\sin\frac{k}{n} - \sin x\right|\right) \frac{\left(\sin\frac{k}{n} - \sin x\right)^N}{N!} \leq$$

$$\omega_1\left(F^{(N)}, \left|\frac{k}{n} - x\right|\right) \frac{\left(\frac{k}{n} - x\right)^N}{N!} \leq \omega_1\left(F^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!}.$$

So if $\frac{k}{n} \geq x$, then

$$|\rho| \leq \omega_1\left(F^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{N! n^{\alpha N}}.$$

ii) Subcase of $\frac{k}{n} \leq x$, then $\sin\frac{k}{n} \leq \sin x$. Hence

$$|\rho| \leq \frac{1}{(N-1)!}\int_{\frac{k}{n}}^{x} \left(\sin s - \sin\frac{k}{n}\right)^{N-1} \left|F^{(N)}\left(\sin s\right) - F^{(N)}\left(\sin x\right)\right| d\sin s =$$

$$\frac{1}{(N-1)!}\int_{\sin\frac{k}{n}}^{\sin x} \left(w - \sin\frac{k}{n}\right)^{N-1} \left|F^{(N)}\left(w\right) - F^{(N)}\left(\sin x\right)\right| dw \leq$$

$$\frac{1}{(N-1)!}\int_{\sin\frac{k}{n}}^{\sin x} \left(w - \sin\frac{k}{n}\right)^{N-1} \omega_1\left(F^{(N)}, |w - \sin x|\right) dw \leq$$

$$\frac{1}{(N-1)!}\omega_1\left(F^{(N)}, \left|\sin x - \sin\frac{k}{n}\right|\right) \int_{\sin\frac{k}{n}}^{\sin x} \left(w - \sin\frac{k}{n}\right)^{N-1} dw \leq$$

$$\frac{1}{(N-1)!}\omega_1\left(F^{(N)}, \left|x - \frac{k}{n}\right|\right) \frac{\left(\sin x - \sin\frac{k}{k}\right)^N}{N} \leq$$

$$\frac{1}{N!}\omega_1\left(F^{(N)}, \frac{1}{n^{\alpha}}\right) \left|x - \frac{k}{n}\right|^N \leq \frac{1}{n^{\alpha N} N!}\omega_1\left(F^{(N)}, \frac{1}{n^{\alpha}}\right).$$

We proved for $\frac{k}{n} \leq x$ that

$$|\rho| \leq \omega_1 \left(F^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!}.$$

So in both cases we got that

$$|\rho| \leq \omega_1 \left(F^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!},$$

when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$.

Also in general ($\frac{k}{n} \geq x$ case)

$$|\rho| \leq \frac{1}{(N-1)!} \left(\int_x^{\frac{k}{n}} \left(\sin \frac{k}{n} - \sin s \right)^{N-1} d \sin s \right) 2 \left\| F^{(N)} \right\|_\infty =$$

$$\frac{1}{(N-1)!} \left(\int_{\sin x}^{\sin \frac{k}{n}} \left(\sin \frac{k}{n} - w \right)^{N-1} dw \right) 2 \left\| F^{(N)} \right\|_\infty =$$

$$\frac{1}{N!} \left(\sin \frac{k}{n} - \sin x \right)^N 2 \left\| F^{(N)} \right\|_\infty \leq \frac{2^{N+1}}{N!} \left\| F^{(N)} \right\|_\infty.$$

Also (case of $\frac{k}{n} \leq x$) we get

$$|\rho| \leq \frac{1}{(N-1)!} \left(\int_{\frac{k}{n}}^x \left(\sin s - \sin \frac{k}{n} \right)^{N-1} d \sin s \right) 2 \left\| F^{(N)} \right\|_\infty =$$

$$\frac{1}{(N-1)!} \left(\int_{\sin \frac{k}{n}}^{\sin x} \left(w - \sin \frac{k}{n} \right)^{N-1} dw \right) 2 \left\| F^{(N)} \right\|_\infty =$$

$$\frac{1}{(N-1)!} \frac{\left(\sin x - \sin \frac{k}{n} \right)^N}{N} 2 \left\| F^{(N)} \right\|_\infty \leq \frac{2^{N+1}}{N!} \left\| F^{(N)} \right\|_\infty.$$

So we proved in general that

$$|\rho| \leq \frac{2^{N+1}}{N!} \left\| F^{(N)} \right\|_\infty.$$

Therefore we derive

$$|M_n(x)| \leq \sum_{\substack{k=\lceil na \rceil \\ \left(k: \left| x - \frac{k}{n} \right| \leq \frac{1}{n^\alpha}\right)}}^{\lfloor nb \rfloor} \Phi(nx-k) |\rho| + \sum_{\substack{k=\lceil na \rceil \\ \left(k: \left| x - \frac{k}{n} \right| > \frac{1}{n^\alpha}\right)}}^{\lfloor nb \rfloor} \Phi(nx-k) |\rho| \leq$$

$$\left(\frac{\omega_1 \left(F^{(N)}, \frac{1}{n^\alpha} \right)}{N! n^{\alpha N}} \right) + \frac{(3.1992)\, 2^{N+1}}{N!} \left\| F^{(N)} \right\|_\infty e^{-n^{(1-\alpha)}}.$$

So that

$$|M_n(x)| \leq \frac{\omega_1\left(F^{(N)}, \frac{1}{n^\alpha}\right)}{N! n^{\alpha N}} + \frac{(3.1992)\, 2^{N+1}}{N!} \left\|F^{(N)}\right\|_\infty e^{-n^{(1-\alpha)}}.$$

Next we estimate

$$\left|G_n^*\left((\sin\cdot - \sin x)^j, x\right)\right| \leq G_n^*\left(|\sin\cdot - \sin x|^j, x\right) \leq$$

$$G_n^*\left(|\cdot - x|^j, x\right) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)\left|\frac{k}{n} - x\right|^j \leq$$

(work as before)

$$\frac{1}{n^{\alpha j}} + (3.1992)(\pi - 2\varepsilon)^j e^{-n^{(1-\alpha)}}.$$

Therefore

$$\left|G_n^*\left((\sin\cdot - \sin x)^j, x\right)\right| \leq \frac{1}{n^{\alpha j}} + (3.1992)(\pi - 2\varepsilon)^j e^{-n^{(1-\alpha)}},$$

$j = 1, ..., N$.

The theorem is proved. ■

We finally give

Theorem 9 *Let $f \in C^N([\varepsilon, \pi-\varepsilon])$, $n, N \in \mathbb{N}$, $\varepsilon > 0$ small, $x \in [\varepsilon, \pi-\varepsilon]$, $0 < \alpha < 1$. Then*

1)

$$\left|G_n(f,x) - \sum_{j=1}^N \frac{\left(f\circ\cos^{-1}\right)^{(j)}(\cos x)}{j!} G_n\left((\cos\cdot - \cos x)^j, x\right) - f(x)\right| \leq \quad (37)$$

$$(5.250312578)\left[\frac{\omega_1\left(\left(f\circ\cos^{-1}\right)^{(N)}, \frac{1}{n^\alpha}\right)}{n^{\alpha N} N!} + \left(\frac{(3.1992)\, 2^{N+1}\left\|\left(f\circ\cos^{-1}\right)^{(N)}\right\|_\infty}{N!}\right) e^{-n^{(1-\alpha)}}\right],$$

2)

$$|G_n(f,x) - f(x)| \leq (5.250312578)\cdot$$

$$\left\{\sum_{j=1}^N \frac{\left|\left(f\circ\cos^{-1}\right)^{(j)}(\cos x)\right|}{j!}\left[\frac{1}{n^{\alpha j}} + (3.1992)(\pi-2\varepsilon)^j e^{-n^{(1-\alpha)}}\right] +\right.$$

$$\left[\frac{\omega_1\left(\left(f\circ\cos^{-1}\right)^{(N)},\frac{1}{n^\alpha}\right)}{N!n^{\alpha N}}+\left(\frac{(3.1992)\,2^{N+1}}{N!}\left\|\left(f\circ\cos^{-1}\right)^{(N)}\right\|_\infty\right)e^{-n^{(1-\alpha)}}\right]\Bigg\}, \tag{38}$$

3) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$ *for some* $x_0\in[\varepsilon,\pi-\varepsilon]$, *it holds*

$$|G_n(f,x_0)-f(x_0)|\leq(5.250312578)\cdot \tag{39}$$

$$\left[\frac{\omega_1\left(\left(f\circ\cos^{-1}\right)^{(N)},\frac{1}{n^\alpha}\right)}{n^{\alpha N}N!}+\left(\frac{(3.1992)\,2^{N+1}\left\|\left(f\circ\cos^{-1}\right)^{(N)}\right\|_\infty}{N!}\right)e^{-n^{(1-\alpha)}}\right].$$

Notice in the last the high speed of convergence of order $n^{-\alpha(N+1)}$.

Proof. Call $F:=f\circ\cos^{-1}$ and let $\frac{k}{n},x\in[\varepsilon,\pi-\varepsilon]$. Then

$$f\left(\frac{k}{n}\right)-f(x)=\sum_{j=1}^{N}\frac{F^{(j)}(\cos x)}{j!}\left(\cos\frac{k}{n}-\cos x\right)^j+$$

$$\frac{1}{(N-1)!}\int_x^{\frac{k}{n}}\left(\cos\frac{k}{n}-\cos s\right)^{N-1}\left(F^{(N)}(\cos s)-F^{(N)}(\cos x)\right)d\cos s.$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi(nx-k)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)=$$

$$\sum_{j=1}^{N}\frac{F^{(j)}(\cos x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\left(\cos\frac{k}{n}-\cos x\right)^j+\frac{1}{(N-1)!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\cdot$$

$$\int_x^{\frac{k}{n}}\left(\cos\frac{k}{n}-\cos s\right)^{N-1}\left(F^{(N)}(\cos s)-F^{(N)}(\cos x)\right)d\cos s.$$

Set here $a=\varepsilon$, $b=\pi-\varepsilon$. Thus

$$G_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)=$$

$$\sum_{j=1}^{N}\frac{F^{(j)}(\cos x)}{j!}G_n^*\left((\cos\cdot-\cos x)^j,x\right)+\Theta_n(x),$$

where

$$\Theta_n(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\lambda,$$

with

$$\lambda := \frac{1}{(N-1)!}\int_x^{\frac{k}{n}} \left(\cos\frac{k}{n} - \cos s\right)^{N-1} \left(F^{(N)}\left(\cos s\right) - F^{(N)}\left(\cos x\right)\right) d\cos s =$$

$$\frac{1}{(N-1)!}\int_{\cos x}^{\cos\frac{k}{n}} \left(\cos\frac{k}{n} - w\right)^{N-1} \left(F^{(N)}\left(w\right) - F^{(N)}\left(\cos x\right)\right) dw.$$

Case of $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\alpha}}$.

i) Subcase of $\frac{k}{n} \geq x$. The function $\cos ine$ is decreasing on $[a,b]$, i.e. $\cos\frac{k}{n} \leq \cos x$.

Then

$$|\lambda| \leq \frac{1}{(N-1)!}\int_{\cos\frac{k}{n}}^{\cos x} \left(w - \cos\frac{k}{n}\right)^{N-1} |\left(F^{(N)}\left(w\right) - F^{(N)}\left(\cos x\right)\right)| dw \leq$$

$$\frac{1}{(N-1)!}\int_{\cos\frac{k}{n}}^{\cos x} \left(w - \cos\frac{k}{n}\right)^{N-1} \omega_1\left(F^{(N)}, |w - \cos x|\right) dw \leq$$

$$\omega_1\left(F^{(N)}, \cos x - \cos\frac{k}{n}\right) \frac{\left(\cos x - \cos\frac{k}{n}\right)^N}{N!} \leq$$

$$\omega_1\left(F^{(N)}, \left|x - \frac{k}{n}\right|\right) \frac{\left|x - \frac{k}{n}\right|^N}{N!} \leq \omega_1\left(F^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!}.$$

So if $\frac{k}{n} \geq x$, then

$$|\lambda| \leq \omega_1\left(F^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!}.$$

ii) Subcase of $\frac{k}{n} \leq x$, then $\cos\frac{k}{n} \geq \cos x$. Hence

$$|\lambda| \leq \frac{1}{(N-1)!}\int_{\cos x}^{\cos\frac{k}{n}} \left(\cos\frac{k}{n} - w\right)^{N-1} \left|F^{(N)}\left(w\right) - F^{(N)}\left(\cos x\right)\right| dw \leq$$

$$\frac{1}{(N-1)!}\int_{\cos x}^{\cos\frac{k}{n}} \left(\cos\frac{k}{n} - w\right)^{N-1} \omega_1\left(F^{(N)}, w - \cos x\right) dw \leq$$

$$\frac{1}{(N-1)!}\omega_1\left(F^{(N)}, \cos\frac{k}{n} - \cos x\right)\int_{\cos x}^{\cos\frac{k}{n}} \left(\cos\frac{k}{n} - w\right)^{N-1} dw \leq$$

$$\frac{1}{(N-1)!}\omega_1\left(F^{(N)}, \left|\frac{k}{n} - x\right|\right) \frac{\left(\cos\frac{k}{n} - \cos x\right)^N}{N} \leq$$

$$\frac{1}{N!}\omega_1\left(F^{(N)}, \left|\frac{k}{n} - x\right|\right) \left|\frac{k}{n} - x\right|^N \leq \frac{1}{N!}\omega_1\left(F^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N}}.$$

We proved for $\frac{k}{n} \leq x$ that

$$|\lambda| \leq \omega_1\left(F^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{N! n^{\alpha N}}.$$

So in both cases we got that

$$|\lambda| \leq \omega_1\left(F^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!},$$

when $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}$.

Also in general ($\frac{k}{n} \geq x$ case)

$$|\lambda| \leq \frac{1}{(N-1)!}\left(\int_{\cos\frac{k}{n}}^{\cos x}\left(w - \cos\frac{k}{n}\right)^{N-1} dw\right) 2\left\|F^{(N)}\right\|_\infty \leq$$

$$\frac{1}{N!}\left(\cos x - \cos\frac{k}{n}\right)^N 2\left\|F^{(N)}\right\|_\infty \leq \frac{2^{N+1}}{N!}\left\|F^{(N)}\right\|_\infty.$$

Also (case of $\frac{k}{n} \leq x$) we get

$$|\lambda| \leq \frac{1}{(N-1)!}\left(\int_{\cos x}^{\cos\frac{k}{n}}\left(\cos\frac{k}{n} - w\right)^{N-1} dw\right) 2\left\|F^{(N)}\right\|_\infty =$$

$$\frac{1}{N!}\left(\cos\frac{k}{n} - \cos x\right)^N 2\left\|F^{(N)}\right\|_\infty \leq \frac{2^{N+1}}{N!}\left\|F^{(N)}\right\|_\infty.$$

So we proved in general that

$$|\lambda| \leq \frac{2^{N+1}}{N!}\left\|F^{(N)}\right\|_\infty.$$

Therefore we derive

$$|\Theta_n(x)| \leq \sum_{\substack{k=\lceil na\rceil \\ \left(k:\left|x-\frac{k}{n}\right|\leq\frac{1}{n^\alpha}\right)}}^{\lfloor nb\rfloor} \Phi(nx-k)|\lambda| + \sum_{\substack{k=\lceil na\rceil \\ \left(k:\left|x-\frac{k}{n}\right|>\frac{1}{n^\alpha}\right)}}^{\lfloor nb\rfloor} \Phi(nx-k)|\lambda| \leq$$

$$\left(\omega_1\left(F^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}\right) + (3.1992)\frac{2^{N+1}}{N!}\left\|F^{(N)}\right\|_\infty e^{-n^{(1-\alpha)}}.$$

So that

$$|\Theta_n(x)| \leq \frac{\omega_1\left(F^{(N)}, \frac{1}{n^\alpha}\right)}{n^{\alpha N}N!} + (3.1992)\frac{2^{N+1}}{N!}\left\|F^{(N)}\right\|_\infty e^{-n^{(1-\alpha)}}.$$

Next we estimate

$$\left|G_n^*\left((\cos\cdot - \cos x)^j, x\right)\right| \leq G_n^*\left(|\cos\cdot - \cos x|^j, x\right) \leq$$

$$G_n^* \left(\left| \cdot - x \right|^j, x \right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi \left(nx - k \right) \left| \frac{k}{n} - x \right|^j \leq$$

(work as before)

$$\frac{1}{n^{\alpha j}} + (3.1992) \left(\pi - 2\varepsilon \right)^j e^{-n^{(1-\alpha)}}.$$

Therefore

$$\left| G_n^* \left(\left(\cos \cdot - \cos x \right)^j, x \right) \right| \leq \frac{1}{n^{\alpha j}} + (3.1992) \left(\pi - 2\varepsilon \right)^j e^{-n^{(1-\alpha)}},$$

$j = 1, ..., N$.

The theorem is proved. ■

4 Complex Neural network Approximations

We make

Remark 4 *Let $X := [a, b]$, $\mathbb{R}$ and $f : X \to \mathbb{C}$ with real and imaginary parts $f_1, f_2 : f = f_1 + i f_2$, $i = \sqrt{-1}$. Clearly f is continuous iff f_1 and f_2 are continuous.*

Also it holds

$$f^{(j)} (x) = f_1^{(j)} (x) + i f_2^{(j)} (x), \tag{40}$$

for all $j = 1, ..., N$, given that $f_1, f_2 \in C^N (X)$, $N \in \mathbb{N}$.

We denote by $C_B (\mathbb{R}, \mathbb{C})$ the space of continuous and bounded functions $f : \mathbb{R} \to \mathbb{C}$. Clearly f is bounded, iff both f_1, f_2 are bounded from $\mathbb{R}$ into $\mathbb{R}$, where $f = f_1 + i f_2$.

Here we define

$$G_n (f, x) := G_n (f_1, x) + i G_n (f_2, x), \tag{41}$$

and

$$\overline{G}_n (f, x) := \overline{G}_n (f_1, x) + i \overline{G}_n (f_2, x). \tag{42}$$

We observe here that

$$\left| G_n (f, x) - f (x) \right| \leq \left| G_n (f_1, x) - f_1 (x) \right| + \left| G_n (f_2, x) - f_2 (x) \right|, \tag{43}$$

and

$$\left\| G_n (f) - f \right\|_\infty \leq \left\| G_n (f_1) - f_1 \right\|_\infty + \left\| G_n (f_2) - f_2 \right\|_\infty. \tag{44}$$

Similarly we get

$$\left| \overline{G}_n (f, x) - f (x) \right| \leq \left| \overline{G}_n (f_1, x) - f_1 (x) \right| + \left| \overline{G}_n (f_2, x) - f_2 (x) \right|, \tag{45}$$

and

$$\left\| \overline{G}_n (f) - f \right\|_\infty \leq \left\| \overline{G}_n (f_1) - f_1 \right\|_\infty + \left\| \overline{G}_n (f_2) - f_2 \right\|_\infty. \tag{46}$$

We present

Theorem 10 *Let $f \in C([a,b],\mathbb{C})$, $f = f_1 + if_2$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $x \in [a,b]$. Then*

i)

$$|G_n(f,x) - f(x)| \leq (5.250312578)\cdot \tag{47}$$

$$\left[\left(\omega_1\left(f_1, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2, \frac{1}{n^\alpha}\right)\right) + (6.3984)\left(\|f_1\|_\infty + \|f_2\|_\infty\right) e^{-n^{(1-\alpha)}}\right] =: \psi_1,$$

and

ii)

$$\|G_n(f) - f\|_\infty \leq \psi_1. \tag{48}$$

Proof. Based on Remark 4 and Theorem 4. ■

We give

Theorem 11 *Let $f \in C_B(\mathbb{R},\mathbb{C})$, $f = f_1 + if_2$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}$. Then*

i)

$$\left|\overline{G}_n(f,x) - f(x)\right| \leq \left(\omega_1\left(f_1, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2, \frac{1}{n^\alpha}\right)\right) + \tag{49}$$

$$(6.3984)\left(\|f_1\|_\infty + \|f_2\|_\infty\right) e^{-n^{(1-\alpha)}} =: \psi_2,$$

ii)

$$\left\|\overline{G}_n(f) - f\right\|_\infty \leq \psi_2. \tag{50}$$

Proof. Based on Remark 4 and Theorem 5. ■

Next we present a result of high order complex neural network approximation.

Theorem 12 *Let $f : [a,b] \to \mathbb{C}$, $[a,b] \subset \mathbb{R}$, such that $f = f_1 + if_2$. Assume $f_1, f_2 \in C^N([a,b])$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$. Then*

i)

$$|G_n(f,x) - f(x)| \leq (5.250312578)\cdot \tag{51}$$

$$\left\{\sum_{j=1}^{N} \frac{\left(\left|f_1^{(j)}(x)\right| + \left|f_2^{(j)}(x)\right|\right)}{j!}\left[\frac{1}{n^{\alpha j}} + (3.1992)(b-a)^j e^{-n^{(1-\alpha)}}\right] + \right.$$

$$\left[\frac{\left(\omega_1\left(f_1^{(N)}, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\alpha}\right)\right)}{n^{\alpha N} N!} + \right.$$

$$\left.\left.\left(\frac{(6.3984)\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)(b-a)^N}{N!}\right) e^{-n^{(1-\alpha)}}\right]\right\},$$

ii) assume further $f_1^{(j)}(x_0) = f_2^{(j)}(x_0) = 0$, $j = 1, ..., N$, *for some* $x_0 \in [a, b]$, *it holds*

$$|G_n(f, x_0) - f(x_0)| \leq (5.250312578) \cdot \tag{52}$$
$$\left[\frac{(\omega_1\left(f_1^{(N)}, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\alpha}\right))}{n^{\alpha N} N!} + \right.$$
$$\left. \left(\frac{(6.3984)\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)(b-a)^N}{N!} \right) e^{-n^{(1-\alpha)}} \right],$$

notice here the extremely high rate of convergence at $n^{-(N+1)\alpha}$,

iii)

$$\|G_n(f) - f\|_\infty \leq (5.250312578) \cdot$$
$$\left\{ \sum_{j=1}^{N} \frac{\left(\left\|f_1^{(j)}\right\|_\infty + \left\|f_2^{(j)}\right\|_\infty\right)}{j!} \left[\frac{1}{n^{\alpha j}} + (3.1992)(b-a)^j e^{-n^{(1-\alpha)}} \right] + \right.$$
$$\left[\frac{\left(\omega_1\left(f_1^{(N)}, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\alpha}\right)\right)}{n^{\alpha N} N!} + \right.$$
$$\left. \left. \left(\frac{(6.3984)\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)(b-a)^N}{N!} \right) e^{-n^{(1-\alpha)}} \right] \right\}. \tag{53}$$

Proof. Based on Remark 4 and Theorem 6. ■

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Hardy inequality in block weighted spaces

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Abstract

In this study, we prove the classical Hardy inequality in block weighted sequence space. Also, by a non-negative function $K = K(x, y)$ we define corresponding continuous case of the block sequence space. Indeed, by a non-negative Lebesgue measurable function K on $(0, \infty) \times (0, \infty)$, we define

$$L_K^p := \Big\{ f \quad : \quad \int_0^\infty \Big(\int_0^\infty K(x,y)|f(x)|dx \Big)^p dy < \infty \Big\}.$$

The space L_K^p equipped by the norm $\|f\|_{p,K} = \Big(\int_0^\infty \Big(\int_0^\infty K(x,y)|f(x)|dx \Big)^p dy \Big)^{\frac{1}{p}}$ is a normed linear space. Finally, for K which is homogeneous of degree θ we prove that

$$\|f\|_{p,K} \leq C(p,q,\theta) \Big(\int_0^\infty x^{p(1+\theta)} |f^p(x)|dx \Big)^{\frac{1}{p}}.$$

Keywords: Integral inequality; Hardy's inequality; Block weighted sequence space; Nürlond's operator.

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1 Introduction

The classical Hardy inequality reads:

$$\int_0^\infty \Big(\frac{\int_0^x f(t)dt}{x} \Big)^p dx < \Big(\frac{p}{p-1} \Big)^p \int_0^\infty f^p(x)\, dx, \qquad (p > 1) \qquad (1-1)$$

where, f is nonnegative function such that $f \in L^p(R_+)$ and $R_+ = (0, \infty)$. The almost dramatic period of research for at least 10 years until G. H. Hardy [2] stated and proved (1-1) was described in details in [4].

Another important inequality is the following:

If $p > 1$ and f is a nonnegative function such that $f \in L^p(R_+)$, then

$$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y} dx \Big)^p dy < \frac{\pi}{sin(\frac{\pi}{p})} \int_0^\infty f^p(y) dy. \qquad (1-2)$$

It was early known that these inequalities are in fact equivalent. Moreover, (1-2) is sometimes called Hilbert's inequality even if Hilbert himself only considered the case $p = 2$ (L^p spaces were not defined at that time).
We also note that (1-1) can be interpreted as the Hardy operator $H : (Hf)(x) := \frac{1}{x}\int_0^x f(t)dt$, maps L^p into L^p with the operator norm $q = \frac{p}{p-1}\Big($since, it is known that $\Big(\frac{p}{p-1}\Big)^p$ is the sharp constant in (1-1)$\Big)$. Similarly, (1-2) may be interpreted as also the operator $A : (Af)(y) := \int_0^\infty \frac{f(x)}{x+y}dx$ maps L^p into l^p with the operator norm $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$.

In 1928 Hardy [4] proved an estimate for some integral operators as a generalization of the Schur test, from which the first "weighted" modification of Hardy's inequality (1-1) followed, namely the inequality

$$\int_0^\infty \Big(\frac{\int_0^x f(t)dt}{x}\Big)^p x^\alpha\, dx < \Big(\frac{p}{p-\alpha-1}\Big)^p \int_0^\infty f^p(x)x^\alpha\, dx, \qquad (p>1) \qquad (1-3)$$

valid , with $p > 1$ and $\alpha < p - 1$, for all measurable non-negative functions f (see [3] Theorem 330), where the constant $\Big(\frac{p}{p-\alpha-1}\Big)^p$ is the best possible. The prehistory of (1-1) up to the time when Hardy finally proved (1-1) in 1925 in [3] can be found in [6]. After that the inequality has been developed and applied in almost unbelievable ways. See for instance the books [6] , [7] devoted to this subject and also the historical article [5] and references given therein.

In this study, at first we prove a weighted Hardy inequality in discrete form and then verify Hardy inequality in weighted block sequence space. Finally, by a non-negative and homogeneous function $K = K(x, y)$ in definition 2.1 we obtain some Hardy-type inequalities.

2 New results

Suppose that $w = (w_n)_{n=1}^\infty$ is a non-negative sequence of real numbers . For $p \in R\backslash\{0\}$, let l_w^p denotes the space of all real sequences $x = \{x_k\}_{k=1}^\infty$, such that

$$\|x\|_{w,p} := \Big(\sum_{k=1}^\infty w_k|x_k|^p\Big)^{\frac{1}{p}} < \infty.$$

For the continuous case, L_w^p denotes the space of all functions f, such that

$$\|f\|_{w,p} := \Big(\int_0^\infty w(x)f^p(x)dx\Big)^{\frac{1}{p}} < \infty.$$

Lashkaripour and Foroutannia in [8], defined the weighted block sequence space as follows. Assume that $F = (F_n)$ is a partition of positive integers where each F_n is a finite interval of N and

$$\max F_n < \min F_{n+1} \qquad (n = 1, 2, ...).$$

The weighted block sequence space $l^p_{w,F}$ is defined as

$$l^p_{w,F} := \Big\{x = (x_n) : \sum_{n=1}^{\infty} w_n| < x, F_n > |^p < \infty\Big\},$$

where, $< x, F_n >= \sum_{j\in F_n} x_j$. The norm on $l^p_{w,F}$ is denoted by $\|.\|_{p,w,F}$ and is defined by

$$\|x\|_{p,w,F} := \Big(\sum_{n=1}^{\infty} w_n| < x, F_n > |^p\Big)^{\frac{1}{p}}.$$

Note that with the above-mentioned definition $l^p_{w,F}$ is not a normed sequence space. Indeed, one may consider $x = (1, -1, 0, 0, ...)$, $F_1 = \{1, 2\}$, $F_2 = \{3, 4\}$, ... and $w_n = 1$ then, $\|x\|_{p,w,F} = 0$ whereas $x \neq 0$.
We reform the above-mentioned definition as

$$l^p_{w,F} := \Big\{x = (x_n) : \sum_{n=1}^{\infty} w_n\Big(\sum_{j\in F_n} |x_j|\Big)^p < \infty\Big\},$$

and

$$\|x\|_{w,F} := \Big(\sum_{n=1}^{\infty} w_n\Big(\sum_{j\in F_n} |x_j|\Big)^p\Big)^{\frac{1}{p}}.$$

Of course, for non-negative sequences the two definitions coincide.

G. Bennett in [1] for a matrix A with non-negative entries and $p > 0$, defined the sequence space

$$l_{A(p)} = \Big\{x = (x_n) : \sum_n \Big(\sum_k a_{n,k}|x_k|\Big)^p < \infty\Big\}.$$

For $p \geq 1$ with the norm

$$\|x\|_{A(p)} = \Big(\sum_n \Big(\sum_k a_{n,k}|x_k|\Big)^p\Big)^{\frac{1}{p}},$$

$l_{A(p)}$ is a normed sequence space.

By a partition $F = (F_n)$, we correspond a matrix $A = (a_{n,k})$ such that $a_{n,k} = 1$, for $k \in F_n$ and $a_{n,k} = 0$, otherwise. One may easily verify that

$$\|x\|_{w,A(p)} = \|x\|_{p,w,F},$$

where,

$$\|x\|_{w,A(p)} = \Big(\sum_n w_n(\sum_k a_{n,k}|x_k|)^p\Big)^{\frac{1}{p}}.$$

For any partition, the corresponding matrix is a quasi-summability matrix, which is an upper triangular matrix which has column-sums 1.
For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$, is a partition of positive integers, $l^p_{w,I} = l_{w,I(p)} = l^p_w$,

and $\|x\|_{w,p,I} = \|x\|_{I(p),w} = \|x\|_{w,p}$.

In light of above considerations, we define continuous form of the space $l_{B(p)}$ as follows:

Definition 2.1 Let $p > 1$ and $K = K(x, y)$ be non-negative Lebesgue measurable function in $(0, \infty) \times (0, \infty)$, we define

$$L_K^p := \Big\{ f \quad : \quad \int_0^\infty \Big(\int_0^\infty K(x,y)|f(x)|dx \Big)^p dy < \infty \Big\}$$

The space L_K^p equipped by the norm $\|f\|_{p,K} = \Big(\int_0^\infty \Big(\int_0^\infty K(x,y)|f(x)|dx \Big)^p dy \Big)^{\frac{1}{p}}$ is a normed linear space. In special case by taking $K(x, y) = 1$, for $x = y$ and 0 in otherwise, L_K^p is reduced to the usual space L^p. Also, by considering $K(x, y) = w(y)$, for $x = y$ and 0 in otherwise, the weighted space L_w^P is obtained.
By taking K in above-mentioned definition as:

$$K(x,y) = \begin{cases} y^{\frac{\beta}{p}-1} & (x \le y), \\ 0 & (x > y), \end{cases}$$

we obtain in Theorem 2.12 below that

$$\|f\|_{p,K}^p < \Big(\frac{p}{p-\beta-1}\Big)^p \int_0^\infty |f^p(x)|x^\beta \, dx. \qquad (\beta < p-1)$$

Note that this corresponds to (1-1) and (1-3), where $\beta = 0$ and $\beta = \alpha$, respectively.

Lemma 2.2. (Young inequality; see [9, p. 30 or p. 49]) If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

for all a, $b > 0$.

In the following, inspired by Elliot's and Ingham's proof for the classical Hardy's inequality we prove a weighted Hardy inequality(see [6, p. 158]).

Theorem 2.3. Let $a_n \ge 0$ and $0 < w_n \le 1$. If $p > 1$, then

$$\sum_{n=1}^\infty \Big(\frac{A_n}{n} \Big)^p w_n < q^p \sum_{n=1}^\infty a_n^p w_n,$$

where, $A_n = \sum_{k=1}^n a_k$.
proof. Let $\alpha_n = \frac{A_n}{n}$ and $\alpha_0 = 0$, then we have

$$\alpha_n^p - q\alpha_n^{p-1} a_n = \alpha_n^p - q\alpha_n^{p-1}\Big(n\alpha_n - (n-1)\alpha_{n-1} \Big)$$

$$= \alpha_n^p(1-nq) + (n-1)q\alpha_n^{p-1}\alpha_{n-1}.$$

Now, by applying Lemma 2.2 one may obtain

$$\alpha_n^p - q\alpha_n^{p-1}a_n \leq \alpha_n^p(1-nq) + (n-1)q\Big(\frac{\alpha_n^p}{q} + \frac{\alpha_{n-1}^p}{p}\Big)$$

$$= (q-1)\Big((n-1)\alpha_{n-1}^p - n\alpha_n^p\Big).$$

Since, $0 < w_n \leq 1$ we have

$$\alpha_n^p w_n - q\alpha_n^{p-1}a_n w_n \leq (q-1)\Big((n-1)\alpha_{n-1}^p - n\alpha_n^p\Big).$$

Summing from 1 to N yields

$$\sum_{n=1}^{N}\Big(\frac{A_n}{n}\Big)^p w_n - q\sum_{n=1}^{N}\Big(\frac{A_n}{n}\Big)^{p-1}a_n w_n \leq (1-q)N\alpha_N^p \leq 0,$$

so from Hölder's inequality we infer that

$$\sum_{n=1}^{N}\Big(\frac{A_n}{n}\Big)^p w_n \leq q\Big(\sum_{n=1}^{N}\Big(\frac{A_n}{n}\Big)^p w_n\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{\infty}a_n^p w_n\Big)^{\frac{1}{p}}.$$

Division by the last factor completes the proof.

Proposition 2.4. (Schur, see [1] Proposition 7.1) Fix p, $1 < p < \infty$, and suppose that A is a matrix with non-negative entries. If

$$\sup_n \sum_{k=1}^{\infty} a_{n,k} = R < \infty$$

and

$$\sup_k \sum_{n=1}^{\infty} a_{n,k} = C < \infty,$$

then A maps l^p into l^p and $\|Ax\|_p \leq R^{\frac{1}{p^*}}C^{\frac{1}{p}}\|x\|_p$.

In the following, we generalize the above-mentioned proposition to the weighted sequence spaces for upper triangular matrices.

Lemma 2.5. Suppose that $1 < p < \infty$, that $w = (w_n)$ is a non-negative increasing sequence and let $B = (b_{n,k})$ be an upper triangular matrix with non-negative entries. If we take

$$\sup_{i\geq 1}\sum_{j=1}^{\infty} b_{i,j} = R_B < \infty, \qquad \inf_{j\geq 1}\sum_{i=1}^{j} b_{i,j} = C_B < \infty,$$

then for $x \geq 0$, we have

$$\| x \|_{w,B(p)} \leq R_B^{\frac{1}{p^*}} C_B^{\frac{1}{p}} \| x \|_{w,p} .$$

Proof. By applying Hölder's inequality, we have

$$\begin{aligned}\| x \|^p_{w,B(p)} &= \sum_{i=1}^{\infty} w_i \Big(\sum_{j=1}^{\infty} b_{i,j} x_j \Big)^p \\ &= \sum_{i=1}^{\infty} w_i \Big(\sum_{j=i}^{\infty} b_{i,j} x_j \Big)^p \\ &\leq \sum_{i=1}^{\infty} w_i \Big(\Big(\sum_{j=i}^{\infty} b_{i,j} \Big)^{p-1} \Big(\sum_{j=i}^{\infty} b_{i,j} x_j^p \Big) \Big) \\ &\leq R_B^{p-1} \sum_{i=1}^{\infty} w_i \Big(\sum_{j=i}^{\infty} b_{i,j} x_j^p \Big).\end{aligned}$$

Now, since (w_n) is increasing, one may obtain

$$\begin{aligned}\| x \|^p_{w,B(p)} &\leq R_B^{p-1} \sum_{j=1}^{\infty} w_j \Big(\sum_{i=1}^{j} b_{i,j} \Big) x_j^p \\ &\leq R_B^{p-1} C_B \| x \|_{w,p} .\end{aligned}$$

Theorem 2.6. Let $a_n \geq 0$ and (w_n) be an increasing sequence satisfying $0 < w_n \leq 1$. If $p > 1$ and $B = (b_{n,k})$ is an upper triangular matrix with non-negative entries, then

$$\|ha_n\|_{w,B(p)} \leq R_B^{\frac{1}{p^*}} C_B^{\frac{1}{p}} q \|a_n\|_{p,w},$$

where, $ha_n = \frac{\sum_{k=1}^{n} a_k}{n}$ is the discrete Hardy operator.
Proof. By using Lemma 2.5 and Theorem 2.3 the assertion is proved.

Corollary 2.7. Let $a_n \geq 0$ and $0 < w_n \leq 1$ be an increasing sequence. If $p > 1$ and $F = (F_n)$ is a partition of natural numbers then

$$\|ha_n\|_{p,w,F} \leq N^{\frac{1}{p^*}} q \|a_n\|_{p,w},$$

where, $ha_n = \frac{\sum_{k=1}^{n} a_k}{n}$ and N is the largest cardinal number of F_n's.

Note that for the case of weighted block sequence space the right-hand side constant necessarily, is not the best possible. Indeed, it depends on the weight of that space. For instance, in the following we prove a special case.

Theorem 2.8. Let $a_n \geq 0$ and $w_n = \frac{1}{n^\alpha}$, $\alpha > 0$. If $p > 1$ and $F = (F_n)$ is a partition of natural numbers such that, $F_n = \{nN - N + 1, nN - N + 2, ..., nN\}$ then

$$\|ha_n\|_{p,w,F} \leq N^{\frac{1}{p^*} + \frac{\alpha}{p}} q \|a_n\|_{p,w},$$

where, $ha_n = \frac{\sum_{k=1}^{n} a_k}{n}$. The constant factor is the best possible.

Proof. By applying Hölder's inequality we have

$$\|ha_n\|_{p,w,F}^p = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \Big(\sum_{j\in F_n} ha_j \Big)^p$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^\alpha} N^{\frac{p}{p^*}} \Big(\sum_{j\in F_n} (ha_j)^p \Big).$$

Since, for $j \in F_n$, we have $j \leq nN$ so

$$\|ha_n\|_{p,w,F}^p \leq N^{\frac{p}{p^*}+\alpha} \sum_{n=1}^{\infty} \sum_{j\in F_n} \frac{1}{j^\alpha} (ha_j)^p$$

$$\leq N^{\frac{p}{p^*}+\alpha} \sum_{n=1}^{\infty} \frac{1}{n^\alpha} (ha_n)^p$$

$$= N^{\frac{p}{p^*}+\alpha} \|ha_n\|_{p,w}^p$$

$$\leq q^p N^{\frac{p}{p^*}+\alpha} \|a_n\|_{p,w}^p.$$

Fix n, for $j \in F_n$, take $a_k = 1$ for $1 \leq k \leq j$ and $a_k = 0$, otherwise. Then

$$\|ha_n\|_{p,w,F}^p = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \Big(\sum_{j\in F_n} ha_j \Big)^p$$

$$= \frac{N^p}{n^\alpha}.$$

On the other hand

$$\|ha_n\|_{w,p}^p \leq q^p \|a_n\|_{w,p,F}^p$$

$$\leq q^p \sum_{k\in F_n} \frac{1}{k^\alpha}$$

$$\leq q^p \frac{N^{1-\alpha}}{(n-1)^\alpha}.$$

So

$$\|ha_n\|_{p,w,F} \geq q N^{\frac{1}{p^*}+\frac{\alpha}{p}} \Big(\frac{n-1}{n} \Big)^{\frac{\alpha}{p}} \|ha_n\|_{w,p}.$$

By tending n to infinity one may obtain

$$\|ha_n\|_{p,w,F} \geq q N^{\frac{1}{p^*}+\frac{\alpha}{p}} \|ha_n\|_{w,p}.$$

Theorem 2.9. Let $p > 1$ and $K = K(x, y)$ be non-negative Lebesgue measurable function on $(0, \infty) \times (0, \infty)$. Suppose that K is homogeneous of degree θ. Then

$$L^p_{x^{p(1+\theta)}} \subseteq L^p_K.$$

Moreover,

$$\|f\|_{p,K} \leq C(p,q,\theta)\Big(\int_0^\infty x^{p(1+\theta)}|f^p(x)|dx\Big)^{\frac{1}{p}},$$

where

$$C(p,q,\theta) = \Big(\int_0^\infty K(t,1)t^{-\frac{1}{p}}dt\Big)^{\frac{1}{q}}\Big(\int_0^\infty K(1,t)t^{(p-1)(1+\theta)-\frac{1}{q}}dt\Big)^{\frac{1}{p}}.$$

Providing all integrals be finite.

Proof. By applying Hölder's inequality we have

$$\begin{aligned}
\|f\|_{p,K}^p &= \int_0^\infty \Big(\int_0^\infty K(x,y)|f(x)|dx\Big)^p dy \\
&= \int_0^\infty \Big(\int_0^\infty \Big(K^{\frac{1}{p}}(x,y)|f(x)|\Big(\frac{x}{y}\Big)^{\frac{1}{pq}}\Big)\Big(K^{\frac{1}{q}}(x,y)\Big(\frac{y}{x}\Big)^{\frac{1}{pq}}\Big)dx\Big)^p dy \\
&\leq \int_0^\infty \Big(\int_0^\infty \Big(K(x,y)|f^p(x)|\Big(\frac{x}{y}\Big)^{\frac{1}{q}}dx\Big)\Big(\int_0^\infty K(x,y)\Big(\frac{y}{x}\Big)^{\frac{1}{p}}dx\Big)^{p-1}dy.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^\infty K(x,y)\Big(\frac{y}{x}\Big)^{\frac{1}{p}}dx &= \int_0^\infty K(yt,y)t^{-\frac{1}{p}}ydt \\
&= y^{1+\theta}\int_0^\infty K(t,1)t^{-\frac{1}{p}}dt.
\end{aligned}$$

So

$$\begin{aligned}
\|f\|_{p,K}^p &\leq \Big(\int_0^\infty K(t,1)t^{-\frac{1}{p}}dt\Big)^{p-1}\int_0^\infty y^{(p-1)(1+\theta)}\Big(\int_0^\infty K(x,y)|f^p(x)|\Big(\frac{x}{y}\Big)^{\frac{1}{q}}dx\Big)dy \\
&= \Big(\int_0^\infty K(t,1)t^{-\frac{1}{p}}dt\Big)^{p-1}\int_0^\infty |f^p(x)|\Big(\int_0^\infty K(x,y)y^{(p-1)(1+\theta)}\Big(\frac{x}{y}\Big)^{\frac{1}{q}}dy\Big)dx \\
&= \Big(\int_0^\infty K(t,1)t^{-\frac{1}{p}}dt\Big)^{p-1}\int_0^\infty |f^p(x)|\Big(\int_0^\infty K(x,xt)(xt)^{(p-1)(1+\theta)}t^{-\frac{1}{q}}xdt\Big)dx \\
&= \Big(\int_0^\infty K(t,1)t^{-\frac{1}{p}}dt\Big)^{p-1}\Big(\int_0^\infty K(1,t)t^{(p-1)(1+\theta)-\frac{1}{q}}dt\Big)\Big(\int_0^\infty x^{p(1+\theta)}|f^p(x)|dx\Big).
\end{aligned}$$

Corollary 2.10. Suppose that $p>1$, that $K(x,y)$ is non-negative and homogeneous of degree -1, also assume that $K(x,y)=K(y,x)$. Then

$$\int_0^\infty \Big(\int_0^\infty K(x,y)|f(x)|dx\Big)^p dy \leq C\int_0^\infty |f(x)|^p dx,$$

where

$$C = \int_0^\infty K(t,1)t^{-\frac{1}{p}}dt.$$

Proof. By assumptions on K we have

$$\begin{aligned}
\int_0^\infty K(t,1)t^{-\frac{1}{p}}dt &= \int_0^\infty t^{-1}K(t,1)t^{1-\frac{1}{p}}dt \\
&= \int_0^\infty K(1,t^{-1})t^{\frac{1}{q}}dt.
\end{aligned}$$

$$= \int_0^\infty K(1,z) z^{\frac{-1}{q}} dz.$$

Note that Corollary 2.10 corresponds to Theorem 319 in [3]. Also, B. Yang in [10] by defining the integral operator T as:

$$(Tf)(y) := \int_0^\infty K(x,y) f(x) dx, \qquad y \in (0,\infty),$$

and assuming some conditions on K, found the norm of operator T. He supposed that $K(x,y)$ is continuous in $(0,\infty)\times(0,\infty)$, satisfying $K(x,y) = K(y,x) > 0$. Then, for $\epsilon(\geq 0)$ small enough and $x > 0$, Yang by setting $\widetilde{K}_\epsilon(r,x)$ as

$$\widetilde{K}_\epsilon(r,x) := \int_0^\infty K(x,t)\Big(\frac{x}{t}\Big)^{\frac{1+\epsilon}{r}} dt \qquad (r = p,q),$$

proved the following assertions:

(i) if $\widetilde{K}_0(r,x) = \int_0^\infty K(x,t)\Big(\frac{x}{t}\Big)^{\frac{1}{r}} dt = K_0(p)(r = p,q; x > 0)$, and $K_0(p)$ is a constant independent of x, then $T \in B(L^r(0,\infty) \to L^r(0,\infty))$ and $\|T\|_r \leq K_0(p)(r = p,q)$;
(ii) if $\widetilde{K}_\epsilon(r,x) = K_\epsilon(p)(r = p,q; x > 0)$ is independent of x, and $K_\epsilon(p) = K_0(p) + o(1)(\epsilon \to 0^+)$, then $\|T\|_r = K_0(p)(r = p,q)$.

Remark 2.11. Note that similar to discrete case, constant factor necessarily, is not the best possible. Indeed, it depends on K. For instance, in the following we give a new proof for inequality (1-3), which the constant factor is the best possible.

Theorem 2.12. Let $p > 1$ and $\beta < p - 1$. Also, suppose that

$$K(x,y) = \begin{cases} y^{\frac{\beta}{p}-1} & (x \leq y), \\ 0 & (x > y), \end{cases}$$

and

$$\int_0^\infty x^\beta |f^p(x)| dx < \infty.$$

Then

$$\|f\|_{p,K}^p < \Big(\frac{p}{p-\beta-1}\Big)^p \int_0^\infty |f^p(x)| x^\beta \, dx.$$

Proof. By applying Hölder's inequality we have

$$\begin{aligned} \|f\|_{p,K}^p &= \int_0^\infty \Big(\int_0^\infty K(x,y)|f(x)|dx\Big)^p dy \\ &= \int_0^\infty \Big(\int_0^\infty \Big(K^{\frac{1}{p}}(x,y)|f(x)|\Big(\frac{x}{y}\Big)^s\Big)\Big(K^{\frac{1}{q}}(x,y)\Big(\frac{y}{x}\Big)^s\Big)dx\Big)^p dy \\ &\leq \int_0^\infty \Big(\int_0^\infty \Big(K(x,y)|f^p(x)|\Big(\frac{x}{y}\Big)^{sp} dx\Big)\Big(\int_0^\infty K(x,y)\Big(\frac{y}{x}\Big)^{sq} dx\Big)^{p-1} dy. \end{aligned}$$

Note that

$$\begin{aligned}\int_0^\infty K(x,y)\Big(\frac{y}{x}\Big)^{sq}dx &= \int_0^\infty K(yt,y)t^{-sq}ydt\\ &= y^{\frac{\beta}{p}}\int_0^1 t^{-sq}dt\\ &= \frac{y^{\frac{\beta}{p}}}{1-sq},\end{aligned}$$

providing, $s<\frac{1}{q}$. So

$$\begin{aligned}\|f\|_{p,K}^p &\le \frac{1}{(1-sq)^{p-1}}\int_0^\infty |f^p(x)|\Big(\int_0^\infty y^{\frac{\beta}{q}}K(x,y)\Big(\frac{x}{y}\Big)^{sp}dy\Big)dx\\ &= \frac{1}{(1-sq)^{p-1}}\Big(\int_0^\infty t^{\beta-1-sp}dt\Big)\int_0^\infty x^\beta|f^p(x)|dx\\ &= \frac{1}{(sp-\beta)(1-sq)^{p-1}}\int_0^\infty x^\beta|f^p(x)|dx,\end{aligned}$$

providing, $\beta<sp<p-1$. One may easily verify that, the factor of integral attains its minimum at $s=\frac{1+\beta}{pq}$, so we have

$$\|f\|_{p,K}^p \le \Big(\frac{p}{p-\beta-1}\Big)^p\int_0^\infty |f^p(x)|x^\beta\,dx.$$

If inequality mentioned in Theorem 2.12 takes the form of equality, then there exist constants c_1, c_2 such that

$$c_1|f^p(x)|\Big(\frac{x}{y}\Big)^{sp} = c_2\Big(\frac{y}{x}\Big)^{sq}.$$

This leads us to

$$\begin{aligned}|f^p(x)| &= \frac{c_2}{c_1}\Big(\frac{y}{x}\Big)^{spq}\\ &= \frac{c_2}{c_1}\Big(\frac{y}{x}\Big)^{1+\beta}.\end{aligned}$$

So

$$x^\beta|f^p(x)| = \frac{c_2}{c_1}\Big(\frac{y^{1+\beta}}{x}\Big).$$

This contradicts to

$$\int_0^\infty x^\beta|f^p(x)|dx<\infty.$$

Let (a_n) be a non-negative sequence with $a_1>0$ and $A_n=a_1+...a_n$.Two matrices are associated with the sequence (a_n). The first is called the Nörlund matrix $N_a=(a_n,k)_{n,k\ge0}$ is defined as follows:

$$a_{n,k}=\begin{cases}\frac{a_{n-k}}{A_n} & k\le n,\\ 0 & k>n,\end{cases}$$

and the second is called the weighted mean matrix $M_a=(a_{n,k})_{n,k\ge0}$ which is defined as the following:

$$a_{n,k}=\begin{cases}\frac{a_k}{A_n} & k\le n,\\ 0 & k>n.\end{cases}$$

In light of above-mentioned definitions, by a weight function $w(x) = x^\alpha$ $(\alpha \neq -1)$ on $(0, \infty)$ we define continuous Nörlund and continuous weighted mean operators as the following:

$$(T_N f)(y) := \int_0^\infty N(x, y) f(x) dx, \qquad y \in (0, \infty),$$

and

$$(T_M f)(y) := \int_0^\infty M(x, y) f(x) dx, \qquad y \in (0, \infty),$$

where

$$N(x, y) = \begin{cases} \frac{(x-y)^\alpha}{\int_0^x z^\alpha dz} & y \leq x, \\ 0 & y > x, \end{cases}$$

$$M(x, y) = \begin{cases} \frac{y^\alpha}{\int_0^x z^\alpha dz} & y \leq x, \\ 0 & y > x. \end{cases}$$

Corollary 2.13. Suppose that $p > 1$ and $\alpha > \frac{-1}{p}$. Then

$$\|f\|_{p,M} \leq \frac{p(1+\alpha)}{1+p\alpha} \|f\|_p, \qquad (2-1)$$

$$\|f\|_{p,N} \leq (1+\alpha)\beta(1+\alpha, \frac{1}{p}) \|f\|_p, \qquad (2-2)$$

where $\beta(.,.)$ is beta function.

Proof. Note that $M = M(x, y)$ and $N = N(x, y)$ are non-negative and homogenous function of degree -1. Now, by applying Theorem 2.9 we have

$$\|f\|_{p,M} \leq C(p, q, -1) \|f\|_p,$$

where

$$\begin{aligned} C(p, q, -1) &= \Big(\int_0^\infty M(t, 1) t^{-\frac{1}{p}} dt \Big)^{\frac{1}{q}} \Big(\int_0^\infty M(1, t) t^{-\frac{1}{q}} dt \Big)^{\frac{1}{p}}. \\ &= \Big(\int_1^\infty (1+\alpha) t^{-1-\alpha-\frac{1}{p}} dt \Big)^{\frac{1}{q}} \Big(\int_0^1 (1+\alpha) t^{\alpha-\frac{1}{q}} dt \Big)^{\frac{1}{p}} \\ &= \frac{p(1+\alpha)}{1+p\alpha}. \end{aligned}$$

For the second inequality we have

$$\begin{aligned} C(p, q, -1) &= \Big(\int_0^\infty N(t, 1) t^{-\frac{1}{p}} dt \Big)^{\frac{1}{q}} \Big(\int_0^\infty N(1, t) t^{-\frac{1}{q}} dt \Big)^{\frac{1}{p}} \\ &= \left((1+\alpha) \int_1^\infty \frac{(t-1)^\alpha}{t^{1+\alpha+\frac{1}{p}}} dt \right)^{\frac{1}{q}} \left((1+\alpha) \int_0^1 (1-t)^\alpha t^{-\frac{1}{q}} dt \right)^{\frac{1}{p}} \end{aligned}$$

$$= (1+\alpha)\beta(1+\alpha, \frac{1}{p}).$$

Remark 2.14. Inequalities mentioned in the above corollary mean that the operators T_M and T_N are bounded in $L^p(0,\infty)$ and

$$\|T_M\|_{L^p\to L^p} \leq \frac{p(1+\alpha)}{1+p\alpha}$$

and

$$\|T_N\|_{L^p\to L^p} \leq (1+\alpha)\beta(1+\alpha, \frac{1}{p}).$$

Remark 2.15. By taking $K(x,y) = \frac{1}{x+y}$ in Theorem 2.9 we have

$$\begin{aligned} C(p,q,-1) &= \int_0^\infty \frac{t^{-\frac{1}{p}}}{1+t}dt \\ &= \beta(\frac{1}{p}, \frac{1}{q}) \\ &= \frac{\pi}{sin(\frac{\pi}{p})}. \end{aligned}$$

So, Theorem 2.9 asserts that

$$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y}dx\Big)^p dy < \frac{\pi}{sin(\frac{\pi}{p})}\int_0^\infty f^p(y)dy.$$

This means that, Theorem 2.9 is a generalization of Hilbert's inequality. Indeed, Hardy et al. in [3] prove the following statements:
Suppose that $p>1$, $p\prime = \frac{p}{p-1}$, and that $K(x,y)$ has the following properties:
(i) K is non-negative, and homogeneous of degree -1,

$$(ii) \qquad \int_0^\infty K(x,1)x^{-\frac{1}{p}}dx = \int_0^\infty K(1,y)y^{-\frac{1}{p\prime}}dx = k,$$

and either (iii) $K(x,1)x^{-\frac{1}{p}}$ is a strictly decreasing function of x, and $K(1,y)y^{-\frac{1}{p\prime}}$ of y, or more generally, $(iii\prime)$ $K(x,1)x^{-\frac{1}{p}}$ decreases from $x=1$ onwards, while the interval (0,1) can be divided into two parts, $(0,\xi)$ and $(\xi,1)$, of which one may be null, in the first of which it decreases and in the second of which it increases, and $K(1,y)y^{-\frac{1}{p\prime}}$ has similar properties. Finally suppose that, when only the less stringent condition $(iii\prime)$ is satisfied,

$$(iv) \qquad K(x,x) = 0.$$

Then

$$\sum\sum K(m,n)a_m b_n < k\Big(\sum a_m^p\Big)^{\frac{1}{p}}\Big(\sum b_n^{p\prime}\Big)^{\frac{1}{p\prime}},$$

unless (a_m) or (b_n) is null,

$$\sum_n \Big(\sum_m K(m,n)a_m\Big)^p < k^p \sum_m a_m^p,$$

unless (a_m) is null,

$$\sum_m \Big(\sum_n K(m,n) b_n \Big)^{p'} < k^{p'} \sum_n b_n^{p'},$$

unless (b_n) is null (see [3 Theorem 318]). For the corresponding theorem for integrals see [3 Theorem 319]. Note that by applying Hölder's inequality one may easily prove that the above-mentioned three inequalities are equivalent.

They extend the above-mentioned assertions as follows:

Suppose that $K(x,y)$ is a strictly decreasing function of x and y, and also satisfies conditions (i) and (ii), that $\lambda_m > 0$, $\mu_n > 0$,

$$\Lambda_m = \lambda_1 + \lambda_2 + ... + \lambda_m, \qquad M_n = \mu_1 + \mu_2 + ... + \mu_n,$$

and that $p > 1$. Then

$$\sum\sum K(\Lambda_m, M_n) \lambda_m^{\frac{1}{p'}} \mu_n^{\frac{1}{p}} a_m b_n < k \Big(\sum a_m^p \Big)^{\frac{1}{p}} \Big(\sum b_n^{p'} \Big)^{\frac{1}{p'}},$$

unless (a_m) or (b_n) is null.

If $K(x,y) = \frac{1}{x+y}$, the following inequality is obtained (see [3, Theorem 321]):

$$\sum\sum \frac{\lambda_m^{\frac{1}{p'}} \mu_n^{\frac{1}{p}}}{\Lambda_m + M_n} a_m b_n < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \Big(\sum a_m^p \Big)^{\frac{1}{p}} \Big(\sum b_n^{p'} \Big)^{\frac{1}{p'}}.$$

In light of above considerations, we shall prove the following corollary of Theorem 2.9.

Corollary 2.16. By taking $w(x) = x^\alpha$ $(\alpha \neq -1)$ and defining $W(x) = \int_0^x w(t)dt$, then

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{\alpha}{q}} y^{\frac{\alpha}{p}}}{W(x) + W(y)} f(x) g(y) < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \Big(\int_0^\infty f^p(x) dx \Big)^{\frac{1}{p}} \Big(\int_0^\infty g^q(y) dy \Big)^{\frac{1}{q}}.$$

Proof. By taking

$$K(x,y) = \frac{x^{\frac{\alpha}{q}} y^{\frac{\alpha}{p}}}{W(x) + W(y)} = (1+\alpha) \frac{x^{\frac{\alpha}{q}} y^{\frac{\alpha}{p}}}{x^{1+\alpha} + y^{1+\alpha}},$$

in Theorem 2.9, we have

$$\begin{aligned} C(p,q,-1) &= \Big(\int_0^\infty K(t,1) t^{-\frac{1}{p}} dt \Big)^{\frac{1}{q}} \Big(\int_0^\infty K(1,t) t^{-\frac{1}{q}} dt \Big)^{\frac{1}{p}} \\ &= (1+\alpha) \Big(\int_0^\infty \frac{t^{\frac{\alpha}{q} - \frac{1}{p}}}{1+t^{1+\alpha}} dt \Big)^{\frac{1}{q}} \Big(\int_0^\infty \frac{t^{\frac{\alpha}{p} - \frac{1}{q}}}{1+t^{1+\alpha}} dt \Big)^{\frac{1}{p}}. \end{aligned}$$

Now, by substituting $z = t^{1+\alpha}$ and noting that

$$\beta(a,b) = \int_0^\infty \frac{z^{a-1}}{(1+z)^{a+b}} dz,$$

we obtain

$$C(p,q,-1) = \beta(\frac{1}{p}, \frac{1}{q})$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}.$$

This completes the proof.

Suppose that (w_n) and (v_m) are two non-negative sequences. Also, assume that $M_w = (\frac{w_k}{W_n})_{n,k\geq 1}$ and $M_v = (\frac{v_k}{V_n})_{n,k\geq 1}$ are the corresponding their weighted mean matrices, respectively. In this case, let $M_{v,w} = (a_{i,j})_{i,j\geq 1}$ denotes the product of M_w and M_v. Then

$$a_{i,j} = \frac{v_j}{W_i}\sum_{k=j}^{i}\frac{w_k}{V_k}.$$

Now, we define the weighted mean operator and the nürlond's operator of two functions $w(t)$ and $v(t)$ as follows:

$$M_v^w(x,y) = \begin{cases} \frac{v(y)}{W(x)}\int_y^x \frac{w(t)}{V(t)}dt & y \leq x, \\ 0 & y > x, \end{cases}$$

$$N_v^w(x,y) = \begin{cases} \frac{\int_y^x w(x-t)v(t-y)dt}{W(x)V(y)} & y \leq x, \\ 0 & y > x. \end{cases}$$

Corollary 2.17. Suppose that $w(t) = t^\alpha$ and $v(t) = t^\gamma$, that $\gamma, \alpha > \frac{-1}{p}$. Then

$$\left(\int_0^\infty \left(\int_0^\infty M_v^w(x,y)|f(x)|dx\right)^p dy\right)^{\frac{1}{p}} \leq C_M(p,q,-1)\|f\|_p,$$

where

$$C_M(p,q,-1) = \frac{p(1+\alpha)(1+\gamma)}{(p\alpha+1)(p\gamma+1)}.$$

Proof. The assertion is a consequence of Theorem 2.9.

Corollary 2.18. Suppose that $w(t) = t^\alpha$ and $v(t) = t^\gamma$, that $\frac{-1}{p} < \gamma < \frac{-1}{q}$ and $\alpha > -1$. If $1 < p < 2$ then

$$\left(\int_0^\infty \left(\int_0^\infty N_v^w(x,y)|f(x)|dx\right)^p dy\right)^{\frac{1}{p}} \leq C_N(p,q,-1)\|f\|_p,$$

where

$$C_N(p,q,-1) = (1+\alpha)(1+\gamma)\beta(1+\alpha,1+\gamma)\beta(-\gamma-\frac{1}{q},\alpha+\gamma+2).$$

Proof. The assertion is a consequence of Theorem 2.9 and that

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad a,b>0.$$

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Univariate mixed Caputo fractional Ostrowski inequalities

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Abstract

Very general univariate mixed Caputo fractional Ostrowski inequalities are presented. One of them is proved sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$.

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1 Introduction

In 1938, A. Ostrowski [8] proved the following important inequality:

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{a-b} \int_a^b f(t)dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty , \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This paper is greatly motivated and inspired also by the following result.

Theorem 2 *(see [1]). Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then it holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)dy - f(x) \right| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right). \tag{2}$$

Inequality (2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, while when n is even the optimal function is

$$\overline{f}(y) := |y-x|^{n+\alpha} (b-a), \quad \alpha > 1.$$

Clearly inequality (2) generalizes inequality (1) for higher order derivatives of f.

Also in [2], see Chapters 24-26, we presented a complete theory of left fractional Ostrowski inequalities.

Here we combine both right and left Caputo fractional derivatives and produce Ostrowski inequalities.

For the concepts of right and left Caputo fractional Calculus we use here, we refer to [3]-[7], and [9].

2 Main Results

We make

Remark 3 *Let $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Let $f \in AC^m([a,b])$, $x_0 \in [a,b]$ (i.e. $f^{(m-1)} \in AC([a,b])$). Thus $f \in AC^m([a,x_0])$, and $f \in AC^m([x_0,b])$. Consequently, by [4, p.40],*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} D^{\alpha}_{*x_0} f(J) dJ, \quad \forall x \in [x_0, b].$$

And by [3],

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (J-x)^{\alpha-1} D^{\alpha}_{x_0-} f(J) dJ, \quad \forall x \in [a, x_0].$$

Here $D^{\alpha}_{*x_0} f$, $D^{\alpha}_{x_0-} f$ are the left and right Caputo fractional derivatives of order α.

Assume $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Then

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} D^{\alpha}_{*x_0} f(J) dJ, \quad \forall x \in [x_0, b] \tag{3}$$

and

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (J-x)^{\alpha-1} D^{\alpha}_{x_0-} f(J) dJ, \quad \forall x \in [a, x_0]. \tag{4}$$

Hence

$$\begin{aligned} |f(x) - f(x_0)| &\leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} \left| D^{\alpha}_{*x_0} f(J) \right| dJ \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_{x_0}^{x} (x-J)^{\alpha-1} dJ \right) \left\| D^{\alpha}_{*x_0} f \right\|_{\infty, [x_0, b]} \\ &= \frac{1}{\Gamma(\alpha)} \frac{(x-x_0)^{\alpha}}{\alpha} \left\| D^{\alpha}_{*x_0} f \right\|_{\infty, [x_0, b]}. \end{aligned} \tag{5}$$

Hence

$$|f(x)-f(x_0)| \leq \frac{\left\|D^{\alpha}_{*x_0} f\right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}, \quad \forall x \in [x_0,b]. \tag{6}$$

Similarly it holds

$$\begin{aligned}
|f(x)-f(x_0)| &\leq \frac{1}{\Gamma(\alpha)}\left(\int_x^{x_0}(J-x)^{\alpha-1}dJ\right)\left\|D^{\alpha}_{x_0-} f\right\|_{\infty,[a,x_0]} \\
&= \frac{1}{\Gamma(\alpha)}\frac{(x_0-x)^{\alpha}}{\alpha}\left\|D^{\alpha}_{x_0-} f\right\|_{\infty,[a,x_0]} \\
&= \frac{\left\|D^{\alpha}_{x_0-} f\right\|_{\infty,[\alpha,x_0]}}{\Gamma(\alpha+1)}(x_0-x)^{\alpha},
\end{aligned}$$

that is

$$|f(x)-f(x_0)| \leq \frac{\left\|D^{\alpha}_{x_0-} f\right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-x)^{\alpha}, \ \forall x \in [a,x_0]. \tag{7}$$

Next we see that

$$\begin{aligned}
\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x_0)\right| &= \frac{1}{b-a}\left|\int_a^b (f(x)-f(x_0))dx\right| \\
&\leq \frac{1}{b-a}\int_a^b |f(x)-f(x_0)|\,dx \qquad (8) \\
&= \frac{1}{b-a}\left\{\int_a^{x_0}|f(x)-f(x_0)|\,dx + \int_{x_0}^b |f(x)-f(x_0)|\,dx\right\} \\
&\leq \frac{1}{(b-a)\Gamma(\alpha+1)}\left\{\left\|D^{\alpha}_{x_0-} f\right\|_{\infty,[a,x_0]}\int_a^{x_0}(x_0-x)^{\alpha}dx\right. \\
&\left. + \left\|D^{\alpha}_{*x_0}\right\|_{\infty,[x_0,b]}\int_{x_0}^b (x-x_0)^{\alpha}dx\right\} \\
&= \frac{1}{(b-a)\Gamma(\alpha+1)}\left\{\left\|D^{\alpha}_{x_0-} f\right\|_{\infty,[a,x_0]}\left(\frac{(x_0-a)^{\alpha+1}}{\alpha+1}\right)\right. \\
&\left. + \left\|D^{\alpha}_{*x_0} f\right\|_{\infty,[x_0,b]}\left(\frac{(b-x_0)^{\alpha+1}}{\alpha+1}\right)\right\} \\
&= \frac{1}{(b-a)\Gamma(\alpha+2)}\left\{\left\|D^{\alpha}_{x_0-} f\right\|_{\infty,[a,x_0]}(x_0-a)^{\alpha+1}\right. \qquad (9) \\
&\left. + \left\|D^{\alpha}_{*x_0} f\right\|_{\infty,[x_0,b]}(b-x_0)^{\alpha+1}\right\}.
\end{aligned}$$

We have proved that

Theorem 4 *Let* $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$, *and* $\left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}$, $\left\| D^{\alpha}_{*x_0} \right\|_{\infty,[x_0,b]} < \infty$, $x_0 \in [a,b]$. *Assume* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f(x_0) \right| \leq \frac{1}{(b-a)\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]} (x_0-a)^{\alpha+1} + \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} (b-x_0)^{\alpha+1} \right\} \tag{10}$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}, \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} \right\} (b-a)^{\alpha}.$$

We also make

Remark 5 *As before we have* ($\alpha \geq 1$)

$$|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} \left| D^{\alpha}_{*x_0} f(J) \right| dJ$$
$$\leq \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \int_{x_0}^{x} \left| D^{\alpha}_{*x_0} f(J) \right| dJ$$
$$\leq \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left\| D^{\alpha}_{*x_0} \right\|_{L_1([x_0,b])}.$$

That is we get

$$|f(x) - f(x_0)| \leq \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left\| D^{\alpha}_{*x_0} f \right\|_{L_1([x_0,b])}, \quad \forall x \in [x_0, b]. \tag{11}$$

Similarly we derive ($\alpha \geq 1$)

$$|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (J-x)^{\alpha-1} \left| D^{\alpha}_{x_0-} f(J) \right| dJ$$
$$\leq \frac{1}{\Gamma(\alpha)} (x_0-x)^{\alpha-1} \int_{x}^{x_0} \left| D^{\alpha}_{x_0-} f(J) \right| dJ$$
$$\leq \frac{1}{\Gamma(\alpha)} (x_0-x)^{\alpha-1} \left\| D^{\alpha}_{x_0-} f \right\|_{L_1([a,x_0])}.$$

That is

$$|f(x) - f(x_0)| \leq \frac{(x_0-x)^{\alpha-1}}{\Gamma(\alpha)} \left\| D^{\alpha}_{x_0-} f \right\|_{L_1([a,x_0])}, \quad \forall x \in [a, x_0]. \tag{12}$$

As before we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x_0)\right| \leq \frac{1}{b-a}\int_a^b |f(x)-f(x_0)|\,dx$$
$$= \frac{1}{b-a}\left\{\int_a^{x_0} |f(x)-f(x_0)|\,dx + \int_{x_0}^b |f(x)-f(x_0)|\,dx\right\} \tag{13}$$
$$\leq \frac{1}{(b-a)\Gamma(\alpha)}\left\{\left(\int_a^{x_0}(x_0-x)^{\alpha-1}dx\right)\left\|D^{\alpha}_{x_0-}f\right\|_{L_1([a,x_0])}\right.$$
$$\left. + \left(\int_{x_0}^b (x-x_0)^{\alpha-1}dx\right)\left\|D^{\alpha}_{*x_0}f\right\|_{L_1([x_0,b])}\right\}$$
$$= \frac{1}{(b-a)\Gamma(\alpha+1)}\left\{(x_0-a)^{\alpha}\left\|D^{\alpha}_{x_0-}f\right\|_{L_1([a,x_0])}\right. \tag{14}$$
$$\left. + (b-x_0)^{\alpha}\left\|D^{\alpha}_{*x_0}f\right\|_{L_1([x_0,b])}\right\}.$$

We have proved

Theorem 6 *Let $\alpha \geq 1$, $m = \lceil \alpha \rceil$, and $f \in AC^m([a,b])$. Assume that $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$ and $D^{\alpha}_{x_0-}f \in L_1([a,x_0])$, $D^{\alpha}_{*x_0}f \in L_1([x_0,b])$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x_0)\right| \leq \frac{1}{(b-a)\Gamma(\alpha+1)}\left\{(x_0-a)^{\alpha}\left\|D^{\alpha}_{x_0-}f\right\|_{L_1([a,x_0])}\right.$$
$$\left. +(b-x_0)^{\alpha}\left\|D^{\alpha}_{*x_0}f\right\|_{L_1([x_0,b])}\right\} \tag{15}$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\max\left\{\left\|D^{\alpha}_{x_0-}f\right\|_{L_1([a,x_0])}, \left\|D^{\alpha}_{*x_0}f\right\|_{L_1([x_0,b])}\right\}$$
$$\cdot (b-a)^{\alpha-1}.$$

We further make

Remark 7 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > 1 - \frac{1}{p}$. Working as before on (3) and (4) and using Hölder's inequality we obtain*

$$|f(x)-f(x_0)| \leq \frac{1}{\Gamma(\alpha)}\frac{(x-x_0)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}}\left\|D^{\alpha}_{*x_0}f\right\|_{L_q([x_0,b])}, \ \forall x \in [x_0,b]. \tag{16}$$

And also it holds

$$|f(x)-f(x_0)| \leq \frac{1}{\Gamma(\alpha)}\frac{(x_0-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}}\left\|D^{\alpha}_{x_0-}f\right\|_{L_q([a,x_0])}, \ \forall x \in [a,x_0]. \tag{17}$$

We see that

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x_0)\right| \leq \frac{1}{b-a}\left\{\int_a^{x_0}|f(x)-f(x_0)|\,dx + \int_{x_0}^b |f(x)-f(x_0)|\,dx\right\}$$

$$\leq \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}}\left\{\left(\int_a^{x_0}(x_0-x)^{\alpha-1+\frac{1}{p}}dx\right)\right.$$

$$\left.\left\|D^{\alpha}_{x_0-}f\right\|_{L_q([a,x_0])} + \left(\int_{x_0}^b (x-x_0)^{\alpha-1+\frac{1}{p}}dx\right)\|D_{*x_0}f\|_{L_q([x_0,b])}\right\} \tag{18}$$

$$= \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(\alpha+\frac{1}{p})} \tag{19}$$

$$\left\{(x_0-a)^{\alpha+\frac{1}{p}}\left\|D^{\alpha}_{x_0-}f\right\|_{L_q([a,x_0])} + (b-x_0)^{\alpha+\frac{1}{p}}\|D_{*x_0}f\|_{L_q([x_0,b])}\right\}.$$

We have proved

Theorem 8 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1,\ \alpha>1-\frac{1}{p},\ m=\lceil\alpha\rceil,\ \alpha>0,$ *and* $f\in AC^m([a,b])$. *Assume that* $f^{(k)}(x_0)=0,\ k=1,...,\ m-1,\ x_0\in[a,b]$. *Assume* $D^{\alpha}_{x_0-}f\in L_q([a,x_0])$, *and* $D^{\alpha}_{*x_0}f\in L_q([x_0,b])$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x_0)\right| \leq \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(\alpha+\frac{1}{p})}$$

$$\left\{(x_0-a)^{\alpha+\frac{1}{p}}\left\|D^{\alpha}_{x_0-}f\right\|_{L_q([a,x_0])}\right. \tag{20}$$

$$\left.+\ (b-x_0)^{\alpha+\frac{1}{p}}\left\|D^{\alpha}_{*x_0}f\right\|_{L_q([x_0,b])}\right\}$$

$$\leq \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(\alpha+\frac{1}{p})}$$

$$\max\left\{\left\|D^{\alpha}_{x_0-}f\right\|_{L_q([a,x_0])},\right. \tag{21}$$

$$\left.\left\|D^{\alpha}_{*x_0}f\right\|_{L_q([x_0,b])}\right\}\cdot(b-a)^{\alpha-1/q}.$$

Corollary 9 *Let* $p=q=2,\ \alpha>\frac{1}{2},\ m=\lceil\alpha\rceil,\ \alpha>0,$ *and* $f\in AC^m([a,b])$. *Assume* $f^{(k)}(x_0)=0,\ k=1,...,m-1,\ x_0\in[a,b]$. *Assume* $D^{\alpha}_{x_0-}f\in L_2([a,x_0])$, *and*

$D^{\alpha}_{*x_0} f \in L_2([x_0, b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f(x_0) \right| \leq \frac{1}{(b-a)\Gamma(\alpha)(2\alpha-1)^{1/2}(\alpha+\frac{1}{2})} \cdot \tag{22}$$

$$\left\{ (x_0-a)^{\alpha+\frac{1}{2}} \left\| D^{\alpha}_{x_0-} f \right\|_{L_2([a,x_0])} + (b-x_0)^{\alpha+\frac{1}{2}} \left\| D^{\alpha}_{*x_0} f \right\|_{L_2([x_0,b])} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)\sqrt{2\alpha-1}(\alpha+\frac{1}{2})} \cdot$$

$$\max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{L_2([a,x_0])}, \left\| D^{\alpha}_{*x_0} f \right\|_{L_2([x_0,b])} \right\} \cdot (b-a)^{\alpha-\frac{1}{2}}.$$

We further make

Remark 10 *Here* $\alpha > 0,\ a \leq x_0 \leq b$. *Let*

$$\overline{f}(x) := \begin{cases} (x-x_0)^{\alpha}, & x \in [x_0, b], \\ (x_0-x)^{\alpha}, & x \in [a, x_0]. \end{cases} \tag{23}$$

See $\overline{f}$ *is in* $AC^m([x_0, b])$, *and in* $AC^m([a, x_0])$.

See that $\overline{f}^{(k)}_{-}(x_0) = \overline{f}^{(k)}_{+}(x_0) = 0,\ k = 0, 1, ...,\ m-1$.

Hence there exists $\overline{f}^{(m-1)}$ *at* x_0, *also* $\overline{f}^{(m-1)} \in AC[a, b]$.

That is $\overline{f} \in AC^m[a, b]$.

We find that

$$\left\| D^{\alpha}_{x_0-} \overline{f} \right\|_{\infty,[a,x_0]} = \Gamma(\alpha+1), \tag{24}$$

and

$$\left\| D^{\alpha}_{*x_0} \overline{f} \right\|_{\infty,[x_0,b]} = \Gamma(\alpha+1). \tag{25}$$

Consequently

$$R.H.S.(10) = \frac{\Gamma(\alpha+1)(\alpha+1)}{(b-a)\Gamma(\alpha+2)(\alpha+1)} \left\{ (x_0-a)^{\alpha+1} + (b-x_0)^{\alpha+1} \right\}$$

$$= \frac{1}{(b-a)(\alpha+1)} \left\{ (x_0-\alpha)^{\alpha+1} + (b-x_0)^{\alpha+1} \right\}$$

Also we see that

$$L.H.S(10) = \frac{1}{b-a} \left\{ \int_a^{x_0} (x_0-x)^{\alpha} dx + \int_{x_0}^b (x-x_0)^{\alpha} dx \right\}$$

$$= \frac{1}{b-a)} \left\{ \frac{(x_0-a)^{\alpha+1}}{\alpha+1} + \frac{(b-x_0)^{\alpha+1}}{\alpha+1} \right\}$$

$$= \frac{1}{(b-a)(\alpha+1)} \left\{ (x_0-a)^{\alpha+1} + (b-x_0)^{\alpha+1} \right\}. \tag{26}$$

Therefore inequality (10) is sharp and attained.

We have proved

Proposition 11 *Inequality (10) is sharp, in particular it is attained.*

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Fixed point theorems for $\tau - \varphi-$concave-convex operators and applications *

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Abstract

In this paper, we introduce the concept of a $\tau - \varphi-$concave–convex operator. Meanwhile by the properties of cone and monotone iterative technique, we prove some new existence and uniqueness theorems of fixed points for mixed monotone operators under more extensive conditions in ordered Banach spaces. Finally, we apple the results to a nonlinear Hammerstein integral equation. The results given in this work generalize and extend some known results (e.g., [Z. Q. Zhao, Existence and uniqueness of fixed points for some mixed monotone operators. Nonlinear Anal., 73 (2010)1481–1490.]).

Key words: $\tau - \varphi-$concave-convex operator; mixed monotone operator; fixed point; integral equation

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1 Preliminaries

It is well known that mixed monotone operators are a class of important operators, which are extensively used in nonlinear differential and integral equations (see [1–3]). In the past several decades, many authors investigated the existence and uniqueness of mixed monotone operators and obtained a few fixed point theorems for mixed monotone operators in ordered Banach spaces, (see [4–9] and the references therein). Recently, some authors focused on mixed monotone operators with certain concavity and convexity (see [10–13]). In 2004, Xu [14] introduced the $\phi-$ concave$(-\varphi)$ –convex operator and found a general method to deal with such a class of operators with certain concavity and convexity. In 2010, Zhao in [15] introduced the e-concave-convex operator and obtained some new

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existence and uniqueness theorems of fixed points for mixed monotone operators without any compact or continuous assumptions or the assumption for the coupled-upper-lower solutions in ordered Banach spaces. The main difference between a $\phi-$ concave $(-\varphi)-$convex operator and an e-concave-convex operator is that , for a $\phi-$ concave $(-\varphi)-$convex operator $A(x,y)$, there is just one variation (x or y) with a coefficient t, but for the e-concave-convex operator $A_1(x,y)$, not only the variation x has a coefficient t, but also y has a coefficient $\frac{1}{t}$, where $t \in (0,1)$. Hence they are different concepts. Moreover, the introduction of an e-concave-convex operator initiates a new research field of the fixed pint theory for binary operators.

Now, a natural problem arises: can we copy with such a class of binary operators as the e-concave-convex operator with certain concavity and convexity by a general method as well as being done in [14]?

In this work, in order to solve this problem, we present the concept of a $\tau-\varphi-$concave-convex operator which includes the e-concave-convex operator as an especial case. Without demanding the assumptions of the existence of coupled upper-lower solutions or compactness or continuity, we prove the existence and uniqueness of fixed points for $\tau-\varphi-$concave-convex operators. Our results in essence extend and generalize the corresponding ones in [15]. As an application, we apply our main fixed point theorem to a class of nonlinear Hammerstein integral equations.

For the convenience of readers and the discussion of the following sections, we state here some definitions, notations and suggest ones refer to [1,2,3, 14, 15].

Suppose E is a real Banach space with norm $\|\cdot\|$. A nonempty convex closed set P is called a cone if it satisfies the following conditions:

(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;

(ii) $x \in P$ and $-x \in P$ implies $x = \theta$, where θ denotes the zero element of E.

A partial order in E is given by $u \leq v$ iff $v - u \in P$. If $u \leq v$ and $u \neq v$, then we denote $u < v$. P is called normal if there exists a constant N such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N \parallel y \parallel$, the smallest constant N satisfying the above inequality is called the normality constant of P. P is called a solid cone if int(P)(the set of interior of P) is non-empty.

Assume that $D \subseteq E$. We say that an operator $A : D \times D \to E$ is mixed monotone if $A(u,v)$ is non-decreasing in its first argument u and is non-increasing in its second argument v, i.e., $u_1 \leq u_2$ $(u_1, u_2 \in D)$ implies $A(u_1, v) \leq A(u_2, v)$ for any $v \in D$ and $v_1 \leq v_2$ $(v_1, v_2 \in D)$ implies $A(u, v_1) \geq A(u, v_2)$ for any $u \in D$. $(u,v) \in D \times D$ is said to be a coupled upper- lower solution of A if $u \leq v$, $u \leq A(u,v), A(v,u) \leq v$. $(u^*, v^*) \in D \times D$ is said to be a coupled quasi- fixed point of A if $A(u^*, v^*) = u^*$ and $A(v^*, u^*) = v^*$. $x^* \in D$ is called a fixed point of A if $A(x^*, x^*) = x^*$.

Definition 1.1. *(see [14]) Let E be a real Banach space and $D \subset E$. $A : D \times D \to E$ is said to be $\phi-$ concave$(-\varphi)-$ convex, if there exist a function $\phi : (0,1] \times D \to (0,\infty)$ and a function $\varphi : (0,1] \times D \to (0,\infty)$ such that $(t,x) \in (0,1] \times D$ implies $t < \phi(t,x)\varphi(t,x) \leq 1$, and also A satisfies the following two conditions:*
(H_1) $A(tx,y) \geq \phi(t,x)A(x,y)$, $\forall t \in (0,1)$, $\forall (x,y) \in D \times D$;
(H_2) $A(x,Ty) \leq \frac{1}{\varphi(t,x)}A(x,y)$, $\forall t \in (0,1)$, $\forall (x,y) \in D \times D$.

Let P be a cone in E. For $e \in P^+ = P \setminus \{\theta\}$, set

$$P_e = \{x \in E | \text{ there exist positive numbers } \alpha, \beta \text{ such that } \alpha e \leq x \leq \beta e\}.$$

In [15], an $e-$ concave-convex operator is introduced as follows.

Definition 1.2. *(see [15]) Suppose that $A : P_e \times P_e \to P_e$ and $e \in P^+$. Suppose that there exists an $\eta(u,v,t) > 0$ such that*

$$A(tu, t^{-1}v) \geq t(1+\eta(u,v,t))A(u,v), \forall u, v \in C_e \text{ and } 0 < t < 1.$$

Then, A is called an $e-$ concave-convex operator.

Now, we give the definition of $\tau - \varphi-$concave–convex operators.

Definition 1.3. *Let E be a real Banach space, P be a cone in E. We say an operator $A : P \times P \to P$ is $\tau - \varphi-$concave–convex if there exist a function $\tau(t)$ defined on an interval (a,b) and a map $\varphi : (a,b) \times P \times P \to (0,+\infty)$ such that*
(C_1) $\tau : (a,b) \to (0,1)$ is a surjection;
(C_2) $A\left(\tau(t)x, \frac{1}{\tau(t)}y\right) \geq \varphi(t,x,y)A(x,y)$, $\forall t \in (a,b)$, $\forall (x,y) \in P \times P$;
(C_3) $\varphi(t,x,y) > \tau(t), \forall t \in (a,b)$, $\forall (x,y) \in P \times P$.

When $\tau(t) = t$, we say that A is a $\varphi-$ concave– convex operator.

Remark 1.1. *The $\tau - \varphi-$concave–convex operator as well as the $\varphi-$ concave– convex operator is a generalization of the $e-$ concave- convex operator which is introduced in [15].*

Remark 1.2. *Condition (C_2) in Definition 1.3 implies that*

$$A\left(\frac{1}{\tau(t)}x, \tau(t)y\right) \leq \frac{1}{\varphi\left(t, \frac{1}{\tau(t)}x, \tau(t)y\right)}A(x,y), \ \forall t \in (a,b), \forall (x,y) \in P \times P. \tag{1.1}$$

2 Main results

In this section, we present our main results.

Theorem 2.1. *Let E be a real Banach space and P be a normal cone in E. Suppose that an operator $A : P \times P \to P$ is mixed monotone and $\tau - \varphi-$concave-convex. In addition, suppose that there exists $h \in P^+$ such that $A(h,h) \in P_h$ and for any $t \in (a,b)$ and $x,\ y \in P_h$, $\varphi(t,x,y) \geq \varphi(t,h,h)$. Then operator A has a unique fixed point x^* in P_h and for any initial $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \cdots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0 (n \to \infty)$.

Proof. Since $A(h,h) \in P_h$, we can choose a sufficiently small number $\lambda_0 \in (0,1)$ such that

$$\lambda_0 h \leq A(h,h) \leq \frac{1}{\lambda_0} h.$$

It follows from (C_1) that there exists $t_0 \in (a,b)$ such that $\tau(t_0) = \lambda_0$, hence

$$\tau(t_0) h \leq A(h,h) \leq \frac{1}{\tau(t_0)} h. \tag{2.1}$$

From (C_3), we know that $\varphi(t_0,h,h) > \tau(t_0)$. Thus, $\varphi(t_0,h,h)/\tau(t_0) > 1$ and we can take a positive integer k such that

$$\left(\frac{\varphi(t_0,h,h)}{\tau(t_0)}\right)^k \geq \frac{1}{\tau(t_0)}, \tag{2.2}$$

which implies

$$\left(\frac{\tau(t_0)}{\varphi(t_0,h,h)}\right)^k \leq \tau(t_0). \tag{2.3}$$

Set $u_0 = [\tau(t_0)]^k h$, $v_0 = 1/[\tau(t_0)]^k h$. Clearly, $u_0, v_0 \in P_h$ and $u_0 = [\tau(t_0)]^{2k} v_0 < v_0$. Take any $r \in [0, [\tau(t_0)]^{2k}]$, then $r \in (0,1)$ and $u_0 \geq r v_0$. By the mixed monotonicity of A, we get $A(u_0, v_0) \leq A(v_0, u_0)$. Combining condition (C_2) with (2.1) and (2.2), we obtain

$$\begin{aligned}
A(u_0,v_0) &= A\left([\tau(t_0)]^k h, \frac{1}{[\tau(t_0)]^k} h\right) = A\left(\tau(t_0)\cdot[\tau(t_0)]^{k-1} h, \frac{1}{\tau(t_0)}\cdot\frac{1}{[\tau(t_0)]^{k-1}} h\right) \\
&\geq \varphi\left(t_0, [\tau(t_0)]^{k-1} h, \frac{1}{[\tau(t_0)]^{k-1}} h\right) A\left([\tau(t_0)]^{k-1} h, \frac{1}{[\tau(t_0)]^{k-1}} h\right) \\
&\geq \varphi\left(t_0, [\tau(t_0)]^{k-1} h, \frac{1}{[\tau(t_0)]^{k-1}} h\right) \cdots \varphi\left(t_0, \tau(t_0) h, \frac{1}{\tau(t_0)} h\right) \varphi(t_0,h,h) A(h,h) \\
&\geq [\varphi(t_0,h,h)]^k \cdot \tau(t_0) h \geq [\tau(t_0)]^k h = u_0.
\end{aligned}$$

For $t \in (a,b)$, from (1.1) and (2.3), we have

$$\begin{aligned} A(v_0,u_0) &= A\left(\frac{1}{[\tau(t_0)]^k}h, [\tau(t_0)]^k h\right) = A\left(\frac{1}{\tau(t_0)}\cdot\frac{1}{[\tau(t_0)]^{k-1}}h, \tau(t_0)\cdot[\tau(t_0)]^{k-1}h\right) \\ &\le \frac{1}{\varphi\left(t_0, \frac{1}{[\tau(t_0)]^{k-1}}h, [\tau(t_0)]^{k-1}h\right)} A\left(\frac{1}{[\tau(t_0)]^{k-1}}h, [\tau(t_0)]^{k-1}h\right) \\ &\le \frac{1}{\varphi\left(t_0, \frac{1}{[\tau(t_0)]^{k-1}}h, [\tau(t_0)]^{k-1}h\right)} \cdots \frac{1}{\varphi\left(t_0, \frac{1}{\tau(t_0)}h, \tau(t_0)h\right)} \cdot \frac{1}{\varphi(t_0,h,h)} A(h,h) \\ &\le \frac{1}{[\varphi(t_0,h,h)]^k}\cdot\frac{1}{\tau(t_0)}h \le \frac{1}{[\tau(t_0)]^k}h = v_0. \end{aligned}$$

Hence we get

$$u_0 \le A(u_0,v_0) \le A(v_0,u_0) \le v_0. \tag{2.4}$$

Construct successively the sequences

$$u_n = A(u_{n-1},v_{n-1}), v_n = A(v_{n-1},u_{n-1}), \quad n = 1,2,\cdots.$$

By the mixed monotonicity of A, we have

$$u_1 = A(u_0,v_0) \le A(v_0,u_0) = v_1.$$

By induction, we get $u_n \le v_n, \ \ n = 1,2\cdots$. Then from (2.4) and the mixed monotonicity of A, we have

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots v_n \le \cdots v_1 \le v_0. \tag{2.5}$$

Noting that $u_0 \ge rv_0$, we get $u_n \ge u_0 \ge rv_0 \ge rv_n, \ \ n = 1,2,\cdots$. Set

$$r_n = \sup\{r > 0 | u_n \ge rv_n\}, \quad n = 1,2,\cdots,$$

then we have $u_n \ge r_n v_n, \ \ n = 1,2,\cdots$ and

$$u_{n+1} \ge u_n \ge r_n v_n \ge r_n v_{n+1}, \quad n = 1,2,\cdots.$$

So $r_{n+1} \ge r_n$, namely, $\{r_n\}$ is increasing with $\{r_n\} \subset (0,1]$. Assume $r_n \to r^*$ as $n \to \infty$. Next we prove $r^* = 1$.

Indeed, suppose to the contrary that $0 < r^* < 1$. From (C_1), there is $t_1 \in (a,b)$ such that $\tau(t_1) = r^*$. Distinguish two cases:

Case one: There exists a positive integer N such that $r_N = r^*$. In this case, we get $r_n = r^*$ for all $n \ge N$. Therefore, for all $n \ge N$, we have

$$\begin{aligned} u_{n+1} &= A(u_n,v_n) \ge A\left(r^* v_n, \frac{1}{r^*}u_n\right) = A\left(\tau(t_1)v_n, \frac{1}{\tau(t_1)}u_n\right) \\ &\ge \varphi(t_1,v_n,u_n)A(v_n,u_n) \ge \varphi(t_1,h,h)A(v_n,u_n) = \varphi(t_1,h,h)v_{n+1}. \end{aligned}$$

By the definition of r_n, we obtain $r_{n+1} = r^* \geq \varphi(t_1, h, h) > \tau(t_1) = r^*$, which is a contradiction.

Case two: For all positive integer n, $r_n < r^*$. Then we get $0 < r_n/r^* < 1$. By (C_1), there exist $s_n \in (a, b)$ such that $\tau(s_n) = r_n/r^*$. So we have

$$\begin{aligned} u_{n+1} &= A(u_n, v_n) \geq A(r_n v_n, \frac{1}{r_n} u_n) = A\left(\frac{r^n}{r^*} r^* v_n, \frac{r^*}{r^n}\frac{1}{r^*} u_n\right) \\ &= A\left(\tau(s_n) \cdot r^* v_n, \frac{1}{\tau(s_n)} \cdot \frac{1}{r^*} u_n\right) \geq \varphi(s_n, r^* v_n, \frac{1}{r^*} u_n) A(r^* v_n, \frac{1}{r^*} u_n) \\ &\geq \varphi(s_n, h, h) A(r^* v_n, \frac{1}{r^*} u_n) = \varphi(s_n, h, h) A\left(\tau(t_1) v_n, \frac{1}{\tau(t_1)} u_n\right) \\ &\geq \varphi(s_n, h, h)\varphi(t_1, h, h) A(v_n, u_n) = \varphi(s_n, h, h)\varphi(t_1, h, h) v_{n+1}. \end{aligned}$$

By the definition of r_n, we have

$$r_{n+1} \geq \varphi(s_n, h, h)\varphi(t_1, h, h) > \tau(s_n)\varphi(t_1, h, h) = \frac{r_n}{r^*}\varphi(t_1, h, h).$$

Let $n \to \infty$, we obtain $r^* \geq (r^*/r^*)\varphi(t_1, h, h) > \tau(t_1) = r^*$, which is also a contradiction. Hence $\lim_{n\to\infty} r_n = 1$. For any natural number m, we have

$$\theta \leq u_{n+m} - u_n \leq v_n - u_n \leq v_n - r_n v_n \leq (1 - r_n) v_n \leq (1 - r_n) v_0;$$

$$\theta \leq v_n - v_{n+m} \leq v_n - u_n \leq (1 - r_n) v_0.$$

By the normality of P, we get, as $n \to \infty$,

$$\| u_{n+m} - u_n \| \leq N \| (1 - r_n) v_0 \| \to 0;$$

$$\| v_n - v_{n+m} - \| \leq N \| (1 - r_n) v_0 \| \to 0.$$

Here N is the normality constant of P. So $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Since E is complete, there exist u^* and v^* such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. From (2.5), we have $u^*, v^* \in [u_0, v_0]$ and for any n, $u_n \leq u^* \leq v^* \leq v_n$. Therefore,

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - r_n) v_0.$$

Since P is normal,

$$\| v^* - u^* \| \leq N \| (1 - r_n) v_0 \| \to 0$$

as $n \to \infty$. Thus $u^* = v^*$. Let $x^* = u^* = v^*$, we have, by the mixed monotonicity of A,

$$u_{n+1} = A(u_n, v_n) \leq A(x^*, x^*) \leq A(v_n, u_n) = v_{n+1}.$$

Take $n \to \infty$, the normality of P implies $x^* = A(x^*, x^*)$. Namely, x^* is a fixed point of A.

In the following, we prove that x^* is the unique fixed point of A in P_h. In fact, assume $\widetilde{x}$ is also a fixed point of A in P_h. By the definition of P_h, there exist positive numbers $\lambda_1, \lambda_2, \eta_1, \eta_2$ such that

$$\lambda_1 h \le x^* \le \lambda_2 h, \quad \eta_1 h \le \widetilde{x} \le \eta_2 h.$$

Then,

$$\widetilde{x} \ge \eta_1 h \ge \frac{\eta_1}{\lambda_2}\lambda_2 h \ge \frac{\eta_1}{\lambda_2}x^*.$$
$$x^* \ge \lambda_1 h \ge \frac{\lambda_1}{\eta_2}\eta_2 h \ge \frac{\lambda_1}{\eta_2}\widetilde{x}.$$

Put $\gamma_1 = \sup\{\gamma > 0 : \widetilde{x} \ge \gamma x^*, x^* \ge \gamma\widetilde{x}\}$. Clearly, $0 < \gamma_1 < +\infty$. Furthermore, we prove $\gamma_1 \ge 1$. Suppose to the contrary that $0 < \gamma_1 < 1$, by (C_1), there exists $t_2 \in (a, b)$ such that $\tau(t_2) = \gamma_1$. Thus,

$$\begin{aligned}\widetilde{x} &= A(\widetilde{x}, \widetilde{x}) \ge A\left(\gamma_1 x^*, \frac{1}{\gamma_1}x^*\right) = A\left(\tau(t_2)x^*, \frac{1}{\tau(t_2)}x^*\right)\\ &\ge \varphi(t_2, x^*, x^*)A(x^*, x^*) = \varphi(t_2, x^*, x^*)x^*.\end{aligned}$$

Because $\varphi(t_2, x^*, x^*) > \tau(t_2) = \gamma_1$, this is in contradiction with the definition of γ_1. Thus $\gamma_1 \ge 1$ and then we have $\widetilde{x} \ge x^*$ and $x^* \ge \widetilde{x}$. Hence $\widetilde{x} = x^*$, that is A has a unique fixed point in P_h.

For any initial $x_0, y_0 \in P_h$, we can choose a small number $e_1, e_2 \in (0, 1)$ such that

$$e_1 h \le x_0 \le \frac{1}{e_1}h, \quad e_2 h \le y_0 \le \frac{1}{e_2}h.$$

By (C_1) again, there exist $t_3, t_4 \in (a, b)$ such that $\tau(t_3) = e_1, \tau(t_4) = e_2$. Hence,

$$\tau(t_3)h \le x_0 \le \frac{1}{\tau(t_3)}h, \quad \tau(t_4)h \le y_0 \le \frac{1}{\tau(t_4)}h.$$

We can choose a sufficiently large positive integer m such that

$$\left[\frac{\varphi(t_3, h, h)}{\tau(t_3)}\right]^m \ge \frac{1}{\tau(t_3)}, \quad \left[\frac{\varphi(t_4, h, h)}{\tau(t_4)}\right]^m \ge \frac{1}{\tau(t_4)}.$$

Let $\alpha = \min\{\tau(t_3), \tau(t_4)\}$, then $\alpha \in (0, 1)$. By (C_1), there exists $\mu_1 \in (a, b)$ such that $\tau(\mu_1) = \alpha$. Put $\overline{u}_0 = [\tau(\mu_1)]^m h$, $\overline{v}_0 = (1/[\tau(\mu_1)]^m)h$. Then $\overline{u}_0, \overline{v}_0 \in P_h$ and $\overline{u}_0 < x_0, y_0 < \overline{v}_0$. Let

$$\overline{u}_n = A(\overline{u}_{n-1}, \overline{v}_{n-1}), \overline{v}_n = A(\overline{v}_{n-1}, \overline{u}_{n-1})$$

and

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}).$$

By a proof similar to that of the existence of x^*, we can prove that there exists $y^* \in P_h$ such that $\lim_{n\to\infty}\overline{u}_n = \lim_{n\to\infty}\overline{v}_n = y^*$ and $A(y^*, y^*) = y^*$. The uniqueness of fixed point of A implies $y^* = x^*$. By induction, we can get $\overline{u}_n \le x_n, y_n \le \overline{v}_n \quad n = 1, 2, \cdots$. Since P is normal, we have $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$. □

Remark 2.1. *If the hypothesis (C_3) in Definition1.3 is replaced by*
(C_3') $\varphi(t,x,y)$ is non-decreasing in x and is non-increasing in y, and the other conditions are unchangeable in Theorem 2.1, then the conclusions of Theorem 2.1 are also true.

Remark 2.2. *If we suppose that operator $A: P_h \times P_h \to P_h$, then $A(h,h) \in P_h$ is clear. So we have the following corollaries.*

Corollary 2.1. *Let E be a real Banach space and P be a normal cone in E. Suppose $h > \theta$ and that an operator $A: P_h \times P_h \to P_h$ is mixed monotone and $\tau - \varphi-$concave-convex. In addition, suppose for any $t \in (a,b)$ and $x,\ y \in P_h$, $\varphi(t,x,y) \geq \varphi(t,h,h)$. Then operator A has a unique fixed point x^* in P_h and for any initial $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \cdots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0 (n \to \infty)$.

Corollary 2.2. *(see [15]) Let E be a real Banach space and P be a normal cone in E. Suppose $h > \theta$ and that $A: P_h \times P_h \to P_h$ is a mixed monotone operator. Suppose that there exists an $\eta(t,u,v) > 0$ such that*

$$A\left(tu, \frac{1}{t}v\right) \geq t(1+\eta(t,u,v))A(u,v), \quad \forall\ u, v \in P_h, \quad 0 < t < 1. \tag{2.6}$$

Then A has a unique fixed point x^* in P_h and for any initial $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \cdots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0 (n \to \infty)$.

Proof. In Theorem 2.1, put $\tau(t) = t$ and $\varphi(t,u,v) = t(1+\eta(t,u,v))$, then all the conditions in Theorem 2.1 are satisfied. Thus, from Theorem 2.1, we prove Corollary 2.2. □

Remark 2.3. *The corresponding results in [15] turn out to be special cases of our main results, (see Theorem 3.1 and 3.2, Corollary 4.1 and 4.2 in [15]).*

Remark 2.4. *In our results, as well as in [15], we need not assume that the operator A is compact and continuous or that the condition of the coupled upper–lower solutions holds. Thus our results in essence improve and extend relevant results in [11-13].*

3 Applications

In this section, we investigate the existence of solutions for a class of nonlinear Hammerstein integral equations on unbounded regions by using some of the fixed-point theorems given above.

Example : Considering the following nonlinear integral equation:

$$x(t) = (Ax)(t) = \int_{\mathbb{R}^n} K(t,s)\left[x^{\alpha}(s) + x^{-\beta}(s)\right] ds, \tag{3.1}$$

where $\alpha, \beta \in (0,1)$ are positive constants.

Conclusion: Suppose that $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ is a positive and continuous function. Then equation (3.1) has a unique positive solution $x^*(t)$. Moreover, constructing successively the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with

$$x_n(t) = \int_{\mathbb{R}^n} K(t,s)\left[x_{n-1}^{\alpha}(s) + y_{n-1}^{-\beta}(s)\right] ds,$$

$$y_n(t) = \int_{\mathbb{R}^n} K(t,s)\left[y_{n-1}^{\alpha}(s) + x_{n-1}^{-\beta}(s)\right] ds$$

for any positive continuous functions $x_0(t), y_0(t)$ defined on $\mathbb{R}^n$, we have, as $n \to \infty$,

$$\sup_{t\in\mathbb{R}^n} |x_n(t) - x^*(t)| \to 0 \quad \text{and} \quad \sup_{t\in\mathbb{R}^n} |y_n(t) - x^*(t)| \to 0.$$

Proof. Let $E = C_B(\mathbb{R}^n)$ be the space of continuous functions defined on $\mathbb{R}^n$. For any $x \in E$, define $\|x\| = \sup_{t\in\mathbb{R}^n} |x(t)|$, then E is a Banach space. Denote by $P = C_B^+(\mathbb{R}^n)$ the set of all nonnegative functions on $C_B(\mathbb{R}^n)$, then P is a normal solid cone in E. Hence, giving a positive continuous function $h(t)$ on $\mathbb{R}^n$, we know that there exist positive constants $\lambda_1, \lambda_2 \in (0,1)$ such that, for any $t \in \mathbb{R}^n$, $\lambda_1 h(t) \le x_0(t) \le (1/\lambda_1)h(t)$ and $\lambda_2 h(t) \le y_0(t) \le (1/\lambda_2)h(t)$.

Clearly, equation (3.1) can be denoted by $x = A(x,x)$, where $A(x,y) = A_1(x) + A_2(y)$, $A_1(x) = \int_{\mathbb{R}^n} K(t,s)x^{\alpha}(s)ds$, $A_2(y) = \int_{\mathbb{R}^n} K(t,s)y^{-\beta}(s)ds$. Then $A : P \times P \to P$.

In following, we prove that A satisfies all the conditions of Theorem 2.1.

Assume that P defines a partial order $\le$ in E, then $A : P \times P \to P$ is a mixed monotone operator. Set

$$P_h = \{x \in E : \quad \text{there exist} \quad \lambda, \mu > 0 \quad \text{such that} \quad \lambda h(t) \le x(t) \le \mu h(t), \quad \forall t \in \mathbb{R}^n\}.$$

Since $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ is a positive and continuous function, $A(h,h) \in P_h$. Let $\alpha_0 = \min\{\alpha, \beta\}$, then $\alpha_0 \in (0,1)$. For any map $\tau(t) : (a,b) \to (0,1)$, where $a < b$, let $\varphi(t) = \tau(t)^{\alpha_0}$, then A is a $\tau - \varphi-$ concave-convex operator. Thus, by Theorem 2.1, we know that equation (3.1) has a unique positive solution $x^*(t)$. □

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Expressions and Iterative Methods for the Weighted Group Inverses of Linear Operators on Banach Space

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Abstract

In this note, some expressions and characterizations for the weighted group inverses $A^{\sharp}_{W}$ of operator A by using the technique of block operator matrix are given, three iterative methods for computing $A^{\sharp}_{W}$ are established, and the necessary and sufficient conditions for iterative convergence to $A^{\sharp}_{W}$ are discussed.

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Key words: weighted group inverse, operator matrix, expression, iterative method

1 Introduction

The Drazin inverse and the W-weighted Drazin inverse are applied to varions fields, for example, applications in singular differential and difference equations, Markov chains, iterative methods and numerical analysis.

Recently, a new weighted group inverse of the rectangular matrix had been defined in [1], the weighted group inverse $A^{\sharp}_{W}$ is not the special case of the W-weighted Drazin inverse, which was defined by Cline and Greville in [2]. Some properties, expressions and the uniqueness have been proved by Cen in [1]. A constructive perturbation bound of the Drazin inverse of a square matrix was derived in [3]. The algebra perturbation and analytical perturbation of the weighted group inverse $A^{\sharp}_{W}$ were studied in [5]. several approximate methods for the W-weighted Drazin inverse of bounded linear operators on Banach space were presented in [6].

In this note, we mainly consider the weighted group inverse of operators on Banach space. For any operator $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, we denote its range, null space and spectral

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radius by $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\rho(A)$, respectively. Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, if there exists some $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, such that $ABA = A$ holds, then B is an inner generalized inverse of A, and the operator A is called relatively regular.

Definition 1.1. *[1] Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. If $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ satisfies the system of operator equations (W-system of operator equations)*

$$(W1)AWXWA = A, (W2)XWAWX = X, (W3)AWX = XWA$$

Then X is called the weighted group inverse of A with W, and denoted by $A^{\sharp}_{W}$. If $A^{\sharp}_{W}$ exists, then it is unique.

Definition 1.2. *[4] Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. If there exist: a Banach space Z, and operators $P \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ and $Q \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, such that P is left invertible, Q is right invertible and*

$$A = PQ \tag{1.1}$$

Then (1.1) *is the full-rank decomposition of A.*

It is easy to see that an operator $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has the full-rank decomposition, if and only if A is relatively regular.

The following lemma is needed in what follows.

Lemma 1.1. *Let $A \in \mathcal{L}(\mathcal{X})$ and $A^{\sharp}$ exists. Then $X = R(A) \oplus N(A)$, and*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A) \end{pmatrix}. \tag{1.2}$$

where A_1 is invertible.

Moreover,

$$A^{\sharp} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A) \end{pmatrix}. \tag{1.3}$$

Proof. The result can be proved similarly as in [4]. □

2 Existence conditions, expressions and the characterization of the weighted group inverse $A^{\sharp}_{W}$

In this section, we will give some existence conditions for the weighted group inverse $A^{\sharp}_{W}$, and also give several expressions for the inverse $A^{\sharp}_{W}$ by using operator matrix blocks.

Theorem 2.1. *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, and $A = PQ$ is the full-rank decomposition of A, where $P \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ and $Q \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. then the following statements are equivalent:*
(i) *$A^{\sharp}_{W}$ exists;*
(ii) *$(AW)^{\sharp}$ and $(WA)^{\sharp}$ exist, and $R(AW) = R(A), N(WA) = N(A)$;*
(iii) *$R(WA) \oplus N(A) = X, N(AW) \oplus R(A) = Y$;*
(iv) *QWP is invertible.*

Proof. (i) $\Rightarrow$ (ii): If $A^{\sharp}_{W}$ exists, then we set X_0 is the solution to the W-system of operator equations. From $(W1)$, $AWX_0WA = A$, we get $R(AW) = R(A), N(WA) = N(A)$.

From $(W1)$ and $(W3)$, $(AW)^2X_0 = A$, $X_0(WA)^2 = A$, hence, $R((AW)^2) = R(A) = R(AW)$, $N((WA)^2) = N(A) = N(WA)$.

From $(W1)$, we have $AWX_0WAW = AW, WAWX_0WA = WA$, combining $(W3)$, we get $X_0W(AW)^2 = AW, (WA)^2WX_0 = WA$, therefore, $N((AW)^2) = N(AW), R((WA)^2) = R(WA)$. Hence $ind(AW) = ind(WA) = 1$, i.e., $(AW)^{\sharp}$ and $(WA)^{\sharp}$ exist.

(ii) $\Rightarrow$ (iii): $(AW)^{\sharp}$ exists is equivalent to $R(AW) \oplus N(AW) = Y$, $(WA)^{\sharp}$ exists is equivalent to $R(WA) \oplus N(WA) = X$, since $R(AW) = R(A), N(WA) = N(A)$, then $R(WA) \oplus N(A) = X, N(AW) \oplus R(A) = Y$.

(iii) $\Rightarrow$ (iv): Consider an arbitrary $x \in Z$ satisfying $QWPx = 0$. then $PQWPx = 0$, which implies $AWPx = 0$, then $Px \in N(AW)$. Since $Px = PQQ_rx = AQ_rx \in R(A)$, where Q_r is the right inverse of Q. So $Px \in N(AW) \cap R(A) = 0$. Since P is left invertible, hence $x = 0$ and QWP is invertible.

(iv) $\Rightarrow$ (i): If QWP is invertible, let $X_0 = P(QWP)^{-2}Q$, it is easy to show that X_0 is a solution to the W-system of operator equations, then $A^{\sharp}_{W}$ exists. □

By Theorem 2.1, and From Lemma 1.1,we have the following result:

Theorem 2.2. *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is relatively regular, $W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, If $A^{\sharp}_{W}$ exists, then:*
(i)$A^{\sharp}_{W} = [(AW)^{\sharp}]^2A$;
(ii)$A^{\sharp}_{W} = A[(WA)^{\sharp}]^2$;
(iii)$A^{\sharp}_{W} = (AW)^{\sharp}A(WA)^{\sharp}$;
(iv)$(AW)^{\sharp} = A^{\sharp}_{W}W$;
(v)$(WA)^{\sharp} = WA^{\sharp}_{W}$;
(vi)$A^{\sharp}_{W} = (WAW)^{(1,2)}_{R(A),N(A)}$.

Next, we present the characterizations of the weighted group inverse $A^{\sharp}_{W}$ by using the above expressions of $A^{\sharp}_{W}$.

Theorem 2.3. *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is relatively regular, $W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, If $A^{\sharp}_{W}$ exists. Then there exist a unique operator $X \in \mathcal{L}(\mathcal{Y})$ such that*

$$AWX = 0, XAW = 0, X^2 = X, AW + X \text{ is invertible} \tag{2.1}$$

and a unique operator $Y \in \mathcal{L}(\mathcal{X})$ such that

$$WAY = 0, YWA = 0, Y^2 = Y, WA + Y \text{ is invertible} \tag{2.2}$$

Further, we have

$$X = I - A^{\sharp}_{W}WAW \tag{2.3}$$

and

$$Y = I - WAWA_W^{\sharp} \tag{2.4}$$

Proof. By Theorem 2.2, we get

$$I - A_W^{\sharp}WAW = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

It is easy to verify that $X = I - A_W^{\sharp}WAW$ satisfies (2.1), which shows the existence of such operator X.
To show the uniqueness, let X_0 be an operator which satisfies (2.1) and let X_0 be partitioned as

$$X_0 = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

with respect to the space decomposition $N(AW) \oplus R(A) = Y$.
By Equation (2.1),

$$0 = AWX_0 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} A_1X_{11} & A_1X_{12} \\ 0 & 0 \end{pmatrix}$$

$$0 = X_0AW = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}A_1 & 0 \\ X_{21}A_1 & 0 \end{pmatrix}$$

so that $X_{11} = 0$, $X_{12} = 0$ and $X_{21} = 0$.

Since $AW + X_0$ is invertible, we get that X_{22} is invertible. From the assumption that $X^2 = X$, we have $X_{22}^2 = X_{22}$, that is, $(X_{22} - I)X_{22} = 0$. Thus, $X_{22} - I = 0$, i.e., $X_{22} = I$.

Finally, we get

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

Thus, we obtain $X_0 = X = I - A_W^{\sharp}WAW$. This shows that X unique exists.

The second statement is proved in a similar manner. □

3 The iterative methods for computing $A_W^{\sharp}$

In this section, we construct three iterative methods for computing $A_W^{\sharp}$, and prove the convergence by using operator matrix blocks.

Lemma 3.1. *Let $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is relatively regular, $W \in \mathcal{L}(\mathcal{Y},\mathcal{X})$, If $A_W^\sharp$ exists, then*
(1)$P_1 = WA(WA)^\sharp = (WA)^\sharp WA = P_{R(WA),N(A)}$;
(2)$P_2 = AW(AW)^\sharp = (AW)^\sharp AW = P_{R(A),N(AW)}$.

Theorem 3.1. *Let $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is relatively regular, $W \in \mathcal{L}(\mathcal{Y},\mathcal{X})$, If $A_W^\sharp$ exists. Define the sequence $(X_k)_k$ in $\mathcal{B}(\mathcal{X},\mathcal{Y})$ in the following way:*

$$\left\{ \begin{array}{rcl} R_k &=& P_1 - WAWX_k, \\ X_{k+1} &=& X_k(I+R_k), k = 0,1,2,\ldots. \end{array} \right. \tag{3.1}$$

Then the iteration (3.1) *converges to $A_W^\sharp$ if and only if $\mathcal{R}(X_0) = R(A), \rho(R_0) < 1$.*

Proof. By Theorem 2.3, we have $A_W^\sharp P_1 = A_W^\sharp P_{R(WA),N(A)} = A_W^\sharp$, and

$$P_1 = WA(WA)^\sharp = \left(\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right) : \left(\begin{array}{c} R(WA) \\ N(A) \end{array} \right) \to \left(\begin{array}{c} R(WA) \\ N(A) \end{array} \right)$$

Since,

$$\begin{array}{rcl} P_1R_k &=& P_1(P_1 - WAWX_k) \\ &=& P_1 - P_1WAWX_k \\ &=& P_1 - P_{R(WA),N(A)}WAWX_k \\ &=& P_1 - WAWX_k = R_k. \end{array}$$

Hence,

$$\begin{array}{rcl} R_k &=& P_1 - WAWX_k = P_1 - WAWX_{k-1}(I+R_{k-1}) \\ &=& P_1 - WAWX_{k-1} - WAWX_{k-1}R_{k-1} \\ &=& R_{k-1} - WAWX_{k-1}R_{k-1} \\ &=& P_1R_{k-1} - WAWX_{k-1}R_{k-1} \\ &=& (P_1 - WAWX_{k-1})R_{k-1} \\ &=& R_{k-1}^2 = \cdots = R_0^{2^k}. \end{array}$$

and

$$\begin{array}{rcl} X_{k+1} &=& X_k(I+R_k) \\ &=& X_{k-1}(I+R_{k-1})(I+R_k) \\ &=& \cdots = X_0(I+R_0)(I+R_1)\cdots(I+R_k) \\ &=& X_0(I+R_0)(I+R_0^2)\cdots(I+R_0^{2^k}) \\ &=& X_0(I+R_0+R_0^2+\cdots+R_0^{2^{k+1}-1}) \end{array}$$

Then,

$$X_{k+1}(I-R_0) = X_0(I-R_0^{2^{k+1}}) \tag{3.2}$$

SUFFICIENCY If $\rho(R_0) < 1$, then $R_0^{2^k} \to 0$ $(k \to \infty)$ and $I - R_0$ is invertible. From (3.2), we obtain $X_\infty = X_0(I - R_0)^{-1}$. From $R(X_0) = R(A)$, by Theorem 2.3, we can set

$$X_0 = \begin{pmatrix} X_{11} & X_{12} \\ 0 & 0 \end{pmatrix}$$

Then,

$$\begin{aligned} R_0 &= P_1 - WAWX_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} W_{11} & 0 \\ 0 & W_{22} \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I - W_{11}A_1X_{11} & -W_{11}A_1X_{12} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$I - R_0 = \begin{pmatrix} W_{11}A_1X_{11} & W_{11}A_1X_{12} \\ 0 & I \end{pmatrix}.$$

Since $I - R_0$ is invertible, then $W_{11}A_1X_{11}$ is invertible, which implies that X_{11} is invertible. Now

$$(I - R_0)^{-1} = \begin{pmatrix} X_{11}^{-1}A_1^{-1}W_{11}^{-1} & -X_{11}^{-1}X_{12} \\ 0 & I \end{pmatrix}.$$

It is easy to verify

$$\begin{aligned} X_\infty &= X_0(I - R_0)^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11}^{-1}A_1^{-1}W_{11}^{-1} & -X_{11}^{-1}X_{12} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A_1^{-1}W_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1^{-2}A_{11} & 0 \\ 0 & 0 \end{pmatrix} = A_W^\sharp. \end{aligned}$$

NECESSITY If X_k converges to $A_W^\sharp$, then $R_k \to P_1 - WAWA_W^\sharp = P_1 - WAW(AW)^\sharp A(WA)^\sharp = P_1 - WA(WA)^\sharp = 0 (k \to 0)$. Hence, we have $R_0^{2^k} \to 0 (k \to 0)$ and then $\rho(R_0) < 1$ and $I - R_0$ is invertible. From (3.2), we obtain $A_W^\sharp(I - R_0) = X_0$. So $R(X_0) = R(A_W^\sharp) = R(A)$. □

The iterative method defined in Theorem 3.1 is the Newton-Raphson method for computing $A_W^\sharp$. We now turn to study the pth-order iterative method for computing $A_W^\sharp$, where $p \geq 2$ is an integer.

Theorem 3.2. *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is relatively regular, $W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, If $A_W^\sharp$ exists. Define the sequence $(X_k)_k$ in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ in the following way:*

$$\begin{cases} R_k &= P_1 - WAWX_k, \\ X_{k+1} &= X_k(I + R_k + \cdots + R_k^{p-1}), \ p \geq 2, k = 0, 1, 2, \ldots. \end{cases} \tag{3.3}$$

Then the iteration (3.3) converges to $A_W^\sharp$ if and only if $\mathcal{R}(X_0) = R(A), \rho(R_0) < 1$.

Proof. Since

$$A_W^\sharp P_1 = A_W^\sharp, P_1 R_k = R_k, (k \geq 0).$$

and

$$\begin{aligned} R_{k+1} &= P_1 - WAWX_{k+1} \\ &= P_1 - WAWX_k(I + R_k + \cdots + R_k^{p-1}) \\ &= P_1 - WAWX_k - WAWX_k R_k(I + R_k + \cdots + R_k^{p-2}) \\ &= R_k - WAWX_k R_k - WAWX_k R_k^2(I + R_k + \cdots + R_k^{p-3}) \\ &= (P_1 - WAWX_k)R_k - WAWX_k R_k^2(I + R_k + \cdots + R_k^{p-3}) \\ &= R_k^2 - WAWX_k R_k^2(I + R_k + \cdots + R_k^{p-3}) \\ &= \cdots = R_k^p = \cdots = R_0^{p^{k+1}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} X_{k+1} &= X_k(I + R_k + \cdots + R_k^{p-1}) \\ &= X_0(I + R_0 + \cdots + R_0^{p-1})(I + R_1 + \cdots + R_1^{p-1}) \cdots (I + R_k + ... + R_k^{p-1}) \\ &= X_0(I + R_0 + \cdots + R_0^{p-1})(I + R_0^p + \cdots + (R_0^p)^{p-1}) \cdots (I + R_0^{p^k} + \cdots + (R_0^{p^k})^{p-1}) \\ &= X_0(I + R_0 + \cdots + R_0^{p^{k+1}-1}). \end{aligned}$$

Hence,

$$X_{k+1}(I - R_0) = X_0(I - R_0^{p^{k+1}}) \tag{3.4}$$

SUFFICIENCY If $\rho(R_0) < 1$, then $R_0^{p^k} \to 0$ $(k \to \infty)$ and $I - R_0$ is invertible. From (3.4), we obtain $X_\infty = X_0(I - R_0)^{-1}$. Similarly to Theorem 3.1, we have

$$X_\infty = X_0(I - R_0)^{-1} = \begin{pmatrix} A_1^{-2}A_{11} & 0 \\ 0 & 0 \end{pmatrix} = A_W^\sharp$$

NECESSITY Similar to the prove of the necessity of Theorem 3.1. □

The iterative method defined in Theorem 3.2 is the hyper power method for computing $A_W^\sharp$. when $p = 2$, it is the Newton-Raphson method. Similar to the prove of the of Theorem 3.1, we have the Euler-Knopp method:

Theorem 3.3. *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is relatively regular, $W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, If $A_W^\sharp$ exists. Define the sequence $(X_k)_k$ in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ in the following way:*

$$X_k = X_0 \sum_{j=0}^{k} (P_1 - WAWX_0)^j, k = 0, 1, 2, \ldots. \tag{3.5}$$

Then the iteration (3.5) *converges to $A_W^\sharp$ if and only if $\mathcal{R}(X_0) = R(A), \rho(R_0) < 1$.*

4 Example

Example: $A = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $W = \begin{pmatrix} 2 & 1 \end{pmatrix}$, $AW = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$, $WA = 5$, in order to meet $R(X_0) = R(A)$, $\rho(P_1 - WAWX_0) < 1$, we can set $X_0 = \alpha A$, where α satisfies

$$0 < \alpha < \frac{2}{25}$$

Let $\alpha = 0.06$, for the Newton-Raphson iteration, we have

k	X_k	R_k
0	$\begin{pmatrix} 0.0600 \\ 0.1800 \end{pmatrix}$	-0.5000
1	$\begin{pmatrix} 0.0300 \\ 0.0900 \end{pmatrix}$	0.2500
2	$\begin{pmatrix} 0.0375 \\ 0.1125 \end{pmatrix}$	0.0625
3	$\begin{pmatrix} 0.0398 \\ 0.1195 \end{pmatrix}$	0.0039
4	$\begin{pmatrix} 0.0400 \\ 0.1200 \end{pmatrix}$	$1.5259e-005$

$X_5 = X_4$, choose $A_W^\sharp = X_4$. By identity $A_W^\sharp = A[(WA)^\sharp]^2$, we can compute that $A_W^\sharp = \begin{pmatrix} 0.04 \\ 0.12 \end{pmatrix}$, which is consistent with X_4.

And for the Euler-Knopp iteration, we have

k	X_k	R_k
0	$\begin{pmatrix} 0.0600 \\ 0.1800 \end{pmatrix}$	-0.5000
1	$\begin{pmatrix} 0.0300 \\ 0.0900 \end{pmatrix}$	0.2500
2	$\begin{pmatrix} 0.0450 \\ 0.1350 \end{pmatrix}$	-0.1250
3	$\begin{pmatrix} 0.0375 \\ 0.1125 \end{pmatrix}$	0.0625
4	$\begin{pmatrix} 0.0412 \\ 0.1238 \end{pmatrix}$	-0.0313
5	$\begin{pmatrix} 0.0394 \\ 0.1181 \end{pmatrix}$	0.0156

6	$\begin{pmatrix} 0.0403 \\ 0.1209 \end{pmatrix}$	-0.0078
7	$\begin{pmatrix} 0.0398 \\ 0.1195 \end{pmatrix}$	0.0039
8	$\begin{pmatrix} 0.0401 \\ 0.1202 \end{pmatrix}$	-0.0020
9	$\begin{pmatrix} 0.0400 \\ 0.1199 \end{pmatrix}$	$9.7656e-004$
10	$\begin{pmatrix} 0.0400 \\ 0.1201 \end{pmatrix}$	$-4.8828e-004$
11	$\begin{pmatrix} 0.0400 \\ 0.1200 \end{pmatrix}$	$2.4414e-004$

$X_{12} = X_{11}$, choose $A_W^\sharp = X_{11}$. Then $A_W^\sharp$ is consistent with X_{11}.

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Briot-Bouquet strong differential subordination

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Abstract

The concept of differential subordination was introduced in [2] by S.S. Miller and P.T. Mocanu and developed in [3]. The concept of strong differential subordination was introduced in [1] by J.A. Antonino and S. Romaguera. In [4] the authors developed the concept of strong differential subordination. In [5], the authors introduced the class $\mathcal{H}^*[a,n,\xi]$ and studied strong differential subordinations for the functions of the class. In this paper we define the concept of Briot-Bouquet strong differential subordination for the functions of the class $\mathcal{H}^*[a,n,\xi]$.

Keywords: analytic function, differential subordination, strong differential subordination, univalent function, dominant, best dominant, convex function, differential equation.
2000 Mathematical Subject Classification: 30C80, 30C45, 30A20.

1 Introduction and preliminaries

Let U denote the unit disc of the complex plane $U=\{z\in\mathbb{C}:\ |z|<1\}$ and $\overline{U}=\{z\in\mathbb{C}:\ |z|\leq 1\}$.

Let $\mathcal{H}(U\times\overline{U})$ denote the class of analytic functions in $U\times\overline{U}$. In [5] the authors introduced the class $\mathcal{H}^*[a,n,\xi]=\{f\in\mathcal{H}(U\times\overline{U})\mid f(z,\xi)=a+a_n(\xi)z^n+a_{n+1}(\xi)z^{n+1}+\ldots,\ z\in U,\ \xi\in\overline{U}\}$, with $a_k(\xi)$ holomorphic functions in $\overline{U}$, $k\geq n$, and $\mathcal{H}_u(U)=\{f\in\mathcal{H}^*[a,n,\xi]:\ f(\cdot,\xi)$ univalent in U for all $\xi\in\overline{U}\}$.

We define $K^*(\alpha)=\left\{f\in\mathcal{H}^*[a,n,\xi]\mid \text{Re}\,\frac{zf''(z,\xi)}{f'(z,\xi)}+1>\alpha,\ z\in U,\ \xi\in\overline{U}\right\}$, the class of convex functions of order α, where $\alpha<1$ and let $K^*=K^*(0)$.

In order to prove our main results we use the following definitions and lemmas.

Definition 1.1 *[5] Let $H(z,\xi)$, $f(z,\xi)$ be analytic in $U\times\overline{U}$. The function $f(z,\xi)$ is said to be strongly subordinate to $H(z,\xi)$, or $H(z,\xi)$ is said to be strongly superordinate to $f(z,\xi)$, if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$ such that $f(z,\xi)=H(w(z),\xi)$, for all $\xi\in\overline{U}$. In such a case we write $f(z,\xi)\prec\prec H(z,\xi)$, $z\in U$, $\xi\in\overline{U}$.*

Remark 1.1 *(i) Since $f(z,\xi)$ is analytic in $U\times\overline{U}$, for all $\xi\in\overline{U}$ and univalent in U, for all $\xi\in\overline{U}$, Definition 1.1 is equivalent to $H(0,\xi)=f(0,\xi)$, for all $\xi\in\overline{U}$ and $f(U\times\overline{U})\subset H(U\times\overline{U})$.*

(ii) If $H(z,\xi)\equiv H(z)$ and $f(z,\xi)\equiv f(z)$ then the strong subordination becomes the usual notion of subordination.

Definition 1.2 *[5] We denote by Q the set of functions $q(\cdot,\xi)$ that are analytic and injective, as function of z on $\overline{U}\setminus E(q)$ where*

$$E(q)=\{\zeta\in\partial U:\ \lim_{z\to\zeta}q(z,\xi)=\infty\}$$

and are such that $q'(\zeta,\xi)\neq 0$ for $\zeta\in\partial U\setminus E(q)$, $\xi\in\overline{U}$. The subclass of Q for which $q(0,\xi)=a$ is denoted by $Q(a)$.

Definition 1.3 *[5] Let Ω_ξ be a set in $\mathbb{C}$, $q(\cdot,\xi)\in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega_\xi,q(\cdot,\xi)]$ consists of those functions $\psi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r,s,t;z,\xi)\notin\Omega_\xi, \tag{A}$$

whenever $r=q(\zeta,\xi)$, $s=m\zeta q'(\zeta,\xi)$, $\text{Re}\,\frac{t}{s}+1\geq m\text{Re}\left[\frac{\zeta q''(\zeta,\xi)}{q'(\zeta,\xi)}+1\right]$, $z\in U$, $\zeta\in\partial U\setminus E(q)$, $\xi\in\overline{U}$ and $m\geq n$.

We write $\Psi_1[\Omega_\xi, q(\cdot,\xi)]$ *as* $\Psi[\Omega_\xi, q(\cdot,\xi)]$.

In the special case when Ω_ξ *is a simply connected domain,* $\Omega_\xi \neq \mathbb{C}$, *and* $h(\cdot,\xi)$ *is a conformal mapping of* $U \times \overline{U}$ *onto* Ω_ξ *we denote this class by* $\Psi_n[h(\cdot,\xi), q(\cdot,\xi)]$.

If $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, *then the admissibility condition (A) reduces to*

$$\psi(r, s; z.\xi) \notin \Omega_\xi, \tag{A'}$$

whenever $r = q(\zeta,\xi)$, $s = \zeta q'(\zeta,\xi)$, $z \in U$, $\zeta \in \partial U \setminus E(q)$, $\xi \in \overline{U}$,*and* $m \geq n$.

If $\psi : \mathbb{C} \times U \times \overline{U} \to \mathbb{C}$, *then the admissibility condition (A) reduces to*

$$\psi(r; z, \xi) \notin \Omega_\xi \tag{A"}$$

whenever $r = q(\zeta,\xi)$, $z \in U$, $\xi \in \overline{U}$, $\zeta \in \partial U \setminus E(q)$.

Lemma 1.1 *[5, Th. 3] Let* $h(\cdot,\xi) \in H_u(U)$, *for all* $\xi \in \overline{U}$ *and* $q(,\xi) \in H_u(U)$, *for all* $\xi \in \overline{U}$, *with* $q(0,\xi) = a$, *and set* $q_\rho(z,\xi) = q(\rho z,\xi)$. *Let* $\psi : C^3 \times U \times \overline{U} \to C$ *satisfy one of the following conditions:*

(i) $\psi \in \Psi_n[h(\cdot,\xi), q_\rho(\cdot,\xi)]$, *for some* $\rho \in (0,1)$, *or*

(ii) there exists $\rho_0 \in (0,1)$ *such that* $\psi \in \Psi_n[h_\rho(\cdot,\xi), q_\rho(\cdot,\xi)]$ *for all* $\rho \in (\rho_0, 1)$.

If $p(\cdot,\xi) \in H^*[a,n,\xi]$, $\psi(p(z,\xi), zp'^2 p''(z,\xi); z, \xi)$ *is analytic in* $U \times \overline{U}$ *and*

$$\psi(p(z,\xi), zp'(z,\zeta), z^2 p''(z,\xi); z, \xi) \prec\prec h(z,\xi),$$

then

$$p(z,\xi) \prec\prec q(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Lemma 1.2 *[5, Th. 5] Let* $h(\cdot,\xi) \in H_u(U)$, *for all* $\xi \in \overline{U}$ *and let* $\psi : C^3 \times U \times \overline{U} \to C$. *Suppose that the differential equation*

$$\psi(q(z,\xi), nzq'(z,\xi), n(n-1)zq'(z,\zeta) + n^2 z^{2n} q''(z,\xi); z, \xi) = h(z,\xi),\ z \in U,\ \xi \in \overline{U},$$

has a solution $q(\cdot,\xi)$, *with* $q(0,\xi) = a$, *and one of the following conditions is satisfied:*

(i) $q(\cdot,\xi) \in Q$ *and* $\psi \in \Psi_n[h(\cdot,\xi), q(\cdot,\xi)]$,

(ii) $q(\cdot,\xi) \in H_u(U)$, *for all* $\xi \in \overline{U}$ *and* $\psi \in \Psi_n[h(\cdot,\xi), q_\rho(\cdot,\xi)]$

(iii) $q(\cdot,\xi) \in H_u(U)$, *for all* $\xi \in \overline{U}$ *and there exists* $\rho_0 \in (0,1)$ *such that*

$$\psi \in \Psi_n[h_\rho(\cdot,\xi), q_\rho(\cdot,\xi)] \text{ for all } \rho \in (\rho_0, 1).$$

If $p(\cdot,\xi) \in H^*[a,n,\xi]$, $\psi(p(z,\xi), zp'(z,\zeta), z^2 p''(z,\xi); z, \xi)$ *is analytic in* $U \times \overline{U}$ *and* $p(\cdot,\xi)$ *satisfies*

$$\psi(p(z,\xi), zp'(z,\zeta), z^2 p''(z,\xi); z, \xi) \prec\prec h(z,\xi), \quad z \in U,\ \xi \in \overline{U},$$

then

$$p(z,\xi) \prec\prec q(z,\xi), \quad z \in U,\ \xi \in \overline{U},$$

and $q(\cdot,\xi)$ *is the best dominant.*

2 Main results

Definition 2.1 *Let* $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, *the functions* $h(\cdot,\xi) \in \mathcal{H}_u(U)$, *for all* $\xi \in \overline{U}$, *with* $h(0,\xi) = a$, *and let* $p \in \mathcal{H}^*[a,n,\xi]$ *satisfy*

$$p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} \prec\prec h(z,\xi), \quad z \in U,\ \xi \in \overline{U}. \tag{1}$$

This first-order strong differential subordination is called Briot-Bouquet strong differential subordination. The name derives from the fact that a differential equation of the form

$$q(z,\xi) + \frac{zq'(z,\xi)}{\beta q(z,\xi) + \gamma} = h(z,\xi), \quad z \in U,\ \xi \in \overline{U}, \tag{2}$$

is called a differential equation of Briot-Bouquet type.

Remark 2.1 *(i) Strong differential subordination (1) is equivalent to: Let $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, and let Ω_ξ be any set in the complex plane $\mathbb{C}$, the function $p(\cdot, \xi) \in \mathcal{H}^*[a, n, \xi]$ and $h(\cdot, \xi) \in \mathcal{H}_u(U)$ for all $\xi \in \overline{U}$ satisfies*

$$\left\{ p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} \right\} \subset \Omega_\xi, \quad z \in U,\ \xi \in \overline{U}, \tag{1'}$$

where $h(U \times \overline{U}) = \Omega_\xi$.

(ii) More general, let $p(\cdot, \xi) \in \mathcal{H}^*[a, n, \xi]$ and $h(\cdot, \xi) \in \mathcal{H}^*[a, n, \xi]$.

$$p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} \prec\prec h(z,\xi) \ \Rightarrow\ p(z,\xi) \prec\prec q(z,\xi),\ z \in U,\ \xi \in \overline{U}. \tag{1"}$$

Related to the strong differential subordination (1) we can formulate the next two problems.

Problem 2.1 *Given $h(\cdot, \xi) \in \mathcal{H}_u(U)$, for all $\xi \in \overline{U}$ with $h(0, \xi) = p(0, \xi)$ what are the conditions for the function $q(\cdot, \xi) \in \mathcal{H}_u(U)$, for all $\xi \in \overline{U}$, with $q(0, \xi) = h(0, \xi)$, for all $\xi \in \overline{U}$ and find this function such that implication (1") holds. Moreover, find the best dominant $q(\cdot, \xi)$, for all $\xi \in \overline{U}$ of this strong differential subordination.*

Problem 2.2 *Given $q(\cdot, \xi) \in \mathcal{H}_u(U)$, for all $\xi \in \overline{U}$ with $q(0, \xi) = p(0, \xi)$ find the function $h(\cdot, \xi) \in \mathcal{H}_u(U)$, for all $\xi \in \overline{U}$ with $h(0, \xi) = q(0, \xi)$, such that implication (1") holds.*

This paper provides certain results which respond to the above formulated problems.

Theorem 2.3 *Let $\beta, \gamma \in C$, $\beta \neq 0$, and let convex function $h(\cdot, \xi) \in H^*[a, n, \xi]$ satisfies*

$$\operatorname{Re}\left[\beta h(z,\xi) + \gamma\right] > 0, \quad z \in U,\ \xi \in \overline{U}. \tag{3}$$

If $p(\cdot, \xi)$ is analytic in $U \times \overline{U}$, with $p(0, \xi) = h(0, \xi)$, for all $\xi \in \overline{U}$ then

$$p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} \prec\prec h(z,\xi), \quad z \in U,\ \xi \in \overline{U}, \tag{4}$$

implies

$$p(z,\xi) \prec\prec h(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Proof. Let $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, $r = p(z, \xi)$, $s = zp'(z, \xi)$ then $\psi(r, s; z, \xi) = r + \frac{s}{\beta r + \gamma}$, and strong differential subordination (4) becomes

$$\psi(p(z,\xi), zp'(z,\xi); z, \xi) \prec\prec h(z,\xi), \tag{5}$$

implies $p(z, \xi) \prec\prec q(z, \xi), \quad z \in U,\ \xi \in \overline{U}$.

We will use part (ii) Lemma 1.1, with $q(z, \xi) = h(z, \xi)$ and $h_\rho(z, \xi) = h(\rho z, \xi)$ to prove this result. We only need to show that $\psi \in \Psi[h_\rho(\cdot, \xi), h_\rho(\cdot, \xi)]$ for $0 < \rho < 1$.

In this case, admissibility condition (A) reduces to showing

$$\psi_0 \equiv \psi(h_\rho(\zeta,\xi), m\zeta h'_\rho(\zeta,\xi)) = h_\rho(\zeta,\xi) + \frac{m\zeta h'_\rho(\zeta,\xi)}{\beta h_\rho(\zeta,\xi) + \gamma} \notin h_\rho(U \times \overline{U}), \tag{6}$$

when $|\zeta| = 1$, $\xi \in \overline{U}$ and $m \geq n$.

Equality (6) is equivalent to

$$\frac{\psi_0 - h_\rho(\zeta,\xi)}{\zeta h'_\rho(\zeta,\xi)} = \frac{m}{\beta h_\rho(z,\xi) + \gamma}, \tag{7}$$

when $|\zeta| = 1$, $\xi \in \overline{U}$, and $m \geq 1$.

Let

$$\lambda = \frac{\psi_0 - h_\rho(\zeta,\xi)}{\zeta h'_\rho(\zeta,\xi)} = \frac{m}{\beta h_\rho(z,\xi) + \gamma}. \tag{8}$$

From Theorem 2.3 we have $\operatorname{Re}\lambda = \operatorname{Re}\frac{m}{\beta h_\rho(z,\xi)+\gamma} > 0$, i.e. $|\arg\lambda| < \frac{\pi}{2}$, which is equivalent to $|\arg[\psi_0 - h_\rho(\zeta,\xi)] - \arg\zeta h'_\rho(\zeta,\xi)| < \frac{\pi}{2}$ since $h_\rho(U \times \overline{U})$ is convex, $h_\rho(\zeta, \xi) \in h_\rho(\partial U, \overline{U})$ and $\zeta h'_\rho(\zeta, \xi)$ is the outer normal to $h_\rho(\partial U, \overline{U})$ at $h_\rho(\zeta, \xi)$, we conclude that $\psi_0 \notin h_\rho(U \times \overline{U})$ and $p(z, \xi) \prec\prec q(z, \xi)$, $z \in U$, $\xi \in \overline{U}$. ■

Example 2.1 *Let* $\beta = 4, \gamma = 2$, $h(z,\xi) = 1 + \frac{\xi}{4}z$, Re $\left[\frac{zh''(z)}{h'(z)} + 1\right] > 0$ *and* Re $\left[4\left(1 + \frac{\xi}{4}z\right) + 2\right] > 0$. *If* $p(\cdot,\xi)$ *is analytic in* $U \times \overline{U}$ *with* $p(0,\xi) = h(0,\xi)$, *for all* $\xi \in \overline{U}$ *then*

$$p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} \prec\prec 1 + \frac{\xi}{4}z, \quad z \in U,\ \xi \in \overline{U},$$

implies

$$p(z,\xi) \prec\prec 1 + \frac{\xi}{4}z, \quad z \in U,\ \xi \in \overline{U}.$$

Theorem 2.4 *Let* $\beta, \gamma \in C$, $\beta \neq 0$ *and let convex function* $h(\cdot,\xi) \in H^*[a,n,\xi]$. *Suppose that the Briot-Bouquet differential equation*

$$q(z,\xi) + \frac{nzq'(z,\xi)}{\beta q(z,\xi) + \gamma} = h(z,\xi), \quad [q(0,\xi) = h(0,\xi) = a] \tag{9}$$

has a solution q *with* $q(0,\xi) = a$ *that satisfy* $q(z,\xi) \prec\prec h(z,\xi)$, $z \in U,\ \xi \in \overline{U}$.

If $p \in H^*[a,n,\xi]$ *then*

$$p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} \prec\prec h(z,\xi) \tag{10}$$

implies

$$p(z,\xi) \prec\prec q(z,\xi), \quad z \in U,\ \xi \in \overline{U},$$

and $q(\cdot,\xi)$ *is the best dominant.*

Proof. We will use part (iii) of Lemma 1.2 to prove this result. Let $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, $\psi(r,s;z,t) = r + \frac{s}{\beta r + \gamma}$, $r = p(z,\xi)$, $s = zp'(z,\xi)$, then strong differential subordination (10) becomes

$$\psi(p(z,\xi), zp'(z,\xi); z, \xi) \prec\prec h(z,\xi), \quad z \in U,\ \xi \in \overline{U}, \tag{11}$$

and we only need to show that $\psi \in \Psi_n[h_\rho(\cdot,\xi), q_\rho(\cdot,\xi)]$, for $0 < \rho < 1$.

In this case, admissibility condition (A) reduces to showing that $\psi_0 \equiv \psi(q_\rho(\zeta,\xi), m\zeta q'_\rho(\zeta,\xi); z, \xi) = q_\rho(\zeta,\xi) + m\frac{\zeta q'_\rho(\zeta,\xi)}{\beta h_\rho(\zeta,\xi) + \gamma} \notin h_\rho(U \times \overline{U})$, when $|\zeta| = 1$, $\xi \in \overline{U}$, and $m \geq n$. Using (9) we obtain $\psi_0 = q_\rho(\zeta,\xi) + \frac{m}{n}[h_\rho(\zeta,\xi) - q_\rho(\zeta,\xi)]$, $|\zeta| = 1,\ \xi \in \overline{U}$.

From $q(z,\xi) \prec h(z,\xi)$ we obtain $q_\rho(\zeta,\xi) \in h_\rho(U \times \overline{U})$.

Using this together with the fact that $h_\rho(U \times \overline{U})$ is a convex domain and $\frac{m}{n} \geq 1$, we deduce that $\psi_0 \notin h_\rho(U \times \overline{U})$. Therefore $\psi \in \Psi_n[h_\rho(\cdot,\xi), q_\rho(\cdot,\xi)]$, and by Lemma 1.2 we conclude that $p(z,\xi) \prec\prec q(z,\xi)$, and that $q(\cdot,\xi)$ is the best dominant. ■

We can also use Theorem 2.4 to obtain a strong subordination relation between univalent solutions of Briot-Bouquet equation of the form (9).

Theorem 2.5 *Let* $h(\cdot,\xi)$, $\xi \in \overline{U}$ *convex, with* $h(0,\xi) = a$ *and let* m *and* n *be positive integers. Let* $q_m(\cdot,\xi)$ *and* $q_n(\cdot,\xi)$, $\xi \in \overline{U}$ *be univalent solutions of differential equation*

$$q(z,\xi) + \frac{nzq'(z,\xi)}{\beta q(z,\xi) + \gamma} = h(z,\xi), \quad z \in U,\ \xi \in \overline{U},\ \beta, \gamma \in \mathbb{C},\ \beta \neq 0, \tag{12}$$

for $n = m$ *and* n *respectively, with* $q_n(z,\xi) \prec h(z,\xi)$, $z \in U$, $\xi \in \overline{U}$. *If* $m > n$, *then*

$$q_m(z,\xi) \prec\prec q_n(z,\xi).$$

Proof. By hypothesis $q_m(z,\xi) + \frac{mzq'_m(z,\xi)}{\beta q_m(z,\xi) + \gamma} = h(z,\xi)$, $z \in U,\ \xi \in \overline{U}$. Let

$$p(z,\xi) = q_m(z^m, \xi), \tag{13}$$

we obtain $p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} = q_m(z^m,\xi) + \frac{mz^m q'_m(z^m,\xi)}{\beta q_m(z^m,\xi) + \gamma} = h(z^m,\xi) \prec\prec h(z,\xi)$, $z \in U,\ \xi \in \overline{U}$.

Since $p(.,\xi) \in \mathcal{H}^*[a,m,\xi] \subset \mathcal{H}^*[a,n,\xi]$, $q_n(z,\xi) \prec\prec h(z,\xi)$ and $q_n(\cdot,\xi)$ satisfies (12) we can apply Theorem 2.4 to obtain $p(z,\xi) \prec\prec q_n(z,\xi)$, and from (13), we have $q_m(z^m,\xi) \prec\prec p(z,\xi)$, which implies $q_m(z,\xi) \prec\prec q_n(z,\xi)$, $z \in U,\ \xi \in \overline{U}$. ■

We can combine Theorem 2.3 and 2.4 and we obtain the following theorem.

Theorem 2.6 *Let $h(\cdot,\xi)$ be convex in U, with $h(0,\xi)=a$ and* $\mathrm{Re}\,[\beta h(z,\xi)+\gamma]>0$, $z\in U,\ \xi\in\overline{U},\ \beta,\gamma\in\mathbb{C},\ \beta\neq 0$. *Let n be a positive integer and suppose that differential equation*

$$q(z,\xi)+\frac{nzq'(z,\xi)}{\beta q(z,\xi)+\gamma}=h(z,\xi),\quad z\in U,\ \xi\in\overline{U}, \tag{14}$$

has a univalent solution $q(\cdot,\xi)$. If $p\in H^*[a,n,\xi]$ satisfies differential subordination

$$p(z,\xi)+\frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma}\prec\prec h(z,\xi),\quad z\in U,\ \xi\in\overline{U}, \tag{15}$$

then

$$p(z,\xi)\prec\prec q(z,\xi)\prec\prec h(z,\xi),\quad z\in U,\ \xi\in\overline{U},$$

and $q(\cdot,\xi)$ is the best dominant.

Proof. If we replace z by z^n in (14) and let $\widetilde{q}(z,\xi)=q(z^n,\xi)$, we obtain $\widetilde{q}(z,\xi)+\frac{z\widetilde{q}'(z,\xi)}{\beta\widetilde{q}(z,\xi)+\gamma}=h(z^n,\xi)\prec\prec h(z,\xi)$, $z\in U,\ \xi\in\overline{U}$, then the function $\widetilde{q}\in\mathcal{H}^*[a,n,\xi]$ satisfies strong differential subordination (15). Since $\widetilde{q}(\cdot,\xi)$ satisfies the conditions of Theorem 2.3, we deduce that $\widetilde{q}(z,\xi)\prec\prec h(z,\xi)$ or equivalently $q(z^n,\xi)\prec\prec h(z,\xi)$. Since this implies $q(z,\xi)\prec\prec h(z,\xi)$, we can combine this result with Theorem 2.4 to obtain $p(z,\xi)\prec\prec q(z,\xi)\prec\prec h(z,\xi)$, and q is the best dominant. ■

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L_p-left Caputo fractional Landau inequalities

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Abstract

Here we establish left Caputo fractional L_p-Landau type inequalities. We give applications on $\mathbb{R}_+$ at the end.

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Keywords and Phrases: Landau inequality, Caputo fractional derivative.

1 Introduction

Let $p \in [1,\infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$, and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1], [3].

The research on these inequalities started by E. Landau [9] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2}, \tag{2}$$

are the best constants in (1).

In 1932, G.H. Hardy and J.E. Littlewood [6] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2}, \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G.H. Hardy, E. Landau and J.E. Littlewood [7] showed that the best constant $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1,\infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1,\infty)$. Infact in [5] and [8], was shown that

$$C_p(\mathbb{R}) \leq \sqrt{2}.$$

In this article we prove fractional Landau inequalities with respect to $\|\cdot\|_p$, $p > 1$, involving the left Caputo fractional derivative.

We need

Definition 1 *([4], p. 38) Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $f \in AC^n([A,B])$ (i.e. $f^{(n-1)} \in AC([A,B])$, $A, B \in \mathbb{R}$). We define the left Caputo fractional derivative*

$$D^{\nu}_{*A} f(x) := \frac{1}{\Gamma(n-\nu)} \int_A^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{5}$$

$\forall\ x \in [A,B]$, and $D^{\nu}_{*A} f(x) := 0$, if $x < A$. Here Γ is the gamma function.

Definition 2 *Let $0 < \nu \leq 1$, $f \in AC([A,B])$, we define*

$$D^{\nu}_{*A} f(x) = \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f'(t)\, dt, \tag{6}$$

$\forall\ x \in [A,B]$.

Notice $D^{n}_{*A} f(x) = f^{(n)}$, for $n \in \mathbb{N}$.

We make

Remark 3 *Let $f' \in AC([A,B])$, $0 < \nu \leq 1$, then $\lceil \nu + 1 \rceil = 2$, and*

$$D^{\nu}_{*A} f'(x) = \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f''(t)\, dt =$$

$$\frac{1}{\Gamma(2-(\nu+1))} \int_A^x (x-t)^{2-(\nu+1)-1} f''(t)\, dt = D^{\nu+1}_{*A} f(x).$$

Hence it holds

$$D^{\nu}_{*A} f'(x) = D^{\nu+1}_{*A} f(x). \tag{7}$$

2 Main results

We use

Theorem 4 *([2], p. 620) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \nu \leq 1$, $f \in AC([A,B])$, $A, B \in \mathbb{R}$, $A < B$. Assume $D^{\nu}_{*A} f \in L_q([A,B])$. Then*

$$\left| \frac{1}{B-A} \int_A^B f(x)\, dx - f(A) \right| \leq \frac{\|D^{\nu}_{*A} f\|_{L_q([A,B])}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B-A)^{\nu-1+\frac{1}{p}}. \tag{8}$$

We make

Remark 5 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $1-\frac{1}{p}<\nu\leq 1$, $f'\in AC([A,B])$, $A,B\in\mathbb{R}$, $A<B$. Assume $D_{*A}^{\nu}f'\in L_q([A,B])$. Then*

$$\left|\frac{1}{B-A}\int_A^B f'(x)\,dx-f'(A)\right|\leq\frac{\|D_{*A}^{\nu}f'\|_{L_q([A,B])}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(B-A)^{\nu-1+\frac{1}{p}}. \tag{9}$$

Equivalently we have: For $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $1-\frac{1}{p}<\nu\leq 1$, $f\in AC^2([A,B])$, $A,B\in\mathbb{R}$, $A<B$. Assume $D_{*A}^{\nu+1}f\in L_q([A,B])$. Then

$$\left|\frac{1}{B-A}\left(f(B)-f(A)\right)-f'(A)\right|\leq \tag{10}$$

$$\frac{\left\|D_{*A}^{\nu+1}f\right\|_{L_q([A,B])}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(B-A)^{\nu-1+\frac{1}{p}}.$$

Therefore it holds

$$|f'(A)|-\frac{1}{B-A}|f(B)-f(A)|\leq\frac{\left\|D_{*A}^{\nu+1}f\right\|_{L_q([A,B])}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(B-A)^{\nu-1+\frac{1}{p}}. \tag{11}$$

We also make

Remark 6 *Let $1-\frac{1}{p}<\nu\leq 1$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $f\in AC^2([A,b])$, $\forall b>A$. Assume $D_{*A}^{\nu+1}f\in L_q([A,+\infty))$ (thus $D_{*A}^{\nu+1}\in L_q([A,b])$). Let $A<a<b$. Then $f\in AC^2([a,b])$ and $D_{*A}^{\nu+1}f\in L_q([a,+\infty))$ and $D_{*A}^{\nu+1}\in L_q([a,b])$. Here*

$$D_{*a}^{\nu+1}f(x)=\frac{1}{\Gamma(1-\nu)}\int_a^x(x-t)^{-\nu}f''(t)\,dt. \tag{12}$$

If $f''(t)\geq 0$ a.e., then

$$D_{*A}^{\nu+1}f(x)\geq D_{*a}^{\nu+1}f(x)\geq 0,\ \text{a.e., for } x\geq a.$$

Thus

$$\infty>\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}\geq\left\|D_{*A}^{\nu+1}f\right\|_{q,[a,+\infty)}\geq\left\|D_{*a}^{\nu+1}f\right\|_{q,[a,+\infty)}.$$

So it is not strange to assume that

$$\left\|D_{*a}^{\nu+1}f\right\|_{q,[a,+\infty)}\leq\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}, \tag{13}$$

$\forall\ a\geq A$ (it is obvious when $\nu\in\mathbb{N}$).

We need

Remark 7 *Let $a,b\in[A,+\infty)$, $a<b$. Then we get*

$$|f'(a)|\leq\frac{1}{b-a}|f(b)-f(a)|+\frac{\left\|D_{*a}^{\nu+1}f\right\|_{L_q([a,b])}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-1+\frac{1}{p}}. \tag{14}$$

We also assume that $\|f\|_{\infty,[A,+\infty)}<\infty$.

Therefore

$$|f'(a)|\leq\frac{2\|f\|_{\infty,[A,+\infty)}}{b-a}+\frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-1+\frac{1}{p}}, \tag{15}$$

$\forall\ a,b\in[A,+\infty)$, $a<b$.

The R.H.S.(15) *depends only on $b-a$.*

Therefore

$$\|f'\|_{\infty,[A,+\infty)}\leq\frac{2\|f\|_{\infty,[A,+\infty)}}{b-a}+\frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-1+\frac{1}{p}}. \tag{16}$$

We may call $t=b-a>0$. Thus

$$\|f'\|_{\infty,[A,+\infty)}\leq\frac{2\|f\|_{\infty,[A,+\infty)}}{t}+\frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}t^{\nu-1+\frac{1}{p}}, \tag{17}$$

$\forall\ t\in(0,\infty)$.

Notice that $0<\nu-1+\frac{1}{p}<1$. Call

$$\widetilde{\mu}:=2\|f\|_{\infty,[A,+\infty)}, \tag{18}$$

$$\widetilde{\theta}:=\frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)},$$

both are positive; and

$$\widetilde{\nu}:=\nu-1+\frac{1}{p},\quad(\widetilde{\nu}\in(0,1)). \tag{19}$$

We consider the function

$$\widetilde{y}(t)=\frac{\widetilde{\mu}}{t}+\widetilde{\theta}t^{\widetilde{\nu}},\quad t\in(0,\infty). \tag{20}$$

The only critical number here is

$$\widetilde{t_0} = \left(\frac{\widetilde{\mu}}{\widetilde{\nu}\widetilde{\theta}} \right)^{\frac{1}{\widetilde{\nu}+1}}, \tag{21}$$

and $\widetilde{y}$ has a global minimum at $\widetilde{t_0}$, which is

$$\widetilde{y}\left(\widetilde{t_0}\right) = \left(\widetilde{\theta}\widetilde{\mu}^{\widetilde{\nu}}\right)^{\frac{1}{\widetilde{\nu}+1}} \left(\widetilde{\nu}+1\right) \widetilde{\nu}^{-\left(\frac{\widetilde{\nu}}{\widetilde{\nu}+1}\right)}. \tag{22}$$

Consequently, we get

$$\widetilde{y}\left(\widetilde{t_0}\right) = \left(\frac{\left\| D_{*A}^{\nu+1} f \right\|_{q,[A,+\infty)}}{\Gamma\left(\nu\right)\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}} \left(\nu+\frac{1}{p}\right)} \right)^{\frac{1}{\nu+\frac{1}{p}}} \cdot$$

$$\left(2\left\|f\right\|_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \left(\nu+\frac{1}{p}\right) \left(\nu-1+\frac{1}{p}\right)^{-\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}. \tag{23}$$

We have proved that

$$\left\|f'\right\|_{\infty,[A,+\infty)} \leq \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}} \right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \frac{1}{\left(\Gamma\left(\nu\right)\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}} \cdot \tag{24}$$

$$\frac{1}{\left(p\left(\nu-1\right)+1\right)^{\frac{1}{(p\nu+1)}}} \cdot \left(\left\|f\right\|_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \left(\left\|D_{*A}^{\nu+1} f\right\|_{q,[A,+\infty)}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}.$$

We have established the following L_q result, left Caputo fractional Landau L_q inequality:

Theorem 8 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \nu \leq 1$, $f \in AC^2\left([A,b]\right)$, $\forall b > A$. Assume $D_{*A}^{\nu+1} f \in L_q\left([A,+\infty)\right)$, $\left\|f\right\|_{\infty,[A,+\infty)} < \infty$, and*

$$\left\|D_{*a}^{\nu+1} f\right\|_{q,[a,+\infty)} \leq \left\|D_{*A}^{\nu+1} f\right\|_{q,[A,+\infty)}, \tag{25}$$

$\forall\ a \geq A$. Then

$$\left\|f'\right\|_{\infty,[A,+\infty)} \leq \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}} \right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \tag{26}$$

$$\frac{1}{\left(\Gamma\left(\nu\right)\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}} \left(p\left(\nu-1\right)+1\right)^{\frac{1}{(p\nu+1)}}} \cdot$$

$$\left(\left\|f\right\|_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \left(\left\|D_{*A}^{\nu+1} f\right\|_{q,[A,+\infty)}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}.$$

We give

Corollary 9 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $1-\frac{1}{p}<\nu\leq 1$, $f\in AC^2([0,b])$, $\forall b>0$. *Assume* $D_{*0}^{\nu+1}f\in L_q([0,+\infty))$, $\|f\|_{\infty,\mathbb{R}_+}<\infty$, *and*

$$\left\|D_{*a}^{\nu+1}f\right\|_{q,[a,+\infty)}\leq\left\|D_{*0}^{\nu+1}f\right\|_{q,\mathbb{R}_+},\tag{27}$$

$\forall\ a\geq 0$. *Then*

$$\|f'\|_{\infty,\mathbb{R}_+}\leq\left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}\cdot\tag{28}$$

$$\frac{1}{\left(\Gamma(\nu)\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}\left(p(\nu-1)+1\right)^{\frac{1}{(p\nu+1)}}}\cdot$$

$$\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}\cdot\left(\left\|D_{*0}^{\nu+1}f\right\|_{q,\mathbb{R}_+}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}.$$

We mention the case of $p=q=2$.

Corollary 10 *Let* $\frac{1}{2}<\nu\leq 1$, $f\in AC^2([0,b])$, $\forall b>0$. *Assume* $D_{*0}^{\nu+1}f\in L_2(\mathbb{R}_+)$, $f\in L_\infty(\mathbb{R}_+)$, *and*

$$\left\|D_{*a}^{\nu+1}f\right\|_{2,[a,+\infty)}\leq\left\|D_{*0}^{\nu+1}f\right\|_{2,\mathbb{R}_+},\tag{29}$$

$\forall\ a\geq 0$. *Then*

$$\|f'\|_{\infty,\mathbb{R}_+}\leq\left(\frac{2\nu+1}{\nu-\frac{1}{2}}\right)^{\left(\frac{\nu-\frac{1}{2}}{\nu+\frac{1}{2}}\right)}.\tag{30}$$

$$\frac{1}{\left(\Gamma(\nu)\right)^{\frac{1}{\left(\nu+\frac{1}{2}\right)}}(2\nu-1)^{\frac{1}{(2\nu+1)}}}\cdot$$

$$\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\left(\frac{\nu-\frac{1}{2}}{\nu+\frac{1}{2}}\right)}\cdot\left(\left\|D_{*0}^{\nu+1}f\right\|_{2,\mathbb{R}_+}\right)^{\frac{1}{\left(\nu+\frac{1}{2}\right)}}.$$

When $\nu=1$ we get:

Corollary 11 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $f\in AC^2([0,b])$, $\forall b>0$. *Assume* $f''\in L_q(\mathbb{R}_+)$, $f\in L_\infty(\mathbb{R}_+)$. *Then*

$$\|f'\|_{\infty,\mathbb{R}_+}\leq(2(p+1))^{\frac{1}{p+1}}\cdot\tag{31}$$

$$\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{p+1}}\cdot\left(\|f''\|_{q,\mathbb{R}_+}\right)^{\frac{p}{p+1}}.$$

We finish with

Corollary 12 *Let* $f\in AC^2([0,b])$, $\forall b>0$. *Assume* $f''\in L_2(\mathbb{R}_+)$, $f\in L_\infty(\mathbb{R}_+)$. *Then*

$$\|f'\|_{\infty,\mathbb{R}_+}\leq\sqrt[3]{6}\cdot\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{3}}\cdot\left(\|f''\|_{2,\mathbb{R}_+}\right)^{\frac{2}{3}}.\tag{32}$$

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Superconvergence of Tensor-Product Quadratic Pentahedral Elements for Variable Coefficient Elliptic Equations*

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Abstract

For a second order variable coefficient elliptic boundary value problem in 3D, we derive the weak estimate of the first type for tensor-product quadratic pentahedral finite elements. Combined the estimate for the discrete derivative Green's function, we show that the derivatives of the finite element approximation and the corresponding interpolant are superclose in the sense of the L^∞-norm.

Mathematics subject classification: 65N30.
Key words: tensor-product quadratic pentahedral elements, superconvergence, discrete derivative Green's function, variable coefficient elliptic equation

I. INTRODUCTION AND PRELIMINARIES

There have been many studies concerned with superconvergence of finite element methods in 3D. Books and survey papers have been published. For the literature, we refer to [1–12] and references therein. As for pentahedral elements, superconvergence results are achieved in [3, 6, 7, 9, 12]. Among of them, [7, 12] give how to locate superconvergent points of the pentahedral finite element approximation. Recently, we obtained supercloseness properties of pentahedral elements in the pointwise sense of the L^∞-norm for the Poisson equation (see [9]). The present article shall discuss maximum norm error estimates for derivatives of tensor-product quadratic pentahedral elements for a second order variable coefficient elliptic equation.

In this article, we shall use letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notation for the Sobolev spaces and their norms.

The model problem considered in this article is

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{3} a_i\partial_i u + a_0 u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{1.1}$$

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Here $\Omega = [0,1]^2 \times [0,1] = \Omega_{xy} \times \Omega_z \subset \mathcal{R}^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. We also assume that the given functions $a_{ij}, a_i \in W^{1,\infty}(\Omega)$, $a_0 \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. In addition, we write $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$, which are usual partial derivatives.

To discretize the problem, one proceeds as follows. The domain Ω is firstly partitioned into subcubes of side h, and each of these is then subdivided into two pentahedra (triangular prisms). We denote by $\{\mathcal{T}^h\}$ a uniform family of pentahedral partitions as above. Thus $\bar{\Omega} = \cup_{e \in \mathcal{T}^h} \bar{e}$. Obviously, we can write $e = D \times L$ (see Fig. 1), where D and L represent a triangle parallel to the xy-plane and a one-dimensional interval parallel to the z-axes, respectively.

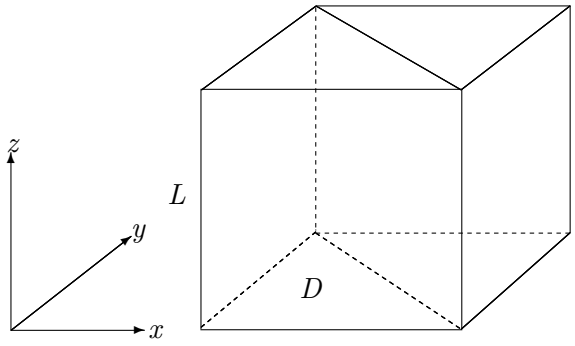


FIG. 1. Pentahedral Elements

We introduce the tensor-product quadratic polynomial space denoted by $\mathcal{P}$, that is,

$$q(x,y,z) = \sum_{(i,j,k)\in\mathcal{I}} a_{ijk} x^i y^j z^k, \ a_{ijk} \in \mathcal{R}, \ q \in \mathcal{P},$$

where $\mathcal{P} = \mathcal{P}_{xy} \otimes \mathcal{P}_z$, $\mathcal{P}_{xy}$ stands for the quadratic polynomial space with respect to (x,y), and $\mathcal{P}_z$ the quadratic polynomial space with respect to z. The indexing set $\mathcal{I}$ satisfies $\mathcal{I} = \{(i,j,k) | i,j,k \geq 0, \ i+j \leq 2, \ k \leq 2\}$. Let Π^e_{xy} be the quadratic interpolation operator with respect to $(x,y) \in D$, and Π^e_z the quadratic interpolation operator with respect to $z \in L$. Thus we may define the tensor-product quadratic interpolation operator by $\Pi^e : H^1_0(e) \to \mathcal{P}(e)$. Obviously, $\Pi^e = \Pi^e_{xy} \otimes \Pi^e_z = \Pi^e_z \otimes \Pi^e_{xy}$.

The corresponding weak form of the problem (1.1) is

$$a(u\,,\,v) = (f\,,\,v) \quad \forall\, v \in H^1_0(\Omega),$$

where

$$\begin{aligned} a(u\,,\,v) &\equiv \int_\Omega \Big(\sum_{i,j=1}^{3} a_{ij} \partial_i u \partial_j v + \sum_{i=1}^{3} a_i \partial_i u v + a_0 u v \Big)\, dxdydz \\ &= \sum_{i,j=1}^{3} (a_{ij} \partial_i u,\, \partial_j v) + \sum_{i=1}^{3} (a_i \partial_i u,\, v) + (a_0 u,\, v), \end{aligned}$$

and

$$(f\,,\,v)=\int_{\Omega} fv\,dxdydz.$$

Define the tensor-product quadratic pentahedral finite element space by

$$S_0^h(\Omega)=\left\{v\in H_0^1(\Omega):v|_e\in\mathcal{P}(e)\ \ \forall\, e\in\mathcal{T}^h\right\}. \tag{1.2}$$

Thus, the finite element method is to find $u_h\in S_0^h(\Omega)$ such that

$$a(u_h\,,\,v)=(f\,,\,v)\ \ \forall\, v\in S_0^h(\Omega).$$

Obviously, there is the following Galerkin orthogonality relation:

$$a(u-u_h\,,\,v)=0\ \ \forall\, v\in S_0^h(\Omega). \tag{1.3}$$

Additionally, from the definitions of Π^e and $S_0^h(\Omega)$, we can define the global tensor-product quadratic interpolation operator $\Pi: H_0^1(\Omega)\to S_0^h(\Omega)$. Here $(\Pi u)|_e=\Pi^e u$. In next section, we will bound the term

$$a(u-\Pi u\,,\,v)=\sum_{i,j=1}^{3}(a_{ij}\partial_i(u-\Pi u),\,\partial_j v)+\sum_{i=1}^{3}(a_i\partial_i(u-\Pi u),\,v)+(a_0(u-\Pi u),\,v). \tag{1.4}$$

II. A WEAK ESTIMATE FOR THE TENSOR-PRODUCT QUADRATIC INTERPOLATION OPERATOR

In this section, we give the following weak estimate for the interpolation operator Π.

Lemma 2.1. *Let $\{\mathcal{T}^h\}$ be a uniform family of pentahedral partitions of Ω, $u\in W^{4,\infty}(\Omega)\cap H_0^1(\Omega)$, and $v\in S_0^h(\Omega)$. Then the tensor-product quadratic interpolation operator Π satisfies the following weak estimate:*

$$|a(u-\Pi u,\,v)|\le Ch^3\|u\|_{4,\infty,\Omega}|v|_{1,1,\Omega}. \tag{2.1}$$

Proof. Obviously, the remainder of the interpolation is as follows:

$$\begin{array}{rcl} u-\Pi u &=& (u-\Pi_{xy}u)+(u-\Pi_z u)+(\Pi_{xy}(u-\Pi_z u)-(u-\Pi_z u))\\ &\equiv& R_{xy}+R_z+R^*, \end{array} \tag{2.2}$$

where $(\Pi_{xy}u)|_e=\Pi_{xy}^e u, (\Pi_z u)|_e=\Pi_z^e u$, and R^* is a high-order term. Thus, we only need to analyze R_{xy} and R_z.

We first bound the term

$$\begin{array}{rcl} a(R_{xy}\,,\,v) &=& \sum_{i,j=1}^{3}(a_{ij}\partial_i R_{xy},\,\partial_j v)+\sum_{i=1}^{3}(a_i\partial_i R_{xy},\,v)+(a_0R_{xy},\,v)\\ &=& \left(\sum_{i,j=1}^{2}(a_{ij}\partial_i R_{xy},\,\partial_j v)+\sum_{i=1}^{2}(a_i\partial_i R_{xy},\,v)\right)\\ &&+\left(\sum_{j=1}^{3}(a_{3j}\partial_3 R_{xy},\,\partial_j v)+\sum_{i=1}^{2}(a_{i3}\partial_i R_{xy},\,\partial_3 v)\right)\\ &&+(a_3\partial_3 R_{xy}+a_0R_{xy},\,v)\\ &\equiv& J_1+J_2+J_3. \end{array} \tag{2.3}$$

By the weak estimate of quadratic triangular elements (see [13]),

$$\begin{aligned} |J_1| &\leq \int_{\Omega_z} \left| \int_{\Omega_{xy}} \Big(\sum_{i,j=1}^{2} a_{ij} \partial_i R_{xy} \partial_j v + \sum_{i=1}^{2} a_i \partial_i R_{xy} v \Big)\, dxdy \right| dz \\ &\leq Ch^3 \int_{\Omega_z} \|u\|_{4,\infty,\Omega_{xy}} |v|_{1,1,\Omega_{xy}}\, dz \\ &\leq Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}, \end{aligned}$$

namely,

$$|J_1| \leq Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.4}$$

Consider the integral

$$I_{3j} \equiv \int_{\Omega} a_{3j} \partial_3 R_{xy} \partial_j v\, dxdydz,\ j = 1, 2, 3.$$

Obviously, by the interpolation error estimate, we have

$$|I_{3j}| \leq C \int_{\Omega} |\partial_3 u - \Pi_{xy}(\partial_3 u)||\partial_j v|\, dxdydz \leq Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.5}$$

As for the integral

$$I_{i3} \equiv \int_{\Omega} a_{i3} \partial_i R_{xy} \partial_3 v\, dxdydz,\ i = 1, 2.$$

Since $v \in S_0^h(\Omega)$, integration by parts yields

$$I_{i3} = -\int_{\Omega} v \partial_3 (a_{i3} \partial_i R_{xy})\, dxdydz = -\int_{\Omega} v \partial_3 a_{i3} \partial_i R_{xy}\, dxdydz - \int_{\Omega} v a_{i3} \partial_3 \partial_i R_{xy}\, dxdydz.$$

Applying integration by parts again, we have

$$I_{i3} = \int_{\Omega} R_{xy} \partial_i (v \partial_3 a_{i3})\, dxdydz + \int_{\Omega} \partial_3 R_{xy} \partial_i (v a_{i3})\, dxdydz.$$

By the interpolation error estimate and the Poincaré inequality, we derive

$$|I_{i3}| \leq Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.6}$$

From (2.5) and (2.6),

$$|J_2| \leq Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.7}$$

Applying the interpolation error estimate and the Poincaré inequality directly gives

$$|J_3| \leq Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.8}$$

From (2.3), (2.4), (2.7), and (2.8),

$$|a(R_{xy}\,,\, v)| \leq Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.9}$$

Next we bound the term

$$\begin{aligned} a(R_z\,,\,v) &= \textstyle\sum_{i,j=1}^{3}(a_{ij}\partial_i R_z,\,\partial_j v)+\sum_{i=1}^{3}(a_i\partial_i R_z,\,v)+(a_0R_z,\,v)\\ &= \textstyle\sum_{i=1}^{2}\sum_{j=1}^{3}(a_{ij}\partial_i R_z,\,\partial_j v)\\ &\quad +\textstyle\sum_{j=1}^{2}(a_{3j}\partial_3 R_z,\,\partial_j v)\\ &\quad +(a_{33}\partial_3 R_z,\,\partial_3 v)\\ &\quad +\Big(\textstyle\sum_{i=1}^{2}(a_i\partial_i R_z,\,v)+(a_0R_z,\,v)\Big)\\ &\quad +(a_3\partial_3 R_z,\,v)\\ &\equiv K_1+K_2+K_3+K_4+K_5. \end{aligned} \tag{2.10}$$

Applying the interpolation error estimate directly gives

$$|K_1|\le Ch^3\|u\|_{4,\infty,\Omega}|v|_{1,1,\Omega}. \tag{2.11}$$

Integration by parts yields

$$K_2=\sum_{j=1}^{2}(R_z,\,\partial_3(v\partial_j a_{3j}))+\sum_{j=1}^{2}(\partial_j R_z,\,\partial_3(va_{3j})).$$

Applying the interpolation error estimate and the Poincaré inequality gives

$$|K_2|\le Ch^3\|u\|_{4,\infty,\Omega}|v|_{1,1,\Omega}. \tag{2.12}$$

To bound the term K_3, we need the following expansion

$$a_{33}(Q)=a_{33}(Q_0)+O(h_e)\equiv a_{33}^0+O(h_e)\ \ \forall\, Q=(x,y,z)\in e,$$

where $h_e=diam(e)$, and $Q_0\in e$ is a fixed point. Thus,

$$K_3=\sum_e\left(\int_e a_{33}^0\partial_3R_z\partial_3v\,dxdydz+\int_e O(h_e)\partial_3R_z\partial_3v\,dxdydz\right).$$

It is easy to see that

$$\left|\int_e O(h_e)\partial_3R_z\partial_3v\,dxdydz\right|\le Ch^3\|u\|_{3,\infty,\Omega}|v|_{1,1,e},$$

whereas (see [9]),

$$\int_e a_{33}^0\partial_3R_z\partial_3v\,dxdydz=0.$$

Summing over all elements yields

$$|K_3|\le Ch^3\|u\|_{3,\infty,\Omega}|v|_{1,1,\Omega}. \tag{2.13}$$

Applying the interpolation error estimate and the Poincaré inequality gives

$$|K_4|\le Ch^3\|u\|_{4,\infty,\Omega}|v|_{1,1,\Omega}. \tag{2.14}$$

As for the term K_5, integration by parts gives

$$K_5 = -(R_z,\, \partial_3(a_3 v)).$$

Thus, applying the interpolation error estimate and the Poincaré inequality gives

$$|K_5| \le Ch^3 \|u\|_{3,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.15}$$

From (2.10)–(2.15),

$$|a(R_z\,,\, v)| \le Ch^3 \|u\|_{4,\infty,\Omega} |v|_{1,1,\Omega}. \tag{2.16}$$

Thus, combining (2.9) and (2.16) completes the proof of the result (2.1).

III. SUPERCONVERGENCE OF THE TENSOR-PRODUCT QUADRATIC PENTAHEDRAL ELEMENT

To derive superconvergence estimates, for every $Z \in \Omega$, we need to give the definition of the discrete derivative Green's function $\partial_{Z,\ell} G_Z^h \in S_0^h(\Omega)$ as follows.

$$a(v, \partial_{Z,\ell} G_Z^h) = \partial_\ell v(Z) \;\forall\, v \in S_0^h(\Omega). \tag{3.1}$$

where $\ell \in \mathcal{R}^3$ and $|\ell| = 1$. $\partial_\ell v(Z)$ stands for the onesided directional derivative

$$\partial_\ell v(Z) = \lim_{|\Delta Z| \to 0} \frac{v(Z + \Delta Z) - v(Z)}{|\Delta Z|},\ \Delta Z = |\Delta Z| \ell.$$

Lemma 3.1. *For $\partial_{Z,\ell} G_Z^h$ the discrete derivative Green's function defined by* (3.1), *we have the following estimate:*

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1,\Omega} \le C |\ln h|^{\frac{4}{3}}. \tag{3.2}$$

The proof of the result (3.2) may be found in [10].

Finally, we give the following maximum norm supercloseness property.

Theorem 3.1. *Let $\{\mathcal{T}^h\}$ be a uniform family of pentahedral partitions of Ω, $u \in W^{4,\infty}(\Omega) \cap H_0^1(\Omega)$. For u_h the tensor-product quadratic pentahedral finite element approximation, and Πu the corresponding interpolant to u. Then we have the following superconvergence estimate:*

$$|u_h - \Pi u|_{1,\infty,\Omega} \le Ch^3 \left|\ln h\right|^{\frac{4}{3}} \|u\|_{4,\infty,\Omega}. \tag{3.3}$$

Proof. For every $Z \in \Omega$ and unit vector $\ell \in \mathcal{R}^3$, from (1.3) and (3.1),

$$\partial_\ell \left(u_h - \Pi u\right)(Z) = a(u_h - \Pi u,\, \partial_{Z,\ell} G_Z^h) = a(u - \Pi u,\, \partial_{Z,\ell} G_Z^h).$$

Hence, from (2.1),

$$\left|\partial_\ell \left(u_h - \Pi u\right)(Z)\right| \le Ch^3 \|u\|_{4,\infty,\Omega} \left|\partial_{Z,\ell} G_Z^h\right|_{1,1,\Omega}. \tag{3.4}$$

Combining (3.2) and (3.4), we immediately obtain the result (3.3).

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ON SUPPORTS OF EQUILIBRIUM MEASURES WITH CONCAVE SIGNED EQUILIBRIA

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Abstract. Let $\Sigma \subset \mathbb{R}$ be compact. In this paper, we study the support of the equilibrium measure for a class of external fields $Q : \Sigma \to \mathbb{R}$, whose associated signed equilibrium measure has a positive part with concave density supported on at most two intervals. We prove that the support of the equilibrium measure is at most two intervals. Our proof uses the iterated balayage algorithm. As a corollary we obtain by a constructive method that the equilibrium measure of any two intervals has convex density. A non-trivial counterpart of the results to the unit circle is also presented.

Key words. Logarithmic potential theory, external fields, equilibrium measure, equilibrium support

AMS subject classifications. 31A15, 30C15, 78A30

1. Introduction.

In recent years minimal energy problems with external fields have found many applications in a variety of areas ranging from diverse subjects such as orthogonal polynomials, weighted Fekete points, numerical conformal mappings, weighted polynomial approximation, rational and Páde approximation, integrable systems, random matrix theory and random permutations. Let $\Sigma \subset \mathbb{C}$ be closed. An essential step towards the solution of such minimal energy problems is the determination of the nature of the support of the equilibrium measure $\mu := \mu_Q$ associated with a given external field $Q : \Sigma \to (-\infty, \infty]$. As described by Deift [8, Chapter 6], information that the support consists of finitely many intervals allows one to set up a system of equations for the endpoints, from which the endpoints may be calculated, and thus, the equilibrium measure may be obtained from a Riemann-Hilbert problem or, equivalently, a singular integral equation. It is for this reason, that it is important to have a priori, conditions to ensure that the support S of μ is the union of a finite number of intervals. For more on the various applications of minimal energy problems we refer the reader to the references [1, 2, 3, 6, 7, 8, 9, 15, 17, 18, 19, 20] and those listed there in.

In this paper we establish a sufficient condition that the equilibrium support consists of at most two intervals. The method utilizes the Iterated Balayage Algorithm, introduced and used first in the papers [15, 6, 7]. As a result from our analysis we establish also that the equilibrium measure for any two intervals is convex. In order to formulate our results we introduce some needed definitions and notations from potential theory.

1.1. Potential-Theoretical Preliminaries.

Let $\Sigma = \cup_{i=1}^{n}[a_i, b_i]$ or let Σ be finitely many closed arcs on the unit circle. Let $w = \exp(-Q)$ be a function (called weight), where $Q : \Sigma \to (-\infty, \infty]$ is continuous in an extended sense but not identically infinity.

Given a Borel probability measure ν supported on Σ, its logarithmic potential and logarithmic energy are respectively given by

$$U^{\nu}(z) := \int \log \frac{1}{|z-t|} d\nu(t), \quad I(\nu) := \int\int \log \frac{1}{|s-t|} d\nu(s) d\nu(t).$$

The weighted energy of ν associated with the external field Q (or weight w) is defined as

$$I_w(\nu) := I(\nu) + 2\int Q(x)\, d\nu(x).$$

It is well known (see [11], [19, Theorem 1.3]), that under the above assumptions there exists a unique *equilibrium measure* $\mu := \mu_w$ associated with Q, that solves the minimal energy problem

$$I_w(\mu) = \mathcal{E}_w := \min_{\nu \in \mathcal{P}(\Sigma)} I_w(\nu) \tag{1.1}$$

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where $\mathcal{P}(\Sigma)$ denotes the class of all Borel probability measures supported on Σ. The support of the equilibrium measure is denoted by S_μ or S_Q. This measure is completely characterized by the Gauss variational conditions

$$\begin{cases} U^\mu(x) + Q(x) = F, & x \in S_Q, \\ U^\mu(x) + Q(x) \geq F, & x \in \Sigma, \end{cases} \tag{1.2}$$

with some constant F. We note that under the assumptions we have, the logarithmic potential $U^\mu(z)$ is continuous in $\mathbb{C}$. In the particular case when $Q \equiv 0$ the unique minimizer μ_Σ of (1.1) is called an *equilibrium measure of* Σ.

Next we introduce the notion of *signed equilibrium* (see [4]).

DEFINITION 1. *Given a compact subset $E \subset \mathbb{C}$ and an external field Q, we call a signed measure η_E supported on E, and of total mass $\eta_E(E) = 1$, a signed equilibrium on E associated with Q, if*

$$U^{\eta_E}(x) + Q(x) = F_E \qquad \text{for all } x \in E. \tag{1.3}$$

The choice of the normalization $\eta_E(E) = 1$ is just for convenience in the applications here.

The next subsection is a brief summary of an essential tool in our analysis, called the Iterated Balayage Algorithm (see [15, 6, 7]). For the rest of the paper we will assume that Σ is a union of finitely many intervals and that the external field Q has δ-Hölder continuous first derivative for some $\delta > 0$.

1.2. The Iterated Balayage Algorithm (IBA). We recall the notion of *balayage* onto a compact set (see [16, Chapter IV]). Let M be a compact subset of the complex plane with positive logarithmic capacity and such that the complement $\overline{\mathbb{C}} \setminus M$ is regular. Then, if ν is any finite positive Borel measure on $\mathbb{C}$ with compact support, there exists a unique measure $\hat{\nu}$ supported on M such that $\|\nu\| = \|\hat{\nu}\|$, and for some constant C,

$$U^{\hat{\nu}}(z) = U^{\nu}(z) + C, \qquad z \in M.$$

The measure $\hat{\nu}$ is called the balayage of ν onto M and we denote it by $\mathrm{Bal}(\nu; M)$. For a signed measure $\sigma = \sigma^+ - \sigma^-$, the balayage is defined as $\mathrm{Bal}(\sigma; M) := \mathrm{Bal}(\sigma^+; M) - \mathrm{Bal}(\sigma^-; M)$.

The *Iterated Balayage Algorithm (IBA)*, presents an iterative method to solve the variational problem (1.2). Given an external field Q on $\Sigma_0 := \Sigma$ one proceeds as follows. The first step is to solve the integral equation

$$\int_{\Sigma_0} \log|x - t| v_0(t) dt = Q(x) - F_0, \qquad x \in \Sigma_0, \tag{1.4}$$

subject to the condition

$$\int_{\Sigma_0} v_0(t) dt = 1. \tag{1.5}$$

Here, F_0 is a fixed constant.

Since $Q \in C^{1+\delta}(\Sigma_0)$ (recall that Σ_0 consists of finitely many intervals), it can be shown that (1.4)–(1.5) has a unique solution $v_0(t)$, given by the Sokhotski-Plemelj formula (see [12, p. 425]). In the simplest case when $\Sigma_0 = [a, b]$ the formula for v_0 is (see [12, p. 428]),

$$v(x) = v_0(x) := \frac{1}{\pi\sqrt{(b-x)(x-a)}} \left[1 + \frac{1}{\pi} P.V. \int_a^b \frac{Q'(t)}{t-x} \sqrt{(b-t)(t-a)} dt \right], \quad a < x < b, \tag{1.6}$$

where the above integral is a Cauchy principle value integral.

In view of Definition 1, v_0 is the density of the signed equilibrium on Σ_0. If v_0 happens to be non-negative on Σ_0 then it is the density of the equilibrium measure with external field Q and we are done. If not, then we put

$$d\sigma_0(t) := v_0(t) dt,$$

so that σ_0 is a signed measure of Σ_0. Clearly,

$$U^{\sigma_0}(x) + Q(x) = F_0 \qquad \text{for all } x \in \Sigma_0$$

Let $\sigma_0 = \sigma_0^+ - \sigma_0^-$ be the Jordan decomposition of σ_0 and let

$$\Sigma_1 := \operatorname{supp}(\sigma_0^+).$$

From [15, Lemma 3], $\mu \leq \sigma_0^+$ and $\operatorname{supp}(\mu) \subset \Sigma_1$, so that in determining μ and its support we may restrict ourselves in (1.1) to $\Sigma = \Sigma_1$.

The next step is to determine the signed equilibrium associated with Q on Σ_1, which is to solve the singular equation

$$\int_{\Sigma_1} \log|x-t| d\sigma_1(t) = Q(x) - F_1, \qquad x \in \Sigma_1 \tag{1.7}$$

subject to the condition

$$\int_{\Sigma_1} d\sigma_1 = 1. \tag{1.8}$$

Alternatively, using balayage one derives that the signed measure $\sigma_1 := \sigma_0^+ - \operatorname{Bal}(\sigma_0^-, \Sigma_1)$ satisfies

$$U^{\sigma_1}(x) + Q(x) = F_1 \qquad \text{for all } x \in \Sigma_1,$$

provided Σ_1 is regular (which will be the case in our applications), and by uniqueness it is the solution to (1.7)-(1.8). If Σ_1 is not regular the equality holds q.e. and still uniqueness holds, but we will not elaborate on this generality here.

To describe this process, an operator J was introduced in [15, 6, 7] on all finite signed measures σ on $[a,b]$ with $\int d\sigma = 1$ and $\operatorname{cap}(\operatorname{supp}(\sigma^+)) > 0$ as follows

$$J(\sigma) := \sigma^+ - \operatorname{Bal}(\sigma^-; \operatorname{supp}(\sigma^+)) = \operatorname{Bal}(\sigma; \operatorname{supp}(\sigma^+)).$$

Since the operator J sweeps the negative part of the measure σ onto the support of the positive part, we have that $J(\sigma)^+ \leq \sigma^+$.

The *IBA scheme*

$$\mathcal{I}(\sigma_0) = \{(\Sigma_k, \sigma_k)\}_{k=0}^{\infty}$$

is obtained as the iterates of the operator J applied to a signed (equilibrium) measure σ_0 supported on the set Σ_0, i.e.,

$$\Sigma_k := \operatorname{supp}(\sigma_{k-1}^+), \qquad \sigma_k := J(\sigma_{k-1}) = J^k(\sigma_0), \quad k = 1, 2, \ldots. \tag{1.9}$$

The measures σ_k are signed measures which have a Jordan decomposition $\sigma_k = \sigma_k^+ - \sigma_k^-$. It follows that

$$\sigma_0^+ \geq \sigma_1^+ \geq \cdots \geq \mu, \tag{1.10}$$

and that

$$\Sigma_0 \supset \Sigma_1 \supset \Sigma_2 \supset \cdots \supset S. \tag{1.11}$$

Even for more general Σ_0 one expects from (1.10) and (1.11) that the sequence $\{\sigma_k^+\}_{k=0}^{\infty}$ converges in the weak* topology to the equilibrium measure μ, but this has not been proven yet. If it holds, then we say that the IBA converges. Besides presenting a possible algorithm for numerical calculations, the iterated Balayage algorithm can also be used to prove rigorous results on the support of μ in certain situations. The main difficulty in proving that the iterated Balayage algorithm converges generally, lies in the fact that one has to show that the negative parts σ_k^- tend to zero as k tends to ∞. This can be shown, if one can control the limiting set $\Sigma^* := \cap_{k=1}^{\infty} \Sigma_k$. See for example [15, 6, 7].

Note that the IBA scheme is derived similarly if Σ_0 is union of finitely many arcs of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, in which case one uses [12, p. 425]. Finally, we remark that recently a continuous version of this algorithm has been used to solve the equilibrium problem for minimal Riesz energy problems for axis-symmetric external fields on the unit sphere $\mathbb{S}^d$ (see [5]).

1.3. Balayage and Kelvin Transform. There is a natural relationship between balayage measures and equilibrium measures. Let us recall that the Kelvin transform $K : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with center z_0 and radius R is given as the inversion with respect to the circle $\{|z - z_0| = R\}$, namely if $z^* = K(z)$, then z^* lies on the ray stemming from z_0 passing through z and distances satisfy $|z - z_0||z^* - z_0| = R^2$. The distance distortion is given by

$$|z^* - x^*| = \frac{R^2|z - x|}{|z - z_0||x - z_0|}. \tag{1.12}$$

To any measure ν we associate its Kelvin transform $\nu^* = K(\nu)$ as $d\nu^*(x^*) = d\nu(x)$. Observe, that both the point and measure conversion are self-inverse.

Given a compact set A and a point $z_0 \notin A$ we find the balayage $\widehat{\delta_{z_0}}$ of the Dirac-delta measure δ_{z_0} using Riesz's approach [16, Chapter IV, §5], namely if $A^* = K(A)$, then the following relation holds

$$\widehat{\delta_{z_0}} = K(\mu_{A^*}), \tag{1.13}$$

where μ_{A^*} is the equilibrium measure of A^*.

2. Results and examples. Following is our main result.

THEOREM 2. *Let $\Sigma \subseteq \mathbb{R}$ and $w : \Sigma \to [0, \infty)$, $w = \exp(-Q)$ as described in Section 1.1. Suppose that the signed equilibrium on Σ associated with Q exists and is denoted by σ_0. If $\operatorname{supp}(\sigma_0^+)$ consists of at most two intervals A_1, A_2, and σ_0^+ has a concave density on each subinterval, then the equilibrium support $\operatorname{supp}(\mu)$ consists of at most two intervals B_1, B_2, with $B_1 \subset A_1$, $B_2 \subset A_2$, and the equilibrium density is concave on each of these intervals.*

We remark that the theorem remains valid for any $\Sigma \subseteq \mathbb{R}$ compact set as long as σ_0^+ has a support consisting of at most two intervals and σ_0^+ has a concave density. Also, the δ-Hölder continuous first derivative condition on Q must be satisfied only on $\operatorname{supp}(\sigma_0^+)$.

The proof of this theorem relies on a lemma concerning the balayage of a measure onto one or two intervals, which we deem important by itself and include in this section.

LEMMA 3. *Let A be an interval or a union of two intervals, and let ν be a positive measure with compact support in $\mathbb{R}$, such that $\nu(A) = 0$. Then the balayage $\hat{\nu} = \operatorname{Bal}(\nu, A)$ of ν onto A is absolutely continuous with convex density on every subinterval.*

If we consider $\nu = \delta_s$, where δ_s is the Dirac-delta measure with point mass at s, and let $s \to \infty$, then it is well known that densities of the balayage measures $\hat{\delta}_s$ converge to the density of the equilibrium measure μ_A. Therefore, we obtain as a byproduct of our analysis the following corollary.

COROLLARY 4. *The density of the equilibrium measure of the union of any two intervals is convex on every subinterval.*

EXAMPLE 5. Let $\Sigma = [0, 3\pi]$ and $Q(x) := 0.5 \int_0^{3\pi} \ln|x - t| \sin t \, dt$. We can verify that the signed equilibrium is $d\eta = 0.5 \sin t \, dt$. Since it is positive and concave on $[0, \pi] \cup [2\pi, 3\pi]$, we conclude that the equilibrium measure associated with $Q(x)$ is supported on at most two intervals $I_1 \cup I_2$, where $I_1 \subset [0, \pi]$ and $I_2 \subset [2\pi, 3\pi]$, and that the equilibrium density is concave. We remark that by symmetry, in fact the support will consist of two intervals which are symmetric to the point 1.5π. We also remark that this external field is not weak convex in the sense defined in [2, Definition 9], therefore it is an essentially new example in the literature.

EXAMPLE 6. (*Freud weights example*) The following classical Freud weights example $w(x) = e^{-|x|^\lambda}$ (see [19, Example IV.1.15]) provides a nice illustration of our results. For simplicity of the computations, we shall assume that $\lambda = 2$, or equivalently $Q(x) = x^2$, and $\Sigma = \mathbb{R}$. General theory yields that the support of the equilibrium measure associated with $Q(x)$ is compact, so we may restrict ourselves to solving the minimal energy problem on the interval $[-\beta, \beta]$ for some large enough β. Using (1.6) we easily find that the density of the signed equilibrium in this case is given by

$$v_\beta(x) = \frac{2}{\pi}\sqrt{\beta^2 - x^2} - \frac{\beta^2 - 1}{\pi\sqrt{\beta^2 - x^2}}. \tag{2.1}$$

For $\beta > 1$ the function $v_\beta(x)$ is clearly concave, and thus by Theorem 2 the equilibrium support is one interval and the equilibrium measure associated with $Q(x)$ has concave density. Indeed, it is known that $\operatorname{supp}(\mu_Q) = [-1, 1]$ and $d\mu_Q(x) = (2\sqrt{1 - x^2}/\pi)\, dx$. Below we provide an alternative argument to this fact.

Observe that since $v_\beta(1) > 0$ for all $\beta > 1$, the IBA scheme is the collection of nested intervals $\operatorname{supp}(\sigma_k) =: [-\beta_k, \beta_k] \supset [-1, 1]$. It can be calculated that the sequence $\{\beta_k\}$ satisfies the recurrence relation

$$\beta_{k+1} = \sqrt{\frac{\beta_k^2 + 1}{2}}, \quad \beta_0 = \beta.$$

The density of σ_k is simply given by (2.1) with β replaced by β_k. One can easily show that $\beta_k \searrow 1$ as $k \to \infty$, implying that $\operatorname{supp}(\mu_Q) = [-1, 1]$ and $d\mu_Q(x) = (2\sqrt{1 - x^2}/\pi)\, dx$.

The unit circle counterpart of Theorem 2 is not a trivial consequence of the result on the real line and its proof is essentially different, so we formulate it as a separate theorem.

THEOREM 7. *Let $\Sigma \subseteq \mathbb{T}$ and $w : \Sigma \to [0, \infty)$, $w = \exp(-Q)$ as described in Section 1.1. Suppose that the signed equilibrium on Σ associated with Q exists and is denoted by σ_0. If $\operatorname{supp}(\sigma_0^+)$ consists of at most two arcs A_1, A_2, and $d\sigma_0^+ = g(\theta)\, d\theta$ has a concave density $g(\theta)$ on each of the arcs, then the equilibrium support $\operatorname{supp}(\mu)$ consists of at most two arcs B_1, B_2, with $B_1 \subset A_1$, $B_2 \subset A_2$, and the equilibrium density is concave on each of these arcs. Here $d\theta$ indicates the arclength Lebesgue measure on $\mathbb{T}$.*

We remark that the theorem remains valid for any $\Sigma \subseteq \mathbb{T}$ compact set as long as σ_0^+ has a support consisting of at most two arcs and $d\sigma_0^+ = g(\theta)\, d\theta$ has a concave density. Also, the δ-Hölder continuous first derivative condition on Q must be satisfied only on $\operatorname{supp}(\sigma_0^+)$.

The key to the proof of the unit circle case is an analog of Lemma 3.

LEMMA 8. *Let A be an arc or a union of two arcs, and let ν be a positive measure with compact support on $\mathbb{T}$, such that $\nu(A) = 0$. Then the balayage $\hat{\nu} := \operatorname{Bal}(\nu, A)$ of ν onto A is absolutely continuous measure with respect to the Lebesgue arclength measure $d\theta$ and has convex density on every subarc.*

REMARK 9. It is possible to derive Lemma 3 from Lemma 8 through a limiting process, which we illustrate briefly. It is enough to consider only the point mass balayage case. Let $s \in \mathbb{R}$ be fixed, $A = [a, b] \cup [c, d] \subset \mathbb{R}$, and let A_R be the preimage of A under the inversion with center $s + iR$ and radius R. Observe, that A_R consists of two arcs on the circle $|x - s - iR/2| = R/2$, and hence $\operatorname{Bal}(\delta_s, A_R)$ has convex density with respect to the Lebesgue arclength on every subarc. As $R \to \infty$ one can show that the density of $\operatorname{Bal}(\delta_s, A_R)$ approaches that of $\operatorname{Bal}(\delta_s, A)$, which would imply the convexity of the limit. However, instead of following this route, we prefer to prove Lemma 3 directly, as its proof is simpler than that of Lemma 8, as well as illustrative and constructive.

The next theorem establishes another condition which guarantees that the support of the equilibrium measure is an interval. We say that a function $h(x)$ is strictly increasing (decreasing) a.e. on $[-1, 1]$, if there exists a set $G \subset [-1, 1]$ such that G has measure 1 and $h(x)$ is strictly increasing (decreasing) on G.

THEOREM 10. *Let Q be an external field on $[-1, 1]$ as described in Section 1.1. Then the following hold:*

(a) Let $f(x) := (1 - x^2)Q'(x)$.
If f is convex on $[-1, 1]$, then the support S is an interval $[b, 1]$, where $b \in [-1, 1)$.
If f is concave on $[-1, 1]$, then the support S is an interval $[-1, b]$, where $b \in (-1, 1]$.

(b) For $x \in [-1, 1]$ let $g(x) := Q'(x)\sqrt{1 - x^2}$, and let $g(x) = 0$ otherwise. Assume that $g(x)$ is absolutely continuous, $\int_{\mathbb{R}} \int_{\mathbb{R}} |(g'(x + u) - g'(x))/u| du dx < \infty$, and $x \mapsto \int_{\mathbb{R}} (g'(x + u) - g'(x))/u du$ is a continuous function on $(-1, 1)$.
If $\sqrt{1 - x^2} g'(x)$ is strictly increasing a.e. on $[-1, 1]$, then the support S is an interval $[b, 1]$, where $b \in [-1, 1)$.
If $\sqrt{1 - x^2} g'(x)$ is a strictly decreasing a.e. on $[-1, 1]$, then the support S is an interval $[-1, b]$, where $b \in (-1, 1]$.

EXAMPLE 11. Let

$$Q(x) = c \arcsin(x/a),\ x \in [-1, 1],\ a > 1, c \neq 0.$$

We find that

$$f(x) = c \left[\sqrt{a^2 - x^2} - \frac{a^2 - 1}{\sqrt{a^2 - x^2}} \right].$$

Thus, for $c > 0$ the function f is concave and the equilibrium support $S_Q = [-1, b]$. For $c < 0$ the function $f(x)$ is convex and $S_Q = [b, 1]$. Observe that the external field Q is weakly-convex in the sense of [2], so this is an independent verification of the result there.

The remainder of this paper is devoted to the proofs of our results.

3. Proof of the real line case. We will repeatedly make use of the Chebyshev's Integral Inequality (see [13, p. 1092], [14, pp. 43-44]), which we formulate as a separate lemma and provide a short proof for completeness.

LEMMA 12. *(Chebyshev, 1882) Let f, g, h be integrable functions on $[a, b]$ and let $f \geq 0$ on $[a, b]$.*

(a) Suppose that both g and h are monotone increasing (decreasing). Then

$$\int_a^b (fg)(x)\,dx \int_a^b (fh)(x)\,dx \leq \int_a^b f(x)\,dx \int_a^b (fgh)(x)dx, \tag{3.1}$$

provided all integrals exist.

(b) If h increases and g decreases (h decreases and g increases), then

$$\int_a^b (fg)(x)\,dx \int_a^b (fh)(x)\,dx \geq \int_a^b f(x)\,dx \int_a^b (fgh)(x)dx, \tag{3.2}$$

provided all integrals exist.

Proof. Clearly (b) follows from (a) by substituting g with $(-g)$, so let h, g be both monotone increasing (decreasing). Then

$$f(x)f(y)[g(x) - g(y)][h(x) - h(y)] \geq 0 \ \text{ for any } \ x, y \in [a, b].$$

Integrating the inequality yields

$$\int_a^b \int_a^b f(x)f(y)[g(x) - g(y)][h(x) - h(y)]\,dx\,dy \geq 0,$$

which implies (3.1). □

REMARK 13. In the particular case when $h(x) = x$, Lemma 12 has an interesting geometric interpretation. Suppose f, g, are integrable non-negative functions on $[a, b]$ and g is an increasing function on $[a, b]$. Then (3.1) becomes

$$\frac{\int x f(x)\,dx}{\int f(x)\,dx} \leq \frac{\int x f(x) g(x)\,dx}{\int f(x) g(x)\,dx}. \tag{3.3}$$

Imagine that the $[a, b]$ interval is a wire with density $f(x)$. The above inequality says that the center of mass will move to the right if we multiply the density by an increasing non-negative function $g(x)$.

We now continue with the proof of Lemma 3.

Proof of Lemma 3. It is enough to prove the lemma only for Dirac delta measures δ_s, $s \notin A$, because in general we have the representation

$$\frac{d\hat{\nu}}{dt} = \int_{\mathrm{supp}(\nu)} \frac{d\hat{\delta}_s}{dt}\,d\nu(s) =: \int_{\mathrm{supp}(\nu)} f(t, s)\,d\nu(s), \tag{3.4}$$

where $f(t, s) := d\hat{\delta}_s/dt$. Then the convexity of $\hat{\nu}$ can be easily derived from that of $f(t, s)$ by integrating the inequality

$$f(\alpha t + (1 - \alpha)y, s) \leq \alpha f(t, s) + (1 - \alpha) f(y, s), \quad 0 \leq \alpha \leq 1, \quad t, y \in A$$

with respect to $d\nu(s)$ and using (3.4) (recall that $\nu(A) = 0$).

We now consider two cases.

Case 1: The set A consists of one interval, for simplicity $A = [-1, 1]$.

From [19, Chapter II, Corollary 4.12] we have that

$$f(t,s) = \frac{1}{\pi}\frac{\sqrt{s^2-1}}{|s-t|\sqrt{1-t^2}} \tag{3.5}$$

We claim that $f_{tt}(t,s) > 0$ for all $t \in (-1,1)$. Without loss of generality let $s > 1$. Then we find that

$$f_{tt}(t,s) = \frac{\sqrt{s^2-1}}{\pi}\left(\frac{2}{(s-t)^3\sqrt{1-t^2}} + \frac{s+t}{(s-t)^2(1-t^2)^{3/2}} + \frac{3t^2}{(s-t)(1-t^2)^{5/2}}\right),$$

which verifies the claim, since $s+t > 0$ and $s-t > 0$.

Case 2: The set A consists of two intervals, i.e., $A = [a,b] \cup [c,d]$. Assume first that $s \in (b,c)$. Applying Kelvin transform K with center s and radius 1 we obtain $A^* := K(A) = [b^*, a^*] \cup [d^*, c^*]$ (see Section 1.3).

The density of the equilibrium measure of A^* (see [21, Lemma 4.4.1]) is given by

$$\frac{d\mu_{A^*}}{dt^*} = \frac{|t^*-y^*|}{\pi\sqrt{|t^*-a^*||t^*-b^*||t^*-c^*||t^*-d^*|}}, \tag{3.6}$$

where $y^* \in (a^*, d^*)$ is determined from the equation

$$\int_{a^*}^{d^*} \frac{t^*-y^*}{\pi\sqrt{|t^*-a^*||t^*-b^*||t^*-c^*||t^*-d^*|}}\,dt^* = 0. \tag{3.7}$$

Using the distance distortion formula (1.12) and the balayage/equilibrium relation (1.13) we derive the balayage density as

$$\frac{d\hat{\delta}_s}{dt} = f(t,s) := \frac{\sqrt{|s-a||s-b||s-c||s-d|}}{\pi|s-y|}\frac{|t-y|}{|t-s|\sqrt{|t-a||t-b||t-c||t-d|}}, \quad t \in A. \tag{3.8}$$

Let

$$\phi(t) := \log(f(t,s)) = \log\left(\frac{|t-y|}{|t-s|\sqrt{|t-a||t-b||t-c||t-d|}}\right) + c(s).$$

We shall prove that for fixed $s \in (b,c)$, we have $\phi''(x) > 0$ for all $x \in (a,b) \cup (c,d)$. This amounts to the inequality

$$\frac{1}{(x-y)^2} < \frac{1}{(x-s)^2} + \frac{1}{2}\left\{\frac{1}{(x-a)^2} + \frac{1}{(x-b)^2} + \frac{1}{(x-c)^2} + \frac{1}{(x-d)^2}\right\}, \text{ for all } x \in A. \tag{3.9}$$

(Observe that if $y^* = s$, then $y = \infty$ and the factors $|t-y|$ and $|s-y|$ in (3.8) are omitted and (3.9) obviously holds.)

If $y^* \in (a^*, d^*) \setminus \{s\}$, then $y \in (-\infty, a) \cup (d, \infty)$. Without loss of generality we may assume $y \in (-\infty, a)$. Clearly, for $x \in [c,d]$ (3.9) holds. So, let us fix $x \in [a,b]$. From (3.7) we derive that

$$y^* - a^* = \frac{\displaystyle\int_{a^*}^{d^*} \frac{t^*-a^*}{\sqrt{|t^*-a^*||t^*-b^*||t^*-c^*||t^*-d^*|}}\,dt^*}{\displaystyle\int_{a^*}^{d^*} \frac{1}{\sqrt{|t^*-a^*||t^*-b^*||t^*-c^*||t^*-d^*|}}\,dt^*} \tag{3.10}$$

Since $\sqrt{|t^* - c^*|}$ is decreasing on $[a^*, d^*]$ we can apply Lemma 12 (see also Remark 13) to estimate that

$$y^* - a^* \geq \frac{\displaystyle\int_{a^*}^{d^*} \frac{t^* - a^*}{\sqrt{|t^* - a^*||t^* - b^*||t^* - d^*|}}\, dt^*}{\displaystyle\int_{a^*}^{d^*} \frac{1}{\sqrt{|t^* - a^*||t^* - b^*||t^* - d^*|}}\, dt^*} \geq \frac{\sqrt{a^* - b^*}}{\sqrt{d^* - b^*}} \cdot \frac{\displaystyle\int_{a^*}^{d^*} \sqrt{\frac{t^* - a^*}{d^* - t^*}}\, dt^*}{\displaystyle\int_{a^*}^{d^*} \frac{1}{\sqrt{(t^* - a^*)(d^* - t^*)}}\, dt^*} \tag{3.11}$$

The second fraction evaluates to $(d^* - a^*)/2$, thus reducing (3.11) to

$$y^* - a^* \geq \frac{\sqrt{|a^* - b^*|}}{\sqrt{|d^* - b^*|}} \frac{d^* - a^*}{2}, \tag{3.12}$$

which after the Kelvin transformation becomes

$$\frac{|y - a|}{|y - s|} \geq \frac{\sqrt{|d - s||a - b|}}{\sqrt{|a - s||d - b|}} \frac{|d - a|}{2|d - s|}. \tag{3.13}$$

Rewriting (3.13) yields

$$\frac{1}{\sqrt{|y - a|}} \leq \frac{2\sqrt{|y - a||a - s|}}{|y - s|} \cdot \frac{\sqrt{|d - s||d - b|}}{|d - a|} \cdot \frac{1}{\sqrt{|a - b|}} \leq \frac{1}{\sqrt{|a - b|}}, \tag{3.14}$$

where each of the two fractions above is less than 1.

On the other hand, one easily derives that

$$\min_{x \in [a,b]} \frac{1}{2} \left\{ \frac{1}{(a - x)^2} + \frac{1}{(b - x)^2} \right\} = \frac{4}{(b - a)^2}.$$

So, the inequality (3.14) implies (3.9) at least with a factor of four in excess.

$$\frac{1}{(y - x)^2} \leq \frac{1}{(y - a)^2} \leq \frac{1}{(a - b)^2} \leq \frac{1}{8} \left\{ \frac{1}{(a - x)^2} + \frac{1}{(b - x)^2} \right\}.$$

This establishes the lemma when $s \in (b, c)$, and hence for any measure ν with $\operatorname{supp}(\nu) \subset [b, c]$ with $\nu(A) = 0$. If we assume that $s \in (-\infty, a) \cup (d, \infty)$, then the balayage $\mu = \operatorname{Bal}(\delta_s, [a, d])$ has a convex density by Case 1. However, from the properties of balayage measure we have that

$$\operatorname{Bal}(\delta_s, [a, b] \cup [c, d]) = \operatorname{Bal}(\mu, [a, b] \cup [c, d]) = \mu_{|_A} + \operatorname{Bal}(\mu_{|_{[b,c]}}, A).$$

Both measures on the right have convex densities on A, the first, as a restriction of a measure with a convex density on the entire interval $[a, d]$, and the second by what we just proved. Hence, $\hat{\delta_s}$ has convex density and the lemma is proved. □

We are now ready for the

Proof of Theorem 2. The proof is based on the iterated balayage algorithm discussed in Section 1.2. Let $\Sigma_0 := \Sigma$ and σ_0 denote the signed equilibrium associated with Q. By the assumption of the theorem the latter exists, and if $\sigma_0 = \sigma_0^+ - \sigma_0^-$ is the Jordan decomposition of σ_0, the measure σ_0^+ has concave density and its support $\Sigma_1 := \operatorname{supp}(\sigma_0^+)$ consists of at most two intervals. Then from Lemma 3, $\operatorname{Bal}(\sigma_0^-, \Sigma_1)$ is convex and thus, $\sigma_1 = J(\sigma_0) = \sigma_0^+ - \operatorname{Bal}(\sigma_0^-, \Sigma_1)$ has concave density on Σ_1. Therefore, $\Sigma_2 := \operatorname{supp}(\sigma_1^+)$ consists of at most two intervals (nested in at most two intervals that make up Σ_1).

Continuing, in this way we obtain an IBA sequence of nested compact sets

$$\Sigma_0 \supset \Sigma_1 \supset \ldots \Sigma_n \supset \cdots \supset \operatorname{supp}(\mu_Q)\ ,$$

each made of at most two intervals.

Now we show that the density of σ_n, which we will denote by $v_n(x)$, is converging to the density of the equilibrium measure μ_Q associated with the external field $Q(x)$.

Let us assume that for all n we have $\Sigma_n = [a_n, b_n] \cup [c_n, d_n]$, where $[a_n, b_n]$ and $[c_n, d_n]$ are two non-trivial disjoint intervals. If it was not the case, the proof would be the same. (We remark that even if Σ_0 consisted of two intervals, we may "lose" one in the IBA if at one step $v_n \leq 0$ on $[a_n, b_n]$ or on $[c_n, d_n]$.)

Let $\lim a_n = a, \lim b_n = b, \lim c_n = c, \lim d_n = d$. We have

$$\begin{aligned} Q(x) - F_n &= \int_{\Sigma_n} \log|t-x| v_n(t)\, dt \\ &= \left(\int_{\Sigma_{n+1}} + \int_{\Sigma_n \setminus \Sigma_{n+1}} \right) \log|t-x| v_n(t) dt \qquad (3.15) \\ &= \left(\int_{[a_{n+1}, b_{n+1}] \cup [c_{n+1}, d_{n+1}]} + \int_{[a_n, a_{n+1}] \cup [b_{n+1}, b_n] \cup [c_n, c_{n+1}] \cup [d_{n+1}, d_n]} \right) \log|t-x| v_n(t) dt. \end{aligned}$$

We claim that

$$\int_{[a_n, a_{n+1}] \cup [b_{n+1}, b_n] \cup [c_n, c_{n+1}] \cup [d_{n+1}, d_n]} v_n(t) dt \to 0. \qquad (3.16)$$

If not, then we can chose a subsequence, denoted by n for simplicity, and $m_i \leq 0, i = 1, \ldots, 4$, such that

$$\int_{[a_n, a_{n+1}]} v_n(t) dt \to m_1, \int_{[b_{n+1}, b_n]} v_n(t) dt \to m_2, \int_{[c_n, c_{n+1}]} v_n(t) dt \to m_3, \int_{[d_{n+1}, d_n]} v_n(t) dt \to m_4, \sum m_i < 0.$$

Since $v_n\Big|_{[a,b] \cup [c,d]}$ is a bounded decreasing sequence, letting $v := \lim v_n$, we can use the dominated convergence theorem and mean value theorem to derive that for any $x \in (a, b) \cup (c, d)$

$$\lim(Q(x) - F_n) = \int_{[a,b] \cup [c,d]} \log|t-x| v(t) dt + m_1 \log|a-x| + m_2 \log|b-x| + m_3 \log|c-x| + m_4 \log|d-x|. \qquad (3.17)$$

This shows that F_n has a finite limit. We also see that we must have $m_1 = m_2 = m_3 = m_4 = 0$. For example, if $m_1 < 0$, let $x = a$ in (3.15) and let $n \to \infty$. The right-hand side of (3.15) is approaching positive infinity while the left hand-side has finite limit, which is a contradiction.

Let $F := \lim F_n$. From (3.15) we get that for any $x \in (a, b) \cup (c, d)$

$$Q(x) - F = \int_{[a,b] \cup [c,d]} \log|t-x| v(t) dt.$$

From (3.16) it is also clear that $v(t)$ is a probability density function. The equilibrium measure μ_Q minimizes the weighted energy on Σ and therefore on $[a, b] \cup [c, d]$, too. Also, the support of μ_Q is a subset of $[a, b] \cup [c, d]$. It follows that $v(x)$ is the density of the equilibrium measure (see [19, Theorem I.3.3]).

If $a = b$ then $\operatorname{supp}(\mu_Q) = [c, d]$. If $c = d$ then $\operatorname{supp}(\mu_Q) = [a, b]$. Finally, if $a < b$ and $c < d$ then $\operatorname{supp}(\mu_Q) = [a, b] \cup [c, d]$. □

4. Proof of the unit circle case. We proceed first with the proof of Lemma 8.

Proof of Lemma 8. As in the proof of Lemma 3 above, it is enough to verify the Lemma for case of a point mass, so without loss of generality we assume that $\nu = \delta_s$, where $s \in \mathbb{T} \setminus A$.

Given two points on the unit circle $e^{i\phi}$ and $e^{i\theta}$ we denote by $[e^{i\phi}, e^{i\theta}]$ the closed arc that connects the points counterclockwise. In this notation, its complement relative to the unit circle would be the open arc $(e^{i\theta}, e^{i\phi})$. Since the case of an arc easily follows from the case of two arcs as we deform one of the arcs to a point, we set $A =$

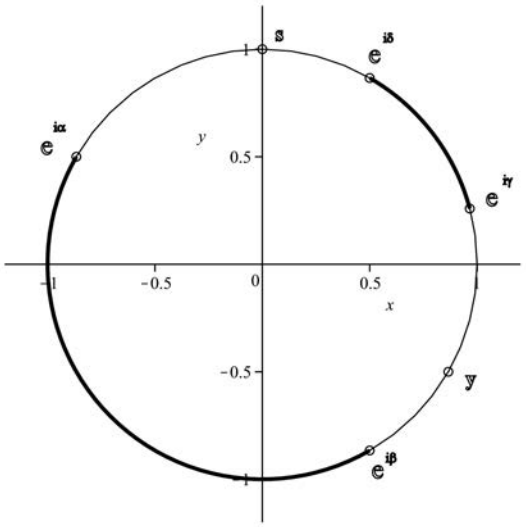


FIG. 4.1. *Circle case*

$[e^{i\alpha}, e^{i\beta}] \cup [e^{i\gamma}, e^{i\delta}]$, where $e^{i\alpha}, e^{i\beta}, e^{i\gamma}, e^{i\delta}$ are points on the unit circle ordered counterclockwise. Without loss of generality we assume that $s = i \in (e^{i\delta}, e^{i\alpha})$ (see Fig. 4.1). For simplicity assume also that all angles below are given in the interval $[\pi/2, 5\pi/2)$, meaning in particular that $\pi/2 < \alpha < \beta < \gamma < \delta < 5\pi/2$.

To find the balayage $\widehat{\nu}$, we observe that after a Kelvin transform centered at s with a radius $\sqrt{2}$ the unit circle is sent to the real line and n $A^* = [a, b] \cup [c, d]$, where $a = \left(e^{i\alpha}\right)^*$, $b = \left(e^{i\beta}\right)^*$, $c = \left(e^{i\gamma}\right)^*$, and $d = \left(e^{i\delta}\right)^*$ (see Fig. 4.2). Using Riesz's approach (Section 1.3) we find $\widehat{\nu} = K(\mu_{A^*})$. Recall that (see (3.6))

$$d\mu_{A^*} = \frac{|t^* - y^*|}{\pi\sqrt{|t^* - a||t^* - b||t^* - c||t^* - d|}}\, dt^*, \quad t^* \in [a, b] \cup [c, d], \tag{4.1}$$

where $t^* = \left(e^{i\theta}\right)^* \in A$ and $y^* \in (b, c)$ is determined from the equation

$$\int_b^c \frac{x^* - y^*}{\pi\sqrt{|x^* - a||x^* - b||x^* - c||x^* - d|}}\, dx^* = 0. \tag{4.2}$$

Since the relationship between the Lebesgue measures on $\mathbb{R}$ and $\mathbb{T}$ is given by

$$\frac{dt^*}{|t^* - s|} = \frac{|d\, t|}{|t - s|} = \frac{d\theta}{|e^{i\theta} - s|},$$

we find that

$$dt = \frac{2d\theta}{|e^{i\theta} - s|^2}. \tag{4.3}$$

Let $y := e^{i\phi} := (y^*)^*$ with $\phi \in (\beta, \gamma)$. Using (1.12) and the formula $|e^{i\xi} - e^{i\zeta}| = 2|\sin\left(\frac{\xi - \zeta}{2}\right)|$, we conclude that (recall the order of the angles)

$$d\widehat{\nu} = C\frac{|\sin\left(\frac{\theta - \phi}{2}\right)|\, d\theta}{\sin\left(\frac{\theta - \pi/2}{2}\right)\sqrt{|\sin\left(\frac{\theta - \alpha}{2}\right)\sin\left(\frac{\theta - \beta}{2}\right)\sin\left(\frac{\theta - \gamma}{2}\right)\sin\left(\frac{\theta - \delta}{2}\right)|}}, \quad \theta \in [\alpha, \beta] \cup [\gamma, \delta], \tag{4.4}$$

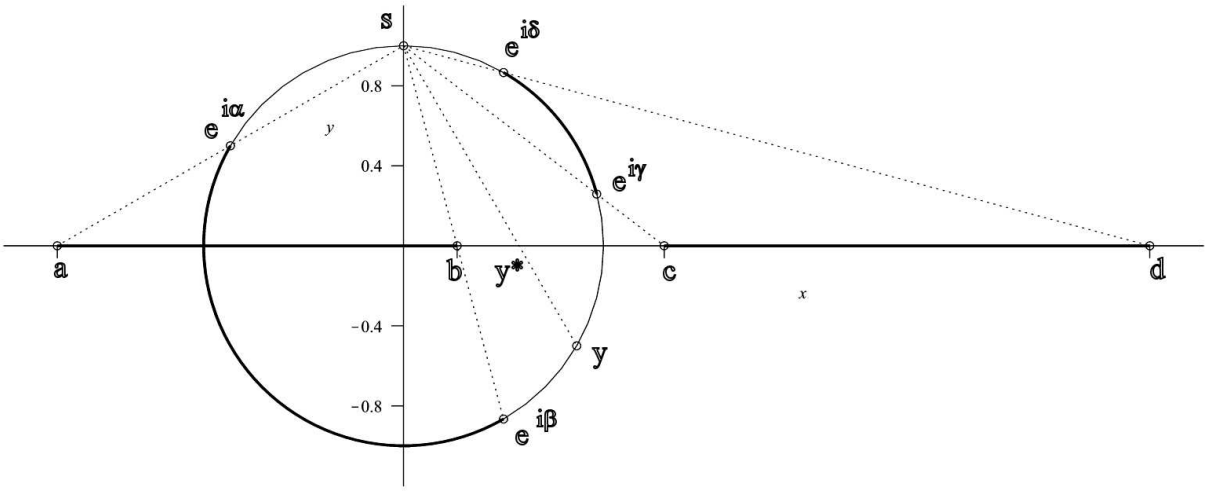


FIG. 4.2. *Kelvin transformation - circle case*

where

$$C = \frac{\sqrt{\sin\left(\frac{\alpha-\pi/2}{2}\right)\sin\left(\frac{\beta-\pi/2}{2}\right)\sin\left(\frac{\gamma-\pi/2}{2}\right)\sin\left(\frac{\delta-\pi/2}{2}\right)}}{2\pi\sin\left(\frac{\phi-\pi/2}{2}\right)}.$$

If $x^* = \left(e^{i\psi}\right)^*$, then

$$x^* - y^* = \frac{\sin\left(\frac{\psi-\phi}{2}\right)}{2\sin\left(\frac{\phi-\pi/2}{2}\right)\sin\left(\frac{\psi-\pi/2}{2}\right)}, \quad x^*, y^* \in [b, c].$$

Indeed, the equality holds with absolute values by (1.12) and since the signs of $x^* - y^*$ and $\sin\left(\frac{\psi-\phi}{2}\right)$ are the same we can remove the absolute value. Therefore, (4.2) implies that ϕ is determined uniquely by

$$\int_\beta^\gamma \frac{\sin\left(\frac{\psi-\phi}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-\delta}{2}\right)|}}\,d\psi = 0, \tag{4.5}$$

which is equivalent to

$$\cot\left(\frac{\phi-\theta}{2}\right) = \frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-\delta}{2}\right)|}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-\delta}{2}\right)|}}\,d\psi} \tag{4.6}$$

for any convenient choice of θ.

Recall that our goal is to show that for any $\pi/2 < \alpha < \beta < \gamma < \delta < 5\pi/2$, the balayage density function

$$N(\theta) := \frac{|\sin\left(\frac{\theta-\phi}{2}\right)|}{\sin\left(\frac{\theta-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta-\beta}{2}\right)\sin\left(\frac{\theta-\gamma}{2}\right)\sin\left(\frac{\theta-\delta}{2}\right)|}}, \quad \theta \in [\alpha,\beta]\cup[\gamma,\delta]$$

is convex, provided ϕ satisfies (4.6). We will actually show more, namely that $(\log(N(\theta)))'' > 0$. The latter is equivalent to verifying for all $\theta \in [\alpha,\beta]\cup[\gamma,\delta]$ the inequality

$$\csc^2\left(\frac{\theta-\phi}{2}\right) \le \csc^2\left(\frac{\theta-\pi/2}{2}\right) + \frac{1}{2}\left[\csc^2\left(\frac{\theta-\alpha}{2}\right) + \csc^2\left(\frac{\theta-\beta}{2}\right) + \csc^2\left(\frac{\theta-\gamma}{2}\right) + \csc^2\left(\frac{\theta-\delta}{2}\right)\right]. \tag{4.7}$$

Let us fix $\theta \in [\alpha,\beta]$ (the case $\theta \in [\gamma,\delta]$ is considered similarly). If $\phi - \theta \ge \pi$, then

$$\csc^2\left(\frac{\theta-\phi}{2}\right) = \frac{4}{|e^{i\phi}-e^{i\theta}|} < \frac{4}{|e^{i\pi/2}-e^{i\theta}|} = \csc^2\left(\frac{\theta-\pi/2}{2}\right)$$

and (4.7) holds trivially. Henceforth, assume that $\phi - \theta < \pi$.

We now perform several perturbations to simplify the problem. First, we study the effect of substituting δ with $5\pi/2$. Let $\phi_1 \in (\beta,\gamma)$ be uniquely determined by

$$\cot\left(\frac{\phi_1-\theta}{2}\right) = \frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-5\pi/2}{2}\right)|}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-5\pi/2}{2}\right)|}}\,d\psi}. \tag{4.8}$$

Observe that the monotonicity of $\cot((\psi-\theta)/2)$ guarantees the existence and uniqueness of ϕ_1. Applying Lemma 12 with

$$h(\psi) = \cot\left(\frac{\psi-\theta}{2}\right), \quad g(\psi) = \sqrt{\frac{\sin\left(\frac{\delta-\psi}{2}\right)}{\sin\left(\frac{5\pi/2-\psi}{2}\right)}} = \sqrt{\cos\left(\frac{5\pi/2-\delta}{2}\right) - \cot\left(\frac{5\pi/2-\psi}{2}\right)\sin\left(\frac{5\pi/2-\delta}{2}\right)}$$

and using that both $h(\psi)$ and $g(\psi)$ are monotone decreasing functions on (β,γ), we conclude that $\cot\left(\frac{\phi-\theta}{2}\right) < \cot\left(\frac{\phi_1-\theta}{2}\right)$, or that $\phi_1 < \phi$.

Similarly, if $\phi_2 \in (\beta,\gamma)$ denotes the unique solution obtained when $\alpha = \theta$ in (4.8), we conclude that $\phi_2 < \phi_1$. Indeed, if

$$\cot\left(\frac{\phi_2-\theta}{2}\right) = \frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{\sqrt{\sin\left(\frac{\psi-\theta}{2}\right)}}{\left(\sin\left(\frac{\psi-\pi/2}{2}\right)\right)^{3/2}\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{\sqrt{\sin\left(\frac{\psi-\theta}{2}\right)}}{\left(\sin\left(\frac{\psi-\pi/2}{2}\right)\right)^{3/2}\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}, \tag{4.9}$$

we again apply Lemma 12, this time with

$$g(\psi)=\sqrt{\frac{\sin\left(\frac{\psi-\alpha}{2}\right)}{\sin\left(\frac{\psi-\theta}{2}\right)}}=\sqrt{\cos\left(\frac{\theta-\alpha}{2}\right)+\cot\left(\frac{\psi-\theta}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)},$$

which is still decreasing function for $\psi\in(\beta,\gamma)$.

Finally, let $\phi_3\in(\beta,\gamma)$ be derived from (4.9) with θ instead of $\pi/2$. Then we have

$$\cot\left(\frac{\phi_3-\theta}{2}\right)=\frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{1}{\sin\left(\frac{\psi-\theta}{2}\right)\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{1}{\sin\left(\frac{\psi-\theta}{2}\right)\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}. \tag{4.10}$$

Applying Lemma 12 with the decreasing function

$$g(\psi)=\left(\frac{\sin\left(\frac{\psi-\pi/2}{2}\right)}{\sin\left(\frac{\psi-\theta}{2}\right)}\right)^{3/2}=\left(\cos\left(\frac{\theta-\pi/2}{2}\right)+\cot\left(\frac{\psi-\theta}{2}\right)\sin\left(\frac{\theta-\pi/2}{2}\right)\right)^{3/2}$$

we obtain $\phi_3<\phi_2$. In Lemma 14 below we will show for the so derived ϕ_3 the inequality

$$\csc^2\left(\frac{\phi_3-\theta}{2}\right)\le 2+\frac{1}{2}\left[\csc^2\left(\frac{\theta-\beta}{2}\right)+\csc^2\left(\frac{\theta-\gamma}{2}\right)\right], \tag{4.11}$$

which coupled with $\theta<\beta<\phi_3<\phi<\gamma$ and $\phi-\theta<\pi$ proves (4.7) so the proof of Lemma 8 will be complete. □

We now prove that (4.11) holds provided ϕ_3 satisfies (4.10). Since we find the inequality interesting in its own right we formulate it as a separate lemma. Without loss of generality we could assume that $\beta=0<\gamma<\theta<2\pi$.

LEMMA 14. *Let* $0<\gamma<\theta<2\pi$. *Then*

$$\left(\frac{\displaystyle\int_0^\gamma \frac{\cot\left(\frac{\theta-t}{2}\right)dt}{\sin\left(\frac{\theta-t}{2}\right)\sqrt{\sin\left(\frac{t}{2}\right)\sin\left(\frac{\gamma}{2}-\frac{t}{2}\right)}}}{\displaystyle\int_0^\gamma \frac{dt}{\sin\left(\frac{\theta-t}{2}\right)\sqrt{\sin\left(\frac{t}{2}\right)\sin\left(\frac{\gamma}{2}-\frac{t}{2}\right)}}}\right)^2\le 1+\frac{1}{2}\left[\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}+\frac{1}{\sin^2\left(\frac{\theta-\gamma}{2}\right)}\right]. \tag{4.12}$$

Proof. First we compute:

$$I_1=\int_0^\gamma \frac{dt}{\sin\left(\frac{\theta-t}{2}\right)\sqrt{\sin\left(\frac{t}{2}\right)\sin\left(\frac{\gamma}{2}-\frac{t}{2}\right)}} \tag{4.13}$$

Making a substitution $y=(\cos(t-\gamma/2)-\cos(\gamma/2))/(1-\cos(\gamma/2))$ in (4.13) leads to

$$\begin{aligned} I_1&=\frac{4\sin\left(\frac{\theta-\gamma/2}{2}\right)}{1-\cos(\frac{\gamma}{2})}\int_0^1\frac{1}{y+c}\frac{dy}{\sqrt{y(1-y)}}=\frac{4\sin\left(\frac{\theta-\gamma/2}{2}\right)}{1-\cos(\frac{\gamma}{2})}\frac{\pi}{\sqrt{c(c+1)}} \\ &=\frac{2\sqrt{2}\pi}{\sqrt{\cos(\gamma/2)-\cos(\theta-\gamma/2)}}, \end{aligned} \tag{4.14}$$

where $c = c(\theta,\gamma) := \dfrac{\cos(\gamma/2)-\cos(\theta-\gamma/2)}{1-\cos(\gamma/2)} > 0.$

Next, we compute similarly,

$$I_2 := \int_0^{\gamma} \frac{\cos((\theta-t)/2)}{\sin^2((\theta-t)/2)} \frac{dt}{\sqrt{\sin(t/2)\sin(\gamma/2-t/2)}}$$
$$= \sqrt{2}\int_0^{\gamma/2} \left[\frac{\cos\left(\frac{\theta-\gamma/2}{2} - u/2\right)}{\sin^2\left(\frac{\theta-\gamma/2}{2} - u/2\right)} + \frac{\cos\left(\frac{\theta-\gamma/2}{2} + y/2\right)}{\sin^2\left(\frac{\theta-\gamma/2}{2} + u/2\right)} \right] \frac{du}{\sqrt{\cos(u)-\cos(\gamma/2)}}.$$

Using the identity

$$\frac{\cos(\alpha-\alpha')}{\sin^2(\alpha-\alpha')} + \frac{\cos(\alpha+\alpha')}{\sin^2(\alpha+\alpha')} = \frac{8\cos(\alpha)\cos(\alpha')(1-1/2(\cos(2\alpha)+\cos(2\alpha')))}{(\cos(2\alpha')-\cos(2\alpha))^2}$$

that holds for any real α and α', and the substitution $y = (\cos u - \cos(\gamma/2))/(1-\cos(\gamma/2))$, we get

$$I_2 = \frac{4\cos((\theta-\gamma/2))}{1-\cos(\gamma/2)} \left[\frac{2(1-\cos(\theta-\gamma/2))}{1-\cos(\gamma/2)} \int_0^1 \frac{1}{(y+c)^2} \frac{dy}{\sqrt{y(1-y)}} - \int_0^1 \frac{1}{y+c} \frac{dy}{\sqrt{y(1-y)}} \right]. \tag{4.15}$$

Since

$$\int_0^1 \frac{1}{(y+c)^2} \frac{dy}{\sqrt{y(1-y)}} = -\frac{d}{dc}\left(\int_0^1 \frac{1}{y+c} \frac{dy}{\sqrt{y(1-y)}} \right) = -\frac{d}{dc} \frac{\pi}{\sqrt{c(1+c)}},$$

we obtain

$$\int_0^1 \frac{dy}{(y+c)^2\sqrt{y(1-y)}} = \frac{\pi(2c+1)}{2\sqrt{c(c+1)}c(c+1)}.$$

Substituting in (4.15) we find that

$$I_2 = \frac{2\sqrt{2}\sin(\theta-\gamma/2)\pi}{(\cos(\gamma/2)-\cos(\theta-\gamma/2))^{3/2}}. \tag{4.16}$$

Thus, we see from (4.14) and (4.16) that

$$(I_2/I_1)^2 = (1/4)(\cot(\theta/2)+\cot((\theta-\gamma)/2))^2 \le (1/2)(\cot^2(\theta/2)+\cot^2((\theta-\gamma)/2)))$$
$$< (1/2)(\csc^2(\theta/2)+\csc^2((\theta-\gamma)/2)) < 1 + 1/2(\csc^2(\theta/2)+\csc^2((\theta-\gamma)/2)),$$

which concludes the proof. □

Proof of Theorem 7. The proof of this theorem follows word for word the argument in Theorem 2, where instead of Lemma 3 we use Lemma 8. □

Proof of Theorem 10. From (1.6) with $[a,b] = [-1,1]$ we can write

$$v(x) = \frac{1}{\pi\sqrt{1-x^2}} \left[1 + \frac{1}{\pi} \int_{-1}^{1} \frac{f(t)-f(x)}{(t-x)\sqrt{1-t^2}}\, dt \right]. \tag{4.17}$$

Here we used the fact that

$$P.V. \int_{-1}^{1} \frac{1}{(t-x)\sqrt{1-t^2}}\, dt = 0,$$

which follows from differentiation of the equilibrium potential on $[-1,1]$

$$\int_{-1}^{1} \log \frac{1}{|x-t|} \frac{dt}{\pi\sqrt{1-t^2}} = \log 2, \quad x \in [-1,1].$$

Proof of part a). If f is identically zero, then Q is constant and $S=[-1,1]$. Note that f cannot be a linear function (unless $f \equiv 0$) because of the δ-Hölder continuity assumption on Q'. So let f be convex but not a linear function. Let $g_t(x) = (f(t)-f(x))/(t-x)$, and $x_1 < x_2$. Then $g_t(x_1) \le g_t(x_2)$ for all $t \in [-1,1]$ and $g_t(x_1) < g_t(x_2)$ holds on a set of positive measure. Integrating over $[-1,1]$ with respect to t, we thus derive that $\sqrt{1-x^2}v(x)$ is strictly increasing. The claim now follows from [15, Theorem 2]. The proof is similar when f is concave.

Proof of part b). Let $\sqrt{1-x^2}g'(x)$ be strictly increasing (strictly decreasing) a.e. on $[-1,1]$. Note that $\sqrt{1-x^2}v(x)$ is strictly increasing (strictly decreasing) function a.e. if the following is positive (negative):

$$\frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}Q'(t)}{t-x}dt = \int_{-1}^{1} \frac{(\sqrt{1-t^2}Q'(t))'}{t-x}dt = \int_{-1}^{1} \frac{\sqrt{1-t^2}(\sqrt{1-t^2}Q'(t))' - \sqrt{1-x^2}(\sqrt{1-x^2}Q'(x))'}{(t-x)\sqrt{1-t^2}}dt.$$

We used that

$$\frac{d}{dx}\int_{-1}^{1} \frac{g(t)}{t-x}dt = \frac{d}{dx}\int_{\mathbb{R}} \frac{g(x+u)-g(x)}{u}du = \int_{\mathbb{R}} \frac{g'(x+u)-g'(x)}{u}du = \int_{-1}^{1} \frac{g'(t)}{t-x}du.$$

We could differentiate inside the parametric integral because of [1, Lemma 13].

The claim of part b) now follows from [15, Theorem 2]. □

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An algorithm for symmetric indefinite linear systems*

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Abstract

In this paper, the new pivoting strategy named boundedly partial pivoting is presented in the process of the incomplete LDL^{T} factorization, based on the maximum weighted matchings in the preprocessing of solving a symmetric indefinite linear system. This new pivoting method is acquired by changing the tri-diagonal pivoting algorithm, and none of permutations of rows and columns are needed. Meanwhile, the lower triangular factor L is bounded. Numerical results demonstrate that the newly proposed pivoting strategy can extend the scope of applications of the LDL^{T} preconditioner, compared with the tri-diagonal pivoting. Moreover, almost the same results are obtained or our boundedly partial pivoting is more efficient with respect to the iteration speed, when both the tri-diagonal pivoting and the boundedly partial pivoting are applicable to incomplete LDL^{T} factorization.

Key words: Symmetric indefinite linear system; Weighted matching; Pivoting; Incomplete LDL^{T} factorization; Boundedly partial pivoting

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1 Introduction

We consider an efficient algorithm to solve symmetric indefinite linear systems $Ax = b$. In [1], Hagemann and Schenk have proposed the preconditioning algorithm based on a combination of the symmetric weighted matchings and the tri-diagonal pivoting technique. In this paper, the preprocessing used is similar to the preprocessing in [1], and during the incomplete LDL^{T} factorization, we put forward a new

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pivoting algorithm based on the tri-diagonal pivoting. To be brief, our preconditioning algorithm is based on a combination of the symmetric weighted matchings and the boundedly partial pivoting technique. The boundedness of the factor L is ensured by the newly proposed pivoting strategy compared to the tri-diagonal pivoting, and therefore the stability of the incomplete factorization is improved.

Here the whole process of the preconditioned iterative solution of a symmetric indefinite linear system is presented. Firstly, the preprocessing technique is applied to the original system $Ax = b$, then we gain the new system

$$\hat{A}\hat{x} = \hat{b},$$

which is also a symmetric indefinite linear system. Secondly, incomplete LDL^{T} technique is applied to factorize $\hat{A}$ in order to construct preconditioner M, which is also indefinite. Namely, the incomplete factorization is

$$M = LDL^{\mathrm{T}} = \hat{A} - E, \tag{1}$$

where E is the error due to the incompleteness of the L factor, and $\hat{A}$ is the scaled and reordered original indefinite linear system:

$$\hat{A} = P_Q Q D_Q A D_Q Q^T P_Q^T. \tag{2}$$

Lastly, the SQMR iterative method solves a system of the form

$$M^{-1}\hat{A}\hat{x} = M^{-1}\hat{b}. \tag{3}$$

Sparse symmetric indefinite linear systems mainly arise in linear and nonlinear optimization,incompressible flow computations, finite element analysis, electromagnetic scattering, shift-invert eigensolvers and so on.

The preconditioned iterative solution of the indefinite linear systems is an active research area, and a variety of approaches have been proposed. Preconditioning is maturing and the choice of a good preconditioner is benefit to accelerate the iterative solution of the indefinite linear systems. In [2], Benzi summarizes the main preconditioning techniques, incomplete factorization methods and sparse approximate inverses, which are applied to accelerate iterative speed of solving sparse symmetric indefinite linear systems. Many preconditioning techniques exploit the inherent block structure of the given problem type. Specially, block and approximate Schur complement approaches [7-10] and indefinite block preconditioners [9, 11-13] have been proposed for saddle-point systems. The direct solvers are mostly chosen to solve general indefinite systems for the poor properties of indefinite problems. Now, since the LU and LDL^{T} factorizations of indefinite matrices may break down or yield bad results because of the small or zero diagonal elements, there exist a number of techniques that try to avoid or alleviate these problems, and pivoting techniques, perturbations of diagonal elements, and block factorizations are the

three basic approaches. A classic pivoting approach for the symmetric dense matrices is the Bunch-Kaufman pivoting [6], which searches for 2×2 diagonal blocks during the factorization. In [6], Duff and Reid introduce the pivoting method which is adapted to sparse matrices. Benzi, Hawa and Tuma [14] study the non-symmetric permutations based on the weighted matchings as a preprocessing step for various incomplete or approximate factorizations. Duff and Gilbert [4] consider the weighted matchings as an approximation order of the pivoting in the context of sparse direct solution methods for symmetric indefinite linear systems. This work is elaborated by Duff and Parlet [3] and Schenk and Gärtner [5]. In [1], the symmetric weighted matchings technique is firstly applied to the preconditioning technique(the incomplete factorization) by Hagemann and Schenk.

Section 2 gives an overview of our preprocessing step, which is similar to the preprocessing in [1]. In section 3, we describe the incomplete LDL^{T} factorization with our pivoting algorithm named by a *boundedly partial pivoting*, which is based on tri-diagonal pivoting, and prove that the newly proposed pivoting can bound the elements of the L factor. Numerical results and comparisons with other techniques are presented in section 4. In closing, we draw a conclusion.

2 Preprocessing

Our preprocessing step is similar to the preprocessing step in [1]. The difference is no more than in the aspect of reordering, and the reordering approach we used is symrcm. The objective of this preprocessing method is to restrict the pivoting during the factorization phase, by obtaining a good pivoting order through the matching in the preprocessing step. Symmetric weighted matchings, symmetric scaling and reordering are involved in the preprocessing. The original system $Ax = b$ is processed by the preprocessing technique and then the new system $\hat{A}\hat{x} = \hat{b}$ is obtained ,with the matched and large entries in or nearby the 1×1 and 2×2 diagonal blocks of the matrix $\hat{A}$. For the details on the preprocessing, the interested reader should consult [1].

We will present the preprocessing step which is similar to the preprocessing step in [1]. The algorithm is given as follows:

Algorithm 2.1. (Pseudo-code for the preprocessing step)

1. $(P_M, D_r, D_c) = build_mps_matching(A)$
2. $match_cycles = get_cycle_repr(P_M)$
3. $split_cycles_\tilde{M} = split_cycle_repr(match_cycles)$
4. $(Q, marker_1 \times 1) = build_diag22_reordering(split_cycles_\tilde{M})$
5. $(A_{comp}, rev_map) = compress_matrix_2 \times 2(A, Q, marker_1 \times 1)$
6. $P_{comp} = build_symrcm_reordering(A_{comp})$
7. $P_Q = expand_reordering_2 \times 2(P_{comp}, marker_1 \times 1, rev_map)$
8. $D_Q = \sqrt{D_r \cdot D_c}$

9. $\hat{A} = P_Q Q D_Q A D_Q Q^T P_Q^T$

Now, we simply explain the meanings of the code in each row.

1. Using the MC64 program package, we obtain the permutation matrix P_M, which corresponds to the maximum weighted matchings, and the row, column scaling matrices D_r, D_c.

2. To gain the cycle representation of the permutation matrix P_M.

3. To split the cycle representation of P_M to the cycle representation $\tilde{M}$, which only contains 1-cycle and 2-cycles.

4. To acquire the permutation matrix Q, which corresponds with the symmetric weighted matchings, by the splitting cycle representation $\tilde{M}$.

5. To get the compressed matrix A_{comp} through compressing the matrix QAQ^T according to the symmetric weighted matchings.

6. To reorder the compressed matrix A_{comp} with the reordering approach – symrcm, and obtain the permutation matrix P_{comp}.

7. To expand the permutation matrix P_{comp} to the corresponding permutation matrix P_Q, which is applied to the matrix QAQ^T.

8. To get the symmetric scaling matrix D_Q by the two-sided scaling matrices D_r and D_c.

9. To acquire the preprocessed matrix $\hat{A}$ through the symmetric scaling matrix D_Q, the permutation matrix Q corresponding to symmetric weighted matchings, and the permutation matrix P_Q corresponding to reordering working on the original matrix A.

Compared to the preprocessing step in [1], the difference is only the code in the sixth row of Algorithm 2.1.

3 Incomplete LDL^{T} factorization with boundedly partial pivoting

3.1 Algorithm

The incomplete LDL^{T} factorization is applied since the systems are symmetric. The largest block pivots we considered are 2×2 pivots, and therefore the factor D is a block diagonal matrix with the 1×1 and 2×2 blocks. In the process of the incomplete factorization, JKI version of Gauss elimination is employed. So the factor L is constructed by column-wise and the dropping strategy can be easily implemented.

In [1], Hagemann and Schenk adopt the tri-diagonal pivoting in the incomplete LDL^{T} factorization since the preprocessed matrix has a heavy diagonal. We note that the cost of the exchanges of rows and columns in the factorization with the tri-diagonal pivoting is avoided, but the boundedness of the L factor can not be proved.

Namely, the factorization is unstable. We will consider the pivoting approach that also has no exchanges of rows and columns and meanwhile makes sure that the factor L is bounded in order to assure the stability of the factorization. Thereby, our approach extends the scope of applications of the LDL^{T} preconditioner.

3.2 Tri-diagonal pivoting

In [15], Bunch proposes the tri-diagonal pivoting aiming at tri-diagonal matrices. Next, we will show this pivoting algorithm. Let

$$B = \begin{bmatrix} \alpha_1 & \beta_2 & & & & \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & \beta_3 & \alpha_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \alpha_{n-1} & \beta_n \\ & & & & \beta_n & \alpha_n \end{bmatrix}.$$

be a symmetric tri-diagonal matrix. Firstly, the tri-diagonal pivoting in the first elimination of B is given as follows.

Algorithm 3.1(Bunch, [15]). Tri-diagonal Pivoting (TP method)

$$\sigma = \max\{\, |\alpha_i|, |\beta_j| : i, j = 2 : n \,\},$$

$$\alpha = (\sqrt{5} - 1)\Big/2 \approx 0.62.$$

If $|\alpha_1|\,\sigma \geq \alpha\beta_2^2$
$S_B = 1$ (use 1×1 pivot)
Else
$S_B = 2$ (use 2×2 pivot)
End

Next, Hagemann and Schenk use the tri-diagonal pivoting which discards the variable σ of Algorithm 3.1. Namely, a constant $\alpha = (\sqrt{5}-1)/2$ is given, and a 2×2 pivot is chosen in step k if

$$\left|a_{kk}^{(k)}\right| \leq \alpha \left|a_{k+1,k}^{(k)}\right|^2, \tag{4}$$

where $a^{(k)}$ designates the entries in the reduced matrix after step k. The constant α is a weight that controls the choice of the pivot.

3.3 Boundedly partial pivoting

We devise a new pivoting approach with no exchanges of rows and columns and the boundedness of the L factor gets guaranteed, based on the tri-diagonal pivoting in [1]. Subsequently, this will be proved and means the valuable stability of the new pivoting approach. The new pivoting approach is named by boundedly partial pivoting. In addition, we must note that the boundedly partial pivoting is proposed in the incomplete factorization under the condition that the system has been preprocessed according to Algorithm 2.1. The algorithm of boundedly partial pivoting is given as follows.

Algorithm 3.2. Boundedly Partial Pivoting (BPP method)

$$\alpha = (\sqrt{5}-1)\Big/2,$$

$$\sigma_k = \max(\left|a^{(k)}_{k:n,k}\right|).$$

If $\left|a^{(k)}_{kk}\right| \geq \alpha\sigma_k^2$
If $\left|a^{(k)}_{kk}\right| \leq \varepsilon_{piv}$, then $a^{(k)}_{kk} = \mathrm{sign}(a^{(k)}_{kk}) \cdot \varepsilon_{piv}$ End
$S = 1$ (choose $a^{(k)}_{kk}$ as 1×1 pivot)
Else If $\left|\det(A^{(k)}_{k:k+1,k:k+1})\right| \leq \delta$
$S = 1$ (let $a^{(k)}_{kk} = \alpha\sigma_k^2$ and choose $a^{(k)}_{kk}$ as 1×1 pivot)
Else
$S = 2$ (choose $\begin{bmatrix} a^{(k)}_{kk} & a^{(k)}_{k,k+1} \\ a^{(k)}_{k+1,k} & a^{(k)}_{k+1,k+1} \end{bmatrix}$ as 2×2 pivot)
End

There are two parameters ε_{piv} and δ in the Algorithm 3.2. And we commonly let ε_{piv}, δ in the subsequent experiments be 10^{-8}, 10^{-3} respectively. Meanwhile, we find that the changes of the value of δ could bring on the changes of the iterative solution rate.

The incomplete factorization with the boundedly partial poviting maybe costs more time and memory in contrast to that with the tri-diagonal pivoting, because that more comparisons of elements in the process of pivoting are needed. This pivoting approach has no exchanges of rows and columns as the tri-diagonal pivoting. Perhaps it is not better than the pivoting strategy with the exchanges of rows and columns, but preferable results can be obtained by combining it with the preprocessing introduced in section 2, and meanwhile there is no cost of the exchanges of rows and columns. Besides, we also make use of the character of the preprocessed matrix that the larger entries mostly concentrate on or round the tri-diagonal of

the matrix. In one word, the LDL^{T} factorization with boundedly partial pivoting is backward stability since the factor L is bounded, and therefore the latter preconditioning SQMR iterative solution is feasible.

3.4 Dropping strategy

The choice of an appropriate dropping strategy is an important part of the incomplete factorization. We use the same dropping strategy as that in [1], based on the average number of elements per column of the matrix. Each column k of the factor may contain at most γ_k elements, where

$$\gamma_k = \gamma \cdot \frac{nnz}{n}, \tag{5}$$

where nnz indicates the number of non-zeros in A and n indicates dimension of A. γ is the only parameter controlling the amount of fill-in in the factors. In our experiments, we also use $\gamma = 8$.

3.5 Summary of the incomplete LDL^{T} factorization method

The following pseudo-code provides a detailed overview of the incomplete factorization phase.

Algorithm 3.3. Incomplete LDL^{T} factorization with boundedly partial pivoting (ILDLT-BPP method).

1. Function $(L, D) = ildlt_blp_factorization(\hat{A})$
2. For $k = 1 : n$
3. $\alpha = (\sqrt{5} - 1)/2; \sigma_k = \max(\left|a^{(k)}_{k:n,k}\right|)$
4. If $\left|a^{(k)}_{kk}\right| \geq \alpha\sigma_k^2$ /*use 1×1 pivot*/ (case 1)
5. keep γ_k largest elements
6. If $\left|a^{(k)}_{kk}\right| \leq \varepsilon_{piv}$, then $a^{(k)}_{kk} = \mathrm{sign}(a^{(k)}_{kk}) \cdot \varepsilon_{piv}$ End
7. $d_{kk} = a^{(k)}_{kk}$; $L_{k+1:n,k} = d^{-1}_{kk} a^{(k)}_{k+1:n,k}$
8. update $a^{(k+1)}_{k+1:n,k+1:n}$ from $a^{(k)}_{k+1:n,k+1:n}$
9. Else
10. If $\left|\det(A^{(k)}_{k:k+1,k:k+1})\right| \leq \delta$ /*use 1×1 pivot */ (case 2)
11. $a^{(k)}_{kk} = \alpha\sigma_k^2$; break
12. Else /*use 2×2 pivot*/ (case 3)
13. keep γ_k largest 1×2 elements
14. $D_k = a^{(k)}_{k:k+1,k:k+1}$; $L_{k+1:n,k:k+1} = a^{(k)}_{k+1:n,k:k+1} \cdot D_k^{-1}$
15. update $a^{(k+2)}_{k+2:n,k+2:n}$ from $a^{(k)}_{k+2:n,k+2:n}$
16. End
17. End

18. End

Theorem 3.1. *The factor L is bounded while we use the Algorithm 3.3-incomplete LDL^{T} factorization with boundedly partial pivoting.*

Proof. Let L denote the lower triangular factor of the incomplete LDL^{T} factorization of $\hat{A}$, namely

$$\hat{A} = LDL^{\mathrm{T}} - E,$$

where D is a block diagonal matrix with 1×1 and 2×2 blocks. And let l_{ij} denote entries of L. For simply to read, we use A instead of $\hat{A}$. If 1×1 pivot elimination is implemented, we will obtain $A^{(k+1)}$ from $A^{(k)}$. Otherwise we will get $A^{(k+2)}$ through 2×2 pivot elimination. The superscript k shows the reduced matrix whose dimension is $n-k+1$ after 1×1 and 2×2 pivot eliminations. Let the entries of $A^{(k)}$ be denoted by $a_{ij}^{(k)}$. We know that the symmetry of the symmetric matrix can be retained after 1×1 and 2×2 pivot eliminations. So $A^{(k+1)}$ and $A^{(k+2)}$ are both symmetric.

Firstly, we consider the case using 1×1 pivot as follows. Through the elimination, the relation of $A^{(k+1)}$ and $A^{(k)}$ is

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)} a_{kj}^{(k)}}{a_{kk}^{(k)}}, \quad i > k,\ j > k.$$

And entries of L are

$$l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}, \quad i > k.$$

Secondly, we consider the case using 2×2 pivot as follows. Through the elimination, the relation of $A^{(k+2)}$ and $A^{(k)}$ is

$$a_{ij}^{(k+2)} = a_{ij}^{(k)} - \begin{bmatrix} a_{ik}^{(k)} & a_{i,k+1}^{(k)} \end{bmatrix} \begin{bmatrix} a_{kk}^{(k)} & a_{k+1,k}^{(k)} \\ a_{k+1,k}^{(k)} & a_{k+1,k+1}^{(k)} \end{bmatrix}^{-1} \begin{bmatrix} a_{kj}^{(k)} \\ a_{k+1,j}^{(k)} \end{bmatrix},$$

$$a_{ij}^{(k+2)} = a_{ij}^{(k)} - \frac{1}{(a_{kk}^{(k)} a_{k+1,k+1}^{(k)} - {a_{k+1,k}^{(k)}}^2)} \begin{bmatrix} a_{ik}^{(k)} & a_{i,k+1}^{(k)} \end{bmatrix} \begin{bmatrix} a_{k+1,k+1}^{(k)} & -a_{k+1,k}^{(k)} \\ -a_{k+1,k}^{(k)} & a_{kk}^{(k)} \end{bmatrix} \begin{bmatrix} a_{kj}^{(k)} \\ a_{k+1,j}^{(k)} \end{bmatrix},$$

$$a_{ij}^{(k+2)} = a_{ij}^{(k)} - \frac{a_{kk}^{(k)} a_{i,k+1}^{(k)} a_{k+1,j}^{(k)} - a_{k+1,k}^{(k)} (a_{ik}^{(k)} a_{k+1,j}^{(k)} + a_{i,k+1}^{(k)} a_{kj}^{(k)}) + a_{k+1,k+1}^{(k)} a_{ik}^{(k)} a_{kj}^{(k)}}{a_{kk}^{(k)} a_{k+1,k+1}^{(k)} - {a_{k+1,k}^{(k)}}^2}, \quad (i > k+1, j > k+1).$$

And entries of L are

$$l_{ik} = \frac{a_{ik}^{(k)} a_{k+1,k+1}^{(k)} - a_{i,k+1}^{(k)} a_{k+1,k}^{(k)}}{a_{kk}^{(k)} a_{k+1,k+1}^{(k)} - {a_{k+1,k}^{(k)}}^2}, \quad i > k+1,$$

$$l_{i,k+1} = \frac{a_{i,k+1}^{(k)} a_{kk}^{(k)} - a_{ik}^{(k)} a_{k+1,k}^{(k)}}{a_{kk}^{(k)} a_{k+1,k+1}^{(k)} - {a_{k+1,k}^{(k)}}^2}, \quad i > k+1,$$

$$l_{k+1,k} = 0.$$

Let μ be the magnitude of the largest entry in $A^{(k)}$ and μ' be the maximum magnitude of any entry in the new reduced matrix ($A^{(k+1)}$ or $A^{(k+2)}$).

Next, we will prove that $\sigma_1, \cdots, \sigma_n$ are all finite positive real numbers. We can easily know $\sigma_k \geq 0$ since $\sigma_k = \max(\left|a^{(k)}_{k:n,k}\right|)$, where $k = 1, \cdots, n$.

Case 1: When $\left|a^{(k)}_{kk}\right| \geq \alpha\sigma_k^2$, choose $a^{(k)}_{kk}$ as 1×1 pivot.

$$\begin{aligned}\left|a^{(k+1)}_{ij}\right| &\leq \left|a^{(k)}_{ij}\right| + \frac{\left|a^{(k)}_{ik}a^{(k)}_{kj}\right|}{\left|a^{(k)}_{kk}\right|} \\ &\leq \left|a^{(k)}_{ij}\right| + \sigma_k\frac{\left|a^{(k)}_{kj}\right|}{\left|a^{(k)}_{kk}\right|} \\ &= \left|a^{(k)}_{ij}\right| + \sigma_k\frac{\left|a^{(k)}_{jk}\right|}{\left|a^{(k)}_{kk}\right|},\end{aligned}$$

$$\mu' \leq \mu + \sigma_k\frac{\sigma_k}{\left|a^{(k)}_{kk}\right|} \leq \mu + \frac{1}{\alpha}.$$

So σ_{k+1} and σ_{k+2} are both finite positive real numbers.

Case 2: When $\left|a^{(k)}_{kk}\right| < \alpha\sigma_k^2$ and $\left|\det(A^{(k)}_{k:k+1,k:k+1})\right| \leq \delta$, implement $a^{(k)}_{k,k} = \alpha\sigma_k^2$ and break, cycle over again and choose $a^{(k)}_{kk}$ as 1×1 pivot. Here, this is the same as the Case 1.

Case 3: When $\left|a^{(k)}_{kk}\right| < \alpha\sigma_k^2$ and $\left|\det(A^{(k)}_{k:k+1,k:k+1})\right| > \delta$, choose $\begin{bmatrix} a^{(k)}_{kk} & a^{(k)}_{k,k+1} \\ a^{(k)}_{k+1,k} & a^{(k)}_{k+1,k+1} \end{bmatrix}$ as 2×2 pivot. According to $\left|\det(A^{(k)}_{k:k+1,k:k+1})\right| > \delta$, we get

$$\left|a^{(k)}_{kk}a^{(k)}_{k+1,k+1} - {a^{(k)}_{k+1,k}}^2\right| > \delta.$$

While $i > k+1, j > k+1$,

$$\begin{aligned}\left|a^{(k+2)}_{ij}\right| &\leq \left|a^{(k)}_{ij}\right| + \left|\frac{a^{(k)}_{kk}a^{(k)}_{i,k+1}a^{(k)}_{k+1,j} - a^{(k)}_{k+1,k}(a^{(k)}_{ik}a^{(k)}_{k+1,j} + a^{(k)}_{i,k+1}a^{(k)}_{kj}) + a^{(k)}_{k+1,k+1}a^{(k)}_{ik}a^{(k)}_{kj}}{a^{(k)}_{kk}a^{(k)}_{k+1,k+1} - {a^{(k)}_{k+1,k}}^2}\right| \\ &< \mu + \frac{\sigma_k\sigma_{k+1}\left|a^{(k)}_{k+1,j}\right| + \sigma_k(\sigma_k\left|a^{(k)}_{k+1,j}\right| + \sigma_{k+1}\left|a^{(k)}_{kj}\right|) + \sigma_{k+1}\sigma_k\left|a^{(k)}_{kj}\right|}{\delta},\end{aligned}$$

$$\begin{aligned}\mu' &< \mu + \frac{\sigma_k\sigma_{k+1}\sigma_{k+1} + \sigma_k(\sigma_k\sigma_{k+1} + \sigma_{k+1}\sigma_k) + \sigma_{k+1}\sigma_k\sigma_k}{\delta} \\ &= \mu + \frac{\sigma_k\sigma_{k+1}^2 + 3\sigma_{k+1}\sigma_k^2}{\delta}.\end{aligned}$$

So σ_{k+2} and σ_{k+3} are both finite positive real numbers.

In summary, we know that $\sigma_1, \cdots, \sigma_n$ are all finite positive real numbers.

Finally, we prove that l_{ik}, $l_{i,k+1}$ are bounded.

In Case 1 and Case 2, 1×1 pivot is chosen and then l_{ik} is

$$l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \Rightarrow |l_{ik}| \leq \frac{1}{\alpha\sigma_k} \text{ (while } i > k \text{ and } \sigma_k \neq 0).$$

When $\sigma_k = 0$, then $\left|a_{kk}^{(k)}\right| = 0$. So

$$\left|a_{kk}^{(k)}\right| \leq \varepsilon_{piv}.$$

Afterwards we implement

$$a_{kk}^{(k)} = \text{sign}(a_{kk}^{(k)}) \cdot \varepsilon_{piv}.$$

In this situation, $|l_{ik}| = 0,\ i > k$. So $l_{ik}(i > k)$ is bounded in Case 1 and Case 2.

In Case 3, 2×2 pivot is chosen and then l_{ik} and $l_{i,k+1}$ are

$$l_{ik} = \frac{a_{ik}^{(k)} a_{k+1,k+1}^{(k)} - a_{i,k+1}^{(k)} a_{k+1,k}^{(k)}}{a_{kk}^{(k)} a_{k+1,k+1}^{(k)} - {a_{k+1,k}^{(k)}}^2}, \quad i > k+1,$$

$$l_{i,k+1} = \frac{a_{i,k+1}^{(k)} a_{kk}^{(k)} - a_{ik}^{(k)} a_{k+1,k}^{(k)}}{a_{kk}^{(k)} a_{k+1,k+1}^{(k)} - {a_{k+1,k}^{(k)}}^2}, \quad i > k+1,$$

$$\begin{aligned} |l_{ik}| &\leq \frac{\sigma_{k+1}\left|a_{ik}^{(k)}\right| + \sigma_k \left|a_{i,k+1}^{(k)}\right|}{\delta} \\ &\leq \frac{\sigma_{k+1}\sigma_k + \sigma_k\sigma_{k+1}}{\delta} = \frac{2\sigma_k\sigma_{k+1}}{\delta}, \\ |l_{i,k+1}| &\leq \frac{\sigma_k \left|a_{i,k+1}^{(k)}\right| + \sigma_{k+1}\left|a_{ik}^{(k)}\right|}{\delta} \\ &\leq \frac{\sigma_k\sigma_{k+1} + \sigma_{k+1}\sigma_k}{\delta} = \frac{2\sigma_k\sigma_{k+1}}{\delta}. \end{aligned}$$

So l_{ik} and $l_{i,k+1}(i > k+1)$ are bounded.

In summary, l_{ik}, $l_{i,k+1}$ are bounded. The proof is completed. □

4 Numerical experiments

We consider the symmetric indefinite matrices which partly come from the matrix market and partly are produced randomly by the function in MATLAB. In this section, the main objective is to compare the boundedly partial pivoting and the tri-diagonal pivoting, to contrast their results, and to draw the conclusion that our pivoting approach is preferable.

4.1 Iterative solution and termination criterion

There are three Krylov subspace methods, which are MINRES, SYMMLQ and SQMR [16], to solve symmetric indefinite systems. Since both the original system and the preconditioner are indefinite, we cannot use methods like SYMMLQ or MINRES, which require a positive definite preconditioner. Therefore, we adopt SQMR method for iterative solution of the preconditioned system. The SQMR method has the benefit of allowing us to solve preconditioned systems $\tilde{A}$, with

$$\tilde{A} = M_1^{-1} A M_2^{-1}, \tag{6}$$

where A is symmetric, and M_1 and M_2 can be arbitrary matrices.

The iterative methods used the following stopping criterion. The current iterate $\hat{x}_n$ was considered good enough, if its residual referring to the preconditioned system $\hat{A}$ was less than 10^{-6}:

$$\left\| M^{-1}\hat{b} - M^{-1}\hat{A}\hat{x}_n \right\| \leq 10^{-6} \left\| M^{-1}\hat{b} \right\|. \tag{7}$$

When using the SQMR iterative method to solve the preconditioned system $M^{-1}\hat{A}\hat{x} = M^{-1}\hat{b}$, all matrices were solved with an artificial right hand side $\hat{b}$, with $\hat{b} = \hat{A} \cdot \vec{1}$, where $\vec{1}$ denotes the column vector of n dimension whose entries are all one. The iterations were started with a zero initial guess.

4.2 Experimental content and Numerical results

In this subsection, we mainly introduce our experimental content and give the numerical results by carrying out relevant programs.

Firstly, we introduce the information of the tested matrices and the meanings of the simple marks. The information of the tested matrices in our experiments is presented in Table 1 and the meanings of simple marks are shown in Table 2.

Note: • n denotes the dimension of matrix and nnz denotes the number of nonzero entries.

• $FLG = 0$ denotes the success of iterative solution, $FLG = 1$ denotes misconvergence of iterative solution, $FLG = 2$ denotes the failure of iterative solution.

Matrix name	n	nnz	Matrix type	Source
bcsstk19	817	6853	Sparse symmetric indefinite matrix	Matrix market
bcsstk22	138	696	Sparse symmetric indefinite matrix	Matrix market
bcsstm27	1224	56126	Sparse symmetric indefinite matrix	Matrix market
randmat1	50	463	Sparse symmetric indefinite matrix	Randomly produced by MATLAB
randmat2	100	2583	Sparse symmetric indefinite matrix	Randomly produced by MATLAB
randmat3	20	142	Sparse symmetric indefinite matrix	Randomly produced by MATLAB

Table 1: Information of the tested matrices

Simple mark	meaning
NPREC	Iterative solution with no preconditioning
NPREP	No preprocessing
ILDLT(0)	Iterative solution with preconditioning of combining zero fill-in incomplete LDL^T factorization and preprocessing
ILDLT-TP	Iterative solution with preconditioning of combining incomplete LDL^T factorization with tri-diagonal pivoting and preprocessing
ILDLT-BPP	Iterative solution with preconditioning of combining incomplete LDL^T factorization with bounded partial pivoting and preprocessing
PIVOT22	The number of 2×2 pivot chosen in the process of incomplete LDL^T factorization
ITERS	Iterative number
FLG	Result mark of iterative solution

Table 2: The meanings of the simple marks

We will analyze from the following three aspects:

(1) Compare the solution results of the NPREC method, the ILDLT(0) method, the ILDLT-TP method and the ILDLT-BPP method (meanings of these four methods are in Table 2). The default values of the parameters γ, ε_{piv} and δ in Table 3 are 8, 10^{-8} and 10^{-3} respectively. In MATLAB, we run corresponding programs and the experimental results obtained are given in Table 3.

(2) We exploit MATLAB's own function $sprandsym(n, dense)$ to randomly produce the sparse symmetric indefinite matrices, where n is to decide the dimension of the produced sparse symmetric matrix and the parameter $dense$, which is valued in the interval [0.2, 0.4], is to control the sparse extent of the sparse symmetric matrix.

Matrix name	Parameter	NPREC	ILDLT(0)	ILDLT-TP	ILDLT-BPP	
bcsstk19	FLG	0	0	0	0	
	ITERS	909	1203	3 $\gamma = 1.7$	3 $\gamma = 1.7$	
	PIVOT22	—	—	0	0	
bcsstk22	FLG	0	0	0	0	
	ITERS	113	25	4 $\gamma = 1$	4 $\gamma = 1$	
	PIVOT22	—	—	0	0	
bcsstm27	FLG	1	1	0	0	
	ITERS	> 10000	> 10000	3 $\gamma = 1.1$	7 $\gamma = 1.1$	3 $\gamma = 1.1$ $\delta = 10^{-4}$
	PIVOT22	—	—	1	58	62
randmat1	FLG	0	2	2	0	
	ITERS	56	—	—	4	
	PIVOT22	—	—	—	23	
randmat2	FLG	0	2	2	0	
	ITERS	142	—	—	4	
	PIVOT22	—	—	—	48	
randmat3	FLG	0	2	0	0	
	ITERS	20	—	10 $\gamma = 1$	8 $\gamma = 1$	
	PIVOT22	—	—	7	9	

Table 3: Solution results of the four methods

Matrix name	Solution method	FLG	ITERS	PIVOT22
randmat1	ILDLT-BPP- NPREP	0	14	18
	ILDLT-BPP	0	4	23
randmat2	ILDLT-BPP- NPREP	0	11	45
	ILDLT-BPP	0	4	48
randmat3	ILDLT-BPP- NPREP	0	20	3
	ILDLT-BPP	0	8	9

Table 4: Solution results of containing preprocessing or not

And then judge the matrix indefinite by validating it has both positive and negative eigenvalues. And thus randmat1, randmat2, and randmat3 are all indefinite since they all have both positive and negative eigenvalues. Commonly, the matrices produced by the above technique are indefinite. Next, the numerical results of the comparison between the ILDLT-BPP method and the ILDLT-BPP-NPREP method (meanings of these two methods are in Table 2) to solve the systems corresponding to the three random sparse symmetric indefinite matrices are given in Table 4.

(3) The residual changes in plans will be presented in order to observe and contrast these solution methods. The residual in step k is

$$res_k = \frac{||M^{-1}\hat{b} - \hat{A}\hat{x}_k||}{||M^{-1}\hat{b}||}. \tag{8}$$

In one thing, the residual changes curves of the iterative solution of the system corresponding to the matrix named bcsstk22 will be given in Fig. 1 and Fig. 2.

In the other thing, the residual changes curves of the iterative solution of the system corresponding to the matrix named randmat2 will be given in Fig. 3.

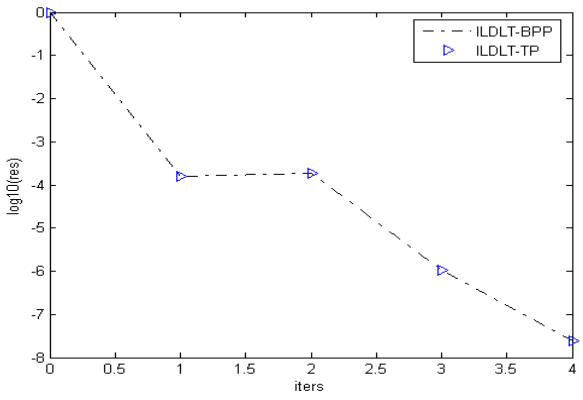


Figure 1: Residual changes curves of ILDLT-BPP and ILDLT-TP for bcsstk22

4.3 Analysis of numerical results

For preparing for the final conclusion, the main work in this subsection is to analyze the data of Table 3 and Table 4 and the residual changes curves in Fig. 1, Fig. 2 and Fig. 3.

At first, from the Table 3, we can clearly see that the ILDLT-TP method and the ILDLT-BPP method are preferable to the NPREC method and the ILDLT(0) method with respect to the iterative number. Therefore, the ILDLT-TP method and the ILDLT-BPP method have the same iterative number for the matrices named bcsstk19 and bcsstk22 with comparing the last two columns. And iterative number

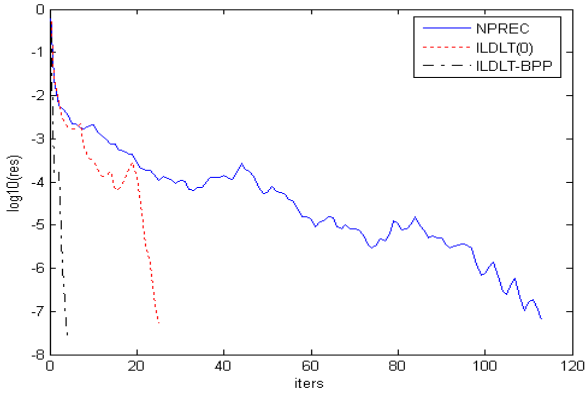


Figure 2: Residual changes curves of NPREC, ILDLT(0) and ILDLT-BPP for bcsstk22

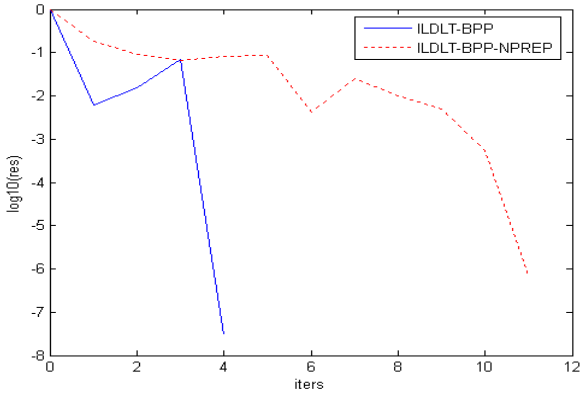


Figure 3: Residual changes curves of ILDLT-BPP and ILDLT-BPP-NPREP for randmat2

of the ILDLT-TP method for solving the system corresponding to the matrix named bcsstm27 is smaller, under the conditions of the default parameters. However, the ILDLT-BPP method is able to get the same iterative number as the ILDLT-TP method by altering the value of the parameter δ. In addition, the iterative number of our ILDLT-BPP method is smaller for the matrix named randmat3.

At second, we still notice that the systems corresponding to the matrices named randmat1 and randmat2 can not be solved by the ILDLT-TP method, nevertheless our ILDLT-BPP method is able to make it and the iterative number is rather small. The reason is that the factor L produced by incomplete LDL^{T} factorization with boundedly partial pivoting is bounded. We have proofed it in the section 3, and then the numerical stability of the factorization gets guaranteed.

Next, to observe the Table 4, in which the results of iterative solution of the ILDLT-BPP-NPREP method and the ILDLT-BPP method are presented. We can easily find that the iterative number of the method without preprocessing is larger.

At last, observe and analyze the three figures. In Fig.1, the two kinds of curves are completely heavy to match, and this explains that the results of iterative solution by the ILDLT-BPP method for the matrix named bcsstk22 are the same as that by the ILDLT-TP method. In Fig.2 and Fig.3, we observe the iterative number corresponding to the point of intersection between curve and straight line $\log_{10}(res) = -6$. The smaller the iterative number is, the quicker the speed of iterative solution will be. From Fig.2, the relation among the ILDLT(0) method, the NPREC method and the ILDLT-BPP method can be easily found. Namely, the ILDLT-BPP method is quicker than the ILDLT(0) method, and meanwhile the ILDLT(0) method is quicker than the NPREC method with respect to the iterative solution rate.

And from Fig. 3, the relation between the ILDLT-BPP-NPREP method and the ILDLT-BPP method is that the iterative solution rate of the later is quite quicker than that of the former. The iterative solution rate is measured by the iterative number, and the smaller the iterative number is, the quicker the speed of the iterative solution will be, vice versa. Here, we need to explain that the iterative number should be an integer.

4.4 Experimental conclusions

From the front two subsections of experimental content and analysis of numerical results, we can draw the conclusion as follows:

(1) Within the scope of the systems which can be solved by the the ILDLT-TP method, our ILDLT-BPP method can be also feasible. Meanwhile, the results of them are probably the same or the iterative solution rate of the ILDLT-TP method is probably quicker under the conditions of the default parameters, but we also can gain the same results of the two methods through altering the value of the parameter δ. Also, the iterative solution rate of the ILDLT-BPP method is possibly quicker.

Therefore, in general, the iterative solution rates of the two kinds of methods are basically the same or our ILDLT-BPP method is indeed preferable to the ILDLT-TP method for some special symmetric indefinite linear systems.

(2) The numerical stability of the incomplete factorization can not be confirmed since the tri-diagonal pivoting in the incomplete factorization of the ILDLT-TP method can not make sure of the boundedness of the L factor. On the contrary, the boundedly partial pivoting in the incomplete factorization of our ILDLT-BPP method is able to guarantee the boundedness of the L factor, and thus the factorization is numerically stable. This will avoid unfeasibility of the last preconditioned SQMR iterative method brought on by the unboundedness of L. Consequently, our ILDLT-BPP method extends the scope of the symmetric indefinite linear systems which can be solved by it.

(3) Within the scope of certain symmetric indefinite linear systems, our ILDLT-BPP method is preferable to the NPREC method and the ILDLT(0) method. At the same time, the preprocessing of the ILDLT-BPP method exerts big function for the iterative solution rate.

5 Conclusions

The main work of this paper is to devise an efficient algorithm concerning the solutions of symmetric indefinite linear systems. The preconditioning of combining the preprocessing, which is similar to the preprocessing in [1], with the boundedly partial pivoting in the incomplete factorization is proposed, and the better results are acquired. The experimental conclusions in section 4 indicates that comparing with the preconditioning of combining the preprocessing and the tri-diagonal pivoting in the incomplete factorization, our method extends the scope of the symmetric indefinite linear systems which could be solved by it, since it warrants the stability of the numerical factorization due to the boundedness of the L factor. For the systems which can be solved by the both methods, the same iterative solution rate is gained or our method is more efficient for the solution of some special symmetric indefinite linear systems.

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ON THE SYMMETRIC PROPERTIES FOR THE SECOND KIND (h,q)-EULER POLYNOMIALS

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Abstract : In [7], we studied the second kind Euler numbers and polynomials. By using these numbers and polynomials, we investigated the alternating sums of powers of consecutive odd integers (see [8]). By applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, which is defined Kim [2], we give recurrence identities the second kind (h,q)-Euler polynomials and the alternating sums of powers of consecutive (h,q)-odd integers.

Key words : the second kind Euler numbers and polynomials, the second kind q-Euler numbers and polynomials, (h,q)-Euler numbers and polynomials, alternating sums

1. Introduction

Euler numbers, Euler polynomials, q-Euler numbers, q-Euler polynomials, the second kind Euler number and the second kind Euler polynomials were studied by many authors (see for details [1-9]). Euler numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In [8], by using the second kind Euler numbers E_j and polynomials $E_j(x)$, we investigated the q-analogue of alternating sums of powers of consecutive odd integers. Let k be a positive integer. Then we obtain

$$T_j(k-1)=\sum_{n=0}^{k-1}(-1)^n(2n+1)^j=\frac{(-1)^{k+1}E_j(2k)+E_j}{2}.$$

In [9], we introduced the second kind (h,q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$. The outline of this paper is as follows. We introduce the second kind (h,q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$. We give some properties of these numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$. In Section 2, we obtain the alternating sums of powers of consecutive (h,q)-odd integers. Finally, we give recurrence identities the second kind (h,q)-Euler polynomials and the alternating sums of powers of consecutive (h,q)-odd integers.

Throughout this paper, we always make use of the following notations: $\mathbb{N}=\{1,2,3,\cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p=p^{-\nu_p(p)}=p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q\in\mathbb{C}$, or p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}$ one normally assume that $|q|<1$. If $q\in\mathbb{C}_p$, we normally assume that $|q-1|_p<p^{-\frac{1}{p-1}}$ so that $q^x=\exp(x\log q)$ for $|x|_p\leq 1$.

$$[x]_q=[x:q]=\frac{1-q^x}{1-q},\ \text{cf. [1, 2, 3, 5] .}$$

Hence, $\lim_{q\to 1}[x]=x$ for any x with $|x|_p\leq 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed odd prime number. For any positive integer N, we set

$$X=\varprojlim_N(\mathbb{Z}/dp^N\mathbb{Z}),\quad X^*=\bigcup_{\substack{0<a<dp\\(a,p)=1}}(a+dp\mathbb{Z}_p),$$

$$a+dp^N\mathbb{Z}_p=\{x\in X\mid x\equiv a\pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

is known to be a distribution on X, cf.[1, 2, 3, 5, 6, 7, 8]. For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by [1, 2, 3]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} g(x)q^x.$$

The bosonic integral was considered from a physical point of view to the bosonic limit $q \to 1$, as follows:

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x) \text{ (see [1, 2, 3])}.$$

We consider the fermionic integral in contrast to the conventional bosonic, which is called fermionic p-adic q-integral on $\mathbb{Z}_p$. That is

$$I_{-q}(g) = \lim_{q\to -q} I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x).$$

The expression for the $I_q(g)$ remains same, so it is tempting to consider the limit $q \to -1$. That is

$$I_{-1}(g) = \lim_{q\to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0\leq x<p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x + 1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

First, we introduce the second kind Euler numbers E_n and polynomials $E_n(x)$. The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x, t) = \frac{2e^t}{e^{2t} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.$$

The second kind (h, q)-Euler polynomials, $E_{n,q}^{(h)}(x)$ are defined by

$$F_q^{(h)}(x, t) = \frac{2e^t}{q^h e^{2t} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{1.3}$$

The second kind (h, q)-Euler numbers $E_{n,q}^{(h)}$ are defined by the generating function:

$$F_q^{(h)}(t) = \frac{2e^t}{q^h e^{2t} + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!}. \tag{1.4}$$

In [9], we introduced the second kind (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$ and investigate their properties. The following elementary properties of the second kind (h, q)-Euler numbers

$E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4)(see, for details, [9]). We, therefore, choose to omit details involved.

Theorem 1.(Witt formula) For $h \in \mathbb{Z}, q \in \mathbb{C}_p$ with $|1-q|_p < 1$, we have

$$\int_{\mathbb{Z}_p} q^{hx}(2x+1)^n d\mu_{-1}(x) = E_{n,q}^{(h)},$$

$$\int_{\mathbb{Z}_p} q^{hy}(x+2y+1)^n d\mu_{-1}(y) = E_{n,q}^{(h)}(x).$$

Theorem 2. For any positive integer n, we have

$$E_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q}^{(h)} x^{n-k}.$$

Theorem 3.(Distribution Relation) For any positive integer m(=odd), we have

$$E_{n,q}^{(h)}(x) = m^n \sum_{a=0}^{m-1} (-1)^a q^{ha} E_{n,q^m}^{(h)} \left(\frac{2a+x+1-m}{m} \right) \text{ for } n \geq 0.$$

2. The alternating sums of powers of consecutive (h,q)-odd integers

In this section, we assume that $q \in \mathbb{C}$, with $|q| < 1$ and $h \in \mathbb{Z}$. By using (1.4), we give the alternating sums of powers of consecutive (h,q)-integers as follows:

$$\sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!} = \frac{2e^t}{q^h e^{2t}+1} = 2\sum_{n=0}^{\infty} (-1)^n q^{nh} e^{(2n+1)t}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n q^{nh} e^{(2n+2k+1)t} + \sum_{n=0}^{\infty} (-1)^{n-k} q^{(n-k)h} e^{(2n+1)t} = 2\sum_{n=0}^{k-1} (-1)^{n-k} q^{(n-k)h} e^{(2n+1)t}.$$

By using (1.3)and (1.4), we obtain

$$-\frac{1}{2}\sum_{j=0}^{\infty} E_{j,q}^{(h)}(2k)\frac{t^j}{j!} + \frac{1}{2}(-1)^{-k} q^{-kh} \sum_{j=0}^{\infty} E_{j,q}^{(h)} \frac{t^j}{j!} = \sum_{j=0}^{\infty} \left((-1)^{-k} q^{-kh} \sum_{n=0}^{k-1} (-1)^n q^{nh} (2n+1)^j \right) \frac{t^j}{j!}.$$

By comparing coefficients of $\dfrac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1} (-1)^n q^{nh} (2n+1)^j = \frac{(-1)^{k+1} q^{kh} E_{j,q}^{(h)}(2k) + E_{j,q}^{(h)}}{2}.$$

By using the above equation we arrive at the following theorem:

Theorem 4. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain

$$T_{j,q}^{(h)}(k-1) = \sum_{n=0}^{k-1} (-1)^n q^{nh} (2n+1)^j = \frac{(-1)^{k+1} q^{kh} E_{j,q}^{(h)}(2k) + E_{j,q}^{(h)}}{2}. \tag{2.1}$$

Remark 5. For the alternating sums of powers of consecutive odd integers, we have

$$\lim_{q \to 1} T_{j,q}^{(h)}(k-1) = \sum_{n=0}^{k-1} (-1)^n (2n+1)^j = \frac{(-1)^{k+1} E_j(2k) + E_j}{2}.$$

3. The symmetry property of the second kind (h,q)-Euler polynomials

In this section, we assume that $q \in \mathbb{C}_p$. In [2], Kim investigated interesting properties of symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli polynomials and Euler polynomials. By using same method of [2], expect for obvious modifications, we obtain recurrence identities the second (h,q)-Euler polynomials and the alternating sums of powers of consecutive (h,q)-odd integers. By using (1.1), we have

$$I_{-1}(g_n) + (-1)^{n-1} I_{-1}(g) = 2\sum_{k=0}^{n-1} (-1)^{n-1-k} g(k), \text{ (see [1-3])}$$

where $n \in \mathbb{N}, g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2\sum_{k=0}^{n-1} (-1)^{n-1-k} g(k) \text{ (see [1], [2], [3], [5])}. \tag{3.1}$$

It will be more convenient to write (3.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2\sum_{k=0}^{n-1} (-1)^{n-1-k} g(k). \tag{3.2}$$

Substituting $g(x) = q^{hx} e^{(2x+1)t}$ into the above, we obtain

$$\int_{\mathbb{Z}_p} q^{h(x+n)} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_{-1}(x) = 2\sum_{j=0}^{n-1} (-1)^j q^{hj} e^{(2j+1)t}. \tag{3.3}$$

After some elementary calculations, we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t}{q^h e^{2t}+1}, \\ &\int_{\mathbb{Z}_p} q^{h(x+n)} e^{(2(x+n)+1)t} d\mu_{-1}(x) = q^{hn} e^{2nt} \frac{2e^t}{q^h e^{2t}+1}. \end{aligned} \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\int_{\mathbb{Z}_p} q^{h(x+n)} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t(1+q^{hn}e^{2nt})}{q^h e^{2t}+1}.$$

From the above, we get

$$\int_{\mathbb{Z}_p} q^{h(x+n)} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_{-1}(x) = \frac{2\int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} q^{hnx} e^{2ntx} d\mu_{-1}(x)}. \tag{3.5}$$

By substituting Taylor series of $e^{(2x+1)t}$ into (3.3), we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty} \left(\int_{\mathbb{Z}_p} q^{h(x+n)} (2x+1+2n)^m d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx} (2x+1)^m d\mu_{-1}(x) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(2\sum_{j=0}^{n-1} (-1)^j q^{hj} (2j+1)^m \right) \frac{t^m}{m!}. \end{aligned}$$

By comparing coefficients $\dfrac{t^m}{m!}$ in the above equation, we obtain

$$q^{hn}\sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p}q^{hx}(2x+1)^k d\mu_{-1}(x)+\int_{\mathbb{Z}_p}q^{hx}(2x+1)^m d\mu_{-1}(x)$$
$$=2\sum_{j=0}^{n-1}(-1)^j q^{hj}(2j+1)^m.$$

By using (2.1), we have

$$q^{hn}\sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p}q^{hx}(2x+1)^k d\mu_{-1}(x)+\int_{\mathbb{Z}_p}q^{hx}(2x+1)^m d\mu_{-1}(x)=2T_{q,m}^{(h)}(n-1). \quad (3.6)$$

By using (3.5) and (3.6), we arrive at the following theorem:

Theorem 6. Let n be odd positive integer. Then we obtain

$$\frac{2\int_{\mathbb{Z}_p}q^{hx}e^{(2x+1)t}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}q^{hnx}e^{2ntx}d\mu_{-1}(x)}=\sum_{m=0}^{\infty}\left(2T_{q,m}^{(h)}(n-1)\right)\frac{t^m}{m!}. \quad (3.7)$$

Let w_1 and w_2 be odd positive integers. By using (3.7), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}q^{h(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2xt}d\mu_{-1}(x)}\\&=\frac{2e^{w_1t}e^{w_2t}e^{w_1w_2xt}(q^{hw_1w_2}e^{2w_1w_2t}+1)}{(q^{hw_1}e^{2w_1t}+1)(q^{hw_2}e^{2w_2t}+1)}\end{aligned} \quad (3.8)$$

By using (3.7) and (3.8), after elementary calculations, we obtain

$$\begin{aligned}a&=\left(\frac{1}{2}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}q^{hw_2x}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}E_{m,q^{w_1}}^{(h)}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}T_{m,q^{w_2}}^{(h)}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \quad (3.9)$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E_{j,q^{w_1}}^{(h)}(w_2x)w_1^jT_{m-j,q^{w_2}}^{(h)}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}. \quad (3.10)$$

By using the symmetry in (3.9), we have

$$\begin{aligned}a&=\left(\frac{1}{2}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_{-1}(x_2)\right)\left(\frac{2\int_{\mathbb{Z}_p}q^{hw_1x}e^{(2x_1+1)(w_1t)}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}E_{m,q^{w_2}}^{(h)}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}T_{m,q^{w_1}}^{(h)}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E_{j,q^{w_2}}^{(h)}(w_1x)w_2^jT_{m-j,q^{w_1}}^{(h)}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}. \quad (3.11)$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (3.10) and (3.11), we arrive at the following theorem:

Theorem 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^j E_{j,q^{w_2}}^{(h)}(w_1x)T_{m-j,q^{w_1}}^{(h)}(w_2-1)\\&=\sum_{j=0}^{m}\binom{m}{j}w_1^{j}w_2^{m-j} E_{j,q^{w_1}}^{(h)}(w_2x)T_{m-j,q^{w_2}}^{(h)}(w_1-1),\end{aligned}$$

where $E_{k,q}^{(h)}(x)$ and $T_{m,q}^{(h)}(k)$ denote the second kind (h,q)-Euler polynomials and the alternating sums of powers of consecutive (h,q)-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^j x^{j-k} E_{k,q^{w_2}}^{(h)}T_{m-j,q^{w_1}}^{(h)}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j}w_2^{m-k} x^{j-k} E_{k,q^{w_1}}^{(h)}T_{m-j,q^{w_2}}^{(h)}(w_1-1).\end{aligned}$$

By using (3.8), we have

$$\begin{aligned}a&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}q^{hw_2x}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\left(2\sum_{j=0}^{w_1-1}(-1)^jq^{w_2hj}e^{(2j+1)(w_2t)}\right)\\&=\sum_{j=0}^{w_1-1}(-1)^jq^{w_2hj}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_{-1}(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}(-1)^jq^{w_2hj}E_{n,q^{w_1}}^{(h)}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.\end{aligned}\tag{3.12}$$

By using the symmetry property in (3.12), we also have

$$\begin{aligned}a&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\left(\frac{2\int_{\mathbb{Z}_p}q^{hw_1x}e^{(2x_1+1)(w_1t)}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\left(2\sum_{j=0}^{w_2-1}(-1)^jq^{w_1hj}e^{(2j+1)(w_1t)}\right)\\&=\sum_{j=0}^{w_2-1}(-1)^jq^{w_1hj}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_{-1}(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^jq^{w_1hj}E_{n,q^{w_2}}^{(h)}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.\end{aligned}\tag{3.13}$$

By comparing coefficients $\dfrac{t^n}{n!}$ in the both sides of (3.12) and (3.13), we have the following theorem.

Theorem 9. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^jq^{w_2hj}E_{n,q^{w_1}}^{(h)}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\\&=\sum_{j=0}^{w_2-1}(-1)^jq^{w_1hj}E_{n,q^{w_2}}^{(h)}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n.\end{aligned}\tag{3.14}$$

Observe that if $q \to 1$, then (3.14) reduces to Theorem 10 in [8].

Substituting $w_1 = 1$ into (3.14), we arrive at the following corollary.

Corollary 10. Let w_2 be odd positive integer. Then we obtain

$$E_{n,q}^{(h)}(x) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j q^{hj} E_{n,q^{w_2}}^{(h)} \left(\frac{x+2j+1-w_2}{w_2} \right).$$

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A Generalisation of Chui's Window Functions And its Boundedness Property

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Keywords window functions, wavelets, Chui's window functions.

Abstract Definition of a window function given by Charles K. Chui in $L^2(\Re)$ is extended to $L^2(\Re^n)$ and it is proved that every window function belonging to $L^2(\Re^n)$ is also $L^1(\Re^n)$ integrable. This result is very useful in wavelet theory in $\Re^n$, particularly to find the wavelet inversion formula in $L^2(\Re^n)$.

Introduction

In the year 1992 [2] Chui defined a window function as a function f in $L^2(\Re)$ such that xf is also in $L^2(\Re)$. He proved that this window function is also $L^1(\Re)$ integrable. His this definition of window function is different from the definition of the window function given in books of signal processing [3]. The $L^1(\Re)$ integrability of the window function defined by Chui is a very important property. The Fourier transform $\hat{f}(\lambda)$ of the window function f defined by Chui is a continuous function of λ which enables us to show that a window function f is compatible if and only if $\hat{f}(\lambda)|_{\lambda=0} = \hat{f}(0) = 0$. This important result enables one to prove the inversion formula for the continuous wavelet transform of a $L^2(\Re)$ function with respect to the kernel ψ which is a window function satisfying the compatibility condition [1]. Our objective is to extend the concept of window function by Chui in $L^2(\Re^n)$ and prove that every $L^2(\Re^n)$ window function is $L^1(\Re^n)$ integrable.
we say that $f \in L^2(\Re^n)$ is a window function if and only if $x_i f, x_i x_j f, x_i x_j x_k f, \cdots, x_1 x_2 \cdots x_n f$ all belongs to $L^2(\Re^n)$; here lower suffixes in a term having more than one suffix are all different. This again enables one to show that

[1] corresponding author

a window function f in $L^2(\Re^n)$ is compatible i.e.

$$\int_{\Re^n} |\hat{f}(\lambda_1, \lambda_2, \cdots, \lambda_n)| / |\lambda_1 \lambda_2 \cdots \lambda_n| \, d\lambda_1 d\lambda_2 \cdots d\lambda_n < \infty$$

if and only if

$$\int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_i, \cdots, x_n) dx_i = 0$$

$\forall i = 1, 2, 3, \cdots, n$ and then f becomes a basic regular wavelet [4]. This result enables us to prove wavelet inversion formula in $L^2(\Re^n)$ with respect to the wavelet kernel $\psi(x_1, x_2, \cdots, x_n) \in L^2(\Re^n)$. This inversion formula has been proved by Pandey [4], Bogess and Narcowich [1].
Our objective is simply to give a different proof than that given in [4] by Pandey to prove the fact that a window function $f(x_1, x_2, \cdots, x_n) \in L^2(\Re^n)$ is $L^1(\Re^n)$ integrable. This result though simple (trivial) to prove for $n = 1$ is definitely diffcult and non-trivial for $n > 1$ and will find applications in wavelet analysis and so deserves to be brought to the knowledge of the mathematical community working in the area of wavelet analysis. We have also given another proof of this fact which is clearer than the proof given by Pandey in [4].
We now recall the definition of a window function $f \in L^2(\Re^n)$ as follows: Let $f \in L^2(\Re^n)$, we say that f is a window function belonging to $L^2(\Re^n)$ if $x_i f, x_i x_j f, x_i x_j x_k f, \cdots, x_1 x_2 \cdots x_n f \in L^2(\Re^n)$, $i, j, k, \cdots$ all assume values $1, 2, 3, \cdots, n$ and any two lower suffixes in a term are different.
We now state and prove our main result as follows :

Theorem 1 *Let $f \in L^2(\Re^n)$ be a window function, then $f \in L^1(\Re^n)$.*

Proof: The proof for the case $n = 1$ though simple (trivial) and is known, but for the sake of completeness we give the proof for the case $n = 1$ also.
$n = 1$; $f \in L^2(\Re)$
A posible bound for $\int_{\Re} |f(x)| dx$ can be given by using Holder's inequality as follows:

$$\begin{aligned}
\int_{\Re} |f(x)| dx &\leq \int_{|x|\leq 1} |f(x)| dx + \int_{|x|\geq 1} |f(x)| dx \\
&\leq \sqrt{\int_{-1}^{1} |f(x)|^2 dx} \sqrt{\int_{-1}^{1} 1^2 dx} + \sqrt{\int_{|x|\geq 1} |xf(x)|^2 dx} \sqrt{\int_{|x|\geq 1} \frac{1}{x^2} dx} \\
&\leq \|f\|_2 \sqrt{2} + \|xf\|_2 \sqrt{2} \\
&\leq (\|f\|_2 + \|xf\|_2)\sqrt{2} \qquad (1)
\end{aligned}$$

The R.H.S. expression in (1) is bounded as f is a window function, therefore $\int_{\Re} |f(x)| dx$ is bounded.

3

Case 2 $n = 2$. A possible bound for $\int_{\Re^2} |f(x_1, x_2)|dx_1dx_2$ is given by

$$\int_{\Re^2} |f(x_1,x_2)|dx_1dx_2 = \int_{\substack{|x_1|\le 1\\|x_2|\le 1}} |f(x_1,x_2)|dx_1x_2 + \int_{\substack{|x_1|\le 1\\|x_2|\ge 1}} |f(x_1,x_2)|dx_1x_2$$
$$+ \int_{\substack{|x_1|\ge 1\\|x_2|\le 1}} |f(x_1,x_2)|dx_1x_2 + \int_{\substack{|x_1|\ge 1\\|x_2|\ge 1}} |f(x_1,x_2)|dx_1x_2$$

Now by Holder's inequality

$$\begin{aligned}
\int_{\Re^2} |f(x_1,x_2)|dx_1dx_2 \quad \le \quad & \sqrt{\int_{-1}^{1}\int_{-1}^{1} |f(x_1,x_2)|^2 dx_1dx_2}\sqrt{\int_{-1}^{1}\int_{-1}^{1} 1^2 dx_1dx_2} \\
& + \int_{\substack{|x_1|\le 1\\|x_2|\ge 1}} |x_2 f(x_1,x_2)|\frac{1}{|x_2|}dx_1dx_2 \\
& + \int_{\substack{|x_1|\ge 1\\|x_2|\le 1}} |x_1 f(x_1,x_2)|\frac{1}{|x_1|}dx_1dx_2 \\
& + \int_{\substack{|x_1|\ge 1\\|x_2|\ge 1}} |x_1x_2 f(x_1,x_2)|\frac{1}{|x_1x_2|}dx_1dx_2 \\
& \le 2\|f\|_2 + \|x_2 f\|_2 \sqrt{\int_{\substack{|x_1|\le 1\\|x_2|\ge 1}} \frac{1}{{x_2}^2}dx_1dx_2} \\
& + \|x_1 f\|_2 \sqrt{\int_{\substack{|x_1|\ge 1\\|x_2|\le 1}} \frac{1}{{x_1}^2}dx_1dx_2} \\
& + \|x_1x_2 f\|_2 \sqrt{\int_{\substack{|x_1|\ge 1\\|x_2|\ge 1}} \frac{1}{{x_1}^2{x_2}^2}dx_1dx_2} \\
& \le 2\|f\|_2 + 2\|x_1 f\|_2 + 2\|x_2 f\|_2 + 2\|x_1x_2 f\|_2 \\
& \le 2^{\frac{2}{2}}[\|f\|_2 + \|x_1 f\|_2 + \|x_2 f\|_2 + \|x_1x_2 f\|_2] \qquad (2)
\end{aligned}$$

The result is proved for $n = 2$.
To prove the result for any $n \ge 1$, we calculate a possible bound on $\int_{\Re^n} |f(x)|dx$ as follows. Let $x \in \Re^n$ and $x_1, x_2, \cdots, x_n$ be components of x. Let $x_{j_1}, x_{j_2}, \cdots, x_{j_n}$ be permutations of $x_1, x_2, \cdots, x_n$ then the space $\Re^n$ can be written as union of various orthants i.e. $\Re^n$ can be expressed as the following union

$$[x \in \Re^n; |x_1| \le 1, |x_2| \le 1, |x_3| \le 1, \cdots, |x_n| \le 1]$$
$$\bigcup$$
$$\bigcup_P [x \in \Re^n; |x_{j_1}| \le 1, |x_{j_2}| \ge 1, |x_{j_3}| \ge 1, \cdots, |x_{j_n}| \ge 1]$$
$$\bigcup$$
$$\bigcup_P [x \in \Re^n; |x_{j_1}| \le 1, |x_{j_2}| \le 1, |x_{j_3}| \ge 1, \cdots, |x_{j_n}| \ge 1]$$

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$$\bigcup$$

$$\bigcup_P [x \in \Re^n; |x_{j_1}| \leq 1, |x_{j_2}| \leq 1, |x_{j_3}| \leq 1, |x_{j_4}| \geq 1, \cdots, |x_{j_n}| \geq 1]$$

$$\bigcup$$

$$\vdots$$

$$\bigcup_P \left[x \in \Re^n; |x_{j_1}| \leq 1, |x_{j_2}| \leq 1, |x_{j_3}| \leq 1, \cdots, |x_{j_{n-1}}| \leq 1, |x_{j_n}| \geq 1\right]$$

$$\bigcup$$

$$[x \in \Re^n; |x_1| \geq 1, |x_2| \geq 1, |x_3| \geq 1, \cdots, |x_n| \geq 1]$$

Where P is various permutations of $x_1, x_2, \cdots, x_n$.
Note that the union

$$\bigcup_P \left[x \in \Re^n; |x_{j_1}| \leq 1, |x_{j_2}| \leq 1, \cdots, |x_{j_r}| \leq 1, |x_{j_{r+1}}| \geq 1, |x_{j_{r+2}}| \geq 1, \cdots, |x_{j_n}| \geq 1\right]$$

has ${}^nC_r$ different terms.
Therefore, a possible bound for $\int_{\Re^n} |f(x)|dx$ can be computed as follows:

$$\begin{aligned}
\int_{\Re^n} |f(x)|dx &= \int_{|x_1|\leq 1, |x_2|\leq 1, \cdots |x_n|\leq 1} |f(x)|dx \\
&+ \sum_P \int_{|x_{j_1}|\leq 1, |x_{j_2}|\geq 1, \cdots, |x_{j_n}|\geq 1} |f(x)|dx_{j_1}dx_{j_2}\cdots dx_{j_n} \\
&+ \sum_P \int_{|x_{j_1}|\leq 1, |x_{j_2}|\leq 1, |x_{j_3}|\geq 1, \cdots, |x_{j_n}|\geq 1} |f(x)|dx_{j_1}dx_{j_2}\cdots dx_{j_n} + \\
&\sum_P \int_{|x_{j_1}|\leq 1, |x_{j_2}|\leq 1, |x_{j_3}|\leq 1, |x_{j_4}|\geq 1, \cdots, |x_{j_n}|\geq 1} |f(x)|dx_{j_1}dx_{j_2}\cdots dx_{j_n} \\
&+ \cdots + \int_{|x_1|\geq 1, |x_2|\geq 1, |x_3|\geq 1, \cdots, |x_n|\geq 1} |f(x)|dx \qquad (3)
\end{aligned}$$

Now by Holder's inequality

$$\int_{|x_1|\leq 1, |x_2|\leq 1, \cdots, |x_n|\leq 1} |f(x)|dx \leq \|f(x)\|_2 \sqrt{\int_{|x_1|\leq 1, |x_2|\leq 1, \cdots, |x_n|\leq 1} 1dx} \leq \|f(x)\|_2 \, 2^{n/2}$$

and

$$\int_{|x_{j_1}|\leq 1, |x_{j_2}|\leq 1, \cdots, |x_{j_k}|\leq 1, |x_{j_{k+1}}|\geq 1, |x_{j_{k+2}}|\geq 1, \cdots, |x_{j_n}|\geq 1} |f(x)|dx_{j_1}dx_{j_2}\cdots dx_{j_k}dx_{j_{k+1}}dx_{j_{k+2}}\cdots dx_{j_n}$$

$$\leq \left\|x_{j_{k+1}}x_{j_{k+2}}\cdots x_{j_{k+n}}f\right\|_2 \sqrt{2^k 2^{n-k}}$$

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$$\leq \left\|x_{j_{k+1}} x_{j_{k+2}} \cdots x_{j_{k+n}} f\right\|_2 2^{\frac{n}{2}}$$

Using these results in (3) we get a possible bound for $\int_{\Re^n} |f(x)|dx$ as

$$\int_{\Re^n} |f(x)|dx \leq 2^{\frac{n}{2}}(\|f\|_2 + \sum_{i=1}^{n} \|x_i f(x)\|_2 + \sum_{i,j=1}^{n} \|x_i x_j f(x)\|_2 + \sum_{i,j,k=1}^{n} \|x_i x_j x_k f(x)\|_2$$

$$+ \cdots + \|x_1 x_2 \cdots x_n f(x)\|_2)$$

Here lower suffixes in a term having more than one x_i are all different.
Theorem 1 can also be proved by using the method of induction as follows.
Second proof of theorem 1: We have proved the result for $n = 1$ i.e. it is true for $L^2(\Re)$.
Now assume that the result is true for $L^2(\Re^n)$. Therefore for a window function $f \in L^2(\Re^n)$ we have

$$\int_{\Re^n} |f(x)|dx \leq 2^{\frac{n}{2}}(\|f\|_2 + \sum_{i=1}^{n} \|x_i f(x)\|_2 + \sum_{i,j=1}^{n} \|x_i x_j f(x)\|_2 + \sum_{i,j,k=1}^{n} \|x_i x_j x_k f(x)\|_2$$

$$+ \cdots + \|x_1 x_2 \cdots x_n f(x)\|_2) \tag{4}$$

Here lower suffixes in a term are all different. According to Corollory 3, p.409 0f [5] $L^p(X) \otimes L^p(Y)$ is dense in $L^p(X \times Y)(1 \leq p < \infty)$. The symbol $\otimes$ stands for the tensor product of vector spaces. Take $p = 2$ and let $X = \Re^n$, and $Y = \Re$ and assume further that there are sequences $f_\nu \in L^2(\Re^n)$ and $g_\nu \in L^2(\Re)$ such that the sequences $f_\nu g_\nu \to f \in L^2(\Re^{n+1})$ in the topology of $L^2(\Re^{n+1})$. An arbitrary variable of $\Re^n$ is denoted by $(x_1, x_2, \cdots, x_n)$ that of $\Re^{n+1}$ by $(x_1, x_2, \cdots, x_n, x_{n+1})$and an arbitrary variable in $\Re$ is x_{n+1}. It is simple to show that if $f \in L^2(\Re^{n+1})$ is a window function, then f_ν, g_ν must be window functions belonging to $L^2(\Re^n)$ and $L^2(\Re)$ respectively such that $f_\nu g_\nu \to f$ in the topology of $L^2(\Re^{n+1})$.
In view of the assumption (4) we have

$$\|f_\nu\|_1 \leq 2^{\frac{n}{2}}(\|f_\nu\|_2 + \sum_{i=1}^{n} \|x_i f_\nu\|_2 + \sum_{i,j=1}^{n} \|x_i x_j f_\nu\|_2 + + \sum_{i,j,k=1}^{n} \|x_i x_j x_k f_\nu\|_2$$

$$+ \cdots + \|x_1 x_2 \cdots x_n f_\nu\|_2)$$

By simple calculation we have

$$\begin{aligned}\|f_\nu g_\nu\|_1 &= \|f_\nu\|_1 \|g_\nu\|_1 \\ &\leq 2^{\frac{n}{2}} \left[\|f_\nu\|_2 + \sum_{n}^{i=1} \|x_i f_\nu\|_2 + \cdots + \|x_1 x_2 \cdots x_n f_\nu\|_2\right] \times \\ &\quad 2^{\frac{1}{2}} \left[\|g_\nu\|_2 + \|x_{n+1} g_\nu\|_2\right] \\ &\leq 2^{\frac{n+1}{2}} (\|f_\nu g_\nu\|_2 + \sum_{i=1}^{n+1} \|x_i f_\nu g_\nu\|_2 + \sum_{i,j=1}^{n+1} \|x_i x_j f_\nu g_\nu\|_2 \\ &\quad + \sum_{i,j,k=1}^{n+1} \|x_i x_j x_k f_\nu g_\nu\|_2 + \cdots + \|x_1 x_2 \cdots x_{n+1} f_\nu g_\nu\|_2) \qquad (5)\end{aligned}$$

Let X_{K_1} and X_{K_2} be characteristic functions of compact sets $K_1 \subset \Re^n$ and $K_2 \subset \Re$ respectively. Therefore $X_{K_1} X_{K_2}$ is a characteristic function of $K_1 \times K_2$ in $\Re^{n+1}$, where

$$K_1 \times K_2 = \{(x_1, x_2, \cdots, x_n, x_{n+1}); (x_1, x_2, \cdots, x_n) \in K_1, x_{n+1} \in K_2\}$$

Let $K = K_1 \times K_2$, where $\times$ denotes the cartesian product of K_1 and K_2. The functions f_ν are window functions in $L^2(\Re^n)$ and g_ν are window functions in $L^2(\Re)$ respectively, therefore $f_\nu X_{K_1} g_\nu X_{K_2}$ is a window function in $L^2(\Re^{n+1})$. Therefore by replacing f_ν by $f_\nu X_{K_1}$ and g_ν by $g_\nu X_{K_2}$ in (5), we get

$$\begin{aligned}\|f_\nu g_\nu X_{K_1} X_{K_2}\|_1 &\leq 2^{\frac{n+1}{2}} (\|f_\nu g_\nu X_{K_1} X_{K_2}\|_2 + \sum_{i=1}^{n+1} \|x_i f_\nu g_\nu X_{K_1} X_{K_2}\|_2 \\ &\quad + \sum_{i,j=1}^{n+1} \|x_i x_j f_\nu g_\nu X_{K_1} X_{K_2}\|_2 \\ &\quad + \sum_{i,j,k=1}^{n+1} \|x_i x_j x_k f_\nu g_\nu X_{K_1} X_{K_2}\|_2 \\ &\quad + \cdots + \|x_1 x_2 \cdots x_{n+1} f_\nu g_\nu X_{K_1} X_{K_2}\|_2) \qquad (6)\end{aligned}$$

Now let $\nu \to \infty$ in (6). Then $f_\nu g_\nu \to f$ in $L^2(\Re^{n+1})$. Therefore with $\nu \to \infty$

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in (6) we get

$$\begin{aligned}\|fX_{K_1}X_{K_2}\|_1 \leq\ & 2^{\frac{n+1}{2}}(\|fX_{K_1}X_{K_2}\|_2+\sum_{i=1}^{n+1}\|x_ifX_{K_1}X_{K_2}\|_2\\ &+\sum_{i,j=1}^{n+1}\|x_ix_jfX_{K_1}X_{K_2}\|_2+\sum_{i,j,k=1}^{n+1}\|x_ix_jx_kfX_{K_1}X_{K_2}\|_2\\ &+\cdots\cdots+\|x_1x_2\cdots x_{n+1}fX_{K_1}X_{K_2}\|_2)\\ &\leq 2^{\frac{n+1}{2}}(\|fX_K\|_2+\sum_{i=1}^{n+1}\|x_ifX_K\|_2\\ &+\sum_{i,j=1}^{n+1}\|x_ix_jfX_K\|_2+\sum_{i,j,k=1}^{n+1}\|x_ix_jx_kfX_K\|_2\\ &+\cdots\cdots+\|x_1x_2\cdots x_{n+1}fX_K\|_2)\\ &\leq 2^{\frac{n+1}{2}}(\|f\|_2+\sum_{i=1}^{n+1}\|x_if\|_2+\sum_{i,j=1}^{n+1}\|x_ix_jf\|_2\\ &+\sum_{i,j,k=1}^{n+1}\|x_ix_jx_kf\|_2+\cdots\cdots+\|x_1x_2\cdots x_{n+1}f\|_2) \quad (7)\end{aligned}$$

$where K=K_1\times K_2=\{(x_1,x_2,\cdots,x_n,x_{n+1});(x_1,x_2,\cdots,x_n)\in K_1,x_{n+1}\in K_2\}$

Clearly K is a compact subset of $\Re^{n+1}$, therefore from (7) we get

$$\begin{aligned}\|fX_K\|_1 \leq\ & 2^{\frac{n+1}{2}}(\|f\|_2+\sum_{i=1}^{n+1}\|x_if\|_2+\sum_{i,j=1}^{n+1}\|x_ix_jf\|_2+\sum_{i,j,k=1}^{n+1}\|x_ix_jx_kf\|_2\\ &+\cdots\cdots+\|x_1x_2\cdots x_{n+1}f\|_2) \quad (8)\end{aligned}$$

K is an arbitrary compact subset of $\Re^{n+1}$ and a sequence of compact subset in $\Re^{n+1}$ can be made to tend to the entire space $\Re^{n+1}$ [5], therefore with this limiting operation of $K\to\Re^{n+1}$ in (8) we get

$$\begin{aligned}\|f\|_1 \leq 2^{\frac{n+1}{2}}(&\|f\|_2+\sum_{i=1}^{n+1}\|x_if\|_2+\sum_{i,j=1}^{n+1}\|x_ix_jf\|_2+\sum_{i,j,k=1}^{n+1}\|x_ix_jx_kf\|_2\\ &+\cdots+\|x_1x_2\cdots x_{n+1}f\|_2)\end{aligned}$$

Hence if the theorem 1 is true in $L^2(\Re^n)$ it is also true in $L^2(\Re^{n+1})$. But it is true in $L^2(\Re)$, therefore, by induction it must be true for $L^2(\Re^n)$ for any $n=1,2,3,\cdots$.

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Quaternion Hilbert C^*-modules

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Abstract

Using quaternion analysis we shall present a concept of Hilbert C^*-module which generalize and extend the corresponding definition in the complex setting. We support our definitions and by some examples. Moreover, we give some basic properties of quaternion Hilbert C^*-module.

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1 Introduction

Let $\mathbb{H}$ be the skew field of quaternions. The elements of $\mathbb{H}$ are of the form:

$$a = \sum_{k=0}^{3} a_k e_k, \text{ where } \{a_k : k \in N_3^0 = N_3 \cup \{0\} \text{ and } N_3 = \{1,2,3\}\} \subset \mathbb{R},$$

where, $e_0 = 1$, e_1, e_2, e_3 are called imaginary units and they define arithmetic rules in $\mathbb{H}$. By definition $e_k^2 = -e_0$, $k \in N_3$, where

$$e_1 e_2 = -e_2 e_1 = e_3,\ e_2 e_3 = -e_3 e_2 = e_1,\ e_3 e_1 = -e_1 e_3 = e_2.$$

Sometimes we use i, j and k instead of e_1, e_2 and e_3. Natural operations of addition and multiplication in $\mathbb{H}$ turn it into a skew-field. The main involution in $\mathbb{H}$, the quaternion conjugation, is defined by

$$\overline{e_0} := e_0, \quad \overline{e_k} := -e_k, \text{ for } k \in N_3,$$

and it extends onto $\mathbb{H}$ by $\mathbb{R}$-linearity, i.e., for $a \in \mathbb{H}$

$$\overline{a} := \sum_{k=0}^{3} \overline{a_k e_k} = \sum_{k=0}^{3} a_k \bar{e_k} = a_0 - \sum_{k=1}^{3} a_k e_k.$$

Note that

$$\overline{a}a = a\overline{a} = \sum_{k=0}^{3} a_k^2 = |a|^2_{\mathbb{R}^4} =: |a|^2_{\mathbb{H}}$$

Therefore, for $a \in \mathbb{H} \setminus \{0\}$ the quaternion $a^{-1} := \frac{1}{|a|^2}\overline{a}$ is an inverse to a. Whereas the above mentioned properties are analogous to the complex one-dimensional case, we have for the quaternion conjugation that for any $a, b \in \mathbb{H}$ $\overline{ab} = \overline{b}\ \overline{a}$. Let Ω be a domain in $\mathbb{R}^3$. For more details about Quaternion analysis, we refer to [3], [8], and [9].

Hereafter, V denotes a vector space over the skew filed of quaternions $\mathbb{H}$, that is, V is the additive group and the multiplication of the vectors $x \in V$ on scalars $\alpha,\ \beta \in \mathbb{H}$ from the left and the right satisfies the axioms of the associativity and distributivity.
If there exists the norm $\|x\|_V =: \|x\|$ in V relative to which V is complete, where $\|\alpha x\| = |\alpha|_{\mathbb{H}}\|x\|$,

$$\|x\beta\| = |\beta|_{\mathbb{H}}\|x\| \text{ and } \|x+y\| \leq \|x\| + \|y\|$$

for each $x, y \in V$ and $\alpha, \beta \in \mathbb{H}$, then V is called the Banach space over $\mathbb{H}$ (see [12]).

Definition 1.1 *Let V be a quaternion vector space over $\mathbb{H}$. A quaternion inner product in V is a quaternion-valued mapping $<,>: V \times V \to \mathbb{H}$, such that*

(1) $< x, x > \geq 0$ *and* $< x, x > = 0$ *iff* $x = 0$,

(2) $< x+y, z > = < x, z > + < y, z >$, *where* $x, y, z \in V$

(3) $< x\alpha, y\beta > = \bar{\alpha} < x, y > \beta$ *where* $\alpha, \beta \in \mathbb{H}$

(4) $< x, y > = \overline{< y, x >}$.

Remark 1.1 *If V is complete relative to the norm*

$$\|x\| = \sqrt{< x, x >},$$

then V is called quaternion Hilbert space.

2 C^*-algebras

C^*-algebras are important and interesting objects of study in functional analysis on Hilbert spaces with algebraic methods. These algebras give a link between algebra, topology and analysis, since they possess analytical and topological structures.
In this section, we give a survey of some basic definitions and facts on C^*algebras (complex C^*algebras), which will be used in this paper. C^*-algebras have found several applications in representation theory of groups, topology, geometry, index theory and in many other subjects of mathematics and physics. The C^*-algebra plays a central role in what is called noncommutative geometry, which was pioneered by A. Connes (see [4]). Also, it plays a major role in the $K-$theory (see [1, 6, 7]).

Definition 2.1 *A C^*algebra $\mathcal{A}$ is a complex Banach algebra with the conjugate-linear involution*

$$* : \mathcal{A} \to \mathcal{A},$$

such that

$$(x^*)^* = x, \ \ (xy)^* = y^*x^*, \ \ \|x^*x\| = \|x\|^2$$

for all $x, y \in \mathcal{A}$

Remark 2.1 *The C^*- condition $\|x^*x\| = \|x\|^2$ implies that the involution is an isometry in the sense that $\|x^*\| = \|x\|$ for all $x \in \mathcal{A}$.*

Remark 2.2 *A C^*- algebra is called unital if it possess a unit. It follows easily that $\|1\| = 1$.*

For more details on the general theory of C^*- algebras, we refer to [13, 15].

Example 2.1 *Let X be a locally compact topological space. We denote by $C_0(X)$ the algebra of all complex-valued continues functions on X vanishing at infinity with pointwise addition and multiplication. The algebra $C_0(X)$ with the involution defined by*

$$f^*(x) = \overline{f(x)} \text{ for each } f \in C_0(X) \text{ and } x \in X,$$

with the norm

$$\|f\|_\infty = \sup\{|f(x)|,\ x \in X\}$$

is a commutative C^*- algebra. Also if X is compact, then the constant function 1 is the unit, i.e there is an element $f \in C_0(X)$ satisfy $1f = f1 = f$.

Example 2.2 *Let $\mathcal{H}$ be an arbitrary Hilbert space. We denote by $B(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. $B(\mathcal{H})$ with the operator norm*

$$\|T\| = \sup\{\|T(x)\|,\ x \in \mathcal{H},\ \ \|x\| = 1,\}$$

for each $T \in B(\mathcal{H})$ and the involution given by the adjoint of T. Then, $B(\mathcal{H})$ is a C^*- algebra.

In the above example $B(\mathcal{H})$ need not to be commutative.
Recently, Ludkovsky [12] introduced the definition of quaternion C^*-algebra which generalize the concept of complex C^*-algebra in Clifford setting as follows:

Definition 2.2 *[12] Let $\mathcal{A}_\mathbb{H}$ be a quaternion Banach space over $\mathbb{H}$ with the property*

$$\|ab\| \leq \|a\|\,\|b\|$$

for each $a, b \in \mathcal{A}_\mathbb{H}$, then $\mathcal{A}_\mathbb{H}$ is called a quaternion Banach algebra over $\mathbb{H}$.

Definition 2.3 *[12] A quaternion C^*-algebra $\mathcal{A}_\mathbb{H}$ is a Banach algebra over $\mathbb{H}$ with the involution $* : \mathcal{A}_\mathbb{H} \to \mathcal{A}_\mathbb{H}$, satisfy*

(1) $(a^*)^* = a$, $a \in \mathcal{A}_\mathbb{H}$

(2) $(a + b)^* = a^* + b^*$,

(3) $(ab)^* = b^*a^*$,

(4) $(\alpha a)^* = a^*\bar{\alpha}$, *for each $a, b \in \mathcal{A}_\mathbb{H}$ and $\alpha \in \mathbb{H}$, such that*

$$\|a^*a\| = \|a\|^2.$$

3 Quaternion Hilbert C^*-module

Hilbert C^*-modules are mathematical objects which generalize the notion of a Hilbert space. The Hilbert C^*-modules were first introduced in the work of Kaplansky [16] in 1953, which developed the theory for commutative, unital algebras (though Kaplansky observed that the assumption of a unit element was not "vital"). In the 1970s the theory was extended to non-commutative C^*-algebras independently by Paschke [17].
Hilbert C^*-modules are an often used tool in operator theory and in operator algebra theory. They serve as a major class of examples in operator C^*-module theory. Beside this, the theory of Hilbert C^*-modules is very interesting on its own. Interacting with the theory of operator algebras and including ideas from non-commutative geometry it progresses and produces results and new problems attracting attention. During the last decade many interesting applications of Hilbert C^*-module theory have been found. At the contrary, the pieces of Hilbert C^*-module theory are still rather scattered through the literature. There are some papers, chapters in monographs and lecture notes that give comprehensive representations of parts of the theory (see [1, 2, 5, 10, 11, 14, 18, 19] and others).
Now, we give the following definitions:

Definition 3.1 *[12] Let $\mathcal{A}_{\mathbb{H}}$ be a quaternion C^*-algebra. The quaternion pre-Hilbert C^*-module is a right $\mathcal{A}_{\mathbb{H}}$-module $E_{\mathbb{H}}$ over $\mathcal{A}_{\mathbb{H}}$-valued inner product $<,>$: $E_{\mathbb{H}} \times E_{\mathbb{H}} \to \mathcal{A}_{\mathbb{H}}$ satisfy the following conditions:*

$(i) < x, x > \geq 0$ *and* $< x, x > = 0$ *iff* $x = 0,\ x \in E_{\mathbb{H}}$

$(ii) < x + y, z > = < x, z > + < y, z >,$ *where* $x, y, z \in E_{\mathbb{H}}$

$(iii) < xa, y\, b > = a^* \ < x, y > \ b,$ *where* $a, b \in \mathcal{A}_{\mathbb{H}}$

$(iv) < x, y >^* = < y, x >$.

Definition 3.2 *The norm of an element $x \in E_{\mathbb{H}}$ is defined by*

$$\|x\|_{E_{\mathbb{H}}} = \sqrt{\| < x, x > \|_{\mathcal{A}_{\mathbb{H}}}}\ . \tag{1}$$

Definition 3.3 *The quaternion pre-Hilbert module $E_{\mathbb{H}}$ over $\mathcal{A}_{\mathbb{H}}$ which is complete with the norm (1) is called quaternion Hilbert $\mathcal{A}_{\mathbb{H}}$-module.*

Remark 3.1 *The quaternion Hilbert module generalizes the quaternion Hilbert space which is recently defined in [12]. When $\mathcal{A}_{\mathbb{H}} = \mathbb{H}$, then the quaternion Hilbert module coincides with the quaternion Hilbert space.*

Remark 3.2 *For any $x, y \in E_{\mathbb{H}}$ and $a \in \mathcal{A}_{\mathbb{H}}$, we have*

$$< xa, y > = < y, xa >^* = \big(< y, x > a\big)^* = a^* < y, x >^* = a^* < x, y > . \tag{2}$$

Example 3.1 *$\mathcal{A}_{\mathbb{H}}$ is a quaternion Hilbert $\mathcal{A}_{\mathbb{H}}$-module.*

Proof: We define the inner product $<,>: \mathcal{A}_{\mathbb{H}} \times \mathcal{A}_{\mathbb{H}} \to \mathcal{A}_{\mathbb{H}}$ by

$$< x, y > = x^* y\ ,\ \forall x, y \in \mathcal{A}_{\mathbb{H}}.$$

Firstly, we verify the conditions of Definition 3.1.
$(i)\ < x, x > = x^* x \geq 0$, because for any $x \in \mathcal{A}_{\mathbb{H}}$, we have $x^* x$ is positive (see [12]). Also,

$$< x, x > = 0 \Longleftrightarrow x^* x = 0 \Longleftrightarrow \|x^* x\| = 0 \Longleftrightarrow \|x\|^2 = 0 \Longleftrightarrow x = 0.$$

Hence condition (i) in the definition is proved.
(ii) For $x, y, z \in \mathcal{A}_{\mathbb{H}}$, we have that $< x+y, z > = (x+y)^* z = (x^*+y^*)z = x^* z + y^* z = < x, z > + < y, z >$.
$(iii) < xa, yb > = (xa)^*(yb) = a^* x^*(yb) = a^*(x^* y)b = a^* < x, y > b,\ a, b \in \mathcal{A}_{\mathbb{H}}$.
$(iv) < x, y >^* = (x^* y)^* = y^*(x^*)^* = y^* x = < y, x >$. Therefore, $\mathcal{A}_{\mathbb{H}}$ is a quaternion pre-Hilbert C^*-module.
Now, we proof the completeness of $\mathcal{A}_{\mathbb{H}}$. Since,

$$\|x\|^2 = \| < x, x > \|_{\mathcal{A}_{\mathbb{H}}} = |x^* x\|_{\mathcal{A}_{\mathbb{H}}} = \|x\|^2_{\mathcal{A}_{\mathbb{H}}}$$

Hence, the norm in the quaternion Hilbert module is the same norm in the quaternion C^*-algebra $\mathcal{A}_{\mathbb{H}}$. Because the quaternion C^*-algebra is complete (see [12]), then quaternion pre-Hilbert $\mathcal{A}_{\mathbb{H}}$ is complete. The proof of our example is completely established.

Theorem 3.1 *Let $\mathcal{A}_{\mathbb{H}}$ be a quaternion C^*-algebra. Define*

$$\mathcal{H}_{\mathcal{A}_{\mathbb{H}}} := \left\{ x = (x_i)_{i \geq 1} \in \mathcal{A}_{\mathbb{H}} \times \mathcal{A}_{\mathbb{H}} \times \mathcal{A}_{\mathbb{H}} \times \ldots : \sum_{i=1}^{\infty} x_i^*\, x_i \ converges\ in \mathcal{A}_{\mathbb{H}} \right\}$$

with the norm

$$\big\|<, x, y >\big\|^2 = \left\| \sum_{i=1}^{\infty} x_i^*\, y_i \right\|.$$

Proof: Since, $\mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$ is a quaternion vector space with the following pointwise operations

$$(x_i) + (y_i) = (x_i + y_i) \text{ and } (x_i)\, a = (x_i\, a)).$$

Now, we have the inner product

$$< (x_i), (y_i) >= \sum_{i\geq 1} x_i^*\, y_i,$$

and the norm

$$\|(x_i)\| = \sqrt{\|\sum_{i\geq 1} x_i^*\, x_i\|}$$

are defined in $\mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$. Then, $\mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$ is a pre-quaternion Hilbert C^*-modules.
To show $\mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$ is complete with respect to the norm induced by the inner product

$$< (x_i), (y_i) >= \sum_{i\geq 1} x_i^*\, y_i.$$

Let $(x^m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$, such that every $x^m = (x_i^m)_{i\geq 1}$ be a sequence in $\mathcal{A}_{\mathbb{H}}$. Clearly each $(x_i^m)_{i\geq 1}$ is a Cauchy sequence in $\mathcal{A}_{\mathbb{H}}$. Hence, it converges to some x_i in $\mathcal{A}_{\mathbb{H}}$ for some fixed i.
Now, let $x = (x_i) \in \mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$. Then, for each $N, i_0 \in \mathbb{N}$, we have

$$\begin{aligned}
\left\| \sum_{i=i_0}^{N} (x_i - x_i^m)^*(x_i - x_i^m) \right\| &= \lim_{N\to\infty} \left\| \sum_{i=i_0}^{N} (x_i^N - x_i^m)^*(x_i^N - x_i^m) \right\| \\
&\leq \lim_{N\to\infty} \left\| \sum_{i=1}^{\infty} (x_i^N - x_i^m)^*(x_i^N - x_i^m) \right\| \\
&= \lim_{N\to\infty} \left\| \|x^N - x^m\|^2 \right\|.
\end{aligned}$$

For each $\epsilon > 0$, and $N > m$, we have

$$\left\| \sum_{i=i_0}^{N} (x_i - x_i^m)^*(x_i - x_i^m) \right\| < \epsilon,$$

then, $x - x^m \in \mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$. Moreover,

$$\|x - x^m\|^2 = \left\| \sum_{i=1}^{\infty} (x_i - x_i^m)^*(x_i - x_i^m) \right\| < \epsilon,$$

so $x^m \to x$ in $\mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$. Therefore, $\mathcal{H}_{\mathcal{A}_{\mathbb{H}}}$ is complete. This completes the proof.

Theorem 3.2 *Let $\mathcal{A}_{\mathbb{H}}$ be a quaternion C^*-algebra. Define $\mathcal{A}_{\mathbb{H}}^n$ to be the direct sum of n-copies of $\mathcal{A}_{\mathbb{H}}$ with the inner product by*

$$< x, y >= \sum_{i=1}^{n} x_i^*\, y_i,$$

*where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, $x_i, y_i \in \mathcal{A}_{\mathbb{H}}$ for $i = 1, 2, 3, \ldots, n$.
Then, $\mathcal{A}_{\mathbb{H}}^n$ is a quaternion Hilbert $\mathcal{A}_{\mathbb{H}}-$module.*

Proof: $\mathcal{A}_{\mathbb{H}}^n$ is a quaternion vector space with the following pointwise operations

$$(x_i) + (y_i) = (x_i + y_i) \text{ and } (x_i)\, a = (x_i\, a) \text{ where } 1 \leq i \leq n.$$

and the quaternion inner product

$$< (x_i), (y_i) >= \sum_{i=1}^{n} x_i^*\, y_i,$$

and the norm

$$\|(x_i)\| = \sqrt{\|\sum_{i=1}^{n} x_i^* \, x_i\|}$$

are defined in $\mathcal{A}_{\mathbb{H}}^n$. Then, $\mathcal{A}_{\mathbb{H}}^n$ is a pre-quaternion Hilbert C^*-modules. We can prove the completeness of $\mathcal{A}_{\mathbb{H}}^n$ as in Theorem 3.1, with some simple modifications.

4 Operators over quaternion Hilbert C^*-modules

Let $E_{\mathbb{H}}$, $M_{\mathbb{H}}$ be two quaternion Hilbert modules over the quaternion C^*-algebra $\mathcal{A}_{\mathbb{H}}$. We will denote by $\mathcal{L}_{\mathcal{A}_{\mathcal{H}}}(E_{\mathbb{H}}, M_{\mathbb{H}})$ the set of all bounded operators from $E_{\mathbb{H}}$ to $M_{\mathbb{H}}$ and $\mathcal{L}_{\mathcal{A}_{\mathbb{H}}}(E_{\mathbb{H}})$ denotes the set of all bounded operators from $E_{\mathbb{H}}$ to itself.
Now, we give the following definition.

Definition 4.1 *The operator $T : E_{\mathbb{H}} \to E_{\mathbb{H}}$ is called adjointable if*

$$< x, Ty >=< T^*x, y >, \quad \textit{where } \; x, y \in E_{\mathbb{H}}.$$

Theorem 4.1 *Let $\mathcal{L}_{\mathcal{A}_{\mathbb{H}}}(E_{\mathbb{H}})$ be the set of all bounded operators on the quaternion Hilbert $\mathcal{A}_{\mathbb{H}}$-module. Then, $\mathcal{L}_{\mathcal{A}_{\mathbb{H}}}(E_{\mathbb{H}})$ define a quaternion C^*-algebra with the norm*

$$\|T\| = \sup_{\|x\| \leq 1} \left\{ \|Tx\|, \quad \textit{where } \; x \in E_{\mathbb{H}} \right\}$$

and the involution is adjointable mapping.

Proof: Firstly, we prove that $\mathcal{L}_{\mathcal{A}_{\mathbb{H}}}(E_{\mathbb{H}})$ is a quaternion Banach algebra.
For $T, S \in \mathcal{L}_{\mathcal{A}_{\mathbb{H}}}(E_{\mathbb{H}})$, we conclude that

$$\begin{aligned}
\|TS\| &= \sup_{\|x\| \leq 1} \left\{ \|TS(x)\|, \quad \text{where } \; x \in E_{\mathbb{H}} \right\} \\
&= \sup_{\|S(x)\| \leq 1} \left\{ \|T(S(x))\|, \quad \text{where } \; S(x) \in E_{\mathbb{H}} \right\} \\
&\leq \sup_{\|S(x)\| \leq 1} \left\{ \|T\| \|(S(x))\|, \quad \text{where } \; S(x) \in E_{\mathbb{H}} \right\} \\
&= \|T\| \sup_{\|S(x)\| \leq 1} \left\{ \|(S(x))\|, \quad \text{where } \; S(x) \in E_{\mathbb{H}} \right\} \\
&= (\|T\|)(\|S\|). \qquad (3)
\end{aligned}$$

Thus, $\mathcal{L}_{\mathcal{A}_{\mathbb{H}}}(E_{\mathbb{H}})$ is a quaternion Banach algebra. On the other hand

$$\begin{aligned}
\|T\|^2 &= \sup_{\|x\| \leq 1} \left\{ \|Tx\|^2, \quad \text{where } \; x \in E_{\mathbb{H}} \right\} \\
&= \sup_{\|x\| \leq 1} \left\{ \| < Tx, Tx > \|, \quad \text{where } \; x \in E_{\mathbb{H}} \right\} \\
&= \sup_{\|x\| \leq 1} \left\{ \| < T^*Tx, x > \|, \quad \text{where } \; x \in E_{\mathbb{H}} \right\} \\
&\leq \sup_{\|x\| \leq 1} \left\{ \|T^*Tx\| \|x\|, \quad \text{where } \; x \in E_{\mathbb{H}} \right\} \\
&\leq \sup_{\|x\| \leq 1} \left\{ \|T^*\| \|Tx\| \|x\|, \quad \text{where } \; x \in E_{\mathbb{H}} \right\}
\end{aligned}$$

$$\leq \sup_{\|x\|\leq 1}\left\{(\|T^*\|)(\|T\|)(\|x\|^2),\ \text{ where }\ x\in E_{\mathbb{H}}\right\}$$
$$\leq (\|T^*\|)(\|T\|), \tag{4}$$

which implies that

$$\|T\|\leq\|T^*\|. \tag{5}$$

Similarly, we have

$$\|T^*\|\leq\|(T^*)^*\|=\|T\|. \tag{6}$$

From (5) and (6), we obtain

$$\|T^*\|=\|T\|. \tag{7}$$

Finally, we verify the C^*-conditions. As in (4), we can obtain that

$$\|T\|^2\leq \sup_{\|x\|\leq 1}\left\{\|T^*Tx\|\|x\|,\ \text{ where }\ x\in E_{\mathbb{H}}\right\}$$
$$\leq \sup_{\|x\|\leq 1}\left\{(\|T^*T\|)(\|x\|^2),\ \text{ where }\ x\in E_{\mathbb{H}}\right\}$$
$$\leq \|T^*T\|. \tag{8}$$

From (3) and (7), we have

$$\|T^*T\|=(\|T^*\|)(\|T\|)\leq\|T\|^2. \tag{9}$$

From (8) and (9), we conclude that

$$\|T^*T\|=\|T\|^2.$$

Hence, $\mathcal{L}_{\mathcal{A}_{\mathbb{H}}}(E_{\mathbb{H}})$ is a quaternion C^*- algebra.

Definition 4.2 *Let $E_{\mathbb{H}}$ be a quaternion Hilbert $\mathcal{A}_{\mathbb{H}}-$module, $M_{\mathbb{H}}$ be a quaternion $\mathcal{B}_{\mathbb{H}}-$module. Then, the algebraic tensor product $E_{\mathbb{H}}\bigotimes_{\mathbb{H}}M_{\mathbb{H}}$ is a right module over the algebraic tensor product of C^*- algebra $\mathcal{A}_{\mathbb{H}}\bigotimes_{\mathbb{H}}\mathcal{B}_{\mathbb{H}}$ such that*

$$(e\otimes m)a\otimes b=e\,a\otimes m\,b,\ \text{where}\ e\in E_{\mathbb{H}}\,,\ m\in M_{\mathbb{H}}\ \text{and}\ a\in\mathcal{A}_{\mathbb{H}},\ b\in\mathcal{B}_{\mathbb{H}}.$$

We define the $\mathcal{A}_{\mathbb{H}}\bigotimes_{\mathbb{H}}\mathcal{B}_{\mathbb{H}}$-valued inner product on $E_{\mathbb{H}}\bigotimes_{\mathbb{H}}M_{\mathbb{H}}$ as the mapping

$$<,>:\left(E_{\mathbb{H}}\bigotimes_{\mathbb{H}}M_{\mathbb{H}}\right)\times\left(E_{\mathbb{H}}\bigotimes_{\mathbb{H}}M_{\mathbb{H}}\right)\longrightarrow\mathcal{A}_{\mathbb{H}}\bigotimes_{\mathbb{H}}\mathcal{B}_{\mathbb{H}}$$

by $<e_1\otimes m_1,e_2\otimes m_2>=<e_1,e_2>\otimes<m_1,m_2>$ where $e_1,e_2\in E_{\mathbb{H}}$ and $m_1,m_2\in M_{\mathbb{H}}$.

Theorem 4.2 *Let $\mathcal{A}_{\mathbb{C}}$ be complex C^*- algebra and $E_{\mathbb{C}}$ be the Hilbert C^*-module over $\mathcal{A}_{\mathbb{C}}$, then $E_{\mathbb{C}}\bigotimes_{\mathbb{C}}\mathbb{H}$ is the algebraic tensor product of the complex Hilbert C^*-modules $E_{\mathbb{C}}$ and $\mathbb{H}$ is a quaternion Hilbert C^*-module over $\mathcal{A}_{\mathbb{C}}\bigotimes_{\mathbb{C}}\mathbb{H}$.*

Proof: Define the inner product

$$<,>:\left(E_{\mathbb{C}}\bigotimes_{\mathbb{C}}\mathbb{H}\right)\times\left(E_{\mathbb{C}}\bigotimes_{\mathbb{C}}\mathbb{H}\right)\longrightarrow\mathcal{A}_{\mathbb{C}}\bigotimes_{\mathbb{C}}\mathbb{H}$$

by $<e_1\otimes h_1,e_2\otimes h_2>=<e_1,e_2>\otimes<h_1,h_2>$ where $e_1,e_2\in E_{\mathbb{C}}$ and $h_1,h_2\in\mathbb{H}$.
Also, we define the norm

$$\|e\otimes h\|=\|e\|\otimes\|h\|.$$

Clearly, we have that $E_{\mathbb{C}}\bigotimes_{\mathbb{C}}\mathbb{H}$ is a quaternion pre-Hilbert module. The proof of completeness is very similar to the complex case (see [10]), so it will be omitted.

Theorem 4.3 *Let $E_{\mathbb{H}}$ be a quaternion Hilbert C^*-module over $\mathcal{A}_{\mathbb{H}}$ and $E_{\mathbb{C}} \bigotimes_{\mathbb{C}} \mathbb{H}$ is the algebraic tensor product of the complex Hilbert C^*-modules $E_{\mathbb{C}}$ and $\mathbb{H}$ over $\mathcal{A}_{\mathbb{C}} \bigotimes_{\mathbb{C}} \mathbb{H}$. Then, $E_{\mathbb{H}}$ is isomorphic to $E_{\mathbb{C}} \bigotimes_{\mathbb{C}} \mathbb{H}$.*

Proof: It suffices to show that the mapping

$$\varphi : E_{\mathbb{H}} \longrightarrow E_{\mathbb{C}} \bigotimes_{\mathbb{C}} \mathbb{H},\ \varphi : e \longrightarrow e \otimes e_0,\ \text{(where } e \in E_{\mathbb{H}} \text{ and } e_0 \text{ is the identity element in } \mathbb{H})$$

is an isomorphism. First, we show that φ is an injective mapping. For any $e_1, e_2 \in E_{\mathbb{H}}$ if $\varphi(e_1) = \varphi(e_2)$. Then,

$$e_1 \otimes e_0 = e_2 \otimes e_0 \Longrightarrow e_1 \otimes e_0 - e_2 \otimes e_0 = 0 \Longrightarrow (e_1 - e_2) \otimes e_0 = 0 \Longrightarrow e_1 - e_2 = 0 \Longrightarrow e_1 = e_2.$$

Hence, φ is an injective mapping.
Also, for any

$$e_1 \otimes e_0 \in E_{\mathbb{C}} \bigotimes_{\mathbb{C}} \mathbb{H},$$

we have $\varphi(e) = e_1 \otimes e_0$, so φ is a surjective mapping. Therefore, φ is an isomorphism.

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Trapezoidal method for solving fuzzy initial value problems

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Abstract

In this paper, the trapezoidal method is proposed to solve the "fuzzy initial value problem". The method is obtained by combining the Euler method and a trapezoidal rule calculating integration. Convergence of the proposed methods are also presented in detail. These methods are illustrated by solving some examples.

Keywords: Fuzzy differential equations; Fuzzy Cauchy problem; Euler method; Trapezoidal rule; Trapezoidal method

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1 Introduction

The fuzzy differential equations is a very important topic from the theoretical point of view [10,11] as well as the applied point of view [4]. The use of fuzzy differential equations is a natural way to model dynamical systems under possibilistic uncertainty. The concept of differential equations in a fuzzy environment was formulated by Kaleva [9]. The last few years, several authors have produced a wide range of results in both the theoretical and applied fields. A variety of exact, approximate, and purely numerical methods are available to find the solution of a fuzzy initial value problem(FIVP). For instance, Abbasbandy et al.[1] introduced the Taylor method, Allahviranloo et al.[2,3] introduced the predictor-corrector method. In this paper, a trapezoidal method is introduced.

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2 Preliminaries

2.1. Notations and definitions

A tilde is placed over a symbol to denote a fuzzy set so $\tilde{\alpha}_1$, $\tilde{f}(t), \ldots$

Definition 2.1[5]. A fuzzy number is a function such as $u : R \to [0, 1]$ satisfying the following properties:
(i) u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$,
(ii) u is a convex fuzzy set i.e. $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}\ \forall x, y \in R, \lambda \in [0, 1]$,
(iii) u is upper semi-continuous on R,
(iv)$\overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Definition 2.2[2]. A fuzzy number u in parametric form is a pair $(\underline{u}(\alpha), \overline{u}(\alpha))$ of function $\underline{u}(\alpha)$, $\overline{u}(\alpha)$, $0 \leq \alpha \leq 1$, which satisfies the following requirements:
1. $\underline{u}(\alpha)$ is a bounded monotonic increasing left continuous function,
2. $\overline{u}(\alpha)$ is a bounded monotonic decreasing left continuous function,
3. $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$.

For arbitrary fuzzy numbers $u = (\underline{u}(\alpha), \overline{u}(\alpha)), v = (\underline{v}(\alpha), \overline{v}(\alpha))$ we define addition and multiplication by a scalar k as
1.$\underline{(u + v)}(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha)$,
2.$\overline{(u + v)}(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha)$,
3.$\underline{(ku)}(\alpha) = k\underline{u}(\alpha), \quad \overline{(ku)}(\alpha) = k\overline{u}(\alpha),\ k \geq 0$,
4.$\underline{(ku)}(\alpha) = k\overline{u}(\alpha), \quad \overline{(ku)}(\alpha) = k\underline{u}(\alpha),\ k \leq 0$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded α-level intervals. Meaning if $v \in E$ then the α-level set

$$[v]^\alpha = \{s | v(s) \geq \alpha\}, \quad 0 < \alpha \leq 1,$$

is a closed bounded interval which is denoted by $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$.

Let I be a real interval. A mapping $y : I \to E$ is called a fuzzy process and its α-level set is denoted by

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)], \quad t \in I, \quad \alpha \in (0, 1].$$

Triangular fuzzy numbers are those fuzzy sets in E which are characterized by an ordered triple $(x^l, x^c, x^r) \in R^3$ with $x^l \leq x^c \leq x^r$ such that $[U]^0 = [x^l, x^r]$ and $[U]^1 = \{x^c\}$ then $[U]^\alpha = [x^c - (1 - \alpha)(x^c - x^l), x^c + (1 - \alpha)(x^r - x^c)]$ for any $\alpha \in I$.

Definition 2.3[2]. The supremum metric d_∞ on E is defined by

$$d_\infty(U,V) = \sup\{d_H([U]^\alpha, [V]^\alpha) : \alpha \in I\},$$

and (E, d_∞) is a complete metric space.

Definition 2.4[2]. A mapping $F : T \to E$ is Hukuhara differentiable at $t_0 \in T \subseteq R$ if for some $h_0 > 0$ the Hukuhara differences

$$F(t_0 + \Delta t) \sim_h F(t_0), \quad F(t_0) \sim_h F(t_0 - \Delta t),$$

exist in E for all $0 < \Delta t < h_0$ and if there exists a $F'(t_0) \in E$ such that

$$\lim_{\Delta t \to 0+} d_\infty((F(t_0 + \Delta t) \sim_h F(t_0))/\Delta t, \ F'(t_0)) = 0$$

and

$$\lim_{\Delta t \to 0+} d_\infty((F(t_0) \sim_h F(t_0 - \Delta t))/\Delta t, \ F'(t_0)) = 0$$

the fuzzy set $F'(t_0)$ is called the Hukuhara derivative of F at t_0.

Recall that $U \sim_h V = W \in E$ are defined on level sets, where $[U]^\alpha \sim_h [V]^\alpha = [W]^\alpha$ for all $\alpha \in I$. By consideration of definition of the metric d_∞, all the level set mappings $[F(.)]^\alpha$ are Hukuhara differentiable at t_0 with Hukuhara derivatives $[F'(t_0)]^a$ for each $\alpha \in I$ when $F : T \to E$ is Hukuhara differentiable at t_0 with Hukuhara derivative $[F'(t_0)]^\alpha$.

Definition 2.5[2]. The fuzzy integral

$$\int_a^b y(t)dt, \quad 0 \le a \le b \le 1,$$

is defined by

$$[\int_a^b y(t)dt]^\alpha = [\int_a^b \underline{y}^\alpha(t)dt, \int_a^b \overline{y}^\alpha(t)dt], \tag{2.1}$$

provided the Lebesgue integrals on the right exist.

Theorem 2.1[5]. Let $f : R \to E$ be a function and $f(t) = (\underline{f}^\alpha(t), \overline{f}^\alpha(t))$, for each $\alpha \in [0,1]$. If f is Hukuhara differentiable , then $\underline{f}^\alpha(t), \overline{f}^\alpha(t)$ are differentiable functions and

$$[f(t)]' = ([f^\alpha(t)]', [\overline{f}^\alpha(t)]') \tag{2.2}$$

Theorem 2.2[5]. For $t_0 \in R$, the fuzzy differential equation

$$y'(t) = f(t, y(t)), \ y(t_0) = y_0 \in E,$$

where $f : R \times E \to E$ is supposed to be continuous, is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s))ds, \quad \forall t \in [t_0, t_1] \tag{2.3}$$

on some interval $(t_0, t_1) \subset R$, under the Hukuhara derivate.

2.2. A fuzzy Cauchy problem

Consider the first-order fuzzy differential equation $y'(t) = f(t, y(t))$, where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of crisp variable t and fuzzy variable y, and $y'(t)$ is Hukuhara fuzzy derivative of y. If an initial value $\tilde{y}(t_0) = \tilde{\alpha}_0$ is given, a fuzzy Cauchy problem of first order will be obtained as follows:

$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \le t \le T, \\ \tilde{y}(t_0) = \tilde{\alpha}_0. \end{cases} \tag{2.4}$$

Sufficient conditions for the existence of a unique solution to Eq. (2.4) are:
• Continuity of f,
• Lipschitz condition $d_\infty(f(t, x), f(t, y)) \le L d_\infty(x, y), L > 0$.

3 The Euler and trapezoidal rule

From (2.1) and (2.2), to solve the system as follows:

$$\underline{y}'^{\alpha}(t) = \underline{f}^{\alpha}(t, y(t)), \quad \overline{y}'^{\alpha}(t) = \overline{f}^{\alpha}(t, y(t))$$

we replace the interval $[t_0, T]$ by a set of discrete equally spaced grid points $t_0 < t_l < t_2 < \cdots < t_N = T$ at which the exact solution $(\underline{Y}^{\alpha}(t), \overline{Y}^{\alpha}(t))$ is approximated by some $(\underline{y}^{\alpha}(t), \overline{y}^{\alpha}(t))$. The exact and approximate solutions at $t_n, 0 \le n \le N$ are denoted by $Y_n(\alpha) = [\underline{Y}_n(\alpha), \overline{Y}_n(\alpha)]$ and $y_n(\alpha) = [\underline{y}_n(\alpha), \overline{y}_n(\alpha)]$, respectively. The grid points at which the solution is calculated are $t_n = t_0 + nh, h = \frac{T - t_0}{N}$. Friedman et al. [8] gave the Euler method as follows:

$$\underline{y}_{n+1}(\alpha) = \underline{y}_n(\alpha) + h\underline{f}^{\alpha}(t_n, y_n), \quad \overline{y}_{n+1}(\alpha) = \overline{y}_n(\alpha) + h\overline{f}^{\alpha}(t_n, y_n). \tag{3.1}$$

Applying the trapezoidal rule [7,8]. Define

$$\underline{s}_N(\alpha) = h[\frac{\underline{f}^{\alpha}(t_0, y(t_0)) + \underline{f}^{\alpha}(T, y(T))}{2} + \sum_{i=1}^{N-1} \underline{f}^{\alpha}(t_i, y(t_i))],$$

$$\overline{s}_N(\alpha) = h[\frac{\overline{f}^{\alpha}(t_0, y(t_0)) + \overline{f}^{\alpha}(T, y(T))}{2} + \sum_{i=1}^{N-1} \overline{f}^{\alpha}(t_i, y(t_i))].$$

For arbitrary fixed α [7,8] we have

$$\lim_{N\to\infty} \underline{s}_N(\alpha) = \underline{F}(\alpha) = \int_a^b \underline{f}^{\alpha}(t, y(t))dt, \tag{3.2}$$

$$\lim_{N\to\infty} \overline{s}_N(\alpha) = \overline{F}(\alpha) = \int_a^b \overline{f}^{\alpha}(t, y(t))dt. \tag{3.3}$$

Theorem 3.1[8]. If $f(t)$ is continuous , the convergence of $\underline{s}_N(\alpha), \overline{s}_N(\alpha)$ to $\underline{F}(\alpha), \overline{F}(\alpha)$, respectively, is uniform in α.

From (2.3), (3.1), (3.2) and (3.3), the following results will be obtained, respectively,

$$\begin{cases} \underline{y}^{\alpha}(t_{i+2}) = \underline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+2}} \underline{f}^{\alpha}(t, y(t))dt = \underline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+1}} \underline{f}^{\alpha}(t, y(t))dt + \int_{t_{i+1}}^{t_{i+2}} \underline{f}^{\alpha}(t, y(t))dt \\ \qquad \approx \underline{y}^{\alpha}(t_{i-1}) + h[\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + \underline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))] + h\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) \\ \qquad = \underline{y}^{\alpha}(t_{i-1}) + h\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + 2h\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})), \\ \overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+2}} \overline{f}^{\alpha}(t, y(t))dt = \overline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+1}} \overline{f}^{\alpha}(t, y(t))dt + \int_{t_{i+1}}^{t_{i+2}} \overline{f}^{\alpha}(t, y(t))dt \\ \qquad \approx \overline{y}^{\alpha}(t_{i-1}) + h[\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + \overline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))] + h\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) \\ \qquad = \overline{y}^{\alpha}(t_{i-1}) + h\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + 2h\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})), \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0, \underline{y}^{\alpha}(t_i) = \alpha_1, \underline{y}^{\alpha}(t_{i+1}) = \alpha_2, \overline{y}^{\alpha}(t_{i-1}) = \alpha_3, \overline{y}^{\alpha}(t_i) = \alpha_4, \overline{y}^{\alpha}(t_{i+1}) = \alpha_5. \end{cases} \tag{3.4}$$

and

$$\begin{cases} \underline{y}^{\alpha}(t_{i+2}) = \underline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+2}} \underline{f}^{\alpha}(t, y(t))dt \\ \qquad \approx \underline{y}^{\alpha}(t_{i-1}) + h[\frac{\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + \underline{f}^{\alpha}(t_{i+2}, y(t_{i+2}))}{2} + \underline{f}^{\alpha}(t_i, y(t_i)) + \underline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+2}} \overline{f}^{\alpha}(t, y(t))dt \\ \qquad \approx \overline{y}^{\alpha}(t_{i-1}) + h[\frac{\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + \overline{f}^{\alpha}(t_{i+2}, y(t_{i+2}))}{2} + \overline{f}^{\alpha}(t_i, y(t_i)) + \overline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0, \underline{y}^{\alpha}(t_i) = \alpha_1, \underline{y}^{\alpha}(t_{i+1}) = \alpha_2, \overline{y}^{\alpha}(t_{i-1}) = \alpha_3, \overline{y}^{\alpha}(t_i) = \alpha_4, \overline{y}^{\alpha}(t_{i+1}) = \alpha_5. \end{cases} \tag{3.5}$$

where $h = \frac{t_{i+2}-t_{i-1}}{3}$.

4 The trapezoidal method

The following algorithm is based on (3.4) as a predictor and (3.5) as a corrector.

Algorithm (Hybrid predictor-corrector method). To approximate the solution of following fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \quad t_0 \le t \le T, \\ \underline{y}^{\alpha}(t_0) = \alpha_0, \underline{y}^{\alpha}(t_1) = \alpha_1, \underline{y}^{\alpha}(t_2) = \alpha_2, \\ \overline{y}^{\alpha}(t_0) = \alpha_3, \overline{y}^{\alpha}(t_1) = \alpha_4, \overline{y}^{\alpha}(t_2) = \alpha_5. \end{cases}$$

positive integer N is chosen

Step 1. Let $h = \frac{T-t_0}{N}$,

$$\underline{w}^{\alpha}(t_0) = \alpha_0, \underline{w}^{\alpha}(t_1) = \alpha_1, \underline{w}^{\alpha}(t_2) = \alpha_2,$$

$$\overline{w}^{\alpha}(t_0) = \alpha_3, \overline{w}^{\alpha}(t_1) = \alpha_4, \overline{w}^{\alpha}(t_2) = \alpha_5.$$

Step 2. Let i = 1.
Step 3. Let $w^{(0)}(t_{i+2}) = (\underline{w}^{(0)\alpha}(t_{i+2}), \overline{w}^{(0)\alpha}(t_{i+2}))$ i.e.

$$\begin{cases} \underline{w}^{(0)\alpha}(t_{i+2}) = \underline{w}^{\alpha}(t_{i-1}) + h\underline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) + 2h\underline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \\ \overline{w}^{(0)\alpha}(t_{i+2}) = \overline{w}^{\alpha}(t_{i-1}) + h\overline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) + 2h\overline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \end{cases}$$

Step 4. Let $t_{i+2} = t_0 + (i+2)h$.
Step 5. Let

$$\begin{cases} \underline{w}^{\alpha}(t_{i+2}) = \underline{w}^{\alpha}(t_{i-1}) + h[\frac{\underline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) + \underline{f}^{\alpha}(t_{i+2}, w^{(0)}(t_{i+2}))}{2} + \underline{f}^{\alpha}(t_i, w(t_i)) + \underline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \\ \overline{w}^{\alpha}(t_{i+2}) = \overline{w}^{\alpha}(t_{i-1}) + h[\frac{\overline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) + \overline{f}^{\alpha}(t_{i+2}, w^{(0)}(t_{i+2}))}{2} + \overline{f}^{\alpha}(t_i, w(t_i)) + \overline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \end{cases}$$

Step 6. $i = i + 1$.
Step 7. if $i \le N - 2$ go to step 3.
Step 8. algorithm will be completed and $(\underline{w}^{\alpha}(T), \overline{w}^{\alpha}(T))$ approximates real value of $(\underline{Y}^{\alpha}(T), \overline{Y}^{\alpha}(T))$.

5 Convergence

From Eq.(3.5), the polygon curves

$$\underline{y}(t,h,\alpha) = \left\{[t_0, \underline{y}_0(\alpha)], [t_1, \underline{y}_1(\alpha)], \cdots, [t_N, \underline{y}_N(\alpha)]\right\},$$

$$\overline{y}(t,h,\alpha) = \left\{[t_0, \overline{y}_0(\alpha)], [t_1, \overline{y}_1(\alpha)], \cdots, [t_N, \overline{y}_N(\alpha)]\right\},$$

are the approximates of $\underline{Y}(t,\alpha)$ and $\overline{Y}(t,\alpha)$, respectively, over the interval $t_0 \le t \le t_N$. The following lemma will be applied to show convergence of these approximates, i.e.

$$\lim_{h\to 0} \underline{y}(t,h,\alpha) = \underline{Y}(t,\alpha), \quad \lim_{h\to 0} \overline{y}(t,h,\alpha) = \overline{Y}(t,\alpha).$$

Lemma 5.1. Let a sequence of numbers $\{w_n\}_{n=0}^{N}$ satisfy:

$$|w_{n+2}| \le A\,|w_{n-1}| + B\,|w_n| + C\,|w_{n+1}| + D, \quad 1 \le n \le N-2$$

for some given positive constants A, B, C and D. Then

$$\begin{aligned}
|w_{n+2}| \le{}& \beta\,|w_0| + \gamma\,|w_1| + \delta\,|w_2| + (C^{n-1} + C^{n-1} + \cdots + C + 1)D \\
&+[(n-2)C^{n-3} + (n-3)C^{n-4} + \cdots + 2C + 1]BD \\
&+[(n-3)C^{n-4} + (n-4)C^{n-5} + \cdots + 2C + 1]AD \\
&+[\zeta_1 C^{n-5} + \zeta_2 C^{n-6} + \cdots + \zeta_{n-5}C + \zeta_{n-4}]B^2 D \\
&+[\lambda_1 C^{n-6} + \lambda_2 C^{n-7} + \cdots + \lambda_{n-6}C + \lambda_{n-5}]ABD \\
&+[\mu_1 C^{n-7} + \mu_2 C^{n-8} + \cdots + \mu_{n-7}C + \mu_{n-6}]A^2 D + \cdots,
\end{aligned}$$

where $\beta, \gamma, \delta, \zeta_s, \lambda_t, \mu_k$ are constants for all s, t and k.

The proof, by using mathematical induction is straightforward.

Theorem 5.1. For arbitrary fixed $\alpha : 0 \le \alpha \le 1$, the trapezoidal rule approximates of Eq. (3.5) converge to the exact solutions $\underline{Y}(t,a), \overline{Y}(t,\alpha)$ for $\underline{Y}, \overline{Y} \in C^3[t_0, T]$.

Proof. It is sufficient to show

$$\lim_{h\to 0} \underline{y}_N(\alpha) = \underline{Y}(T,\alpha), \quad \lim_{h\to 0} \overline{y}_N(\alpha) = \underline{Y}(T,\alpha).$$

By using exact value the following results will be obtained:

$$\begin{cases} \underline{Y}_{n+2}(\alpha) = \underline{Y}_{n-1}(\alpha) + \frac{h}{2} f(t_{n-1}, \underline{Y}_{n-1}(\alpha)) + \frac{h}{2} f(t_{n+2}, \underline{Y}_{n+2}(\alpha)) + hf(t_n, \underline{Y}_n(\alpha)) \\ \qquad\qquad +hf(t_{n+1}, \underline{Y}_{n+1}(\alpha)) - \frac{h^3}{4} \underline{Y}'''(\xi_n), \\ \overline{Y}_{n+2}(\alpha) = \overline{Y}_{n-1}(\alpha) + \frac{h}{2} f(t_{n-1}, \overline{Y}_{n-1}(\alpha)) + \frac{h}{2} f(t_{n+2}, \overline{Y}_{n+2}(\alpha)) + hf(t_n, \overline{Y}_n(\alpha)) \\ \qquad\qquad +hf(t_{n+1}, \overline{Y}_{n+1}(\alpha)) - \frac{h^3}{4} \overline{Y}'''(\zeta_n), \end{cases}$$

where $t_{n-1} \le \xi_n, \zeta_n \le t_{n+2}$. Consequently

$$\begin{aligned} \underline{Y}_{n+2}(\alpha) - \underline{y}_{n+2}(\alpha) &= \underline{Y}_{n-1}(\alpha) - \underline{y}_{n-1}(\alpha) + \tfrac{h}{2}(f(t_{n-1}, \underline{Y}_{n-1}(\alpha)) - f(t_{n-1}, \underline{y}_{n-1}(\alpha))) \\ &\quad + \tfrac{h}{2}(f(t_{n+2}, \underline{Y}_{n+2}(\alpha)) - f(t_{n+2}, \underline{y}_{n+2}(\alpha))) + h(f(t_n, \underline{Y}_n(\alpha)) - f(t_n, \underline{y}_n(\alpha))) \\ &\quad + h(f(t_{n+1}, \underline{Y}_{n+1}(\alpha)) - f(t_{n+1}, \underline{y}_{n+1}(\alpha))) - \tfrac{1}{4} h^3 \underline{Y}'''(\xi_n), \\ \overline{Y}_{n+2}(\alpha) - \overline{y}_{n+2}(\alpha) &= \overline{Y}_{n-1}(\alpha) - \overline{y}_{n-1}(\alpha) + \tfrac{h}{2}(f(t_{n-1}, \overline{Y}_{n-1}(\alpha)) - f(t_{n-1}, \overline{y}_{n-1}(\alpha))) \\ &\quad + \tfrac{h}{2}(f(t_{n+2}, \overline{Y}_{n+2}(\alpha)) - f(t_{n+2}, \overline{y}_{n+2}(\alpha))) + h(f(t_n, \overline{Y}_n(\alpha)) - f(t_n, \overline{y}_n(\alpha))) \\ &\quad + h(f(t_{n+1}, \overline{Y}_{n+1}(\alpha)) - f(t_{n+1}, \overline{y}_{n+1}(\alpha))) - \tfrac{1}{4} h^3 \overline{Y}'''(\zeta_n). \end{aligned}$$

Denote $w_n = \underline{Y}_n(\alpha) - \underline{y}_n(\alpha), \quad v_n = \overline{Y}_n(\alpha) - \overline{y}_n(\alpha)$. Then

$$|w_{n+2}| \le \frac{2 + hL_1}{2 - hL_2} |w_{n-1}| + \frac{2hL_3}{2 - hL_2} |w_n| + \frac{2hL_4}{2 - hL_2} |w_{n+1}| + \frac{h^3}{4 - 2hL_2} \underline{M},$$

$$|v_{n+2}| \le \frac{2 + hL_5}{2 - hL_6} |v_{n-1}| + \frac{2hL_7}{2 - hL_6} |v_n| + \frac{2hL_8}{2 - hL_6} |v_{n+1}| + \frac{h^3}{4 - 2hL_6} \overline{M}.$$

Where $\underline{M} = \max_{t_0 \le t \le T} \left| \underline{Y}'''(t, \alpha) \right|$ and $\overline{M} = \max_{t_0 \le t \le T} \left| \overline{Y}'''(t, \alpha) \right|$.

Set $L = max\{L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8\} < \frac{2}{h}$, then

$$|w_{n+2}| \le \frac{2 + hL}{2 - hL} |w_{n-1}| + \frac{2hL}{2 - hL} |w_n| + \frac{2hL}{2 - hL} |w_{n+1}| + \frac{h^3}{4 - 2hL} \underline{M},$$

$$|v_{n+2}| \le \frac{2 + hL}{2 - hL} |v_{n-1}| + \frac{2hL}{2 - hL} |v_n| + \frac{2hL}{2 - hL} |v_{n+1}| + \frac{h^3}{4 - 2hL} \overline{M}.$$

are obtained, so by Lemma (5.1), since $w_0 = v_0 = 0$, $w_1 = v_1 = 0$ and $w_2 = v_2 = 0$ we have:

$$|w_{n+2}| \leq \frac{(\frac{2hL}{2-hL})^n-1}{\frac{2hL}{2-hL}-1}\times\frac{h^3}{4-2hL}\underline{M}+[(n-2)(\frac{2hL}{2-hL})^{n-3}+(n-3)(\frac{2hL}{2-hL})^{n-4}+\cdots+2\times\frac{2hL}{2-hL}+1]\ \frac{2hL}{2-hL}\times\frac{h^3}{4-2hL}\underline{M}$$
$$+[(n-3)(\frac{2hL}{2-hL})^{n-4}+(n-4)(\frac{2hL}{2-hL})^{n-5}+\cdots+2\times\frac{2hL}{2-hL}+1]\ \frac{2+hL}{2-hL}\times\frac{h^3}{4-2hL}\underline{M}$$
$$+[\zeta_1(\frac{2hL}{2-hL})^{n-5}+\zeta_2(\frac{2hL}{2-hL})^{n-6}+\cdots+\zeta_{n-5}\times\frac{2hL}{2-hL}+\zeta_{n-4}](\frac{2hL}{2-hL})^2\ \times\frac{h^3}{4-2hL}\underline{M}$$
$$+[\lambda_1(\frac{2hL}{2-hL})^{n-6}+\lambda_2(\frac{2hL}{2-hL})^{n-7}+\cdots+\lambda_{n-6}\times\frac{2hL}{2-hL}+\lambda_{n-5}]\frac{2+hL}{2-hL}\times(\frac{2hL}{2-hL})\ \times\frac{h^3}{4-2hL}\underline{M}$$
$$+[\mu_1(\frac{2hL}{2-hL})^{n-7}+\mu_2(\frac{2hL}{2-hL})^{n-8}+\cdots+\mu_{n-7}\times\frac{2hL}{2-hL}+\mu_{n-6}](\frac{2+hL}{2-hL})^2\times\frac{h^3}{4-2hL}\underline{M}+\ldots,$$

$$|v_{n+2}| \leq \frac{(\frac{2hL}{2-hL})^n-1}{\frac{2hL}{2-hL}-1}\times\frac{h^3}{4-2hL}\overline{M}+[(n-2)(\frac{2hL}{2-hL})^{n-3}+(n-3)(\frac{2hL}{2-hL})^{n-4}+\cdots+2\times\frac{2hL}{2-hL}+1]\ \frac{2hL}{2-hL}\times\frac{h^3}{4-2hL}\overline{M}$$
$$+[(n-3)(\frac{2hL}{2-hL})^{n-4}+(n-4)(\frac{2hL}{2-hL})^{n-5}+\cdots+2\times\frac{2hL}{2-hL}+1]\ \frac{2+hL}{2-hL}\times\frac{h^3}{4-2hL}\overline{M}$$
$$+[\zeta_1(\frac{2hL}{2-hL})^{n-5}+\zeta_2(\frac{2hL}{2-hL})^{n-6}+\cdots+\zeta_{n-5}\times\frac{2hL}{2-hL}+\zeta_{n-4}](\frac{2hL}{2-hL})^2\ \times\frac{h^3}{4-2hL}\overline{M}$$
$$+[\lambda_1(\frac{2hL}{2-hL})^{n-6}+\lambda_2(\frac{2hL}{2-hL})^{n-7}+\cdots+\lambda_{n-6}\times\frac{2hL}{2-hL}+\lambda_{n-5}]\frac{2+hL}{2-hL}\times(\frac{2hL}{2-hL})\ \times\frac{h^3}{4-2hL}\overline{M}$$
$$+[\mu_1(\frac{2hL}{2-hL})^{n-7}+\mu_2(\frac{2hL}{2-hL})^{n-8}+\cdots+\mu_{n-7}\times\frac{2hL}{2-hL}+\mu_{n-6}](\frac{2+hL}{2-hL})^2\times\frac{h^3}{4-2hL}\overline{M}+\ldots$$

If $h \to 0$ then $w_{n+2} \to 0, v_{n+2} \to 0$, which concludes the proof.

6 Example

Example 6.1[3,6]. Consider the initial value problem

$$y'(t) = -y(t) + t + 1,$$

$$y(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha),$$

$$y(0.01) = (0.01 + (0.985 + 0.015\alpha)e^{-0.01} - (1-\alpha)0.025e^{0.01},$$
$$0.01 + (0.985 + 0.015\alpha)e^{-0.01} + (1-\alpha)0.025e^{0.01}),$$

$$y(0.02) = (0.02 + (0.985 + 0.015\alpha)e^{-0.02} - (1-\alpha)0.025e^{0.02},$$

$$0.02 + (0.985 + 0.015\alpha)e^{-0.02} + (1-\alpha)0.025e^{0.02}).$$

The exact solution at $t = 0.1$ is given by

$$Y(0.1,\alpha) = (0.1 + (0.985 + 0.015\alpha)e^{-0.1} - (1-\alpha)0.025e^{0.1},$$

$$0.1 + (0.985 + 0.015\alpha)e^{-0.1} + (1-\alpha)0.025e^{0.1}).$$

By using the trapezoidal method with N = 10 the following results are obtained:

α	$\underline{y}$	$\underline{Y}$	Error
0	0.96363561497119	0.96363558381353	-3.115766045169721e-008
0.1	0.96775579852993	0.96775576723577	-3.129415648839284e-008
0.2	0.97187598208867	0.97187595065802	-3.143065274713308e-008
0.3	0.97599616564741	0.97599613408026	-3.156714889485102e-008
0.4	0.98011634920615	0.98011631750250	-3.170364526461356e-008
0.5	0.98423653276489	0.98423650092474	-3.184014152335379e-008
0.6	0.98835671632363	0.98835668434699	-3.197663767107173e-008
0.7	0.99247689988236	0.99247686776923	-3.211313415185657e-008
0.8	0.99659708344110	0.99659705119147	-3.224963029957451e-008
0.9	1.00071726699984	1.00071723461372	-3.238612644729244e-008
1.0	1.00483745055858	1.00483741803596	-3.252262281705498e-008
α	$\overline{y}$	$\overline{Y}$	Error
0	1.01889344915856	1.01889412971731	6.805587482450193e-007
0.1	1.01748777493959	1.01748845854918	6.836095900464301e-007
0.2	1.01608210072061	1.01608278738104	6.866604309596625e-007
0.3	1.01467642650163	1.01467711621291	6.897112725390286e-007
0.4	1.01327075228266	1.01327144504477	6.927621138963502e-007
0.5	1.01186507806368	1.01186577387664	6.958129550316272e-007
0.6	1.01045940384470	1.01046010270850	6.988637963889488e-007
0.7	1.00905372962573	1.00905443154037	7.019146377462704e-007
0.8	1.00764805540675	1.00764876037223	7.049654791035920e-007
0.9	1.00624238118777	1.00624308920409	7.080163202388690e-007
1.0	1.00483670696880	1.00483741803596	7.110671618182352e-007

The results of Example 6.1 are shown in Fig. 1.

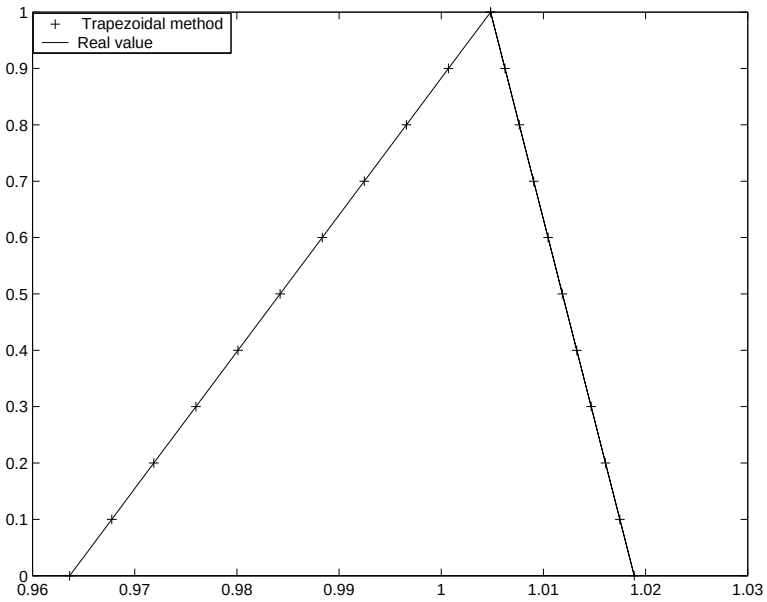


Fig.1 The results of Example 6.1

Example 6.2[3,6]. Consider the initial value problem

$y'(t) = -y(t)$,

$y(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha)$,

$y(0.01) = ((0.985+0.015\alpha)e^{-0.01}-(1-\alpha)0.025e^{0.01}, (0.985+0.015\alpha)e^{-0.01}+(1-\alpha)0.025e^{0.01})$,

$y(0.02) = ((0.985+0.015\alpha)e^{-0.02}-(1-\alpha)0.025e^{0.02}, (0.985+0.015\alpha)e^{-0.02}+(1-\alpha)0.025e^{0.02})$.

The exact solution at $t = 0.1$ is given by

$Y(0.1, \alpha) = ((0.985+0.015\alpha)e^{-0.1}-(1-\alpha)0.025e^{0.1}, (0.985+0.015\alpha)e^{-0.1}+(1-\alpha)0.025e^{0.1})$.

By using the trapezoidal method with N = 10 the following results are obtained:

α	$\underline{y}$	$\underline{Y}$	Error
0	0.86363561497119	0.86363558381353	-3.115766045169721e-008
0.1	0.86775579852993	0.86775576723577	-3.129415671043745e-008
0.2	0.87187598208867	0.87187595065802	-3.143065274713308e-008
0.3	0.87599616564741	0.87599613408026	-3.156714911689562e-008
0.4	0.88011634920615	0.88011631750250	-3.170364526461356e-008
0.5	0.88423653276489	0.88423650092474	-3.184014163437610e-008
0.6	0.88835671632363	0.88835668434699	-3.197663800413864e-008
0.7	0.89247689988236	0.89247686776923	-3.211313415185657e-008
0.8	0.89659708344110	0.89659705119147	-3.224963029957451e-008
0.9	0.90071726699984	0.90071723461372	-3.238612655831474e-008
1.0	0.90483745055858	0.90483741803596	-3.252262292807728e-008
α	$\overline{y}$	$\overline{Y}$	Error
0	0.91889344915856	0.91889412971731	6.805587485780862e-007
0.1	0.91748777493959	0.91748845854918	6.836095900464301e-007
0.2	0.91608210072061	0.91608278738104	6.866604310706848e-007
0.3	0.91467642650163	0.91467711621291	6.897112724280063e-007
0.4	0.91327075228266	0.91327144504477	6.927621138963502e-007
0.5	0.91186507806368	0.91186577387664	6.958129550316272e-007
0.6	0.91045940384470	0.91046010270850	6.988637963889488e-007
0.7	0.90905372962573	0.90905443154037	7.019146376352481e-007
0.8	0.90764805540675	0.90764876037223	7.049654788815474e-007
0.9	0.90624238118777	0.90624308920409	7.080163202388690e-007
1.0	0.90483670696880	0.90483741803596	7.110671615961905e-007

The results of Example 6.2 are shown in Fig. 2.

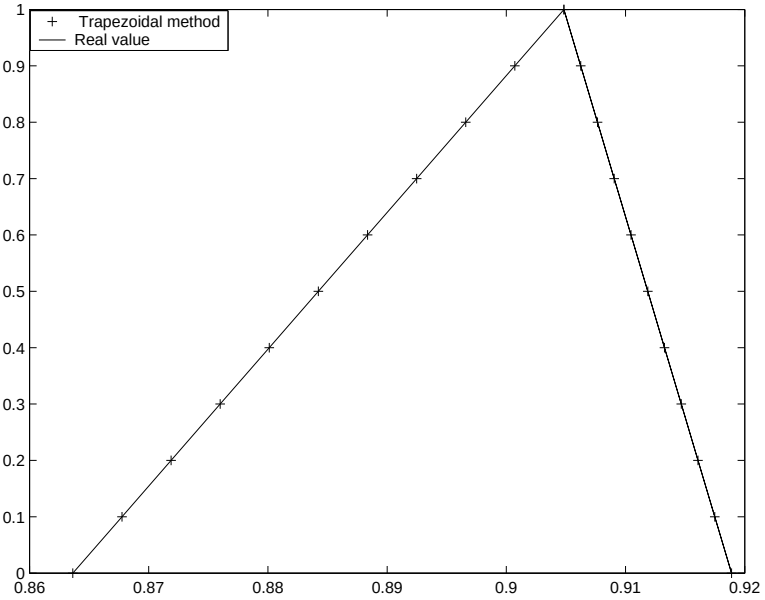


Fig.2 The results of Example 6.2

7 Conclusion

In this paper, The Euler method and the trapezoidal rule are combined, and a trapezoidal method is given. The correctness of the proposed method is shown by some examples. It is worth mentioning that the trapezoidal three-step method can naturally be generalized to trapezoidal m-step methods.

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Fuzzy approximate of homomorphisms

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Abstract. In this paper, we approximate homomorphisms in fuzzy sense by using the fixed point method and by using the direct method.

Keywords: Fuzzy approximate, homomorphisms, fuzzy normed space, Banach algebra, fixed point, functional equation.

1. Introduction and preliminaries

The theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view [1, 7, 8]. Following Cheng and Mordeson [4], Saadati and Vaezpour [12] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [9] and investigated some properties of fuzzy normed spaces [13]. We use the definition of fuzzy normed spaces given in [1, 12] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the Cauchy functional equation in the fuzzy normed algebra setting.

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;

(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

Definition 1.2. (1) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

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(2) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [2]).

Definition 1.3. Let X be an algebra and (X, N) a fuzzy normed space.

(1) The fuzzy normed space (X, N) is called a *fuzzy normed algebra* if

$$N(xy, st) \geq N(x, s) \cdot N(y, t)$$

for all $x, y \in X$ and all positive real numbers s and t.

(2) A complete fuzzy normed algebra is called a *fuzzy Banach algebra.*

Example 1.4. *Let* $(X, \|\cdot\|)$ *be a normed algebra. Let*

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X. \end{cases}$$

Then $N(x,t)$ *is a fuzzy norm on* X *and* $(X, N(x,t))$ *is a fuzzy normed algebra.*

Definition 1.5. Let (X, N_X) and (Y, N) be fuzzy normed algebras. Then a multiplicative $\mathbb{R}$-linear mapping $H : (X, N_X) \to (Y, N)$ is called a *fuzzy algebra homomorphism.*

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called a quadratic functional equation. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [15] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.6. [3, 6] *Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then for each given element* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

Throughout this paper, assume that (X, N_X) is a fuzzy normed algebra and that (Y, N) is a fuzzy Banach algebra.

2. Hyers-Ulam stability of the Cauchy functional equation in fuzzy Banach algebras: fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of the Cauchy functional equation in fuzzy Banach algebras.

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{2}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$N\left(f(rx+ry) - rf(x) - rf(y), t\right) \geq \frac{t}{t+\varphi(x,y)}, \tag{2.1}$$

$$N\left(f(xy) - f(x)f(y), t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{2.2}$$

for all $x, y \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Then $H(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a fuzzy algebra homomorphism $H : X \to Y$ such that

$$N\left(f(x) - H(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $r = 1$ and $y = x$ in (2.1), we get

$$N\left(f(2x) - 2f(x), t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,x)},\ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see the proof of [10, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N\left(g(x) - h(x), \varepsilon t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N\left(Jg(x) - Jh(x), L\varepsilon t\right) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} \\ &= \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.6, there exists a mapping $H : X \to Y$ satisfying the following:

(1) H is a fixed point of J, i.e.,

$$H\left(\frac{x}{2}\right) = \frac{1}{2}H(x) \tag{2.5}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - H(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x)$$

for all $x \in X$;

(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.3) holds.

By (2.1),

$$N\left(2^k f\left(\frac{rx + ry}{2^k}\right) - 2^k r f\left(\frac{x}{2^k}\right) - 2^k r f\left(\frac{y}{2^k}\right), 2^k t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $r \in \mathbb{R}$. So

$$N\left(2^k f\left(\frac{rx + ry}{2^k}\right) - 2^k r f\left(\frac{x}{2^k}\right) - 2^k r f\left(\frac{y}{2^k}\right), t\right) \geq \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Since $\lim_{k\to\infty} \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}\varphi(x,y)} = 1$ for all $x, y \in X$, all $t > 0$ and all $r \in \mathbb{R}$,

$$N\left(H(rx + ry) - rH(x) - rH(y), t\right) = 1$$

for all $x, y \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Thus $H(rx + ry) - rH(x) - rH(y) = 0$. So the mapping $H : X \to Y$ is additive and $\mathbb{R}$-linear.

By (2.2),

$$N\left(4^k f\left(\frac{xy}{4^k}\right) - 2^k f\left(\frac{x}{2^k}\right) \cdot 2^k f\left(\frac{y}{2^k}\right), 4^k t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $r \in \mathbb{R}$. So

$$N\left(4^k f\left(\frac{xy}{4^k}\right) - 2^k f\left(\frac{x}{2^k}\right) \cdot 2^k f\left(\frac{y}{2^k}\right), t\right) \geq \frac{\frac{t}{4^k}}{\frac{t}{4^k} + \frac{L^k}{2^k}\varphi(x, y)}$$

for all $x, y \in X$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{\frac{t}{4^k}}{\frac{t}{4^k}+\frac{L^k}{2^k}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(H(xy) - H(x)H(y), t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus $H(xy) - H(x)H(y) = 0$. So the mapping $H : X \to Y$ is a fuzzy algebra homomorphism, as desired. □

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.1) and (2.2). Then $H(x) := N\text{-}\lim_{k\to\infty} \frac{1}{2^k} f\left(2^k x\right)$ exists for each $x \in X$ and defines a fuzzy algebra homomorphism $H : X \to Y$ such that

$$N\left(f(x) - H(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g\left(2x\right)$$

for all $x \in X$.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Hyers-Ulam stability of the Cauchy functional equation in fuzzy Banach algebras: direct method

In this section, we prove the Hyers-Ulam stability of the Cauchy functional equation in fuzzy Banach algebras by using the direct method.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping such that

$$\lim_{t\to\infty} N\left(f(rx + ry) - rf(x) - rf(y), t\widetilde{\varphi}(x, y)\right) = 1 \tag{3.2}$$

uniformly on X^2 for each $r \in \mathbb{R}$, and

$$\lim_{t\to\infty} N\left(f(xy) - f(x)f(y), t\widetilde{\varphi}(x, y)\right) = 1 \tag{3.3}$$

uniformly on X^2. Then $H(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a fuzzy algebra homomorphism $H : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N\left(f(x + y) - f(x) - f(y), \delta\widetilde{\varphi}(x, y)\right) \geq \alpha \tag{3.4}$$

for all $x, y \in X$, then

$$N\left(f(x) - H(x), \delta\widetilde{\varphi}(x, x)\right) \geq \alpha \tag{3.5}$$

for all $x \in X$.

Furthermore, the fuzzy algebra homomorphism $H : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - H(x), t\widetilde{\varphi}(x, x)) = 1 \tag{3.6}$$

uniformly on X.

Proof. Let $r = 1$ in (3.2). For a given $\varepsilon > 0$, by (3.2), we can find some $t_0 > 0$ such that

$$N\left(f(x+y) - f(x) - f(y), t\varphi(x, y)\right) \geq 1 - \varepsilon \tag{3.7}$$

for all $t \geq t_0$. Letting $y = x$ in (3.7), we get

$$N(f(2x) - 2f(x), t\varphi(x, x) \geq 1 - \epsilon \tag{3.8}$$

for all $x \in X$.

By induction on n, we will show that

$$N\left(f(x) - 2^n f\left(\frac{x}{2^n}\right), t\sum_{k=1}^{n} 2^{k-1}\varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right)\right) \geq 1 - \varepsilon \tag{3.9}$$

for all $t \geq t_0$, all $x \in X$ and all $n \in \mathbb{N}$.

It follows from (3.8) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\varphi\left(\frac{x}{2}, \frac{x}{2}\right)\right) \geq 1 - \epsilon$$

for all $x \in X$.

Thus we get (3.9) for $n = 1$.

Assume that (3.9) holds for $n \in \mathbb{N}$. Then

$$\begin{aligned}
&N\left(f(x) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right), t\sum_{k=1}^{n+1} 2^{k-1}\varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right)\right) \\
&\geq \min\left\{N\left(f(x) - 2^n f\left(\frac{x}{2^n}\right), t_0\sum_{k=1}^{n} 2^{k-1}\varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right)\right),\right. \\
&\qquad \left. N\left(2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right), 2^n t_0\varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right)\right)\right\} \\
&\geq \min\{1 - \varepsilon, 1 - \varepsilon\} = 1 - \varepsilon.
\end{aligned}$$

This completes the induction argument. Letting $t = t_0$ and replacing n and x by p and $\frac{x}{2^n}$ in (3.9), respectively, we get

$$N\left(2^n f\left(\frac{x}{2^n}\right) - 2^{n+p} f\left(\frac{x}{2^{n+p}}\right), 2^n t_0\sum_{k=1}^{p} 2^{k-1}\varphi\left(\frac{x}{2^{n+k}}, \frac{x}{2^{n+k}}\right)\right) \geq 1 - \varepsilon \tag{3.10}$$

for all integers $n \geq 0, p > 0$.

It follows from (3.1) and the equality

$$\sum_{k=1}^{p} 2^{n+k-1}\varphi\left(\frac{x}{2^{n+k}}, \frac{x}{2^{n+k}}\right) = \sum_{k=n+1}^{n+p} 2^{k-1}\varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right)$$

that for a given $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$t_0\sum_{k=n+1}^{n+p} 2^{k-1}\varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right) < \delta$$

for all $n \geq n_0$ and $p > 0$. Now we deduce from (3.10) that

$$\begin{aligned} & N\left(2^n f\left(\frac{x}{2^n}\right) - 2^{n+p} f\left(\frac{x}{2^{n+p}}\right), \delta\right) \\ & \geq N\left(2^n f\left(\frac{x}{2^n}\right) - 2^{n+p} f\left(\frac{x}{2^{n+p}}\right), 2^n t_0 \sum_{k=1}^{p} 2^{k-1} \varphi\left(\frac{x}{2^{n+k}}, \frac{x}{2^{n+k}}\right)\right) \\ & \geq 1 - \varepsilon \end{aligned}$$

for each $n \geq n_0$ and all $p > 0$. Thus the sequence $\left\{2^k f\left(\frac{x}{2^k}\right)\right\}$ is Cauchy in Y. Since Y is a fuzzy Banach space, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges to some $H(x) \in Y$. So we can define a mapping $H : X \to Y$ by $H(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$, namely, for each $t > 0$ and $x \in X$,

$$\lim_{n\to\infty} N\left(2^n f\left(\frac{x}{2^n}\right) - H(x), t\right) = 1.$$

Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$, there is an $n_1 > n_0$ such that $t_0 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \frac{t}{4}$ for all $n \geq n_1$. Hence for each $k \geq n_1$, we have

$$\begin{aligned} N(H(x+y) - H(x) - H(y), t) \geq \min\Big\{ & N\left(H(x+y) - 2^k f\left(\frac{x+y}{2^k}\right), \frac{t}{4}\right), \\ & N\left(H(x) - 2^k f\left(\frac{x}{2^k}\right), \frac{t}{4}\right), N\left(H(y) - 2^k f\left(\frac{y}{2^k}\right), \frac{t}{4}\right), \\ & N\left(2^k f\left(\frac{x+y}{2^k}\right) - 2^k f\left(\frac{x}{2^k}\right) - 2^k f\left(\frac{y}{2^k}\right), \frac{t}{4}\right)\Big\}. \end{aligned}$$

The first three terms on the right-hand side of the above inequality tend to 1 as $k \to \infty$, and the last term is greater than

$$N\left(2^k f\left(\frac{x+y}{2^k}\right) - 2^k f\left(\frac{x}{2^k}\right) - 2^k f\left(\frac{y}{2^k}\right), t_0 2^k \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right),$$

which is greater than or equal to $1 - \varepsilon$. Thus

$$N(H(x+y) - H(x) - H(y), t) \geq 1 - \varepsilon$$

for all $t > 0$. Since $N\left(H(x+y) - H(x) - H(y), t\right) = 1$ for all $t > 0$, by (N_2),

$$H(x+y) - H(x) - H(y) = 0$$

for all $x \in X$. Hence the mapping $H : X \to Y$ is additive.

Let $y = 0$ in (3.2). For a given $\varepsilon > 0$, by (3.2), we can find some $t_0 > 0$ such that

$$N\left(f(rx) - rf(x), t\varphi(x, 0)\right) \geq 1 - \varepsilon \tag{3.11}$$

for all $t \geq t_0$. It follows from (3.11) that $H(rx) = rH(x)$ for all $r \in \mathbb{R}$ and all $x \in X$.

Similarly, it follows from (3.3) that $H(xy) = H(x)H(y)$ for all all $x, y \in X$.

Now let for some positive δ and α, (3.4) holds. Let

$$\varphi_n(x, y) := \sum_{k=1}^{n} 2^k \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right)$$

for all $x, y \in X$. Let $x \in X$. By the same reasoning as in the beginning of the proof, one can deduce from (3.4) that

$$N\left(f(x) - 2^n f(\frac{x}{2^n}), \delta \sum_{k=1}^{n} 2^{n-k} \varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right)\right) \geq \alpha \tag{3.12}$$

for all positive integers n. Let $t > 0$. We have

$$\begin{aligned} & N\left(f(x) - H(x), \delta\varphi_n(x, x) + t\right) \\ & \geq \min\left\{N\left(f(x) - 2^n f\left(\frac{x}{2^n}\right), \delta\varphi_n(x, x)\right), N\left(2^n f\left(\frac{x}{2^n}\right) - H(x), t\right)\right\} \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) and the fact that $\lim_{n\to\infty} N\left(2^n f\left(\frac{x}{2^n}\right) - H(x), t\right) = 1$, we observe that

$$N\left(f(x) - H(x), \delta\varphi_n(x,x) + t\right) \geq \alpha$$

for large enough $n \in \mathbb{N}$. Thanks to the continuity of the function $N(f(x) - H(x), \cdot)$, we see that

$$N\left(f(x) - H(x), \delta\widetilde{\varphi}(x,x) + t\right) \geq \alpha.$$

Letting $t \to 0$, we conclude that

$$N\left(f(x) - H(x), \delta\widetilde{\varphi}(x,x)\right) \geq \alpha.$$

To end the proof, it remains to prove the uniqueness assertion. Let T be another additive mapping satisfying (3.5) and (3.6). Fix $c > 0$. Given $\varepsilon > 0$, by (3.6) for Q and T, we can find some $t_0 > 0$ such that

$$\begin{aligned} N\left(f(x) - H(x), t\widetilde{\varphi}(x,x)\right) &\geq 1-\varepsilon, \\ N\left(f(x) - T(x), t\widetilde{\varphi}(x,x)\right) &\geq 1-\varepsilon \end{aligned}$$

for all $x \in X$ and all $t \geq t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$t_0 \sum_{k=n}^{\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right) < \frac{c}{2}$$

for all $n \geq n_0$. Since

$$\begin{aligned} \sum_{k=n}^{\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right) &= 2^n \sum_{k=n}^{\infty} 2^{(k-n)} \varphi\left(2^{n-k}\frac{x}{2^n}, 2^{n-k}\frac{x}{2^n}\right) \\ &= 2^n \sum_{l=0}^{\infty} 2^l \varphi\left(\frac{1}{2^l}\frac{x}{2^n}, \frac{1}{2^l}\frac{x}{2^n}\right) = 2^n \widetilde{\varphi}\left(2^{-l}\frac{x}{2^n}, 2^{-l}\frac{x}{2^n}\right), \end{aligned}$$

we have

$$\begin{aligned} N(H(x) - T(x), c) &\geq \min\left\{N\left(2^n f\left(\frac{x}{2^n}\right) - H(x), \frac{c}{2}\right), N\left(T(x) - 2^n f\left(\frac{x}{2^n}\right), \frac{c}{2}\right)\right\} \\ &= \min\left\{N\left(f\left(\frac{x}{2^n}\right) - H\left(\frac{x}{2^n}\right), \frac{2c}{2^n}\right), N\left(T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right), \frac{2c}{2^n}\right)\right\} \\ &\geq \min\left\{N\left(f\left(\frac{x}{2^n}\right) - H\left(\frac{x}{2^n}\right), 2^{-n}t_0 \sum_{k=n}^{\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right)\right),\right. \\ &\qquad \left. N\left(T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right), 2^{-n}t_0 \sum_{k=n}^{\infty} 2^k \varphi\left(\frac{x}{2^k}, \frac{x}{2^k}\right)\right)\right\} \\ &= \min\left\{N\left(f\left(\frac{x}{2^n}\right) - H\left(\frac{x}{2^n}\right), t_0\widetilde{\varphi}\left(\frac{x}{2^n}, \frac{x}{2^n}\right)\right),\right. \\ &\qquad \left. N\left(T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right), t_0\widetilde{\varphi}\left(\frac{x}{2^n}, \frac{x}{2^n}\right)\right)\right\} \\ &\geq 1-\varepsilon. \end{aligned}$$

It follows that $N(H(x) - T(x), c) = 1$ for all $c > 0$. Thus $H(x) = T(x)$ for all $x \in X$. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x,y) := \sum_{n=0}^{\infty} 2^{-n} \varphi\left(2^n x, 2^n y\right) < \infty$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* (3.2) *and* (3.3). *Then* $H(x) := N\text{-}\lim_{n\to\infty} 2^{-n} f(2^n x)$ *exists for each* $x \in X$ *and defines a fuzzy algebra homomorphism* $H : X \to Y$ *such that if for some* $\delta > 0, \alpha > 0$

$$N\left(f(x+y) - f(x) - f(y), \delta\widetilde{\varphi}(x,y)\right) \geq \alpha$$

for all $x, y \in X$, *then*

$$N\left(f(x) - H(x), \delta\widetilde{\varphi}(x,x)\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the fuzzy algebra homomorphism $H : X \to Y$ *is a unique mapping such that*

$$\lim_{t\to\infty} N\left(f(x) - H(x), t\widetilde{\varphi}(x,x)\right) = 1$$

uniformly on X.

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On the weighted q-Bernstein polynomials associated with the twisted q-Bernoulli numbers

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Abstract Recently, Kim introduced q-Bernstein polynomials and weighted q-Bernstein polynomials which are different q-Bernstein polynomials of Phillips(see [6]). In this paper we investigate some properties on the weighted q-Bernstein polynomials. From these properties, we derive some interesting new identities between the weighted q-Bernstein polynomials and the twisted q-Bernoulli numbers.

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1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number fields, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of

natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we normally assume $|q| < 1$, and if $q \in \mathbb{C}_p$, then we assume $|q-1|_p < 1$. As a q-number, we define

$$[x]_q = \frac{1-q^x}{1-q}, \text{ (see [1-19])}.$$

Note that $\lim_{q\to 1}[x]_q = x$ (see [1-19]). Let $C[0,1]$ be the set of continuous function on $[0,1]$. For $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{Z}_+$, the weighted q-Bernstein operator of order n for $f \in C[0,1]$ is defined by Kim as follows:

$$\begin{aligned} \mathbb{B}_{n,q}^{(\alpha)}(f \mid x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}^{(\alpha)}(x \mid q), \text{ (see [5, 6, 11, 15, 19])}. \end{aligned} \tag{1}$$

Here $B_{k,n}^{(\alpha)}(x \mid q) = \binom{n}{k}[x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}$ is called the weighted q-Bernstein polynomials of degree n (see [5, 6, 11, 19]). Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$, which is called the bosonic q-integral on $\mathbb{Z}_p$, is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ (see [10])}. \tag{2}$$

In [13], Kim defined the weighted q-Bernoulli numbers as follows:

$$\tilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q(q^\alpha \tilde{\beta}^{(\alpha)} + 1)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{3}$$

with usual convention about replacing $(\tilde{\beta}^{(\alpha)})^n$ by $\tilde{\beta}_{n,q}^{(\alpha)}$.

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then we have

$$q^n I_q(f_n) = I_q(f) + (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l), \text{ see [9, 10]}. \tag{4}$$

Let $C_{p^n} = \{\zeta \mid \zeta^{p^n} = 1\}$ be the cyclic group of order p^n. Then the locally constant space is constructed by

$$T_p = \lim_{n\to\infty} C_{p^n} = C_{p^\infty} = \cup_{n\geq 0} C_{p^n}, \text{ see [1-19]}. \tag{5}$$

By (2), (3) and (4), we get Witt's formula for $\tilde{\beta}_{n,q}^{(\alpha)}$ as follows: for $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,

$$\begin{aligned}\tilde{\beta}_{n,q}^{(\alpha)} &= \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x) \\ &= -\frac{n\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m\alpha+m}[m]_{q^\alpha}^{n-1} + (1-q)\sum_{m=0}^{\infty} q^m [m]_{q^\alpha}^n, \text{ (see [13])}.\end{aligned} \tag{6}$$

In this paper we study some properties of the weighted q-Bernstein polynomials. By using these properties of the weighted q-Bernstein polynomials, we give some interesting identities for the twisted q-Bernoulli numbers.

2. On the weighted q-Bernstein polynomials

In this section we assume that $0 < q < 1$. Let

$$\mathbb{P}_{n,q}^{\alpha} = \left\{ \sum_i a_i^{\alpha} [\alpha x]_q^i \mid a_i^{\alpha} \in \mathbb{R}, \alpha \in \mathbb{N} \right\}$$

be the space of q-polynomials with weight α of degree less than or equal to n.

First, we claim that the weighted q-Bernstein polynomials of degree n defined by (1) are basis for $\mathbb{P}_{n,q}^{\alpha}$.

It is easy to show that the twisted q-Bernstein polynomials of degree n generate the space of q-polynomials with weight α. That is, any q-polynomials of degree less than or equal to n with weight α can be written as a linear combination of the weight q-Bernstein polynomials of degree n.

Let $n, k \in \mathbb{Z}_+$ and $x \in [0,1]$. Then we get

$$B_{k,n}^{(\alpha)}(x \mid q) = \sum_{l=k}^{n} \binom{n}{k}\binom{l}{k}(-1)^{l-k}[x]_{q^\alpha}^k, \alpha \in \mathbb{N}. \tag{7}$$

If there exist constant $C_0^{\alpha}, C_1^{\alpha}, \ldots, C_n^{\alpha}$ such that

$$C_0^{\alpha} B_{0,n}^{(\alpha)}(x \mid q) + C_1^{\alpha} B_{1,n}^{(\alpha)}(x \mid q) + \cdots + C_n^{\alpha} B_{n,n}^{(\alpha)}(x \mid q) = 0$$

holds for all x, then we have

$$\begin{aligned} 0 &= C_0^{\alpha} B_{0,n}^{(\alpha)}(x \mid q) + C_1^{\alpha} B_{1,n}^{(\alpha)}(x \mid q) + \cdots + C_n^{\alpha} B_{n,n}^{(\alpha)}(x \mid q) \\ &= C_0^{\alpha} \sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{i}{0}[x]_{q^\alpha}^i + C_1^{\alpha}\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\binom{i}{1}[x]_{q^\alpha}^i + \cdots \\ &\quad + C_n^{\alpha}\sum_{i=n}^{n}(-1)^{i-n}\binom{n}{i}\binom{i}{n}[x]_{q^\alpha}^i \\ &= C_0^{\alpha} + \left\{\sum_{i=0}^{1} C_i^{\alpha}(-1)^{i-1}\binom{n}{1}\binom{1}{i}\right\}[x]_{q^\alpha} + \cdots + \left\{\sum_{i=0}^{n} C_i^{\alpha}(-1)^{i-n}\binom{n}{n}\binom{n}{i}\right\}[x]_{q^\alpha}^n. \end{aligned}$$

Thus, we have

$$\begin{aligned} & C_0^{\alpha} = 0 \\ & \sum_{i=0}^{1} C_i^{\alpha}(-1)^{i-1}\binom{n}{1}\binom{1}{i} = 0 \\ & \quad \vdots \quad \vdots \quad \vdots \\ & \sum_{i=0}^{n} C_i^{\alpha}(-1)^{i-n}\binom{n}{n}\binom{n}{i} = 0, \end{aligned}$$

which implies that $C_0^{\alpha} = C_1^{\alpha} = \ldots = C_n^{\alpha} = 0$.

For $P_q^{\alpha}(x) \in \mathbb{P}_{n,q}^{\alpha}$, $\mathbb{P}_{n,q}^{\alpha}$ can be written as a linear combination of the weighted q-Bernstein basis functions as follows:

$$P_q^{\alpha}(x) = C_0^{\alpha} B_{0,n}^{(\alpha)}(x \mid q) + C_1^{\alpha} B_{1,n}^{(\alpha)}(x \mid q) + \cdots + C_n^{\alpha} B_{n,n}^{(\alpha)}(x \mid q). \tag{8}$$

By (8), we get

$$P_q^{\alpha}(x) = \left(\begin{array}{c} B_{0,n}^{(\alpha)}(x \mid q), B_{1,n}^{(\alpha)}(x \mid q), \ldots, B_{n,n}^{(\alpha)}(x \mid q) \end{array} \right) \begin{pmatrix} C_0^{\alpha} \\ C_1^{\alpha} \\ \vdots \\ C_n^{\alpha} \end{pmatrix}. \tag{9}$$

From (9), we can derive the following equation.

$$P_q^{\alpha}(x) = \left(\begin{array}{c} 1, [\alpha x]_q, \ldots, [\alpha x]_q^n \end{array} \right) \begin{pmatrix} b_{0,0}^{\alpha} & 0 & \ldots & 0 \\ b_{1,0}^{\alpha} & b_{1,1}^{\alpha} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0}^{\alpha} & b_{n,1}^{\alpha} & \ldots & b_{n,n}^{\alpha} \end{pmatrix} \begin{pmatrix} C_0^{\alpha} \\ C_1^{\alpha} \\ \vdots \\ C_n^{\alpha} \end{pmatrix},$$

where the $b_{i,j}^{\alpha}$ are the coefficients of power basis with weight α that are used to determine the respective weighted q-Bernstein polynomials.

For example,

$$\begin{aligned} B_{0,2}^{(\alpha)}(x \mid q) &= 1 - \frac{2}{[\alpha]_q}[\ \alpha x]_q + \frac{1}{[\alpha]_q^2}[\alpha x]_q^2, \\ B_{1,2}^{(\alpha)}(x \mid q) &= \binom{2}{1}[x]_{q^{\alpha}}[1-x]_{q^{-\alpha}} = \frac{2}{[\alpha]_q}[\ \alpha x]_q - \frac{2}{[\alpha]_q^2}[\alpha x]_q^2, \\ B_{2,2}^{(\alpha)}(x \mid q) &= \binom{2}{2}[x]_{q^{\alpha}}^2 = \frac{1}{[\alpha]_q^2}[\alpha x]_q^2. \end{aligned}$$

In the quadratic case($n = 2$), the matrix representation is given by

$$P_q^{\alpha}(x) = \left(\ 1, [\alpha x]_q, [\alpha x]_q^2 \ \right) \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{-2}{[\alpha]_q} & \dfrac{2}{[\alpha]_q} & 0 \\ \dfrac{1}{[\alpha]_q^2} & \dfrac{-2}{[\alpha]_q^2} & \dfrac{1}{[\alpha]_q^2} \end{pmatrix} \begin{pmatrix} C_0^{\alpha} \\ C_1^{\alpha} \\ C_2^{\alpha} \end{pmatrix}.$$

Remark(see [4]). All results of this section for $q = 1$ are well known in classical case(see Bernstein Polynomials by Joy).

3. On the weighted q-Bernstein polynomials and the twisted q-Bernoulli numbers

In this section we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Let $\zeta \in T_p$. Then we consider the twisted q-Bernoulli polynomials with weighted α as follows:

$$\tilde{\beta}_{n,q,\zeta}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [y+x]_{q^{\alpha}}^n \zeta^y d\mu_q(y), \text{ for } n \in \mathbb{Z}_+. \tag{10}$$

In the special case, $x = 0$, $\tilde{\beta}_{n,q,\zeta}^{(\alpha)}(0) = \tilde{\beta}_{n,q,\zeta}^{(\alpha)}$ are call the n-th twisted q-Bernoulli numbers with weight α.

By simple calculation, we easily get

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n \zeta^x d\mu_q(x) &= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_{q^{\alpha}}^l \zeta^x d\mu_q(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \tilde{\beta}_{l,q,\zeta}^{(\alpha)}. \end{aligned} \tag{11}$$

Taking the bosonic q-integral on $\mathbb{Z}_p$ for one weighted q-Bernstein polynomials in (1), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x \mid q) \zeta^x d\mu_q(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} [x]_{q^{\alpha}}^k [1-x]_{q^{-\alpha}}^{n-k} \zeta^x d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_{q^{\alpha}}^{k+l} \zeta^x d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \tilde{\beta}_{k+l,q,\zeta}^{(\alpha)}. \end{aligned} \tag{12}$$

From (11), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x \mid q)\zeta^x d\mu_q(x) &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n-l}\zeta^x d\mu_q(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\sum_{j=0}^{n-l}\binom{n-l}{j}(-1)^j\tilde{\beta}_{j,q,\zeta}^{(\alpha)}.\end{aligned} \tag{13}$$

By comparing the coefficients on the both sides of (12) and (13), we obtain the following theorem.

Theorem 1. For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{\beta}_{k+l,q,\zeta}^{(\alpha)} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\sum_{j=0}^{n-l}\binom{n-l}{j}(-1)^j\tilde{\beta}_{j,q,\zeta}^{(\alpha)}.$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x \mid q)B_{k,n_2}^{(\alpha)}(x \mid q)\zeta^x d\mu_q(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\int_{\mathbb{Z}_p}[x]_{q^\alpha}^{2k}[1-x]_{q^{-\alpha}}^{n_1+n_2-2k}\zeta^x d\mu_q(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n_1+n_2-l}\zeta^x d\mu_q(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\sum_{j=0}^{n_1+n_2-l}(-1)^j\binom{n_1+n_2-l}{j}\tilde{\beta}_{j,q,\zeta}^{(\alpha)}.\end{aligned} \tag{14}$$

By the binomial theorem and the weighted q-Bernstein polynomials, we easily see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x \mid q)B_{k,n_2}^{(\alpha)}(x \mid q)\zeta^x d\mu_q(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\int_{\mathbb{Z}_p}[x]_{q^\alpha}^{2k+l}\zeta^x d\mu_q(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\tilde{\beta}_{2k+l,q,\zeta}^{(\alpha)}.\end{aligned} \tag{15}$$

By comparing the coefficients on the both sides of (14) and (15), we obtain the following theorem.

Theorem 2. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have

$$\sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \tilde{\beta}^{(\alpha)}_{2k+l,q,\zeta}$$
$$= \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \sum_{j=0}^{n_1+n_2-l} (-1)^j \binom{n_1+n_2-l}{j} \tilde{\beta}^{(\alpha)}_{j,q,\zeta}.$$

Let $s \in \mathbb{N}$ with $s \geq 2$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B^{(\alpha)}_{k,n_1}(x \mid q) \cdots B^{(\alpha)}_{k,n_s}(x \mid q)}_{s-\text{times}} \zeta^x d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} [x]^{sk}_{q^\alpha} [1-x]^{n_1+\cdots+n_s-sk}_{q^{-\alpha}} \zeta^x d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} [1-x]^{n_1+\cdots+n_s-l}_{q^{-\alpha}} \zeta^x d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \sum_{j=0}^{n_1+\cdots+n_s-l} (-1)^j \binom{n_1+\cdots+n_s-l}{j} \tilde{\beta}^{(\alpha)}_{j,q,\zeta}.
\end{aligned} \tag{16}$$

By simple calculation, we easily see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B^{(\alpha)}_{k,n_1}(x \mid q) \cdots B^{(\alpha)}_{k,n_s}(x \mid q)}_{s-\text{times}} \zeta^x d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} \int_{\mathbb{Z}_p} [x]^{sk+l}_{q^\alpha} \zeta^x d\mu_q(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} \tilde{\beta}^{(\alpha)}_{sk+l,q,\zeta}.
\end{aligned} \tag{17}$$

By comparing the coefficients on the both sides of (16) and (17), we obtain the following theorem.

Theorem 3. Let $s \in \mathbb{N}$ with $s \geq 2$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$, we have

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} \tilde{\beta}^{(\alpha)}_{sk+l,q,\zeta}$$
$$= \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \sum_{j=0}^{n_1+\cdots+n_s-l} (-1)^j \binom{n_1+\cdots+n_s-l}{j} \tilde{\beta}^{(\alpha)}_{j,q,\zeta}.$$

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On optimizing restricted edge connectivity of graphs[1]

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Abstract Based on the properties of distance maximal pair of vertices, we present sufficient conditions for triangle-free graphs to be maximally restricted edge connected. Some known results are direct corollaries of these observations.

Keywords Graph; Restricted edge connectivity; distance maximal pair

1 Introduction

All graphs appearing in this work are simple and connected with order at least four. An edge cut S of a connected graph G is restricted if $G - S$ contains no isolated vertices, the minimum cardinality $\lambda'(G)$ over all restricted edge cuts of graph G is called its restricted edge connectivity. It is well-known that $\lambda'(G) \leq \xi(G)$ holds for any graph G that contains restricted edge cuts [1]. Graph G is called maximally restricted edge connected if the equality holds, where $\xi(G)$ denotes the minimum edge degree of graph G.

The observations obtained in [2] show that under some reasonable conditions networks with greater restricted edge connectivity are locally more reliable. And so the optimization of restricted edge connectivity of graphs draws a lot of attention, for advances in this field the readers are suggested to refer to [3-8]. Let $d(u, v)$ represent the distance between two vertices u and v of a graph G. The distance $d(X, Y)$ between two vertex-sets X and Y is defined as the minimum distance between vertices of these two vertex-sets, namely $d(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$. Vertex-set pair (X, Y) is called k-distance maximal if $d(X, Y) = k$ and there exist no vertex-sets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ such that $X_1 \neq X$ or $Y_1 \neq Y$, and $d(X_1, Y_1) = k$. In section two we present the following theorem.

Theorem 1.1 Let G be a triangle-free connected graph that contains restricted edge cuts. If there is an isolated edge in the subgraph induced by any

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3-distance maximal pair of vertex-sets, then G is maximally restricted edge connected.

This observation generalizes the result obtained in [8], which says that if for all 3-distance maximal pairs of vertex sets X and Y of graph G there exists an isolated vertex in the subgraph induced by $X \cup Y$ then G has maximal edge connectivity. A similar condition is also found for bipartite graphs to be maximally restricted edge connected. As is pointed in section two, many well-known results are direct observation of theorem 1.1. For any two disjoint subsets X and Y of the vertex-set of graph G or two of its subgraphs, $[X, Y]$ indicates the set of edges with one end in X and the other end in Y. For other symbols and terminology not specified herein, we follow that of [9].

2 Maximal restricted edge connectivity

Proof of theorem 1.1. Suppose on the contrary that $\lambda'(G) < \xi(G)$. Let $[S, T]$ be a minimum restricted edge cut of G, let $A \subseteq S$ and $B \subseteq T$ be the sets of vertices covered by $[S, T]$. Set $A_0 = S - A$, $B_0 = T - B$. Then $\max\{|A|, |B|\} \leq \lambda'(G) < \xi(G)$. We shall discuss in two different cases.

Case 1. $A_0 \neq \emptyset \neq B_0$.

Noticing that $d(A_0, B_0) \geq 3$, we can choose a 3-distance maximal pair X and Y such that $A_0 \subseteq X$ and $B_0 \subseteq Y$. By condition postulated in theorem 1.1, $G[X \cup Y]$ contains an isolated edge uv. If $u, v \in A_0$ then $N(u), N(v) \subseteq A$, since G is triangle-free it follows that $\xi(G) \leq d(uv) = |N(u)| + |N(v)| - 2 \leq |A| \leq \lambda'(G) < \xi(G)$. This contradiction also occurs when $u, v \in B_0$.

If $u \in A_0$ and $v \in A$ then $N(u) \subseteq A$. Since G is triangle-free it follows that

$$\xi(G) \leq d(u) + d(v) - 2 = |N(u) \cap A| + |N(v) \cap A| + |N(v) \cap B| - 1.$$

Recalling that every vertex $x \in A$ is incident with at least one edge of $[S, T]$, we deduce that $|N(u) \cap A| + |N(v) \cap A| \leq \sum_{x \in A - \{v\}} |N(x) \cap B| + 1$. The Combination of the previous two inequalities implies that $\xi(G) \leq \sum_{x \in A} |N(x) \cap B| = \lambda'(G) < \xi(G)$. This contradiction also occurs when $u \in B_0$ and $v \in B$.

If $u, v \in A$, then $N(u), N(v) \subseteq A \cup B$. And so

$$\begin{aligned} \xi(G) &\leq d(u) + d(v) - 2 \\ &= |N(u) \cap A| + |N(v) \cap A| + |N(u) \cap B| + |N(v) \cap B| - 2 \\ &\leq \sum_{x \in A} |N(x) \cap B| = \lambda'(G) < \xi(G). \end{aligned}$$

This contradiction also occurs when $u, v \in B$. Since uv cannot be an edge of $[S, T]$, case 1 cannot occur.

Case 2. A_0 or $B_0 = \emptyset$, assume without loss of generality that $A_0 = \emptyset$.

Let uv be any edge of A. Since G is triangle-free, it follows that

$$\begin{aligned} \xi(G) &\leq d(u) + d(v) - 2 \\ &= |N(u) \cap A| + |N(v) \cap A| + |N(u) \cap B| + |N(v) \cap B| - 2 \\ &\leq \sum_{x \in A} |N(x) \cap B| = \lambda'(G) < \xi(G). \end{aligned}$$

Theorem 1.1 follows from this contradiction. □

Hellwig and Volkmann show in [5] that if $|N(x) \cap N(y)| \geq 2$ holds for all pairs of nonadjacent vertices x and y, then $\xi(G) = \lambda'(G)$. When G is triangle-free, this observation is a direct result of theorem 1.1 since in this case graph G contians no 3-distance maximal pairs. And so, we present the following corollaries, the proofs of them are omited for simplicity.

Corollary 2.1 If G is a triangle-free connected graph of diameter at most two, then $\xi(G) = \lambda'(G)$. □

Corollary 2.2 Let G be a triangle-free connected graph of order n. If $|N(x) \cap N(y)| \geq 1$ holds for all pairs of nonadjacent vertices x and y, then $\xi(G) = \lambda'(G)$. □

The following corollary is a direct observation of corollary 2.2.

Corollary 2.3 Let G be a triangle-free connected graph of order n. If $n \leq 2\delta + 1$, then $\xi(G) = \lambda'(G)$. □

For any subset X of the vertex-set of bipartite graph $G = (V', V''; E)$, let $X' = X \cap V'$, $X'' = X \cap V''$. A pair of vertex-sets X and Y of a bipartite graph G is called (k, k)-distance maximal if $d(X^{'}, Y^{'}) = k$, $d(X^{''}, Y^{''}) = k$ and there exist no vertex set $X_1 \supseteq X$ and $Y_1 \supseteq Y$ such that $X_1 \neq X$ or $Y_1 \neq Y$, and $d(X_1^{'}, Y_1^{'}) = d(X_1^{''}, Y_1^{''}) = k$.

Theorem 2.4 Let G be a bipartite graph that contains restricted edge cuts. If $G[X \cup Y]$ contains an isolated edge for any $(4,4)$-distance maximal pair of vertex-sets X and Y, then G is maximally restricted edge connected.

Proof Suppose no the contrary that $\lambda'(G) < \xi(G)$ for a bipartite graph $G = (V', V'', E)$,Let $[S,T]$ be a minimum restricted edge cut, $A \subseteq S$ and $B \subseteq T$ be the sets of vertices incident with edges of $[S,T]$, $A_0 = S - A$ and $B_0 = T - B$. $A_0^{'} = V^{'} \bigcap A_0$,$A_0^{''} = V^{''} \bigcap A_0$,$B_0^{'} = V^{'} \bigcap B_0$, $B_0^{''} = V^{''} \bigcap B_0$, Then $|A|, |B| \leq \lambda'(G) < \xi(G)$.

Case 1. $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$.

Noticing that $d(A_0^{'}, B_0^{'}) \geq 4$ and $d(A_0^{''}, B_0^{''}) \geq 4$, we choose a (4,4)-distance maximal pair X and Y such that $A_0 \subseteq X$ and $B_0 \subseteq Y$. Then there is an isolated edge $e = uv$ in the subgraph induced by $X \cup Y$. If $e = uv \subseteq A_0$, then $\xi(G) \leq d(e) = |N(u)| + |N(v)| - 2 \leq |A| \leq \lambda'(G) < \xi(G)$. This inequalty also holds when $e = uv \subseteq B_0$. If $u \in A_0$ and $v \in A$, since $(N(v) - u) \subset A \cup B$ it follows that

$$\begin{aligned} \xi(G) &\leq d(u) + d(v) - 2 = |N(u) \cap A| + |N(v) \cap A| + |N(v) \cap B| - 1 \\ &\leq \sum_{x \in A - \{v\}} |N(x) \cap B| + |N(v) \cap B| \\ &= \sum_{x \in A} |N(x) \cap B| = \lambda'(G) < \xi(G). \end{aligned}$$

Similar contradiction occurs when $u \in B_0$ and $v \in B$. If $u, v \in A$, then

$$\begin{aligned} \xi(G) &\leq d(u) + d(v) - 2 \\ &= |N(u) \cap A| + |N(v) \cap A| + |N(u) \cap B| + |N(v) \cap B| - 2 \\ &\leq \sum_{x \in A} |N(x) \cap B|. \end{aligned}$$

Similar contradiction occurs when $e = uv \subseteq B$.

Case 2. A_0 or $B_0 = \emptyset$, assume without loss of generality that $A_0 = \emptyset$.

Let uv be any edge of A. Since G is triangle-free, it follows that

$$\begin{aligned} \xi(G) &\leq d(u) + d(v) - 2 \\ &= |N(u) \cap A| + |N(v) \cap A| + |N(u) \cap B| + |N(v) \cap B| - 2 \\ &\leq \sum_{x \in A} |N(x) \cap B| = \lambda'(G) < \xi(G). \end{aligned}$$

Theorem 2.4 follows from this contradiction. □

The following corollary is a direct result of theorem 2.4, since in this case graph G contains no (4, 4)-distance maximal pairs.

Corollary 2.5 Let G be a bipartite graph that contains restricted edge cuts. If G has diameter at most three, then $\xi(G) = \lambda'(G)$. □

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Dynamics of the max-type difference equation $x_{n+1} = \max\{\frac{A}{x_n}, x_{n-k}\}$

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Abstract In this paper, we study the following max-type difference equation

$$x_{n+1} = \max\{\frac{A}{x_n}, x_{n-k}\}, \quad n = 0, 1, \ldots,$$

where $A \in (-\infty, +\infty)$, $k \in \{1, 2, \cdots\}$ and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_0 \in (-\infty, +\infty)$. We show that every well-defined solution of this equation is eventually periodic with period $k+1$, which confirms the truth of the conjecture proposed by Elsayed and Stević [Nonlinear Analysis 71(2009), 910- 922].

AMS Subject Classification: 39A10; 39A11.

Keywords: Max-type equation, Well-defined solution, Periodicity.

1 Introduction

In the recent years, there has been a lot of interest in studying the global behavior of max-type difference equations (see [1-24]). In [1,3-5,8,10], the second order max-type difference equation

$$x_{n+1} = \max\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\}, \quad n = 0, 1, \ldots, \tag{1.1}$$

has been studied for positive coefficients A_n, which are periodic with period k. The case $k = 1$ was studied in ref.[1], the case $k = 2$ was studied in ref.[3], the case $k = 3$ was studied in refs

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[4,10] and the more difficult case $k = 4$ was studied in ref.[8]. Chen [5] found that every positive solution of (1.1) is eventually periodic with period 2 when $\{A_n\}_{n=0}^{+\infty}$ is a periodic sequence with period $k \geq 2$ and $A_n \in (0, 1)$ for all $n \geq 0$. These results were also included in the recent monograph [11] along with other related references. In [20], we studied global behavior of (1.1) when $\{A_n\}_{n=0}^{+\infty}$ is a sequence of positive real numbers with $0 \leq \inf A_n \leq \sup A_n < 1$.

Liu and Yang in [14] investigated the quantitative bounds for the positive solutions to the following max-type difference equation

$$x_{n+1} = \max\{A, \frac{x_n^p}{x_{n-k}^{p^{k+1}}}\}, \quad n = 0, 1, \cdots,$$

where $A, p \in (0, +\infty)$ and k is a positive integer.

Simsek et al. [16] studied the behavior of the solutions of the following system of difference equations

$$x_{n+1} = \max\{\frac{A}{x_n}, \frac{y_n}{x_n}\}, \qquad y_{n+1} = \max\{\frac{A}{y_n}, \frac{x_n}{y_n}\},$$

where the constant A and the initial conditions are positive real numbers.

Gelişken and Çinar in [9] investigated the difference equation

$$x_n = \max\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^{\alpha}}\}, \quad n = 0, 1, \cdots,$$

where $A > 0$, $0 < \alpha < 1$ and initial conditions are positive real numbers, and proved that every positive solution of this difference equation approaches 1 or is eventually periodic with period 2.

Yalçinkaya et al. in [24] considered the following second-order max-type difference equation

$$x_{n+1} = \max\{\frac{1}{x_n}, Ax_{n-1}\}, \quad n = 0, 1, \cdots, \tag{1.2}$$

Equation(1.2) is not difficult for handling since, by the change $y_n = x_n x_{n-1}$, it is transformed into one of the following first-order difference equations

$$y_{n+1} = \max\{1, Ay_n\} \qquad \text{or} \qquad y_{n+1} = \min\{1, Ay_n\}.$$

Using these equations, it is easy to see that for the case $A = 1$, every solution of (1.2) is eventually periodic with period 2.

In this paper, we consider the max-type equation

$$x_{n+1} = \max\{\frac{A}{x_n}, x_{n-k}\}, \quad n = 0, 1, \cdots, \tag{1.3}$$

where $A \in \mathbf{R} = (-\infty, +\infty)$, $k \in \{1, 2, \cdots\}$ and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_0 \in \mathbf{R}$. In [12], Iričanin and Elsayed showed that every well-defined solution of (1.3) for $k = 3$ and $A \geq 0$ is eventually periodic with period 4.

Elsayed and Stević [7] showed that every well-defined solution of (1.3) for $k = 2$ is eventually periodic with period 3 and proposed the following conjecture.

Conjecture 1 Assume $A \in \mathbf{R}$ and $k \in \{1, 2, \cdots\}$. Show that every well-defined solution of (1.3) is eventually periodic with period $k + 1$.

In this paper, we shall confirm that the above conjecture is true. Our main result is the following theorem.

Theorem 1 Let $A \in \mathbf{R}$ and $k \in \{1, 2, \cdots\}$, then every well-defined solution of of (1.3) is eventually periodic with period $k + 1$.

2 Proof of Theorem 1

Let $\{x_n\}_{n=-k}^{+\infty}$ be a well-defined solution of (1.3). If $A = 0$, then equation (1.3) becomes $x_{n+1} = x_{n-k}$, from which it follows that $\{x_n\}_{n=-k}^{+\infty}$ is periodic with period $k + 1$. Hence, in the sequel we will consider the case $A \neq 0$.

It follows from (1.3) that for any $n \geq 0$ and any $i \geq 0$,

$$x_{(n+1)(k+1)+i} = \max\{\frac{A}{x_{n(k+1)+i+k}}, x_{n(k+1)+i}\} \geq x_{n(k+1)+i}.$$

Then $\{x_{n(k+1)+i}\}_{n=0}^{+\infty}$ is increasing for every $i \geq 0$.

Claim 1 There exist $0 \leq \gamma \leq k$ and $N \geq 0$ such that for any $n \geq N$,

$$x_{(n+1)(k+1)+\gamma} = x_{n(k+1)+\gamma}.$$

Proof Indeed, if for every $0 \leq j \leq k$, there exists a sequence $0 \leq s_1^j < s_2^j < \cdots < s_n^j < \cdots$ such that

$$x_{s_n^j(k+1)+j} < x_{(s_n^j+1)(k+1)+j} \quad \text{for any} \quad n \geq 1,$$

then we take $m_0 < m_1 < \cdots < m_{2k+2}$ such that $x_{m_j(k+1)+j} < x_{(m_j+1)(k+1)+j}$ for any $j \in \{0, 1, \cdots, 2k + 2\}$.

Let

$$r_j = \begin{cases} m_{2k+2}, & \text{if } \ j = 2k + 2, \\ \max\{m_j \leq r \leq r_{j+1} : x_{r(k+1)+j} < x_{(r+1)(k+1)+j}\}, & \text{if } \ j \in \{0, 1, \cdots, 2k + 1\}, \end{cases}$$

then

$$\begin{aligned}
x_{(r_{2k+2}+1)(k+1)+2k+2} &= \max\{\frac{A}{x_{(r_{2k+2}+1)(k+1)+2k+1}}, x_{r_{2k+2}(k+1)+2k+2}\} \\
&= \frac{A}{x_{(r_{2k+2}+1)(k+1)+2k+1}} \\
&= \frac{A}{x_{(r_{2k+1}+1)(k+1)+2k+1}} \\
&= \frac{A}{\max\{\frac{A}{x_{(r_{2k+1}+1)(k+1)+2k}}, x_{r_{2k+1}(k+1)+2k+1}\}} \\
&= x_{(r_{2k+1}+1)(k+1)+2k} \\
&= x_{(r_{2k}+1)(k+1)+2k} \\
&\cdots\cdots\cdots\cdots \\
&= x_{(r_0+1)(k+1)} \\
&\leq x_{r_{2k+2}(k+1)+2k+2},
\end{aligned}$$

which is a contradiction. Claim 1 is proven.

Claim 2 There exist $N_1 \geq N$ such that for any $n \geq N_1$,

$$\frac{A}{x_{N(k+1)+\gamma}} \leq x_{(n-1)(k+1)+\gamma+1},$$

where N and γ are as in Claim 1.

Proof Indeed, if $\frac{A}{x_{N(k+1)+\gamma}} > x_{(n-1)(k+1)+\gamma+1}$ for all $n \geq N$, then it follows from Claim 1 and equation (1.3) that for $n \geq N$,

$$\begin{aligned}
x_{n(k+1)+\gamma+1} &= \max\{\frac{A}{x_{n(k+1)+\gamma}}, x_{(n-1)(k+1)+\gamma+1}\} \\
&= \max\{\frac{A}{x_{N(k+1)+\gamma}}, x_{(n-1)(k+1)+\gamma+1}\} \\
&= \frac{A}{x_{N(k+1)+\gamma}} \\
&> x_{n(k+1)+\gamma+1}.
\end{aligned}$$

This is a contradiction. Claim 2 is proven.

By Claim 2, we see that $x_{(n+1)(k+1)+\gamma+1} = x_{n(k+1)+\gamma+1}$ for all $n \geq N_1$. Thus $x_{(n+1)(k+1)+\gamma+1}$ is a constant eventually. By induction, we can show that $x_{(n+1)(k+1)+\gamma+i}$ is a constant eventually for every $i \in \{0, 1, \cdots, k\}$, which implies that $\{x_n\}_{n=-k}^{+\infty}$ is eventually periodic with period $k+1$.

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Opial-type inequalities for functions and their ordinary and balanced fractional derivatives

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Abstract

Here we present L_p, $p > 0$, fractional Opial type inequalities subject to high order boundary conditions. They involve the right and left Caputo, Riemann-Liouville fractional derivatives. These derivatives are blended together into the balanced Caputo, Riemann-Liouville, respectively, fractional derivative. We give an application to a balanced fractional boundary value problem by proving uniqueness of the solution.

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Key Words and Phrases: Opial inequality, fractional inequality, Caputo fractional derivative, Riemann-Liouville fractional derivative, boundary conditions.

1 Introduction

Here we continue [5].

This article is inspired by the famous theorem of Z. Opial [13], 1960, which follows.

Theorem A. *Let* $x(t) \in C^1([0,h])$ *be such that* $x(0) = x(h) = 0$,*and* $x(t) > 0$ *in* $(0,h)$.*Then*

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h (x'(t))^2 dt. \qquad (*)$$

In $(*)$,*the constant* $\frac{h}{4}$ *is the best possible. Inequality* $(*)$ *holds as equality for the optimal function*

$$x(t) = \begin{cases} ct, & 0 \leq t \leq h/2 \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases}$$

where $c > 0$ *is an arbitrary constant.*

To prove easier Theorem A, Beesack [7] proved the following well-known Opial type inequality which is used very commonly.

This is another inspiration to our work.

Theorem B. *Let $x(t)$ be absolutely continous in $[0,a]$, and $x(0)=0$.*

Then

$$\int_0^a |x(t)x'(t)|\,dt \leq \frac{a}{2}\int_0^a (x'(t))^2 dt. \qquad (**)$$

*Inequality (**) is sharp, it is attained by $x(t)=ct,\ c>0$ is an arbitrary constant.*

See also [12].

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations, also to give upper bounds to their solutions.

By themselves have made a great subject of intensive research and there exists a great literature about them.

Typical and great sources on them are the monographs [1], [3].

2 Preliminaries

We need

Definition 1. *([4], [9], [10], [11], [14]). Let $f \in AC^m([a,b])$ (space of functions from $[a,b]$ into $\mathbb{R}$ with $m-1$ derivative absolutely continuous function on $[a,b], m \in \mathbb{N}$, where $m=\lceil \alpha \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $\alpha>0$.*

We define the right Caputo fractional derivatives of order $\alpha>0$, by

$$D_{b-}^{\alpha} f(x) \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\xi - x)^{m-\alpha-1} f^{(m)}(\xi) d\xi. \qquad (1)$$

We set $D_{b-}^0 f(x) = f(x),\ \forall x \in [a,b]$.

We give the right Caputo fractional Taylor formula with integral remainder.

Theorem 2. *([4]) Let $f \in AC^m([a,b]), x \in [a,b], \alpha>0, m=\lceil \alpha \rceil$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \frac{1}{\Gamma(\alpha)}\int_x^b (\xi - x)^{\alpha-1} D_{b-}^{\alpha} f(\xi) d\xi. \qquad (2)$$

We make

Definition 3. *([8, p.38]) Let $f \in AC^m([a,b]), m \in \mathbb{N}$, where $m=\lceil \alpha \rceil, \alpha>0$. We define the left Caputo fractional derivative of order $\alpha>0$, by*

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt. \qquad (3)$$

We set $D_{*a}^0 f(x) = f(x), \forall x \in [a,b]$.

We give the left Caputo fractional Taylor formula with integral remainder.

Theorem 4. *([8, p.40]) Let $f \in AC^m([a,b]), m \in \mathbb{N}$,where $m = \lceil \alpha \rceil, \alpha > 0, x \in [a,b]$.*

Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} D_{*a}^{\alpha} f(\tau) d\tau. \tag{4}$$

Definition 5. *([5]) Let $f \in AC^m([a,b]), m \in \mathbb{N}$,where $m = \lceil \alpha \rceil, \alpha > 0, x \in [a,b]$.*

We define the balanced Caputo fractional derivative,

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), \text{ for } \frac{a+b}{2} \le x \le b, \\ D_{*a}^{\alpha} f(x), \text{ for } \alpha \le x < \frac{a+b}{2} \end{cases} \tag{5}$$

3 Main Results

We prove

Theorem 6. *Let $\alpha > 0, m = \lceil \alpha \rceil, x \in [a,b], f \in AC^m([a,b]), f^{(k)}(a) = f^{(k)}(b) = 0, k = 0,1,...,m-1; 0 < p < 1$ and $q < 0 : \frac{1}{p} + \frac{1}{q} = 1$.*

Assume $D_{*a}^{\alpha} f \neq 0$ on $\left[a, \frac{a+b}{2}\right)$ and of fixed sign, a.e.; $D_{b-}^{\alpha} f \neq 0$ on $\left[\frac{a+b}{2}, b\right]$ and of fixed sign, a.e.

Then

$$\begin{aligned} &\int_a^b |f(w)| \left|D^{\alpha} f(w)\right| dw \ge \\ &\left(\frac{(b-a)^{((\alpha-1)+\frac{2}{p})}}{\left(2^{(\alpha+\frac{1}{p})}\Gamma(\alpha)((p(\alpha-1)+1)(p(\alpha-1)+2)\right)^{1/p}} \right) \\ &\left[\left(\int_a^{\frac{a+b}{2}} \left|D_{*a}^{\alpha} f(t)\right|^q dt \right)^{2/q} + \left(\int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(t)\right|^q dt \right)^{2/q} \right]. \end{aligned} \tag{6}$$

Proof. Here $\alpha > 0, \mathbb{N} \ni m = \lceil \alpha \rceil, x \in [a,b], f \in AC^m([a,b]), f^{(k)}(a) = f^{(k)}(b) = 0$,for $k = 0,1,...,\ m-1;\ 0 < p < 1$ and $q : \frac{1}{p} + \frac{1}{q} = 1$. Then by (4),

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} D_{*a}^{\alpha} f(t) dt, \forall x \in [a,b].$$

Also by (2),

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} D_{b-}^{\alpha} f(t) dt, \forall x \in [a,b].$$

Assumed $D_{*a}^{\alpha} f \neq 0$ over $\left[\alpha, \frac{a+b}{2}\right)$ and of fixed sign, a.e., and assumed $D_{b-}^{\alpha} f \neq 0$ over $\left[\frac{a+b}{2}, b\right]$ and of fixed sign, a.e. Then we have

$$|f(w)| = \frac{1}{\Gamma(\alpha)} \int_a^w (w-t)^{\alpha-1} \left|D_{*a}^{\alpha} f(t)\right| dt,\ \forall\ w \in \left[a, \frac{a+b}{2}\right),$$

and

$$|f(w)| = \frac{1}{\Gamma(\alpha)} \int_w^b (t-w)^{\alpha-1} \left| D_{b-}^{\alpha} f(t) \right| dt, \ \forall w \in \left[\frac{a+b}{2}, b \right].$$

Therefore by Hölder's inequality $(q = p/(p-1) < 0)$ we get

$$|f(w)| \geq \frac{1}{\Gamma(\alpha)} \left(\int_a^w (w-t)^{p(\alpha-1)} dt \right)^{1/p} \left(\left(\int_a^w \left| D_{*a}^{\alpha} f(t) \right|^q dt \right) \right)^{1/q}.$$

Hence

$$|f(w)| \geq \frac{1}{\Gamma(\alpha)} \frac{(w-a)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left(\int_a^w \left| D_{*a}^{\alpha} f(t) \right|^q dt \right)^{1/q}, \forall w \in \left[a, \frac{a+b}{2} \right).$$

Similarly we obtain

$$|f(w)| \geq \frac{1}{\Gamma(\alpha)} \frac{(b-w)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left(\int_w^b \left| D_{b-}^{\alpha} f(t) \right|^q dt \right)^{1/q}, \forall w \in \left[\frac{a+b}{2}, b \right].$$

Set

$$z_1(w) := \int_a^w \left| D_{*a}^{\alpha} f(t) \right|^q dt, \ z_1(a) = 0,$$

so that $z_1'(w) = \left| D_{*a}^{\alpha} f(w) \right|^q$, and $\left| D_{*a}^{\alpha} f(w) \right| = (z_1'(w))^{1/q}$,all $a \leq w < \frac{a+b}{2}$. Set

$$z_2(w) := \int_w^b \left| D_{b-}^{\alpha} f(t) \right|^q dt, \ z_2(b) = 0,$$

so that $z_2'(w) = -\left| D_{b-}^{\alpha} f(w) \right|^q$,and $\left| D_{b-}^{\alpha} f(w) \right| = (-z_2'(w))^{1/q}$, $\frac{a+b}{2} \leq w \leq b$. Therefore

$$|f(w)| \left| D_{*a}^{\alpha} f(w) \right| \geq \frac{1}{\Gamma(\alpha)} \frac{(w-a)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left(z_1(w) z_1'(w) \right)^{1/q},$$

all $a \leq w < \frac{a+b}{2}$. Hence

$$\int_a^{\frac{a+b}{2}} |f(w)| \left| D_{*a}^{a} f(w) \right| dw \geq \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}}$$

$$\int_a^{\frac{a+b}{2}} (w-a)^{\alpha-1+\frac{1}{p}} (z_1(w) z_1'(w)^{1/q} dw \geq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}} \left(\int_a^{\frac{a+b}{2}} (w-a)^{p(\alpha-1)+1} \right)^{1/p} \left(\int_a^{\frac{a+b}{2}} z_1(w) z_1'(w) dw \right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(p(\alpha-1)+2)^{1/p}} \left(\frac{b-a}{2} \right)^{(\alpha-1)+(2/p)} \left(\frac{z_1^2(w)}{2} \Big|_a^{\frac{a+b}{2}} \right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(p(\alpha-1)+2)^{1/p}} \left(\frac{b-a}{2} \right)^{\left(\alpha-1+\frac{2}{p}\right)} \left(\frac{z_1^2(\frac{a+b}{2})}{2} \right)^{1/q}.$$

Therefore

$$\int_a^{\frac{a+b}{2}} |f(w)| \left|D^{\alpha}_{*a} f(w)\right| dw \geq c^* \left(\int_a^{\frac{a+b}{2}} \left|D^{\alpha}_{*a} f(t)\right|^q dt \right)^{2/q}, \tag{7}$$

where

$$c^* := \frac{1}{2^{1/q}\Gamma(\alpha)\left((p(\alpha-1)+1)\left(p(\alpha-1)+2\right)\right)^{1/p}} \left(\frac{b-a}{2}\right)^{\left(\alpha-1+\frac{2}{p}\right)}. \tag{8}$$

Similarly

$$|f(w)| \left|D^{\alpha}_{b-} f(w)\right| \geq \frac{1}{\Gamma(\alpha)} \frac{(b-w)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}} (z_2(w)(-z_2'(w)))^{1/q},$$

and

$$\begin{aligned}
\int_{\frac{a+b}{2}}^b |f(w)| \left|D^{\alpha}_{b-} f(w) dw\right| &\geq \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}} \int_{\frac{a+b}{2}}^b (b-w)^{(p(\alpha-1)+1)/p} \left(z_2(w)\left(-z_2'(w)\right)\right)^{1/q} dw \\
&\geq \frac{1}{2^{1/q}\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(p(\alpha-1)+2)^{1/p}} \left(\frac{b-a}{2}\right)^{(\alpha-1)+\frac{2}{p}} \\
&\left(\int_{\frac{a+b}{2}}^b \left|D^{\alpha}_{b-} f(t)\right|^q dt \right)^{2/q}.
\end{aligned}$$

I.e.

$$\int_{\frac{a+b}{2}}^b |f(w)| \left|D^{\alpha}_{b-} f(w)\right| dw \geq c^* \left(\int_{\frac{a+b}{2}}^b \left|D^{\alpha}_{b-} f(t)\right|^q dt \right)^{2/q}. \tag{9}$$

adding (7) and (9) we get (6). □

We present and need

Theorem 7. *Let $x \leq s \leq 0$ and $f \in L_\infty([x,0]), r > 0$. Define*

$$G(s) = \int_s^0 (t-s)^r f(t) dt. \tag{10}$$

Then there exists

$$G'(s) = -r \int_s^0 (t-s)^{r-1} f(t) dt. \tag{11}$$

Proof. Fix $s_0 \in [x,0]$ and notice that

$$G(s_0) = \int_{s_0}^{0} (t-s_0)^r f(t)dt = \int_{x}^{0} \chi_{[s_0,0]}(t)(t-s_0)^r f(t)dt.$$

We call

$$g(s,t) := \chi_{[s,0]}(t)(t-s)^r f(t),$$

which is a Lebesgue integrable function for every $s \in [x,0]$.That is, $g(s_0,t) = \chi_{[s_0,0]}(t)(t-s_0)^r f(t)$,all $t \in [x,0]$,and $G(s_0) = \int_x^0 g(s_0,t)dt$. We would like to study if there exists

$$\frac{\partial g(s_0,t)}{\partial s} = f(t)\left(\lim_{h\to 0} \frac{\chi(t)_{[s_0+h,o]}(t-s_0-h)^r - \chi(t)_{[s_0,0]}(t-s_0)^r}{h}\right). \quad (12)$$

We distinguish the following cases:

(1) Let $x \le t < s_0$;then there exist small enough $h > 0$ such that $t < s_0 \pm h$.

That is,

$$\chi_{[s_0\pm h,0]}\ (t) = \chi_{[s_0,0]}(t) = 0.$$

Hence, there exists

$$\frac{\partial g(s_0,t)}{\partial s} = 0, \text{ all } t : x \le t < s_0.$$

(2) Let $s_0 < t \le 0$;then there exist small enough $h > 0$ such that $t > s_0 \pm h$.

That is,

$$\chi_{[s_0\pm h,0]}(t) = \chi_{[s_0,0]}(t) = 1.$$

In that case

$$\begin{aligned}\frac{\partial g(s_0,t)}{\partial s} &= f(t)\left(\lim_{h\to 0} \frac{(t-(s_0+h))^r - (t-s_0)^r}{h}\right)\\ &= -r(t-s_0)^{r-1} f(t),\end{aligned}$$

exists for almost all $t : s_0 < t \le 0$.

(3) Let $t = s_0$. Then we see that

$$\frac{\partial g_+(s_0,s_0)}{\partial s} = f(s_0)\left(\lim_{h\to 0+} \frac{0\cdot(-h)^r - 1\cdot 0^r}{h}\right) = 0.$$

Also we get

$$\frac{\partial g_-(s_0,s_0)}{\partial s} = f(s_0)\left(\lim_{h\to 0-} \frac{1\cdot(-h)^r - 1\cdot 0^r}{h}\right) =$$

$$f(s_0)\left(\lim_{h\to 0-} \frac{(-h)^r}{h}\right) = -f(s_0)\left(\lim_{h\to 0-} \frac{(-h)^r}{(-h)}\right)$$

$$= -f(s_0)\left(\lim_{h\to 0-} (-h)^{r-1}\right) = -f(s_0)\left(\lim_{h\to 0+} h^{r-1}\right).$$

The last limit does not exist if $0 < r < 1$;equals $-f(s_0)$if $r = 1$ and may not exist; and equals 0 if $r > 1$.

In general as a conclusion we get that $\frac{\vartheta g(s_0,t)}{\vartheta s}$exists for almost all $t \in [x, 0]$.

Next we define the difference quotient at s_0,

$$D_{s_0}(h,t) := f(t)\left(\frac{\chi(t)_{[s_0+h,0]}(t-s_0-h)^r - \chi_{[s_0,0]}(t)(t-s_0)^r}{h}\right),$$

for $h \neq 0$,and $D_{s_0}(0,t) := 0$.

Again we distinguish the following cases:

(1) Let $x \leq t < s_0$;then there exist small enough $h > 0$ such that $t < s_0 \pm h$.

Clearly then $D_{s_0}(h,t) = 0$.

(2) Let $s_0 < t \leq 0$; then there exists small enough h>0 such that $t > s_0 \pm h$.In that case

$$D_{s_0}(\pm h,t) = f(t)\left(\frac{(t-(s_0 \pm h)^r - (t-s_0)^r}{\pm h}\right).$$

Call $\rho := t - s_0 > 0$;clearly $0 < \rho \leq |x|$.

Define

$\varphi(h) := \frac{(\rho-h)^r-\rho^r}{h} = \frac{(t-s_0-h)^r-(t-s_0)^r}{h}$,for h close to zero, $r > 0$.

That is

$$D_{s_0}(h,t) = f(t)\varphi(h).$$

If $r = 1$, then $\varphi(h) = -1$ and $D_{s_0}(h,t) = -f(t)$.

We now treat the following subcases:

($\mathbf{2}_{(i)}$) If $r > 1$and $|h|$ small, then by mean value theorem we get

$$|D_{s_0}(h,t)| = |f(t)|\,|\varphi(h)| \leq |f(t)|\,r2^{r-1}\,|x|^{r-1}.$$

That is, for $r \geq 1$ and small $|h|$ we obtain

$$|D_{s_0}(h,t)| \leq r2^{r-1}\,|x|^{r-1}\,|f(t)|.$$

($\mathbf{2}_{(ii)}$) If $0 < r < 1$and $|h|$ small we get the following:

The function $\gamma(\rho) = \rho^r,\ 0 \leq \rho \leq |x|$ is concave and increasing. Let $h > 0$; then

$$|\varphi(h)| = \frac{\rho^r - (\rho-h)^r}{\rho-(\rho-h)} < \rho^{r-1} = (r-s_0)^{r-1},$$

and for $h<0$,again

$$\varphi(h)=\frac{(\rho-h)^r-\rho^r}{(\rho-h)-\rho}<\rho^{r-1}=(t-s_0)^{r-1}.$$

Therefore we obtain

$$|D_{s_0}(h,t)|\leq|f(t)|\,(t-s_0)^{r-1},$$

for $0<r<1$and $|h|$ small.

(3) Case of $t=s_0$; then

$D_{s_0}(h,s_0)=-f(s_0)\,|h|^{r-1}$, for $h<0$,

and

$D_{s_0}(h,s_0)=0$, for $h>0$.

So, if $r\geq 1$ we obtain

$|D_{s_0}(h,s_0)|\leq|f(s_0)|\,|x|^{r-1}$,

for small $|h|$.

If $0<r<1$,then for small $|h|$ with $h<0$,the function

$D_{s_0}(h,s_0)$ may be unbounded.

In conlcusion we get:

(I) For $r\geq 1$,that

$|D_{s_0}(h,t)|\leq r2^{r-1}\,|x|^{r-1}\,\|f\|_\infty<+\infty$,for almost all $t\in[x,0]$.

(II) For $0<r<1$,that

$|D_{s_0}(h,t)|\leq\lambda(t)$, for almost all $t\in[x,0]$,where

$$\lambda(t):=\begin{cases}0, & x\leq t\leq s_0,\\ |f(t)|\,(t-s_0)^{r-1}, & s_0<t\leq 0.\end{cases}$$

Clearly λ is integrable on $[x,0]$.

Then by Theorem 24.5, pp. 193-194 of [2], we get that $\frac{\partial g(s_0,\cdot)}{\partial s}$ defines an integrable function, and there exists

$$\begin{aligned}G'(s_0)&=\int_x^0\frac{\partial g(s_0,t)}{\partial s}dt=\\ &-r\int_{s_0}^0(t-s_0)^{r-1}f(t)dt+\int_x^{s_0}0dt\\ &=-r\int_{s_0}^0(t-s_0)^{r-1}f(t)dt.\end{aligned}$$

That proves the claim.

□

We need

Theorem 8. *([3] Theorem 7.7, p.117)*
Let $0 \leq s \leq x$ *and* $f \in L_\infty([0,x]), r > 0$. *Define*

$$F(s) := \int_0^s (s-t)^r f(t)dt. \tag{13}$$

Then there exists

$$F'(s) = r\int_0^s (s-t)^{r-1} f(t)dt, \tag{14}$$

$\forall s \in [0,x]$.
We make

Remark 9. *Let* $f \in AC^m([a,b]),\ m \in \mathbb{N}, m = \lceil \alpha \rceil,\ \alpha > 1$. *Assume* $f^{(k)}(a) = f^{(k)}(b) = 0,\ k = 0,1,...,m-1$.
Then we get by (4))

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} D^\alpha_{*a} f(t)dt, \tag{15}$$
$$\forall x \in [a,b],$$

also (by (2))

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} D^\alpha_{b-} f(t)dt, \tag{16}$$
$$\forall x \in [a,b].$$

Assume that

$$D^\alpha_{*a} f,\ D^\alpha_{b-} f \in L_\infty([a,b]). \tag{17}$$

Here $\alpha \notin \mathbb{N}$ *and* $m-1 < \alpha < m,\ m \in \mathbb{N}$. *So from* $\alpha > m-1$ *we get* $(\alpha-1) - (m-2) > 0$. *Thus for* $\ell = 1,...,m-1$, *we obtain* $0 \leq \ell - 1 \leq m-2$, *and* $(\alpha-1) - (\ell-1) \geq (\alpha-1) - (m-2) > 0$.
Hence $(\alpha-1) - (\ell-1) > 0$ *and* $\alpha - \ell > 0,\ \ell = 1,...,m-1$.
Therefore by Theorem 8 and (15) we get

$$f'(x) = \frac{(\alpha-1)}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-2} D^\alpha_{*a} f(t)dt$$
$$= \frac{1}{\Gamma(\alpha-1)}\int_a^x (x-t)^{\alpha-2} D^\alpha_{*a} f(t)dt.$$

Similarly, we obtain

$$f^{(\ell)}(x) = \frac{1}{\Gamma(\alpha-\ell)}\int_a^x (x-t)^{\alpha-\ell-1} D^\alpha_{*a} f(t)dt,$$
$$\text{for } \ell = 1,...,m-1, \forall x \in [a,b]. \tag{18}$$

Using Theorem 7 and (16), we derive

$$f^{(\ell)}(x)=\frac{(-1)^{\ell}}{\Gamma(\alpha-\ell)}\int_{x}^{b}(t-x)^{\alpha-\ell-1}D_{b-}^{\alpha}f(t)dt,$$
$$\ell=1,,...,m-1,x\in[a,b]. \tag{19}$$

We prove ($p=1,q=\infty$ case).

Theorem 10. *Let $\alpha>0,\ m=\lceil\alpha\rceil,\ x\in[a,b],\ f\in AC^{m}([a,b]),\ f^{(k)}(a)=f^{(k)}(b)=0$, for $k=0,1,...,\ m-1$. Assume*

$$D_{*a}^{\alpha}f\in L_{\infty}\left(\left[a,\frac{a+b}{2}\right)\right),\ D_{b-}^{\alpha}f\in L_{\infty}\left(\left[\frac{a+b}{2},b\right]\right)$$

Then

$$\int_{a}^{b}|f(x)|\,|D^{\alpha}f(x)|\,dx\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\cdot\frac{1}{\Gamma(\alpha+2)}$$
$$\left(\left\|D_{*a}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right)}^{2}+\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}^{2}\right) \tag{20}$$

Proof. Let $\alpha>0,\ m=\lceil\alpha\rceil,\ x\in[a,b],\ f\in AC^{m}([a,b]),\ f^{(k)}(a)=f^{(k)}(b)=0$, for $k=0,1,...,\ m-1$. Assume

$$D_{*a}^{\alpha}f\in L_{\infty}\left(\left[a,\frac{a+b}{2}\right)\right),$$
$$D_{b-}^{\alpha}f\in L_{\infty}\left(\left[\frac{a+b}{2},b\right]\right).$$

Then

$$|f(x)|\leq\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\left\|D_{*a}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right)},\ \ a\leq x<\frac{a+b}{2},$$

and

$$|f(x)|\leq\frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right],}\ \ \frac{a+b}{2}\leq x\leq b.$$

Thus

$$|f(x)|\,|D_{*a}^{\alpha}f(x)|\leq\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\left\|D_{*a}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right)}^{2},\ \forall x\in\left[a,\frac{a+b}{2}\right).$$

Hence

$$\int_{a}^{\frac{a+b}{2}}|f(x)|\,|D_{*a}^{\alpha}f(x)|\,dx\leq\frac{\left\|D_{*a}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right)}^{2}}{\Gamma(\alpha+2)}\cdot\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}}.$$

Similarly

$$|f(x)|\leq\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)},$$
$$\forall x\in\left[\frac{a+b}{2},b\right].$$

And

$$|f(x)|\left|D_{b-}^{\alpha}f(x)\right| \leq \frac{\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}^{2}}{\Gamma(\alpha+1)}(b-x)^{\alpha},\ \forall x\in\left[\frac{a+b}{2},b\right].$$

Hence

$$\int_{\frac{a+b}{2}}^{b}|f(x)|\left|D_{b-}^{\alpha}f(x)\right|dx \leq \frac{\left\|D_{b-}^{\alpha}\right\|_{\infty,\left[\frac{a+b}{2},b\right]}^{2}}{\Gamma(\alpha+2)}\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}}.$$

Consequently it holds,

$$\int_{a}^{b}|f(x)|\left|D^{\alpha}f(x)\right|dx \leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+2)}\left(\left\|D_{*a}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right)}^{2}+\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)$$

establishing the claim. □

We continue with

Theorem 11. *Let* $f\in AC^{m}([a,b]),\ N\ni m=\lceil\alpha\rceil,\ \alpha\geq 1$. *Assume* $f^{(k)}(a)=f^{(k)}(b)=0,\ k=0,1,...,\ m-1$. *Assume* $D_{*a}^{\alpha}f,\ D_{b-}^{\alpha}f\in L_{\infty}([a,b]),\ p,q>1:\ \frac{1}{p}+\frac{1}{q}=1,\ \alpha>m-\frac{1}{p}$.

Then

(i) *Case of* $1<q\leq 2$. *Then*

$$\int_{a}^{b}\left|f^{(\ell)}(w)\right|\left|D^{\alpha}f(w)\right|dw \leq \frac{2^{-(\alpha-\ell+\frac{1}{p})}(b-a)^{(\alpha-\ell-1+\frac{2}{p})}}{\Gamma(\alpha-\ell)[(p(\alpha-\ell-1)+(p(\alpha-\ell-1)+2)]^{1/p}} \tag{21}$$

$$\left(\int_{a}^{b}\left|D^{\alpha}f(w)\right|^{q}dw\right)^{2/q},\ \ell=1,...,\ m-1.$$

(ii) *Case of* $q>2$. *Then*

$$\int_{a}^{b}\left|f^{(\ell)}(w)\right|\left|D^{\alpha}f(w)\right|dw \leq \frac{2^{-\left(\alpha-\ell+\frac{1}{q}\right)}(b-a)^{\left(\alpha-\ell-1+\frac{2}{p}\right)}}{\Gamma(\alpha-\ell)[(p(\alpha-\ell-1)+1)(p(\alpha-\ell-1)+2)]^{1/p}}\left(\int_{a}^{b}\left|D^{\alpha}f(w)\right|^{q}dw\right)^{2/q}, \tag{22}$$

$$\ell=1,,...,\ m-1.$$

(iii) When $p=q=2,\ \alpha > m-\frac{1}{2},\ (\ell=1,...,\ m-1)$. Then

$$\int_a^b \left|f^{(\ell)}(w)\right| |D^\alpha f(w)|\, dw \leq \frac{2^{-\left(\alpha-\ell+\frac{1}{2}\right)}(b-a)^{\alpha-\ell}}{\Gamma(\alpha-\ell)[\sqrt{2(\alpha-\ell)(2\alpha-2\ell-1)}} \left(\int_a^b (D^\alpha f(w))^2 dw\right). \tag{23}$$

Proof. By the assumption of the theorem we have

$$f^{(\ell)}(x) = \frac{1}{\Gamma(\alpha-\ell)}\int_a^x (x-t)^{\alpha-\ell-1} D_{*a}^\alpha f(t)dt,$$
$$\text{for } \ell=1,...,\ m-1;\ \forall x\in[a,b].$$

Similarly it holds

$$f^{(\ell)}(x) = \frac{(-1)^\ell}{\Gamma(\alpha-\ell)}\int_x^b (t-x)^{\alpha-\ell-1} D_{b-}^\alpha f(t)dt,$$
$$\ell=l,...,\ m-1;\ \forall x\in[a,b].$$

Hence

$$\left|f^{(\ell)}(x)\right| \leq \frac{1}{\Gamma(\alpha-\ell)}\int_a^x (x-t)^{\alpha-\ell-1} \left|D_{*a}^\alpha f(t)\right| dt,$$
$$\ell=1,...,\ m-1,\ \forall x\in[a,b].$$

And

$$\left|f^{(\ell)}(x)\right| \leq \frac{1}{\Gamma(\alpha-\ell)}\int_x^b (t-x)^{\alpha-\ell-1} \left|D_{b-}^\alpha f(t)\right| dt,$$
$$\ell=1,...,\ m-1;\ \forall x\in[a,b].$$

Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$.Acting now as in the proof of Theorem 9 of [5], and replacing f by $f^{(\ell)}$ and α by $\alpha-\ell$ we prove our theorem. □

We further present.

Theorem 12. *Let* $\alpha\geq 1,\ \ell=1,...,\ m-1;\ f\in AC^m([a,b]), m=\lceil\alpha\rceil, f^{(k)}(a)=f^{(k)}(b)=0,\ k=0,1,...,\ m-1.$

Assume $D_{*a}^\alpha f, D_{b-}^\alpha f \in L_\infty([a,b])$.*Assume* $D_{*a}^\alpha f\neq 0$ *over* $\left[a,\frac{a+b}{2}\right)$,*and of fixed sign, a.e.;* $D_{b-}^\alpha f\neq 0$ *over* $\left[\frac{a+b}{2},b\right]$ *and of fixed sign, a.e.,* $0<p<1,\ q<$

$0;\ \frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\int_a^b \left|f^{(\ell)}(w)\right| |D^\alpha f(w)|\, dw \geq \left(\frac{(b-a)^{\left((\alpha-\ell-1)+\frac{2}{p}\right)}}{2^{\left(\alpha-\ell+\frac{1}{p}\right)}\Gamma(\alpha-\ell)\binom{(p(\alpha-\ell-1)+1}{(p(\alpha-\ell-1)+2}^{1/p}}\right) \left[\left(\int_a^{\frac{a+b}{2}} |D^\alpha_{*a} f(t)|^q\, dt\right)^{2/q} + \left(\int_{\frac{a+b}{2}}^b \left|D^\alpha_{b-} f(t)\right|^q dt\right)^{2/q}\right] \tag{24}$$

Proof. As in the proof of Theorem 6, by replacing f by $f^{(\ell)}$ and α by $\alpha-\ell$, for $\ell=1,...,\ m-1;\ \alpha\geq 1$. □

Theorem 13. *The $p=1,\ q=\infty$ case again. Let $\alpha\geq 1,\ \ell=1,...,\ m-1;\ f\in AC^m([a,b]),\ m=\lceil\alpha\rceil,\ f^{(k)}(a)=f^{(k)}(b)=0;\ k=0,1,...,\ m-1$. Assume $D^\alpha_{*a}f,\ D^\alpha_{b-}f\in L_\infty([a,b])$.*

Then

$$\int_a^b \left|f^{(\ell)}(x)\right| |D^\alpha f(x)|\, dx \leq \left(\frac{b-a}{2}\right)^{\alpha-\ell+1}\cdot\frac{1}{\Gamma(\alpha-\ell+2)} \left(\left\|D^\alpha_{*a}f\right\|^2_{\infty,\left[a,\frac{a+b}{2}\right)} + \left\|D^\alpha_{b-}f\right\|^2_{\infty,\left[\frac{a+b}{2},b\right]}\right). \tag{25}$$

Proof. As in the proof of Theorem 10, by replacing f by $f^{(\ell)}$ and α by $\alpha-\ell$, for $\ell=1,...,\ m-1$. □

We need [4], [9], [10], [8], p.22)

Definition 14. *Let $\alpha>0,\ m=\lceil\alpha\rceil,\ f\in AC^m([a,b])$. We define the right Reimann-Liouville fractional derivative by*

$$\mathcal{D}^\alpha_{b-}f(x):=\frac{(-1)^m}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m\int_x^b (t-x)^{m-\alpha-1}f(t)dt, \tag{26}$$

$D^0_{b-}f(x):=I(x)$ *(the identity operator).*

We also define the left Reimann-Liouville fractional derivative by

$$\mathcal{D}^\alpha_{a+}f(x):=\frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m\int_a^x (x-t)^{m-a-1}f(t)dt, \tag{27}$$

$$\mathcal{D}^0_{a+}f(x):=I(x).$$

We further define the balanced Reimann-Liouville fractional derivative ([5])

$$\mathcal{D}^\alpha f(x):=\begin{cases}\mathcal{D}^\alpha_{b-}f(x), & \text{for } \frac{a+b}{2}\leq x\leq b,\\ \mathcal{D}^\alpha_{a+}f(x), & \text{for } \alpha\leq x<\frac{a+b}{2}.\end{cases} \tag{28}$$

Remark 15. *Let now* $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *In [6] we have proved that* $D^{\alpha}_{b-} f(x)$, $D^{\alpha}_{*a} f(x)$ *are continuous functions in* $x \in [a,b]$. *Of course* $C^m([a,b]) \subset AC^m([a,b])$, *so that* $f \in AC^m([a,b])$.

Thus by Theorem 9 of [4], we obtain that also $\mathcal{D}^{\alpha}_{b-} f(x)$ *exists and continuous for every* $x \in [a,b]$. *Furthermore if* $f^{(k)}(b) = 0$, $k = 0, 1..., m-1$ *we get*

$$\mathcal{D}^{\alpha}_{b-} f(x) = D^{\alpha}_{b-} f(x), \tag{29}$$

$\forall x \in [a,b]$.

Similarly, by [8], p.39, we get that $\mathcal{D}^{\alpha}_{a+} f(x)$ *exists and continuous in* $x \in [a,b]$. *Furthermore if* $f^{(k)}(\alpha) = 0$, $k = 0, 1, ..., m-1$ *we get*

$$\mathcal{D}^{\alpha}_{a+} f(x) = D^{\alpha}_{*a} f(x), \tag{30}$$

$\forall x \in [a,b]$.

So if $f^{(k)}(\alpha) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$ *we obtain that*

$$\mathcal{D}^{\alpha} f(x) = D^{\alpha} f(x), \tag{31}$$

$\forall x \in [a,b]$.

So by Theorem 11 we obtain the corresponding results for the balanced Reimann-Liouville fractional derivative.

Theorem 16. *Let* $f \in C^m([a,b])$, $N \ni m = \lceil \alpha \rceil$, $\alpha \geq 1$. *Assume* $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$. *Assume* p, $q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > m - \frac{1}{p}$.

(i) *Case of* $1 < q \leq 2$. *Then*

$$\int_a^b \left| f^{(\ell)}(w) \right| |\mathcal{D}^{\alpha} f(w)| \, dw \leq$$
$$\frac{2^{-(\alpha-\ell+\frac{1}{p})} (b-a)^{(\alpha-\ell-1+\frac{2}{p})}}{\Gamma(\alpha-\ell)[(p(\alpha-\ell-1)+1)(p(\alpha-\ell-1)+2)]^{1/p}} \left(\int_a^b |\mathcal{D}^{\alpha} f(w)|^q \, dw \right)^{2/q}, \tag{32}$$

$\ell = 1, ..., m-1$.

(ii) *Case of* $q > 2$. *Then*

$$\int_a^b \left| f^{(\ell)}(w) \right| |\mathcal{D}^{\alpha} f(w)| \, dw \leq$$
$$\frac{2^{(\alpha-\ell+\frac{1}{q})} (b-a)^{(\alpha-\ell-1+\frac{2}{p})}}{\Gamma(\alpha-\ell)[(p(\alpha-\ell-1)+1)(p(\alpha-\ell-1)+2)]^{1/p}} \left(\int_a^b |\mathcal{D}^{\alpha} f(w)|^q \, dw \right)^{2/q},$$
$\ell = 1, ..., m-1$. (33)

(iii) *When* $p=q=2,\ \alpha > m-\frac{1}{2},\ (\ell=1,...,m-1)$. *Then*

$$\int_a^b |f|^{(\ell)}(w)\,|\mathcal{D}^\alpha f(w)|\,dw \leq \frac{2^{-(\alpha-\ell+\frac{1}{2})}(b-a)^{\alpha-\ell}}{\Gamma(\alpha-\ell)[\sqrt{2(\alpha-\ell)(2\alpha-2\ell-1)}} (\int_a^b (\mathcal{D}^\alpha f(w))^2 dw). \tag{34}$$

Remark 17. *For* $f\in C^m([a,b])$ $m=\lceil\alpha\rceil,\ \alpha>0$, *one can rewrite also Theorems 6, 10, 12 and 13 in the language Riemann-Liouville fractional derivatives, as we did in Theorem 16.*

Application 18. *Uniqueness of solution to fractional boundary value problem*

$$(D^\nu f)(t) = F(t,f,f',...,f^{(m-1)}):$$
$$f^{(i)}(a)=a_i, f^{(i)}(b)=b_i, 0\leq i\leq m-1, m:=\lceil\nu\rceil,\ \nu\geq 1, \nu > m-\frac{1}{2} \tag{35}$$

Here $f\in AC^m([a,b]),\ D^\nu_{*a}f,\ D^\nu_{b-}f\in L_\infty([a,b]); F$ *is a continuous function on* $[a,b]\times\mathbb{R}^m,\ t\in[a,b]$. *Also* F *fulfills the Lipschitz condition*

$$\left|F(t,z_0,z_1,...,z_{m-1})-F(t,z_0',z_1',...,z_{m-1}'\right| \leq \sum_{i=0}^{m-1} q_i(t)\,|z_i-z_i'|,$$

where $q_i(t)\geq 0,\ 0\leq i\leq m-1$ *are continuous functions on* $[a,b]$.

Call

$$\phi(a,b) := \sum_{i=0}^{m-1} \|q_i\|_\infty \frac{2^{-(\nu-i+\frac{1}{2})}(b-a)^{\nu-i}}{\Gamma(\nu-i)\sqrt{2(\nu-i)(2\nu-2i-1)}.}$$

Assume that

$$\phi(a,b)<1. \tag{36}$$

Next we prove uniqueness to the solution of (35). Let f_1,f_2 *as above fulfilling (35); that is,*

$$(D^v f_j)(t) = F(t,f_j,f_j',...,f_j^{(m-1)}): f_j^{(i)}(a)=a_i, f_j^{(i)}(b)=b_i, 0\leq i\leq m-1, j=1,2.$$

Call $g:=f_1-f_2$. *Then*

$$(D^\nu g)(t) = F(t,f_1,f_1',...,f_1^{(m-1)}) - F(t,f_2,f_2',...,f_2^{(m-1)}),$$
$$\text{such that } g^{(i)}(a)=0,\ g^{(i)}(b)=0,\ 0\leq i\leq m-1. \tag{37}$$

Here

$$\left|F(t,f_1,f_1',...,f_1^{(m-1)})-F(t,f_2,f_2',...,f_2^{(m-1)})\right|$$
$$\leq \sum_{i=0}^{m-1} q_i(t)\left|f_1^{(i)}(t)-f_2^{(i)}(t)\right| = \sum_{i=0}^{m-1} q_i(t)\left|g^{(i)}(t)\right|.$$

From

$$(D^\nu g(t))^2 = (D^\nu g(t))\{F(t,f_1,f_1',...,f_1^{(m-1)})-F(t,f_2,f_2',...,f_2^{(m-1)})\},$$

we obtain

$$(D^\nu g(t))^2 \leq |D^\nu g(t)|\left(\sum_{i=0}^{m-1} q_i(t)\left|g^{(i)}(t)\right|\right)$$
$$= \sum_{i=0}^{m-1} q_i(t)(|D^\nu g(t)|\left|g^{(i)}(t)\right|).$$

Integrating the last inequality we get

$$\int_a^b (D^\nu g(t))^2 dt \leq \sum_{i=0}^{m-1}\int_a^b q_i(t)\left|g^{(i)}(t)\right||D^\nu g(t)|\,dt$$
$$\leq \sum_{i=0}^{m-1}\|q_i\|_\infty \int_a^b \left|g^{(i)}(t)\right||D^\nu g(t)|\,dt. \tag{38}$$

From inequalities (23) and (13) of [5] we find that

$$\int_a^b \left|g^{(i)}(t)\right||D^\nu g(t)|\,dt$$
$$\leq \frac{2^{-(\nu-i+\frac{1}{2})}(b-a)^{\nu-i}}{\Gamma(\nu-i)\sqrt{2(2\nu-2i-1)(\nu-i)}}\int_a^b ((D^\nu g)(t))^2 dt,\ \textit{for } i=0,1,...,m-1. \tag{39}$$

Combining (38) and (39) we derive

$$\int_a^b ((D^\nu g)(t))^2 dt \leq \phi(a,b)\int_a^b ((D^\nu g(t))^2 dt. \tag{40}$$

If $\int_a^b (D^\nu g(t))^2 dt \neq 0$, *then from (40) we get* $\phi(a,b)\geq 1$, *a contradiction by (36).*

Thus

$$\int_a^b (D^\nu g(t))^2 dt = 0,$$

which implies

$$(D^\nu g(t)^2 = 0,\ a.e.in\ t\in[a,b];$$

that is

$$D^{\nu} g(t) = 0, \ a.e, in \ t \in [a, b].$$

Therefore

$$D^{\nu}_{b-} g(t) = 0, \ for \ \frac{a+b}{2} \leq t \leq b, \ a.e.$$

and

$$D^{\nu}_{*a} g(t) = 0, for \ a \leq t < \frac{a+b}{2}, \ a.e.$$

But

$$g^{(i)}(a) = g^{(i)}(b) = 0, \ for \ all \ 0 \leq i \leq m-1.$$

So from (2) and (4) we get $g(x) = 0$ *on* $\left[\frac{a+b}{2}, b\right]$, *and* $g(x) = 0$ *on* $\left[a, \frac{a+b}{2}\right]$, *respectively.*

Thus $g(x) = 0, \ \forall x \in [a, b]$,*giving us* $f_1(x) = f_2(x), \forall x \in [a, b]$ *that is proving the uniqueness of solution to fractional boundary value problem (35).*

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Two-grid algorithms for linear and nonlinear elliptic problems based on HSS iteration method *

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Abstract: In this paper, based on the idea of Hermitian/skew-Hermitian splitting(HSS) iteration method, we present several two-grid (two level) algorithms for solving a large class of linear and nonlinear second-order elliptic boundary value problems. These algorithms reduce the nonsymmetric linear system or nonlinear system on the fine-grid space to two linear (one symmetric positive definite and one shifted skew-symmetric) systems on the fine-grid space and the original system on a coarse-grid space. Under certain conditions, these algorithms have a convergence property similar to that of the two-grid algorithms proposed by Xu. It is shown, both theoretically and numerically, that our algorithms perform well for some convection-diffusion problems.

Keywords: Two-grid algorithm, Finite element, HSS iteration method, Convection-diffusion problems.

1 Introduction

To solve linear and nonlinear elliptic partial differential equations(PDEs), many authors have investigated the two-grid (two level) method or multilevel method ([9, 10, 11, 15, 16, 23, 29]). By employing two finite element subspaces, $\mathcal{V}_H$ and $\mathcal{V}_h$ (with mesh size $h \ll H$), Xu proposed some two-grid algorithms ([32, 33, 34]). With these algorithms, solving the nonsymmetric system on the fine-grid space is reduced to solving a symmetric and positive definite(SPD) system on the fine-grid space and the original system on a much smaller space; solving the nonlinear systems on the fine-grid space is reduced to solving a linear system on the fine-grid space and the original system on a much smaller space. Two-grid method was investigated by many authors, for example, Axelsson et al. ([1, 2]), Layton et al. ([26, 27]) and other authors ([20, 25, 28]). Additionally, some authors extended this method in other ways, such as the finite difference method ([21, 30]) and the finite volume method ([14, 18]).

Bai, Golub and Ng ([4]) recently proposed the HSS iteration method for solving the large sparse, non-Hermitian and positive definite system of linear equations. The advantage of the HSS method is that it converges unconditionally to the unique solution of the linear equation. Hence, this method has been further investigated in [5, 6, 7, 8, 13, 12]. In this paper, based on the idea of HSS iteration method, we present several new two-grid algorithms for solving linear and nonlinear problems, respectively. With the proposed algorithms, the solution of the nonsymmetric linear system or the nonlinear system on the fine-grid space is reduced to the solutions of two (one SPD and one shifted skew-symmetric) systems on the fine-grid space and the solution of the original system on the coarse-grid space. The coarse-grid space can be quite small corresponding to the fine-grid space by choosing proper parameter. Moreover, these algorithms converge unconditionally to the discretization solution of the original problem even without recourse to the coarse-grid space. Numerical results show that these new algorithms perform well for the convection-diffusion problems.

For simplicity of exposition, we assume that Ω is a convex polygonal domain in R^2. The domain is partitioned by quasi-uniform triangulation. We shall use the standard notation for the Sobolev spaces $\mathcal{W}_p^m(\Omega)$ with norm $\|\cdot\|_{m,p,\Omega}$ and the seminorm $|\cdot|_{m,p,\Omega}$ ([19]). For $p = 2$, we denote $\mathcal{H}^m(\Omega) = \mathcal{W}_2^m(\Omega)$, $\mathcal{H}_0^1(\Omega) = \{v \in \mathcal{H}^1(\Omega) : v\,|_{\partial\Omega} = 0\}$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. Throughout this paper, the letter C denotes a generic positive constant independent of the mesh parameter and may be different at its different occurrences.

The rest of the paper is organized as follows. In Section 2, we introduce the HSS iteration method briefly and present several two-grid algorithms for nonsymmetric elliptic problems. In Section 3, we

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propose a couple of two-grid algorithms for some nonlinear problems and give the error estimate in $\mathcal{H}^1$ norm. Finally, in Section 4, some numerical experiments are given to show the efficiency of the new algorithms.

2 Two-grid algorithms for nonsymmetric linear problems

In this section, by employing two finite element subspaces: a coarse-grid space $\mathcal{V}_H$ and a fine-grid space $\mathcal{V}_h(\subset \mathcal{H}_0^1(\Omega))$, we present a couple of two-grid algorithms for nonsymmetric problems based on the HSS iteration method.

2.1 HSS iteration method and nonsymmetric linear elliptic equation

Consider the linear system

$$\mathrm{A}x = \mathrm{f}, \quad x \in C^n. \tag{2.1}$$

Here, $\mathrm{f} \in C^n$ and $\mathrm{A} \in C^{n\times n}$ is large sparse non-Hermitian and positive definite. Let

$$\mathrm{A} = \mathrm{H} + \mathrm{S},$$

where $\mathrm{H} = \frac{1}{2}(\mathrm{A} + \mathrm{A}^*)$, $\mathrm{S} = \frac{1}{2}(\mathrm{A} - \mathrm{A}^*)$, A^* denotes the conjugate transpose of A. The HSS iteration method proposed in [4] for solving (2.1) is: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$, until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha \mathrm{I} + \mathrm{H})x^{(k+\frac{1}{2})} &= (\alpha \mathrm{I} - \mathrm{S})x^{(k)} + \mathrm{f}, \\ (\alpha \mathrm{I} + \mathrm{S})x^{(k+1)} &= (\alpha \mathrm{I} - \mathrm{H})x^{(k+\frac{1}{2})} + \mathrm{f}, \end{cases}$$

where α is a given positive constant. Bai et al.([4]) have proved that HSS iteration method converges unconditionally to the exact solution of the equation (2.1), and the upper bound of the contraction factor of the HSS iteration is only dependent on the spectrum of the Hermitian part H but is independent of the spectrum of the skew-Hermitian part S or the eigenvectors of the matrices H, S and A.

Consider the nonsymmetric linear equation of the form

$$\begin{cases} \mathcal{L}u \equiv -\nabla \cdot (\mathbf{a}(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u + c(x)u &= f, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega \end{cases} \tag{2.2}$$

under the following assumptions.

Assumption 1: (1) Let $\mathbf{a}(x) \in C^1(\overline{\Omega}, R^{2\times 2})$, $\mathbf{b}(x) \in C^1(\overline{\Omega})^2$, $c(x) \in C^1(\overline{\Omega})$ and $f \in \mathcal{L}^2(\Omega)$, assume that $c(x) - \frac{1}{2}\nabla \cdot \mathbf{b}(x) \geq 0$ for any $x \in \overline{\Omega}$;

(2) Assume that $\mathbf{a}(x)$ is symmetric and there exists a positive constant m such that $\xi^T \mathbf{a}(x)\xi \geq m|\xi|^2$ for any $\xi \in R^2$ and $x \in \overline{\Omega}$.

For brevity, we omit the variable x in the following exposition. The variational form of (2.2) is to find $u \in \mathcal{H}_0^1(\Omega)$ such that

$$\mathcal{A}(u, v) = (f, v) \quad \forall\, v \in \mathcal{H}_0^1(\Omega), \tag{2.3}$$

where the bilinear form is

$$\mathcal{A}(u, v) = \int_\Omega (\mathbf{a}\nabla u \cdot \nabla v + \mathbf{b} \cdot \nabla u v + cuv)dx \quad \forall\, u, v \in \mathcal{H}_0^1(\Omega).$$

From Assumption 1, we know that (2.3) has a unique solution $u \in \mathcal{H}_0^1(\Omega)$([17]).

Set $\mathcal{V}_h \subset \mathcal{H}_0^1(\Omega)$ is a piecewise linear finite element space on a mesh from a regular family of partition. The finite element approximation of u in (2.3) is to find $u_h \in \mathcal{V}_h$ such that

$$\mathcal{A}(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in \mathcal{V}_h. \tag{2.4}$$

Define $P_h : \mathcal{V} \to \mathcal{V}_h$ by

$$\mathcal{A}(P_h u, v_h) = \mathcal{A}(u, v_h) \quad \forall\, v_h \in \mathcal{V}_h.$$

Lemma 2.1 *([17]) Assume that $u \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$ is the solution of (2.3), if h is sufficiently small, then the projection P_h admits the estimate*

$$\begin{aligned}
&\|u - P_h u\|_{0,p} \le Ch^2 \|u\|_{2,p}, \quad 2 \le p < \infty, \\
&\|u - P_h u\|_{0,\infty} \le Ch^2 |\log h| \|u\|_{2,\infty}, \\
&\|u - P_h u\|_{1,p} \le C \inf_{v_h \in \mathcal{V}_h} \|u - v_h\|_{1,p} \le Ch \|u\|_{2,p}, \quad 2 \le p \le \infty.
\end{aligned}$$

To present new two-grid algorithms, we define the operators $\mathcal{H}$ and $\mathcal{S}$ as

$$\begin{cases}
\mathcal{H}u = \dfrac{1}{2}(\mathcal{L} + \mathcal{L}^*)u = -\nabla \cdot (\mathbf{a}\nabla u) - \dfrac{1}{2}(\nabla \cdot \mathbf{b})u + cu, \\
\mathcal{S}u = \dfrac{1}{2}(\mathcal{L} - \mathcal{L}^*)u = \mathbf{b} \cdot \nabla u + \dfrac{1}{2}(\nabla \cdot \mathbf{b})u,
\end{cases}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ and

$$\mathcal{L}^* u = -\nabla \cdot (\mathbf{a}\nabla u + \mathbf{b}u) + cu.$$

The bilinear forms of $\mathcal{H}$ and $\mathcal{S}$ are defined by

$$\begin{cases}
\mathcal{H}(u, v) = \displaystyle\int_\Omega \mathbf{a}\nabla u \cdot \nabla v dx + \int_\Omega (c - \frac{1}{2}\nabla \cdot \mathbf{b}) uv dx, \\
\mathcal{S}(u, v) = \displaystyle\int_\Omega (\mathbf{b} \cdot \nabla u) v dx + \int_\Omega (\frac{1}{2}\nabla \cdot \mathbf{b}) uv dx.
\end{cases} \tag{2.5}$$

Let $\{\varphi_1, \cdots, \varphi_n\}$ be a basis of $\mathcal{V}_h$, i.e., $\mathcal{V}_h = span\{\varphi_1, \cdots, \varphi_n\}$. For any vector $\mathbf{v}_h = (v_1, \cdots, v_n)^T$, define

$$\mathbf{P} : R^n \to \mathcal{V}_h \subset \mathcal{H}_0^1(\Omega), \quad v_h = \mathbf{P}\mathbf{v}_h = \sum_{i=1}^n v_i \varphi_i.$$

Obviously, $\mathbf{P}$ is an isomorphism from R^n to $\mathcal{V}_h$.

Lemma 2.2 *The matrix $\boldsymbol{L}_h = (L_{ij})_{n\times n}$ and the vector $\boldsymbol{f}_h = (f_1, \cdots, f_n)^T$ are defined by*

$$L_{ij} = \mathcal{B}(\varphi_i, \varphi_j), \quad f_i = f(\varphi_i) \quad (i, j = 1, \ldots, n),$$

and the variational problem is to find $u_h \in \mathcal{V}_h$ such that

$$\mathcal{B}(u_h, v_h) = (f, v_h) \quad \forall\ v_h \in \mathcal{V}_h.$$

Then the above variational problem and

$$\boldsymbol{L}_h \boldsymbol{u}_h = \boldsymbol{f}_h$$

are equivalent. If $\langle \boldsymbol{u}_h, \boldsymbol{v}_h \rangle = \sum_i u_i v_i$ is the usual scalar product, then $\mathcal{B}(u_h, v_h) = \mathcal{B}(\mathbf{P}\boldsymbol{u}_h, \mathbf{P}\boldsymbol{v}_h) = \langle \boldsymbol{L}_h \boldsymbol{u}_h, \boldsymbol{v}_h \rangle$ and $(f_h, v_h) = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle$ for any $v_h \in \mathcal{V}_h$, $\boldsymbol{v}_h \in R^n$.

The above lemma is similar to Theorem 8.1.3 in [24]. From Lemma 2.2, we introduce an operator $\mathcal{I}$ and the corresponding bilinear form $\mathcal{I}(u_h, v_h)$ is defined as $\mathcal{I}(\varphi_i, \varphi_j) = I_{ij}$ $(i, j = 1, \ldots, n)$, where $\mathbf{I} = (I_{ij})_{n\times n}$ is the identity matrix. Denote $(\alpha\mathcal{I} \pm \mathcal{H})(u, v) = \alpha\mathcal{I}(u, v) \pm \mathcal{H}(u, v)$ and $(\alpha\mathcal{I} \pm \mathcal{S})(u, v) = \alpha\mathcal{I}(u, v) \pm \mathcal{S}(u, v)$.

2.2 Two-grid algorithms and convergence analysis

Based on the HSS iteration method introduced in above subsection, we present a couple of two-grid algorithms to solve the nonsymmetric linear equation (2.2).

Algorithm 2.1 *1. Find $u_H \in \mathcal{V}_H$ such that*

$$\mathcal{A}(u_H, v_H) = (f, v_H) \quad \forall\ v_H \in \mathcal{V}_H.$$

2. Find $u_h^* \in \mathcal{V}_h$ such that

$$(\alpha\mathcal{I} + \mathcal{S})(u^h, v_h) = (\alpha\mathcal{I} - \mathcal{H})(u_H, v_h) + (f, v_h) \quad \forall\ v_h \in \mathcal{V}_h, \tag{2.6}$$

$$(\alpha\mathcal{I} + \mathcal{H})(u_h^*, v_h) = (\alpha\mathcal{I} - \mathcal{S})(u^h, v_h) + (f, v_h) \quad \forall\ v_h \in \mathcal{V}_h. \tag{2.7}$$

Theorem 2.1 *Suppose that Assumption 1 is satisfied and $u_h^* \in \mathcal{V}_h$ is obtained by Algorithm 2.1. If H is sufficiently small, then*

$$\|u_h - u_h^*\|_1 \le C\rho(\alpha)H\|u\|_2 \quad \textit{and} \quad \|u - u_h^*\|_1 \le C(h + \rho(\alpha)H)\|u\|_2$$

hold for $u \in \mathcal{H}^1(\Omega)$, where $\rho(\alpha) = \rho((\alpha\mathbf{I} + \boldsymbol{H}_h)^{-1}(\alpha\boldsymbol{I} - \boldsymbol{S}_h)(\alpha\boldsymbol{I} + \boldsymbol{S}_h)^{-1}(\alpha\mathbf{I} - \mathbf{H}_h))$, $\mathbf{H}_h$ and $\mathbf{S}_h$ are the matrices of $\mathcal{H}(u_h, v_h)$ and $\mathcal{S}(u_h, v_h)$, respectively.

Proof: It follows from (2.5) and (2.4) that

$$\begin{aligned}
&\mathcal{A}(u_h, v_h) = (\alpha\mathcal{I} + \mathcal{S})(u_h, v_h) - (\alpha I - \mathcal{H})(u_h, v_h) = (f, v_h) \quad \forall\ v_h \in \mathcal{V}_h,\\
&\mathcal{A}(u_h, v_h) = (\alpha\mathcal{I} + \mathcal{H})(u_h, v_h) - (\alpha\mathcal{I} - \mathcal{S})(u_h, v_h) = (f, v_h) \quad \forall\ v_h \in \mathcal{V}_h.
\end{aligned}$$

A direct calculation with (2.6) and (2.7) implies

$$(\alpha\mathcal{I} + \mathcal{S})(u_h - u^h, v_h) = (\alpha\mathcal{I} - \mathcal{H})(u_h - u_H, v_h) \quad \forall\ v_h \in \mathcal{V}_h, \tag{2.8}$$

$$(\alpha\mathcal{I} + \mathcal{H})(u_h - u_h^*, v_h) = (\alpha\mathcal{I} - \mathcal{S})(u_h - u^h, v_h) \quad \forall\ v_h \in \mathcal{V}_h. \tag{2.9}$$

It follows from (2.8), (2.9) and Lemma 2.2 that there exist matrices $\alpha\mathbf{I} \pm \mathbf{S}_h, \alpha\mathbf{I} \pm \mathbf{H}_h$ such that

$$\langle(\alpha\mathbf{I} + \mathbf{S}_h)(\mathbf{u}_h - \mathbf{u}^h), \mathbf{v}_h\rangle = \langle(\alpha\mathbf{I} - \mathbf{H}_h)(\mathbf{u}_h - \mathbf{u}_H), \mathbf{v}_h\rangle, \tag{2.10}$$

$$\langle(\alpha\mathbf{I} + \mathbf{H}_h)(\mathbf{u}_h - \mathbf{u}_h^*), \mathbf{v}_h\rangle = \langle(\alpha\mathbf{I} - \mathbf{S}_h)(\mathbf{u}_h - \mathbf{u}^h), \mathbf{v}_h\rangle. \tag{2.11}$$

Obviously, $\mathbf{S}_h$ is skew-symmetric and $\mathbf{H}_h$ is SPD, so $(\alpha\mathbf{I} \pm \mathbf{S}_h)^{-1}$ exist and $\alpha\mathbf{I} + \mathbf{H}_h$ is also SPD. Let $\mathbf{Q}(\alpha) = (\alpha\mathbf{I} - \mathbf{S}_h)(\alpha\mathbf{I} + \mathbf{S}_h)^{-1}$. Clearly, $\mathbf{Q}(\alpha)$ is a unitary matrix. Combining this with (2.10) and (2.11) yields

$$\begin{aligned}
\langle(\alpha\mathbf{I} + \mathbf{H}_h)(\mathbf{u}_h - \mathbf{u}_h^*), \mathbf{v}_h\rangle &= \langle(\alpha\mathbf{I} - \mathbf{S}_h)(\alpha\mathbf{I} + \mathbf{S}_h)^{-1}(\alpha\mathbf{I} - \mathbf{H}_h)(\mathbf{u}_h - \mathbf{u}_H), \mathbf{v}_h\rangle\\
&= \langle\mathbf{Q}(\alpha)(\alpha\mathbf{I} - \mathbf{H}_h)(\mathbf{u}_h - \mathbf{u}_H), \mathbf{v}_h\rangle.
\end{aligned} \tag{2.12}$$

Define the bilinear form

$$\mathcal{M}(w_h, v_h) = \int_\Omega (\nabla u_h \cdot \nabla v_h + u_h v_h)dx \quad \forall\ w_h, v_h \in \mathcal{V}_h.$$

Obviously, $\|w_h\|_1^2 = \mathcal{M}(w_h, w_h)$. It follows from Lemma 2.2 that there exists a unique SPD matrix $\mathbf{M}_h$ such that

$$\mathcal{M}(w_h, v_h) = \langle\mathbf{M}_h\mathbf{w}_h, \mathbf{v}_h\rangle \quad \forall\ w_h, v_h \in \mathcal{V}_h. \tag{2.13}$$

Setting $\mathbf{v}_h = (\alpha\mathbf{I} + \mathbf{H}_h)^{-1}\mathbf{M}_h(\mathbf{u}_h - \mathbf{u}_h^*)$, by the Schwaz inequality, (2.12) and (2.13), we get

$$\begin{aligned}
&\langle\mathbf{M}_h(\mathbf{u}_h - \mathbf{u}_h^*), \mathbf{u}_h - \mathbf{u}_h^*\rangle\\
&\le |\langle\mathbf{Q}(\alpha)(\alpha\mathbf{I} - \mathbf{H}_h)(\mathbf{u}_h - \mathbf{u}_H), (\alpha\mathbf{I} + \mathbf{H}_h)^{-1}\mathbf{M}_h(\mathbf{u}_h - \mathbf{u}_h^*)\rangle|\\
&= |\langle\mathbf{M}_h^{\frac{1}{2}}(\alpha\mathbf{I} + \mathbf{H}_h)^{-1}\mathbf{Q}(\alpha)(\alpha\mathbf{I} - \mathbf{H}_h)(\mathbf{u}_h - \mathbf{u}_H), \mathbf{M}_h^{\frac{1}{2}}(\mathbf{u}_h - \mathbf{u}_h^*)\rangle|\\
&= |\langle\mathbf{B}_h\mathbf{M}_h^{\frac{1}{2}}(\mathbf{u}_h - \mathbf{u}_H), \mathbf{M}_h^{\frac{1}{2}}(\mathbf{u}_h - \mathbf{u}_h^*)\rangle|\\
&\le \sqrt{\langle\mathbf{M}_h(\mathbf{u}_h - \mathbf{u}_h^*), \mathbf{u}_h - \mathbf{u}_h^*\rangle}\sqrt{\langle\mathbf{B}_h^T\mathbf{B}_h\mathbf{M}_h^{\frac{1}{2}}(\mathbf{u}_h - \mathbf{u}_H), \mathbf{M}_h^{\frac{1}{2}}(\mathbf{u}_h - \mathbf{u}_H)\rangle}\\
&\le \rho(\mathbf{B}_h)\sqrt{\langle\mathbf{M}_h(\mathbf{u}_h - \mathbf{u}_h^*), \mathbf{u}_h - \mathbf{u}_h^*\rangle}\sqrt{\langle\mathbf{M}_h(\mathbf{u}_h - \mathbf{u}_H), (\mathbf{u}_h - \mathbf{u}_H)\rangle},
\end{aligned} \tag{2.14}$$

where $\mathbf{B}_h = \mathbf{M}_h^{\frac{1}{2}}(\alpha\mathbf{I} + \mathbf{H}_h)^{-1}\mathbf{Q}(\alpha)(\alpha\mathbf{I} - \mathbf{H}_h)\mathbf{M}_h^{-\frac{1}{2}}$ and $\rho(\mathbf{B}_h)$ denotes the spectrum radius of $\mathbf{B}_h$. From (2.13), (2.14) and Lemma 2.1, we have

$$\|u_h - u_h^*\|_1 \le \rho(\mathbf{B}_h)\|u_h - u_H\|_1 = \rho((\alpha\mathbf{I} + \mathbf{H}_h)^{-1}\mathbf{Q}(\alpha)(\alpha\mathbf{I} - \mathbf{H}_h))\|u_h - u_H\|_1 \le C\rho(\alpha)H\|u\|_2, \tag{2.15}$$

where $\rho(\alpha) = \rho((\alpha\mathbf{I} + \mathbf{H}_h)^{-1}\mathbf{Q}(\alpha)(\alpha\mathbf{I} - \mathbf{H}_h))$. Since $\mathbf{Q}(\alpha)$ is a unitary matrix, we have

$$\begin{aligned}
\rho(\alpha) &= \rho(\mathbf{Q}(\alpha)(\alpha\mathbf{I} + \mathbf{H}_h)^{-1}(\alpha\mathbf{I} - \mathbf{H}_h))\\
&\le \|\mathbf{Q}(\alpha)(\alpha\mathbf{I} + \mathbf{H}_h)^{-1}(\alpha\mathbf{I} - \mathbf{H}_h)\|_0\\
&\le \|(\alpha\mathbf{I} + \mathbf{H}_h)^{-1}(\alpha\mathbf{I} - \mathbf{H}_h)\|_0,
\end{aligned}$$

where $\|\cdot\|_0$ denotes spectrum norm. Hence,

$$\rho(\alpha) \leq \max_{\lambda_i \in \lambda(\mathbf{H}_h)} |\frac{\alpha - \lambda_i}{\alpha + \lambda_i}| = \tau(\widehat{\alpha}) = \frac{\sqrt{\lambda_{max}} - \sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}} + \sqrt{\lambda_{min}}} = \frac{\sqrt{\kappa(\mathbf{H}_h)} - 1}{\sqrt{\kappa(\mathbf{H}_h)} + 1}$$

holds for $\widehat{\alpha} = \sqrt{\lambda_{max}\lambda_{min}}$, where λ_{min} and λ_{max} are the minimum and the maximum eigenvalues of the matrix $\mathbf{H}_h$, respectively, and $\kappa(\mathbf{H}_h)$ denotes the spectrum condition number of $\mathbf{H}_h$. Hence, $\rho(\alpha) \leq \tau(\widehat{\alpha}) < 1$ for any $\alpha > 0$.

It follows from (2.15) and Lemma 2.1 that

$$\|u - u_h^*\|_1 \leq \|u - u_h\|_1 + \|u_h - u_h^*\|_1 \leq C(h + \rho(\alpha)H)\|u\|_2.$$

This completes the proof.

According to Theorem 2.1, to derive the optimal accuracy, we can take $H = \frac{1}{\rho(\alpha)}h$ for $\mathcal{H}^1$ norm. Moreover, the convergence depends on the parameter α and SPD part $\mathcal{H}$, but does not depend on the skew-Hermitian part $\mathcal{S}$. It has been proved that $\rho(\alpha) < 1$, however, it is difficult to show that how to make $\rho(\alpha)$ sufficiently small by choosing the optimal parameter α.

Since HSS iteration method converges unconditionally to the exact solution of the problem, we shall present another algorithm without coarse grid correction repeatedly.

Algorithm 2.2 *1. Find $u_H \in \mathcal{V}_H$ such that*

$$\mathcal{A}(u_H, v_H) = (f, v_H) \quad \forall\ v_H \in \mathcal{V}_H.$$

2. Set $u_h^0 = u_H$. For any $v_h \in \mathcal{V}_h$ and $k = 0, 1, \ldots, K-1$, find $u_h^k \in \mathcal{V}_h$ such that

$$\begin{aligned}(\alpha\mathcal{I} + \mathcal{S})(u_h^{k+\frac{1}{2}}, v_h) &= (\alpha\mathcal{I} - \mathcal{H})(u_h^k, v_h) + (f, v_h),\\ (\alpha\mathcal{I} + \mathcal{H})(u_h^{k+1}, v_h) &= (\alpha\mathcal{I} - \mathcal{S})(u_h^{k+\frac{1}{2}}, v_h) + (f, v_h).\end{aligned}$$

Theorem 2.2 *Suppose that Assumption 1 is satisfied and $u_h^K \in \mathcal{V}_h$ is obtained by Algorithm 2.2 for $K \geq 1$. Then*

$$\|u_h - u_h^K\|_1 \leq C\rho(\alpha)^K H\|u\|_2 \quad \textit{and} \quad \|u - u_h^K\|_1 \leq C(h + \rho(\alpha)^K H)\|u\|_2$$

hold for $u \in \mathcal{H}^1(\Omega)$, where $\rho(\alpha)$ is defined in Theorem 2.1.

Proof: This proof is similar to that of Theorem 2.1. It follows from Algorithm 2.2 that

$$\begin{aligned}(\alpha I + \mathcal{S})(u_h - u_h^{k+\frac{1}{2}}, v_h) &= (\alpha I - \mathcal{H})(u_h - u_h^k, v_h),\\ (\alpha\mathcal{I} + \mathcal{H})(u_h - u_h^{k+1}, v_h) &= (\alpha\mathcal{I} - \mathcal{S})(u_h - u_h^{k+\frac{1}{2}}, v_h).\end{aligned}$$

Similar to the derivation of (2.15), we get

$$\|u_h - u_h^{k+1}\|_1 \leq \rho(\alpha)\|u_h - u_h^k\|_1.$$

Combining the above inequality with the estimation in Theorem 2.1 implies

$$\|u_h - u_h^K\|_1 \leq \rho(\alpha)^K\|u_h - u_h^0\|_1 \leq C\rho(\alpha)^K H\|u\|_2.$$

This completes the proof.

From the above proposed algorithms, we see that each iteration on the fine-grid space alternates between the SPD part $\mathcal{H}$ and the skew-symmetric part $\mathcal{S}$ of the operator $\mathcal{L}$, analogously to the classical alternating direction implicit (ADI) iteration for solving PDEs ([22]). Consider the convection-diffusion problem

$$\begin{cases} -\epsilon\triangle u + \mathbf{b}(x)\cdot\nabla u + c(x)u &= f, \text{ in } \Omega,\\ u &= 0, \text{ on } \partial\Omega, \end{cases}$$

where $\epsilon > 0$ and $\epsilon \ll \|\mathbf{b}(x)\|$. When these new two-grid algorithms are applied to solve the above problem, from inequality (2.15), we have that $\|u_h - u_h^*\|_1 \leq \rho(\alpha)\|u_h - u_H\|_1 \leq \tau(\widehat{\alpha})\|u_h - u_H\|_1$ for some positive constant $\tau(\widehat{\alpha}) < 1$. Therefore, our algorithms converges unconditionally in $\mathcal{H}^1$ norm no matter how small ϵ is. The numerical experiment in section 4 shows that Algorithm 2.2 is effective for some convection-diffusion problems.

3 Two-grid algorithms for nonlinear elliptic problems

Consider the mildly nonlinear elliptic equation of the form

$$\begin{cases} \mathcal{L}(u)u \equiv -\nabla\cdot(\mathbf{a}(x,u)\nabla u) + \mathbf{b}(x,u)\cdot\nabla u + f(x,u) &= 0, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

under the following assumptions.

Assumption 2: (1) Assume that $\mathbf{a}(x,u) \in C^1(\overline{\Omega}\times R, R^{2\times 2})$, $\mathbf{b}(x,u) \in C^1(\overline{\Omega})^2$, $f(x,u) \in C^1(\overline{\Omega})$ and $\mathcal{L}'(w) : \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is an isomorphism. Here $\mathcal{L}'(w)$ denotes the Frécher derivative of $\mathcal{L}$ at w and is given by $\mathcal{L}'(w)v = -\nabla\cdot(\mathbf{a}(x,w)\nabla v + \mathbf{a}'(x,w)\cdot\nabla wv) + \mathbf{b}'(x,w)\cdot\nabla wv + \mathbf{b}(x,w)\cdot\nabla v + f'(x,w)v$.

(2) Assume that $\mathbf{a}(x,u)$ is symmetric and there exists a positive constant m such that $\xi^T\mathbf{a}(x,u)\xi \geq m|\xi|^2$ for any $\xi \in R^2$, $x \in \overline{\Omega}$, $u \in R$.

From Assumption 2, it can be seen that (3.1) has an isolated solution ([34]). The variational form of (3.1) is to find $u \in \mathcal{H}_0^1(\Omega)$ such that

$$\mathcal{A}(u,v) = 0 \quad \forall\ v \in \mathcal{H}_0^1(\Omega), \tag{3.2}$$

where the natural bilinear form is defined by

$$\mathcal{A}(u,v) = \int_\Omega (\mathbf{a}(u)\nabla u\cdot\nabla v + \mathbf{b}(u)\cdot\nabla uv + f(u)v)dx \quad \forall\ u,v \in \mathcal{H}_0^1(\Omega).$$

The finite element approximation of u in (3.2) is to find $u_h \in \mathcal{V}_h$ such that

$$\mathcal{A}(u_h,v_h) = 0 \quad \forall\ v_h \in \mathcal{V}_h. \tag{3.3}$$

To present new algorithms, for $u,v,w \in \mathcal{W}_\infty^1(\Omega)\cap\mathcal{H}_0^1(\Omega)$, we define

$$\mathcal{A}(w;u,v) = \int_\Omega (\mathbf{a}(w)\nabla u\cdot\nabla v + \mathbf{b}(w)\cdot\nabla uv + f(w)v)dx \quad \forall\ u,v,w \in \mathcal{H}_0^1(\Omega)$$

and

$$\begin{cases} \mathcal{H}(w;u,v) = \displaystyle\int_\Omega \mathbf{a}(w)\nabla u\cdot\nabla v dx + \int_\Omega (f(w) - \frac{1}{2}\nabla\cdot\mathbf{b}(w))uvdx, \\ \mathcal{S}(w;u,v) = \displaystyle\int_\Omega (\mathbf{b}(w)\cdot\nabla u)vdx + \int_\Omega (\frac{1}{2}\nabla\cdot\mathbf{b}(w))uvdx. \end{cases} \tag{3.4}$$

Denote $(\alpha\mathcal{I}\pm\mathcal{H})(w;u,v) = \alpha\mathcal{I}(w;u,v)\pm\mathcal{H}(w;u,v)$ and $(\alpha\mathcal{I}\pm\mathcal{S})(w;u,v) = \alpha\mathcal{I}(w;u,v)\pm\mathcal{S}(w;u,v)$, where the bilinear form $\mathcal{I}(w_h;u_h,v_h) = \mathcal{I}(u_h,v_h)$ satisfies $\mathcal{I}(\varphi_i,\varphi_j) = I_{ij}$ $(i,j=1,\ldots,n)$ and $\mathbf{I} = (I_{ij})_{n\times n}$ is the identity matrix.

Algorithm 3.1 *1. Find $u_H \in \mathcal{V}_H$ such that*

$$\mathcal{A}(u_H,v_H) = 0 \quad \forall\ v_H \in \mathcal{V}_H.$$

2. Find $u_h^* \in \mathcal{V}_h$ such that

$$\begin{aligned} (\alpha\mathcal{I}+\mathcal{S})(u_H;u^h,v_h) &= (\alpha\mathcal{I}-\mathcal{H})(u_H;u_H,v_h) \quad \forall\ v_h \in \mathcal{V}_h, \\ (\alpha\mathcal{I}+\mathcal{H})(u_H;u_h^*,v_h) &= (\alpha\mathcal{I}-\mathcal{S})(u_H;u^h,v_h) \quad \forall\ v_h \in \mathcal{V}_h. \end{aligned}$$

Theorem 3.1 *Suppose that Assumption 2 is satisfied and $u_h^* \in \mathcal{V}_h$ is obtained by Algorithm 3.1. Then*

$$\|u_h - u_h^*\|_1 \leq C(\rho(\alpha)H + H^2) \quad \textit{and} \quad \|u - u_h^*\|_1 \leq C(h + \rho(\alpha)H + H^2),$$

where $\rho(\alpha)$ is defined in Theorem 2.1.

Proof: It follows from (3.3), (3.4) and Algorithm 3.1 that

$$\begin{aligned} &(\alpha\mathcal{I}+\mathcal{H})(u_H;u_h-u_h^*,v_h) \\ &= [(\alpha\mathcal{I}+\mathcal{H})(u_H;u_h,v_h) - (\alpha\mathcal{I}+\mathcal{H})(u_h;u_h,v_h)] \\ &\quad + [(\alpha\mathcal{I}-\mathcal{S})(u_h;u_h,v_h) - (\alpha\mathcal{I}-\mathcal{S})(u_H;u_h,v_h)] \\ &\quad + [(\alpha\mathcal{I}-\mathcal{S})(u_H;u_h,v_h) - (\alpha\mathcal{I}-\mathcal{S})(u_H;u^h,v_h)] \\ &= Q_1 + Q_2 + (\alpha\mathcal{I}-\mathcal{S})(u_H;u_h-u^h,v_h). \end{aligned} \tag{3.5}$$

By the Cauchy-Schwarz inequality and Lemma 2.1, we obtain

$$\begin{aligned}
|Q_1| &= |(\alpha\mathcal{I}+\mathcal{H})(u_H;u_h,v_h)-(\alpha\mathcal{I}+\mathcal{H})(u_h;u_h,v_h)| \\
&\le \Big|\int_\Omega (\mathbf{a}(u_H)-\mathbf{a}(u_h))\nabla u_h\cdot\nabla v_h dx\Big| \\
&\quad + \Big|\int_\Omega [(f(u_H)-f(u_h))-\frac{1}{2}(\nabla\cdot\mathbf{b}(u_H)-\nabla\cdot\mathbf{b}(u_h)]u_h v_h dx\Big| \\
&\le C\int_\Omega |u_H-u_h||\nabla u_h\cdot\nabla v_h|dx + C\int_\Omega |u_H-u_h||u_h v_h|dx \\
&\le C|u_h|_{1,\infty}\|u_h-u_H\|\|v_h\|_1 \\
&\le CH^2\|\|v_h\|_1, \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
|Q_2| &= |(\alpha\mathcal{I}-\mathcal{S})(u_h;u_h,v_h)-(\alpha\mathcal{I}-\mathcal{S})(u_H;u_h,v_h)| \\
&\le \Big|\int_\Omega [\mathbf{b}(u_H)-\mathbf{b}(u_h)]\cdot\nabla u_h v_h dx\Big| + \frac{1}{2}\Big|\int_\Omega [\nabla\cdot\mathbf{b}(u_H)-\nabla\cdot\mathbf{b}(u_h)]u_h v_h dx\Big| \\
&\le C|u_h|_{1,\infty}\|u_h-u_H\|\|v_h\|_1 \\
&\le CH^2\|\|v_h\|_1. \qquad (3.7)
\end{aligned}$$

Again, from (3.3), (3.4) and Algorithm 3.1, we also obtain

$$\begin{aligned}
&(\alpha\mathcal{I}+\mathcal{S})(u_H;u_h-u^h,v_h) \\
&= [(\alpha\mathcal{I}+\mathcal{S})(u_H;u_h,v_h)-(\alpha\mathcal{I}+\mathcal{S})(u_h;u_h,v_h)] \\
&\quad + [(\alpha\mathcal{I}-\mathcal{H})(u_h;u_h,v_h)-(\alpha\mathcal{I}-\mathcal{H})(u_H;u_h,v_h)] \\
&\quad + [(\alpha\mathcal{I}-\mathcal{H})(u_H;u_h,v_h)-(\alpha\mathcal{I}-\mathcal{H})(u_H;u_H,v_h)] \\
&= \widehat{Q}_1+\widehat{Q}_2+(\alpha I-\mathcal{H})(u_H;u_h-u_H,v_h). \qquad (3.8)
\end{aligned}$$

Using the same techniques for (3.6) and (3.7), we have

$$|\widehat{Q}_1|\le CH^2\|\|v_h\|_1 \quad\text{and}\quad |\widehat{Q}_2|\le CH^2\|\|v_h\|_1. \qquad (3.9)$$

It follows from (3.5)-(3.9) that

$$(\alpha\mathcal{I}+\mathcal{S})(u_H;u_h-u^h,v_h)\le CH^2\|v_h\|_1+(\alpha\mathcal{I}-\mathcal{H})(u_H;u_h-u_H,v_h), \qquad (3.10)$$

$$(\alpha\mathcal{I}+\mathcal{H})(u_H;u_h-u_h^*,v_h)\le CH^2\|v_h\|_1+(\alpha\mathcal{I}-\mathcal{S})(u_H;u_h-u^h,v_h). \qquad (3.11)$$

Similar to the derivation of (2.15), the above inequalities imply

$$\|u_h-u_h^*\|_1\le CH^2+\rho(\alpha)\|u_h-u_H\|_1\le C(\rho(\alpha)H+H^2).$$

Then, it follows from Lemma 2.1 ithat

$$\|u-u_h^*\|_1\le\|u-u_h\|_1+\|u_h-u_h^*\|_1\le C(h+\rho(\alpha)H+H^2).$$

This completes the proof.

We see that Algorithm 3.1 reduces a nonlinear system to two (a shifted skew-symmetric and a SPD) linear systems on the fine-grid space and a nonlinear system on the coarse-grid space. If the parameter α is chosen to make $\rho(\alpha)$ sufficiently small, the mesh size of the coarse grid H can be chosen as $\frac{1}{\rho(\alpha)}h$. Therefore, the coarse-grid space $\mathcal{V}_H$ can be chosen much smaller than the fine-grid space $\mathcal{V}_h$ and still achieve optimal approximation. However, for some cases, $\rho(\alpha)$ is close to 1. To make coarse grid H sufficiently small, it need to add the number of iterations on the fine-grid space, and we obtain another two-grid algorithm.

Algorithm 3.2 *1. Find $u_H\in\mathcal{V}_H$ such that*

$$\mathcal{A}(u_H,v_H)=0\quad\forall\ v_H\in\mathcal{V}_H.$$

2. Set $u_h^0 = u_H$. *For any* $v_h \in \mathcal{V}_h$ *and* $k = 0, 1, \ldots, K-1$, *find* $u_h^k \in \mathcal{V}_h$ *such that*

$$(\alpha\mathcal{I} + \mathcal{S})(u_H; u_h^{k+\frac{1}{2}}, v_h) = (\alpha\mathcal{I} - \mathcal{H})(u_H; u_h^k, v_h) \quad \forall\ v_h \in \mathcal{V}_h,$$
$$(\alpha\mathcal{I} + \mathcal{H})(u_H; u_h^{k+1}, v_h) = (\alpha\mathcal{I} - \mathcal{S})(u_H; u_h^{k+\frac{1}{2}}, v_h) \quad \forall\ v_h \in \mathcal{V}_h.$$

Theorem 3.2 *Suppose that Assumption 2 is satisfied and* $u_h^K \in \mathcal{V}_h$ *is obtained by Algorithm 3.2 for* $K \geq 1$. *Then*

$$\|u_h - u_h^K\|_1 \leq C(\rho(\alpha)^K H + H^2) \quad \textit{and} \quad \|u - u_h^K\|_1 \leq C(h + \rho(\alpha)^K H + H^2)$$

hold for $u \in \mathcal{H}^1(\Omega)$, *where* $\rho(\alpha)$ *is defined in Theorem 2.1.*

Proof: Using the same technique for (3.10) and (3.11), from Algorithm 3.2, we have

$$(\alpha\mathcal{I} + \mathcal{S})(u_H; u_h - u_h^{k+\frac{1}{2}}, v_h) \leq CH^2\|v_h\|_1 + (\alpha\mathcal{I} - \mathcal{H})(u_H; u_h - u_h^k, v_h),$$
$$(\alpha\mathcal{I} + \mathcal{H})(u_H; u_h - u_h^{k+1}, v_h) \leq CH^2\|v_h\|_1 + (\alpha I - \mathcal{S})(u_H; u_h - u_h^{k+\frac{1}{2}}, v_h).$$

Similar to the derivation of (2.15), the above inequalities imply

$$\|u_h - u_h^{k+1}\|_1 \leq \rho(\alpha)\|u_h - u_h^k\|_1 + CH^2.$$

Combining the above inequality with the estimation in Theorem 3.1 implies

$$\|u_h - u_h^K\|_1 \leq \rho(\alpha)^K\|u_h - u_h^0\|_1 + C\frac{1}{1-\rho(\alpha)}H^2 \leq C(\rho(\alpha)^K H + H^2).$$

This completes the proof.

4 Numerical experiments

In this section, we present two numerical examples for Algorithm 2.2 and Algorithm 3.2, the first example is for linear equation (2.2) and the second one is for quasi-linear equation (3.1). The experiments were carried out on a PC with Intel CPU T5600 and 2.50 GB memory. For all tests we use the piecewise linear basis functions.

Example 4.1: (standard test problem as in [3, 31].)

$$\begin{cases} -\varepsilon\triangle u + \mathbf{b}\cdot\nabla u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\mathbf{b} = (\cos\varphi, \sin\varphi)^T$ and f is chosen so that $u = x(1-x)\sin(\pi y)$ is the exact solution. The domain $\Omega = [0,1]\times[0,1]$ is uniformly divided into families $\mathcal{T}_H$ and $\mathcal{T}_h$ of triangulations.

In the experiment, we compare Algorithm 2.2 with Algorithm 4.1 proposed in [34] by taking different values of φ and ε. The coarse mesh size and fine mesh size are set as $H = \frac{1}{64}$ and $h = \frac{1}{256}$, respectively, and the choices of parameters α are listed in Table 1. The CPU times are given when the solutions computed by Algorithm 2.2 and Algorithm 4.1 of [34] approximating to the finite element solution u_h. From the numerical results listed in Table 2, Algorithm 2.2 spends more CPU times when $\varepsilon = 1$ since it need to solve shifted skew-symmetric systems on the fine grid. Fortunately([12]) the skew-symmetric systems can be solved to relatively low accuracy by inner iteration schemes, such as the Krylov subspace method. In the experiment, the GMRES method with low accuracy is used, so it does not need to spend too much time to solve the skew-symmetric systems. However, if ε becomes smaller, i.e., equation (4.1) is diffusion-dominated, Algorithm 2.2 can derive better results.

Table 1. The values of α for Algorithm 2.2.

ε	1	1e-2	1e-3	1e-4
α	1.2	2e-2	2.5e-3	8e-4

Table 2. $\mathcal{H}^1$-error for equation (4.1) computed by Algorithm 2.2 and Algorithm 4.1 in [34].

Algorithms		Algorithm 2.2		Algorithm 4.1 in [34]	
φ	ε	$\|u-u_h^*\|_1$	CPU Time(s)	$\|u-u^h\|_1$	CPU Time(s)
0	1	1.13e-3	77.04	1.13e-3	57.04
	1e-2	1.13e-3	79.66	1.35e-3	57.84
	1e-4	1.14e-3	83.41	7.86e-3	86.96
	1e-5	1.48e-3	99.34	7.96e-3	111.89
$\frac{\pi}{16}$	1	1.13e-3	80.15	1.13e-3	59.76
	1e-2	1.13e-3	79.60	1.35e-3	59.83
	1e-4	1.17e-3	82.89	7.87e-3	86.88
	1e-5	1.54e-3	105.77	8.04e-3	113.66
$\frac{\pi}{8}$	1	1.13e-3	81.76	1.13e-3	60.85
	1e-2	1.13e-3	79.68	1.35e-3	60.41
	1e-4	1.20e-3	82.86	7.88e-3	87.39
	1e-5	1.79e-3	105.31	8.18e-3	113.30
$\frac{3\pi}{8}$	1	1.13e-3	81.64	1.13e-3	66.06
	1e-2	1.13e-3	79.54	1.35e-3	60.96
	1e-4	1.20e-3	82.49	7.87e-3	86.89
	1e-5	1.72e-3	104.46	8.04e-3	116.69

Example 4.2: Consider the *stationary viscous Burger'equation*

$$\begin{cases} -\varepsilon\triangle u + \mathbf{b}(u)\cdot\nabla u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $\varepsilon = 0.1$, $\mathbf{b}(u) = (u,u)^T$ and f is so chosen that $u = 50\cos(2\pi x + \frac{\pi}{2})(y^2 - y^3)$ is the exact solution. The domain $\Omega = [0,1]\times[0,1]$ is uniformly divided into families $\mathcal{T}_H$ and $\mathcal{T}_h$ of triangulations. The standard finite element solution u_h on different meshes is computed by Newton-GMRES method and the numerical results are listed in Table 3. In the implementation of Algorithm 3.2, we fix the fine grid h and vary the coarse grid. We solve equation (4.2) exactly on the coarse grid and solve a linear system on the fine grid by HSS iteration method. The numerical results listed in Table 4 shows that u_h^* computed by Algorithm 3.2 can approximate to the standard finite element solution u_h very well.

Table 3. Error of u_h in $\mathcal{H}^1$ norms for equation (4.2).

Mesh	$h=\frac{1}{32}$	$h=\frac{1}{64}$	$h=\frac{1}{128}$	$h=\frac{1}{256}$	$h=\frac{1}{512}$
$\|u-u_h\|_1$	6.60e-1	3.29e-1	1.64e-1	8.22e-2	4.11e-2

Table 4. $\mathcal{H}^1$-error for equation (4.2) computed by Algorithm 3.2.

Meshes	$H=\frac{1}{64}, h=\frac{1}{256}$	$H=\frac{1}{128}, h=\frac{1}{256}$	$H=\frac{1}{128}, h=\frac{1}{512}$	$H=\frac{1}{256}, h=\frac{1}{512}$
$\|u-u_h^*\|_1$	8.24e-2	8.22e-2	4.11e-2	4.11e-2
α	3e-2	2e-2	1.75e-2	1.75e-2

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A note on Nonlinear boundary value problem for first order impulsive integro-differential equations of mixed type [J. Math. Anal. Appl. 325 (2007) 830-842]

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Abstract In this note, we first give two counterexamples to the comparison principles given by L. Chen and J. Sun [J. Math. Anal. Appl. 325 (2007) 830-842]. Then we suggest and show corrected versions of the comparison principles.

Keywords Impulsive differential equations, Nonlinear boundary value problem,

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Counter examples, Comparison principles.

1 Introduction and preliminaries

In recent years, there are vast scale of papers devoted to the nonlinear boundary value problems for impulsive differential equations(see [1]-[7]). This type of equations has become an important aspect of impulsive differential equations for its deep physical background and realistic mathematical model. It is well known that the comparison principles play important roles in the theory of impulsive differential equations, for example, they are essential for developing the monotone iterative method, which permits us to construct a sequence of approximate solutions converging to the extremal solutions of differential equations with nonlinear boundary value conditions (see [1]-[7]). The purpose of this note is to give out two counterexamples on the comparison principles in Lemmas 1 and 2 of [1]. And then we establish two new comparison principles.

In [1], Chen and Sun considered the following nonlinear boundary value problem

$$\begin{cases} x'(t) = f(t, x(t), x(\theta(t)), (Wx)(t), (Sx)(t)), & t \neq t_k, t \in J = [0, T], \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \cdots, p, \\ g(x(0), x(T)) = 0, \end{cases} \tag{1.1}$$

where $f \in C(J \times R^4, R), I_k \in C(R, R), g \in C(R \times R, R), \theta \in C(J, J), \Delta x(t_k) = x(t_k^+) - x(t_k^-)$, in which $x(t_k^+), x(t_k^-)$ denote the right and left limit of $x(t)$ at $t_k, k = 1, 2, \cdots, p$, which are fixed points such that $0 < t_1 < t_2 < \cdots < t_p < T, J_0 = J \setminus \{t_1, t_2, \cdots t_p\}$. The integral part in Eq. (1.1) is defined by

$$(Wx)(t) = \int_0^{\gamma(t)} K(t, s)x(\delta(s))ds, \qquad (Sx)(t) = \int_0^T H(t, s)x(\sigma(s))ds,$$

here $\gamma, \delta, \sigma \in C(J, J), K(t, s) \in C(D, R^+), D = \{(t, s) \in J \times J : 0 \leq s \leq \gamma(t), t \in J\}, H(t, s) \in C(J \times J, R^+), K^* = \max\{K(t, s) : (t, s) \in D\}, H^* = \max\{H(t, s) : (t, s) \in J \times J\}$.

For convenience, we first introduce some definitions: $PC(J) = \{u : J \to R : u$ is continuous for any $t \in J_0$; $u(t_k^+)$,$u(t_k^-)$ exist,$k = 1, 2, \cdots, p$ and $u(t_k^-) = u(t_k)\}$, $PC^1(J) = \{u \in PC(J) : u$ is continuously differentiable for any $t \in J_0$; $u'(0^+)$, $u'(T^-)$, $u'(t_k^+)$ and $u'(t_k^-)$ exist for $k = 1, 2, \cdots p\}$. It is clear that $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\| u \|_{PC(J)} = \sup\{| u(t) | : t \in J\}, \qquad \| u \|_{PC^1(J)} = \| u \|_{PC(J)} + \| u' \|_{PC(J)} .$$

2 Problem and statement

In this section we first recall the comparison principles in [1].

Comparison Principle 2.1 (Lemma 1 of [1]) If $u \in PC^1(J)$ and satisfies

$$\begin{cases} u'(t) \le -M_1 u(t) - M_2 u(\theta(t)) - M_3 (Wu)(t) - M_4 (Su)(t), & t \ne t_k, t \in J, \\ \Delta u(t_k) \le L_k u(t_k), & k = 1, 2, \cdots, p, \\ u(0) \le 0, \end{cases} \tag{2.1}$$

where $M_i \ge 0, i \in \{2, 3, 4\}$, $-1 < L_k \le 0$ for $k = 1, 2 \cdots p$. Assume that $0 \le \theta, \delta, \sigma \le t$ and

$$(1 + \prod_{j=1}^{p} (1 + L_k)^{-1}) \int_0^T [M_2 e^{M_1(t-\theta(t))} + M_3 \int_0^{\gamma(t)} K_1(t, s) ds + M_4 \int_0^T H_1(t, s) ds] dt \le 1, \tag{2.2}$$

where $K_1(t, s) = e^{M_1(t-\delta(s))} K(t, s), H_1(t, s) = e^{M_1(t-\sigma(s))} H(t, s)$. Then $u(t) \le 0$ on J.

Comparison Principle 2.2 (Lemma 2 of [1]) If $u \in PC^1(J)$ and satisfies

$$\begin{cases} u'(t) \ge M_1 u(t) + M_2 u(\theta(t)) + M_3 (Wu)(t) + M_4 (Su)(t), & t \ne t_k, t \in J, \\ \Delta u(t_k) \ge L_k u(t_k), & k = 1, 2, \cdots, p, \\ u(T) \le 0, \end{cases} \tag{2.3}$$

where $M_i \ge 0, i \in \{2, 3, 4\}$, $L_k \ge 0$ for $k = 1, 2 \cdots p$. Assume that $t \le \theta, \delta, \sigma \le T$ and

$$(1 + \prod_{j=1}^{p} (1 + L_k)) \int_0^T [M_2 e^{M_1(\theta(t)-t)} + M_3 \int_0^{\gamma(t)} K_2(t, s) ds + M_4 \int_0^T H_2(t, s) ds] dt \le 1, \tag{2.4}$$

where $K_2(t,s) = e^{M_1(\delta(s)-t)}K(t,s), H_2(t,s) = e^{M_1(\sigma(s)-t)}H(t,s)$. Then $u(t) \leq 0$ on J.

Now we give following counterexamples to Comparison Principles 2.1 and 2.2, respectively.

Counterexample 2.1 In Comparison Principle 2.1, we let $M_1 = M_2 = M_3 = 0, M_4 = 1, L_1 = 0, T = 2, t_1 = 1, H(t,s) \equiv \frac{1}{10}, \theta(t) = \gamma(t) = \delta(t) = \sigma(t) = t$. Then from $H_1(t,s) = e^{M_1(t-\sigma(s))}H(t,s)$, we have $H_1(t,s) \equiv \frac{1}{10}$. So

$$(1+\prod_{j=1}^{p}(1+L_k))\int_0^T [M_2 e^{M_1(t-\theta(t))} + M_3\int_0^{\gamma(t)} K_1(t,s)ds + M_4\int_0^T H_1(t,s)ds]dt = \frac{4}{5} < 1,$$

which shows (2.2) holds.

Set

$$u(t) = \begin{cases} -27t^2 + t, & t \in [0,1], \\ -27t^2 + t - 1, & t \in (1,2]. \end{cases}$$

We easily obtain $u(1^+) < u(1^-),\ u(0) = 0$. Also we have

$$\begin{aligned} u'(t) + \frac{1}{10}\int_0^2 u(s)ds &= -54t + 1 + \frac{1}{10}\int_0^2 (-27s^2 + s)ds \\ &= -54t - 6 < 0 \qquad \forall t \in [0,1]. \end{aligned}$$

$$\begin{aligned} u'(t) + \frac{1}{10}\int_0^2 u(s)ds &= -54t + 1 + \frac{1}{10}\int_0^2 (-27s^2 + s - 1)ds \\ &= -54t - \frac{31}{5} < 0 \qquad \forall t \in (1,2]. \end{aligned}$$

Therefore, $u(t)$ satisfies both (2.1) and (2.2). However, $u(t) > 0$ on $t \in (0, \frac{1}{27})$, which shows that Comparison Principle 2.1 is not true.

Counterexample 2.2 In Comparison Principle 2.2, we let $M_1 = M_2 = M_3 = 0, M_4 = 1, L_1 = 0, T = 1, t_1 = \frac{1}{2}, H(t,s) \equiv \frac{1}{3}, \theta(t) = \gamma(t) = \delta(t) = \sigma(t) = t$. From $H_2(t,s) = e^{M_1(\sigma(s)-t)}H(t,s)$, we have $H_2(t,s) \equiv \frac{1}{3}$. Then we easily obtain

$$(1+\prod_{j=1}^{p}(1+L_k))\int_0^T [M_2 e^{M_1(\theta(t)-t)} + M_3\int_0^{\gamma(t)} K_2(t,s)ds + M_4\int_0^T H_2(t,s)ds]dt = \frac{2}{3} < 1,$$

which shows (2.4) holds.

Set

$$u(t) = \begin{cases} -27t^2 + 53t - 27, & t \in [0, \frac{1}{2}], \\ -27t^2 + 53t - 26, & t \in (\frac{1}{2}, 1]. \end{cases}$$

We easily obtain $u(\frac{1}{2}^+) > u(\frac{1}{2}^-),\ u(1) = 0$. Also we have

$$\begin{aligned} u'(t) - \frac{1}{3}\int_0^1 u(s)ds &= -54t + 53 - \frac{1}{3}\int_0^1 (-27s^2 + 53s - 27)ds \\ &= -54t + 53 + \frac{19}{6} > 0, \qquad \forall t \in [0, \frac{1}{2}], \end{aligned}$$

$$\begin{aligned} u'(t) - \frac{1}{3}\int_0^1 u(s)ds &= -54t + 53 - \frac{1}{3}\int_0^1 (-27s^2 + 53s - 26)ds \\ &= -54t + 53 + \frac{17}{6} > 0, \qquad \forall t \in (\frac{1}{2}, 1], \end{aligned}$$

which shows (2.3) holds.

Therefore, $u(t)$ satisfies both (2.3) and (2.4). However, $u(t) > 0$ on $t \in (\frac{26}{27}, 1)$, which shows that Comparison Principle 2.2 is not true.

After carefully checking the proofs of Lemmas 1 and 2 in [1], we find the causes of the errors of the comparison principles.

Cause 2.1 In lemma 1 of [1], the authors only gave out the proof under the condition $t_* \le t^*$. However, if $t_* > t^*$, the conclusion is not correct. The same problem also occurs in the proof of Lemma 2 (for details, see [1]).

Now we show two new comparison principles as follows.

Theorem 2.1 Suppose that $u \in PC^1(J)$ and

$$\begin{cases} u'(t) \le -M_1 u(t) - M_2 u(\theta(t)) - M_3 (Wu)(t), & t \ne t_k, t \in J, \\ \Delta u(t_k) \le L_k u(t_k), & k = 1, 2, \cdots, p, \\ u(0) \le 0, \end{cases} \tag{2.5}$$

where $M_i \ge 0, i \in \{2, 3\}$, $-1 < L_k \le 0$ for $k = 1, 2 \cdots p$. Assume that $0 \le \theta, \gamma, \delta \le t$ and

$$(1 + \prod_{j=1}^{p}(1 + L_k)^{-1}) \int_0^T [M_2 e^{M_1(t-\theta(t))} + M_3 \int_0^{\gamma(t)} K_1(t, s)ds]dt \le 1, \tag{2.6}$$

where $K_1(t,s) = e^{M_1(t-\delta(s))}K(t,s)$. Then $u(t) \leq 0$ on J.

Proof. Let $v(t) = u(t)e^{M_1 t}$. Then we have

$$\begin{cases} v'(t) \leq -M_2 e^{M_1(t-\theta(t))}v(\theta(t)) - M_3(W_1 v)(t), & t \neq t_k, t \in J, \\ \Delta v(t_k) \leq L_k v(t_k), & k = 1, 2, \cdots p, \\ v(0) \leq 0, \end{cases}$$

where $(W_1 v)(t) = \int_0^{\gamma(t)} K_1(t,s)v(\delta(s))ds$.

Obviously, the functions $v(t)$ and $u(t)$ have the same sign on $t \in J$.

Suppose,to the contrary, that there exists a $t^* \in J$ such that $v(t^*) > 0$. Let $t_* \in [0, t^*)$, such that $v(t_*) = \inf\{v(t) : t \in [0, t^*)\} = -\lambda$, then $\lambda \geq 0$.

If $\lambda = 0$, we have $v'(t) \leq 0$ on $t \in [0, t^*]$. Therefore, $\Delta v(t_k) \leq 0,\ 0 < t_k \leq t^*$. Hence $v(t)$ is non-increasing and $v(t^*) \leq v(0) \leq 0$, which is a contradiction.

If $\lambda > 0$, assume that $t_* \in (t_i, t_{i+1}], t^* \in (t_j, t_{j+1}],\ i, j = 0, 1, 2, \cdots m, i \leq j$. Now,we integrate the first inequality of (2.5) between t_* and t_{i+1} and obtain

$$v(t_{i+1}) - v(t_*) \leq \lambda \int_{t_*}^{t_{i+1}} [M_2 e^{M_1(t-\theta(t))} + M_3 \int_0^{\gamma(t)} K_1(t,s)ds]dt.$$

Similarly we get

$$v(t_{i+2}) - (1 + L_{i+1})v(t_{i+1}) \leq \lambda \int_{t_{i+1}}^{t_{i+2}} [M_2 e^{M_1(t-\theta(t))} + M_3 \int_0^{\gamma(t)} K_1(t,s)ds]dt,$$

$$\vdots$$

$$v(t_j) - (1 + L_{j-1})v(t_{j-1}) \leq \lambda \int_{t_{j-1}}^{t_j} [M_2 e^{M_1(t-\theta(t))} + M_3 \int_0^{\gamma(t)} K_1(t,s)ds]dt,$$

$$v(t^*) - (1 + L_j)v(t_j) \leq \lambda \int_{t_j}^{t^*} [M_2 e^{M_1(t-\theta(t))} + M_3 \int_0^{\gamma(t)} K_1(t,s)ds]dt.$$

By summing up both sides of the inequalities above, we get from $v(t_*) = -\lambda$

$$(1 + \prod_{j=1}^{p}(1 + L_k)^{-1}) \int_0^T [M_2 e^{M_1(t-\theta(t))} + M_3 \int_0^{\gamma(t)} K_1(t,s)ds]dt > 1,$$

which is a contradiction. Therefore, $v(t) \leq 0$ on $t \in J$. The proof is complete.

Theorem 2.2 Suppose that $u \in PC^1(J)$ and

$$\begin{cases} u'(t) \geq M_1 u(t) + M_2 u(\theta(t)) + M_3 (Su)(t), & t \neq t_k, t \in J, \\ \Delta u(t_k) \geq L_k u(t_k), & k = 1, 2, \cdots, p, \\ u(T) \leq 0, \end{cases} \tag{2.7}$$

where $M_i \geq 0, i \in \{2, 3\}$, $L_k \geq 0$ for $k = 1, 2 \cdots p, (Su)(t) = \int_t^T H(t, s) u(\sigma(s)) ds$. Assume that $t \leq \theta, \sigma \leq T$ and

$$(1 + \prod_{j=1}^{p} (1 + L_k)) \int_0^T [M_2 e^{M_1(\theta(t)-t)} + M_3 \int_t^T H_2(t, s) ds] dt \leq 1 \tag{2.8}$$

where $H_2(t, s) = e^{M_1(\sigma(s)-t)} H(t, s)$. Then $u(t) \leq 0$ on J.

Proof. Let $v(t) = u(t) e^{-M_1 t}$. Then we have

$$\begin{cases} v'(t) \geq M_2 e^{M_1(\theta(t)-t)} v(\theta(t)) + M_3 (S_1 v)(t), & t \neq t_k, t \in J, \\ \Delta v(t_k) \geq L_k v(t_k), & k = 1, 2, \cdots p, \\ v(T) \leq 0, \end{cases}$$

where $(S_1 v)(t) = \int_t^T H_2(t, s) v(\sigma(s)) ds$.

Also the functions $v(t)$ and $u(t)$ have the same sign on $t \in J$.

Suppose,to the contrary, that there exists a $t^* \in J$ such that $v(t^*) > 0$. Let $t_* \in (t^*, T]$ such that $v(t_*) = \inf\{v(t) : t \in (t^*, T]\} = -b$. Then $b \geq 0$.

If $b = 0$, we have $v'(t) \geq 0$ on $t \in (t^*, T]$. So $\Delta v(t_k) \geq 0$, $t^* < t_k \leq T$ and $v(T) \geq v(t^*) > 0$, which is a contradiction.

If $b > 0$, assume that $t^* \in (t_i, t_{i+1}], t_* \in (t_j, t_{j+1}]$, $i, j = 0, 1, 2, \cdots m, i \leq j$. Now,we integrate the first inequality of (2.7) between t^* and t_{i+1}, we have

$$v(t_{i+1}) - v(t^*) \geq -b \int_{t^*}^{t_{i+1}} [M_2 e^{M_1(\theta(t)-t)} + M_3 \int_t^T H_2(t, s) ds] dt.$$

Similarly we get

$$v(t_{i+2}) - (1 + L_{i+1}) v(t_{i+1}) \geq -b \int_{t_{i+1}}^{t_{i+2}} [M_2 e^{M_1(\theta(t)-t)} + M_3 \int_t^T H_2(t, s) ds] dt,$$

$$\vdots$$

$$v(t_j) - (1 + L_{j-1})v(t_{j-1}) \geq -b \int_{t_{j-1}}^{t_j} [M_2 e^{M_1(\theta(t)-t)} + M_3 \int_t^T H_2(t,s)ds]dt,$$

$$v(t_*) - (1 + L_j)v(t_j) \geq -b \int_{t_j}^{t_*} [M_2 e^{M_1(\theta(t)-t)} + M_3 \int_t^T H_2(t,s)ds]dt.$$

By summing up both sides of the inequalities above, we have from $v(t_*) = -\lambda$

$$(1 + \prod_{j=1}^{p}(1 + L_k)) \int_0^T [M_2 e^{M_1(\theta(t)-t)} + M_3 \int_t^T H_2(t,s)ds]dt > 1,$$

which is a contradiction. Hence $v(t) \leq 0$ on $t \in J$. The proof is complete.

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On the integrability of solutions of non-autonomous differential equations of second order with multiple variable deviating arguments

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Abstract- This paper considers a non-autonomous differential equation of second order with multiple variable deviating arguments. By using time-varying delay Lyapunov-Krasovskii functional, we propose to find some sufficient conditions for the solutions of the equation considered are quadratic integrable.

1.Introduction

In applied science, second order nonlinear differential equations with and without time lag are used to model some practical problems in biology, chemistry, physics, mechanics, electronics, engineering, economy, control theory, medicine, atomic energy, information theory, etc. (see the books of Ahmad and Rama Mohana Rao [1], Gopalsamy [4], Kolmanovskii and Myshkis [7], Kolmanovskii and Nosov [8], Krasovskii [9] and the references thereof). On the other hand, till now, many theoretical results relative to the stability, boundedness, etc. of solutions of nonlinear differential equations of second order have been obtained in the literature (see the papers of Baker [2], Cheng and Xu [3], Kato ([5], [6]), Malyseva [10], Meng [11], Mureşan [12], Nápoles Valdés [13], Saker [14], Sun and Meng [16], Tunç [17-23], Tunç and Şevli [24], Tunç and Tunç [25], Zhao [26], Zhao et al. [27, 28], Wang [28], Xu [29] and the references cited in these sources).

Meanwhile, first, in 1995, Meng [11] considered ordinary linear differential equation

$$x''(t)+p(t)x'(t)+[q_1(t)+q_2(t)]x(t)=f(t). \tag{1}$$

With the aid of auxiliary functions and integral inequalities, the author obtained some sufficient conditions under which all solutions of Eq. (1) belong to $L^\infty[a,\infty)$ or $L^\infty[a,\infty)\cap L^2[a,\infty)$.

Later, in 2002, Run [29] studied second order time lag differential equation

$$(r(t)x'(t))'+[a(t)+b(t)]x(t)=f[t,x(t),x(\varphi(t))].$$

The author obtained two conclusions about the boundedness and quadratic integrability of the solutions of the above equation.

At the same time, in 2002, Sun and Meng [17] took into consideration nonlinear differential equation of second order without a deviating argument,

$$(r(t)x'(t))'+p(t)x'(t)+[q_1(t)+q_2(t)]x(t)=f(t,x(t)). \tag{2}$$

The authors studied quadratic integrability and boundedness of all solutions of Eq. (2), and the main results in [17] were obtained by application of an auxiliary function, the Lyapunov function.

Later, in 2005, Zhao et al. [27] concerned with the following second order differential equation without a deviating argument

$$(r(t)x'(t))'+p(t)x'(t)+q_1(t)x(t)+q_2(t)x(t-\tau)=f(t,x(t)). \tag{3}$$

Keywords- Non-autonomous; differential equation; second order; quadratic integrable; multiple deviating arguments.
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The authors established some sufficient conditions to show that all solutions of Eq. (3) belong to $L^{\infty}[a,\infty)$ or $L^{2}[a,\infty)$, respectively, and the Lyapunov functional method was employed as main tool.

Finally, in 2007, Zhao et al. [28] considered nonlinear differential equation of second order with a deviating argument

$$(r(t)x'(t))' + p(t)x'(t) + q_1(t)x(t) + q_2(t)x(h(t)) = f(t,x(t)). \tag{4}$$

The authors established a theorem which include some sufficient conditions and guarantee that all solutions of Eq. (4) are integrable.

In this paper, instead of Eq. (4), we consider a non-autonomous differential equation of second order with multiple variable deviating arguments,

$$(r(t)x'(t))' + p(t)x'(t) + q_0(t)x(t) + \sum_{i=1}^{n} q_i(t)x(h_i(t)) = f(t,x(t)), \tag{5}$$

where $t \in [a,\infty)$, $(a>0,\ a \in \Re)$, p, q_0 and q_i are continuous functions and r and h_i are differentiable functions on $[a,\infty)$ with $r(t)>0$, $0 \le h_i(t) \le t$, $\lim_{t\to\infty} h_i(t) \to \infty$, f is a continuous function on $[a,\infty)\times(-\infty,\infty)$ with

$$\left|f(t,x)\right| \le g(t) + k(t)\left|x\right|^{\alpha},\ g(t) \ge 0,\ k(t) \ge 0,\ 0 \le \alpha \le 1, \alpha \in \Re. \tag{6}$$

However, to the best of our knowledge, no author has discussed the integrability of solutions of second order differential equations with multiple deviating arguments. Thus, it is worthwhile to continue to the investigation of the integrability of solutions of nonlinear differentials of second order with multiple deviating arguments. That is to say that this paper is the first attempt on the topic. This work is motivated by the papers of Zhao et al. [28] and the works mentioned above. We here establish a new result on the quadratic integrability of the solutions of Eq. (5) and also an example is considered to illustrate the result obtained. It should be noted that Eq. (5) and the assumptions will be established here are different from that in the papers mentioned above and in the literature. Our result is new and complement to previously known results.

2.Main result

Our main result is given by the following theorem.

Theorem. In addition to the basic assumptions imposed on the functions r, p, q_0, q_i, h_i and f, we assume that the following conditions hold:

(i) $q_0(t)>0,\ q_0'(t)>0,\ q_i(t)>0,\ r'(t)>0,\ p(t)\ge 0,\ h_i'(t)>0$ for all $t \in [a,\infty)$,

$$\left| r^{\frac{1}{2}}(t)q_0'(t)q_0^{-\frac{3}{2}}(t)\right| \le \delta < 4, \ (\delta \text{ is appositive constant}),$$

(ii) $$\int_a^{\infty} \frac{(q_0'(t))^2 q_i(t)}{q_0^{3}(t)C(t)}dt < \infty,\ \int_a^{\infty} \frac{g^2(t)}{\sqrt{q_0(t)}r'(t)}dt < \infty,\ \int_a^{\infty} \frac{(q_0'(t))^2 \sum_{i=1}^{n} q_i(t)}{\lambda^2(t)q_0^{\frac{7}{2}}(t)} < \infty,$$

$$\int_a^\infty r^{\frac{1}{2}}(t)\frac{d(q_0'(t)q_0^{-\frac{3}{2}}(t))}{dt}dt<\infty,\ \int_a^\infty \frac{q_0'(t)k(t)}{q_0^{\frac{\alpha+7}{4}}(t)}dt<\infty,\ \int_a^\infty \frac{k(t)}{q_0^{\frac{\alpha+1}{4}}(t)r^{\frac{1}{2}}(t)}dt<\infty,$$

$$\int_a^\infty p(t)r^{-1}(t)dt<\infty,\ \int_a^\infty C(t)q_0^{-\frac{1}{2}}(t)dt<\infty,\ \text{where}\ \ \sum_{i=1}^n b_i(t)=C(t),$$

$$b_i(h_i(t))=\frac{[\lambda^{-2}(t)+\lambda^2(t)]q_i(t)}{h_i'(t)},\ (i=1,2,...,n),\ \ \lambda^2(t)=\frac{q_0'(t)r^{\frac{1}{2}}(t)}{2q_0^{\frac{3}{2}}(t)}.$$

Then any solution $x(t)$ of Eq. (3) is quadratically integrable and bounded.

Proof. Define the Lyapunov functional

$$V(t)=\sqrt{q_0(t)}x^2(t)+\frac{r(t)}{\sqrt{q_0(t)}}(x'(t))^2+\sum_{i=1}^n\int_{h_i(t)}^t b_i(s)x^2(s)ds. \tag{7}$$

The time derivative of the Lyapunov functional $V(t)$ along solutions of (5) is given by

$$\frac{dV}{dt}=\frac{q_0'(t)}{2\sqrt{q_0(t)}}x^2(t)+2\sqrt{q_0(t)}x(t)x'(t)+\left(\frac{r(t)}{\sqrt{q_0(t)}}\right)'(x'(t))^2+\frac{2r(t)x'(t)x''(t)}{\sqrt{q_0(t)}}$$

$$+\sum_{i=1}^n\{b_i(t)x^2(t)-h_i'(t)b_i(h_i(t))x^2(h_i(t))\}$$

$$=\frac{q_0'(t)}{2\sqrt{q_0(t)}}x^2(t)+2\sqrt{q_0(t)}x(t)x'(t)+\frac{r'(t)}{\sqrt{q_0(t)}}(x'(t))^2-\frac{q_0'(t)r(t)}{2q_0^{\frac{3}{2}}(t)}(x'(t))^2$$

$$+\frac{2r(t)x'(t)x''(t)}{\sqrt{q_0(t)}}+\sum_{i=1}^n\{b_i(t)x^2(t)-h_i'(t)b_i(h_i(t))x^2(h_i(t))\}.$$

It follows from (5) that

$$x''(t)=\frac{1}{r(t)}[f(t,x(t))-r'(t)x'(t)-p(t)x'(t)-q_0(t)x(t)-\sum_{i=1}^n q_i(t)x(h_i(t))].$$

Hence

$$\frac{dV}{dt}\le\frac{q_0'(t)}{2\sqrt{q_0(t)}}x^2(t)-\frac{r'(t)}{\sqrt{q_0(t)}}(x'(t))^2-\frac{q_0'(t)r(t)}{2q_0^{\frac{3}{2}}(t)}(x'(t))^2+\frac{2x'(t)}{\sqrt{q_0(t)}}f(t,x(t))$$

$$-\frac{2x'(t)}{\sqrt{q_0(t)}}\sum_{i=1}^n q_i(t)x(h_i(t))+\sum_{i=1}^n\{b_i(t)x^2(t)-h_i'(t)b_i(h_i(t))x^2(h_i(t))\}$$

by the last estimate and $p(t) \geq 0$.

The assumption $|f(t,x)| \leq g(t) + k(t)|x|^{\alpha}$ implies

$$\frac{dV}{dt} \leq \frac{q_0'(t)}{2\sqrt{q_0(t)}} x^2(t) - \frac{r'(t)}{\sqrt{q_0(t)}} (x'(t))^2 - \frac{q_0'(t) r(t)}{2q_0^{\frac{3}{2}}(t)} (x'(t))^2$$

$$+ \frac{2g(t)}{\sqrt{q_0(t)}} x'(t) + \frac{2k(t)|x(t)|^{\alpha}}{\sqrt{q_0(t)}} x'(t)$$

$$- \frac{2x'(t)}{\sqrt{q_0(t)}} \sum_{i=1}^{n} q_i(t) x(h_i(t)) + \sum_{i=1}^{n} \{b_i(t) x^2(t) - h_i'(t) b_i(h_i(t)) x^2(h_i(t))\}.$$

By the estimate $-ax^2 + bx \leq -\frac{a}{2} x^2 + \frac{b^2}{2a}$, it follows that

$$\frac{dV}{dt} \leq \frac{q_0'(t)}{2\sqrt{q_0(t)}} x^2(t) - \frac{q_0'(t) r(t)}{2q_0^{\frac{3}{2}}(t)} (x'(t))^2 - \frac{2x'(t)}{\sqrt{q_0(t)}} \sum_{i=1}^{n} q_i(t) x(h_i(t))$$

$$- \frac{r'(t)}{2\sqrt{q_0(t)}} (x'(t))^2 + \frac{2g^2(t)}{\sqrt{q_0(t)} r'(t)} + \frac{2k(t)|x(t)|^{\alpha}}{\sqrt{q_0(t)}} x'(t)$$

$$+ \sum_{i=1}^{n} [b_i(t) x^2(t) - h_i'(t) b_i(h_i(t)) x^2(h_i(t))].$$

Let

$$B(t) = \frac{q_0'(t)}{2\sqrt{q_0(t)}} x^2(t) - \frac{q_0'(t) r(t)}{2q_0^{\frac{3}{2}}(t)} (x'(t))^2 - \frac{2x'(t)}{\sqrt{q_0(t)}} \sum_{i=1}^{n} q_i(t) x(h_i(t)).$$

so that

$$\frac{dV}{dt} \leq B(t) + \frac{2k(t)|x(t)|^{\alpha}}{\sqrt{q_0(t)}} x'(t) + \frac{2g^2(t)}{\sqrt{q_0(t)} r'(t)}$$

$$+ \sum_{i=1}^{n} \{b_i(t) x^2(t) - h_i'(t) b_i(h_i(t)) x^2(h_i(t))\}$$

by $q_0(t) > 0$ and $r'(t) > 0$.

It also follows from (5) that

$$r(t)[x'(t)]^2 = [r(t) x(t) x'(t)]' + p(t) x(t) x'(t) + q_0(t) x^2(t)$$

$$+ x(t) \sum_{i=1}^{n} q_i(t) x(h_i(t)) - f(t, x(t)) x(t)$$

and

$$\frac{q_0'(t)}{2\sqrt{q_0(t)}}x^2(t)=\frac{q_0'(t)r(t)}{2q_0^{\frac{3}{2}}(t)}(x'(t))^2-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}[r(t)x(t)x'(t)]'$$

$$-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}p(t)x(t)x'(t)-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}x(t)\sum_{i=1}^{n}q_i(t)x(h_i(t))+\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}f(t,x(t))x(t)$$

so that

$$B(t)=-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}[r(t)x(t)x'(t)]'-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}p(t)x(t)x'(t)$$

$$-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}x(t)\sum_{i=1}^{n}q_i(t)x(h_i(t))+\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}f(t,x(t))x(t)-\frac{2x'(t)}{\sqrt{q_0(t)}}\sum_{i=1}^{n}q_i(t)x(h_i(t)).$$

The estimate $\left|f(t,x)\right|\le g(t)+k(t)\left|x\right|^{\alpha}$ leads

$$B(t)\le-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}[r(t)x(t)x'(t)]'-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}p(t)x(t)x'(t)$$

$$-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}x(t)\sum_{i=1}^{n}q_i(t)x(h_i(t))+\frac{q_0'(t)g(t)}{2q_0^{\frac{3}{2}}(t)}x(t)$$

$$+\frac{q_0'(t)k(t)}{2q_0^{\frac{3}{2}}(t)}\left|x(t)\right|^{\alpha+1}-\frac{2x'(t)}{\sqrt{q_0(t)}}\sum_{i=1}^{n}q_i(t)x(h_i(t)).$$

Let

$$A(t)=-\frac{2x'(t)}{\sqrt{q_0(t)}}\sum_{i=1}^{n}q_i(t)x(h_i(t))-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}x(t)\sum_{i=1}^{n}q_i(t)x(h_i(t)).$$

Using the estimate $-2uv\le\beta^{-2}u^2+\beta^2v^2$, it follows that

$$A(t)\le\sum_{i=1}^{n}q_i(t)\{\lambda^{-2}(t)x^2(h_i(t))+\lambda^2(t)q_0^{-1}(t)(x'(t))^2\}$$

$$+\sum_{i=1}^{n}q_i(t)\{\lambda^2(t)x^2(h_i(t))+4^{-1}\lambda^{-2}(t)(q_0'(t))^2q_0^{-3}(t)x^2(t)\}$$

$$=\{\lambda^2(t)q_0^{-1}(t)(x'(t))^2+4^{-1}\lambda^{-2}(t)(q_0'(t))^2q_0^{-3}(t)x^2(t)\}\sum_{i=1}^{n}q_i(t)$$

$$+\{\lambda^{-2}(t)+\lambda^2(t)\}\sum_{i=1}^{n}q_i(t)x^2(h_i(t)),$$

where

$$\lambda^2(t)=\frac{q_0'(t)\sqrt{r(t)}}{2q_0^{\frac{3}{2}}(t)},\ \lambda^{-2}(t)\frac{(q_0'(t))^2}{4q_0^{\ 3}(t)}=\lambda^2(t)q_0^{\ -1}(t)\frac{q_0(t)}{r(t)}=\frac{\lambda^2(t)}{r(t)}.$$

In view of (7), it is seen that

$$\sqrt{q_0(t)}x^2(t)+\frac{r(t)}{\sqrt{q_0(t)}}(x'(t))^2\le V(t)$$

so that

$$\frac{(q_0'(t))^2\sum_{i=1}^{n}q_i(t)}{4\lambda^2(t)q_0^{\frac{7}{2}}(t)}\sqrt{q_0(t)}x^2(t)+\frac{(q_0'(t))^2\sum_{i=1}^{n}q_i(t)}{4\lambda^2(t)q_0^{\frac{7}{2}}(t)}\frac{r(t)}{\sqrt{q_0(t)}}(x'(t))^2\le\frac{(q_0'(t))^2\sum_{i=1}^{n}q_i(t)}{4\lambda^2(t)q_0^{\frac{7}{2}}(t)}V(t).$$

It also follows from the terms contained on the right hand side of $A(t)$ that

$$\sum_{i=1}^{n}q_i(t)[\frac{(q_0'(t))^2}{4\lambda^2(t)q_0^{\ 3}(t)}x^2(t)+\lambda^2(t)q_0^{\ -1}(t)(x'(t))^2]\le\frac{(q_0'(t))^2\sum_{i=1}^{n}q_i(t)}{4\lambda^2(t)q_0^{\frac{7}{2}}(t)}V(t).$$

Then

$$A(t)\le\frac{(q_0'(t))^2\sum_{i=1}^{n}q_i(t)}{4\lambda^2(t)q_0^{\frac{7}{2}}(t)}V(t)+(\lambda^{-2}(t)+\lambda^2(t))\sum_{i=1}^{n}q_i(t)x^2(h_i(t))$$

so that

$$\frac{dV}{dt}\le\frac{(q_0'(t))^2\sum_{i=1}^{n}q_i(t)}{4\lambda^2(t)q_0^{\frac{7}{2}}(t)}V(t)+(\lambda^{-2}(t)+\lambda^2(t))\sum_{i=1}^{n}q_i(t)x^2(h_i(t))$$

$$-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}\left[r(t)x(t)x'(t)\right]'-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}p(t)x(t)x'(t)$$

$$+\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}g(t)x(t)+\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}k(t)|x(t)|^{\alpha+1}+\frac{2k(t)|x(t)|^{\alpha}}{\sqrt{q_0(t)}}x'(t)$$

$$+\frac{2g^2(t)}{\sqrt{q_0(t)}r'(t)}+\sum_{i=1}^{n}\{b_i(t)x^2(t)-h_i'(t)b_i(h_i(t))x^2(h_i(t))\}.$$

Let

$$b_i(h_i(t))=\frac{[\lambda^{-2}(t)+\lambda^2(t)]q_i(t)}{h_i'(t)}$$

so that

$$\frac{dV}{dt} \le \frac{(q_0'(t))^2 \sum_{i=1}^{n} q_i(t)}{4\lambda^2(t) q_0^{\frac{7}{2}}(t)} V(t) - \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} [r(t)x(t)x'(t)]'$$

$$- \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} p(t)x(t)x'(t) + \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} g(t)x(t) + \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} k(t)|x(t)|^{\alpha+1}$$

$$+ \frac{2k(t)|x(t)|^{\alpha}}{\sqrt{q_0(t)}} x'(t) + \frac{2g^2(t)}{\sqrt{q_0(t)} r'(t)} + \sum_{i=1}^{n} b_i(t) x^2(t).$$

By the estimate $ax^2 + bx \le \frac{3}{2} ax^2 + \frac{b^2}{2a}$, it follows that

$$\frac{dV}{dt} \le \frac{(q_0'(t))^2 \sum_{i=1}^{n} q_i(t)}{4\lambda^2(t) q_0^{\frac{7}{2}}(t)} V(t) - \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} [r(t)x(t)x'(t)]'$$

$$- \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} p(t)x(t)x'(t) + \frac{3}{2} \sum_{i=1}^{n} b_i(t) x^2(t) + \frac{(q_0'(t))^2 g^2(t)}{8 q_0^{3}(t) \sum_{i=1}^{n} b_i(t)}$$

$$+ \frac{2g^2(t)}{\sqrt{q_0(t)} r'(t)} + \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} k(t)|x(t)|^{\alpha+1} + \frac{2k(t)|x(t)|^{\alpha}}{\sqrt{q_0(t)}} |x'(t)|.$$

Making use of the estimates

$$2r^{\frac{1}{2}}(t)|x(t)||x'(t)| \le V(t), \ |x'(t)| \le \frac{q_0^{\frac{1}{4}}(t)}{r^{\frac{1}{2}}} V^{\frac{1}{2}}(t), \ |x(t)|^{\alpha} \le q_0^{\frac{-\alpha}{4}} V^{\frac{\alpha}{2}}(t),$$

we get

$$\frac{dV}{dt} \le [\frac{(q_0'(t))^2 \sum_{i=1}^{n} q_i(t)}{4\lambda^2(t) q_0^{\frac{7}{2}}(t)} + \frac{1}{4} \frac{q_0'(t) p(t)}{q_0^{\frac{3}{2}}(t) r^{\frac{1}{2}}(t)} + \frac{3}{2} \sum_{i=1}^{n} b_i(t) q_0^{-\frac{1}{2}}(t)] V(t)$$

$$- \frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)} [r(t)x(t)x'(t)]' + \frac{2g^2(t)}{\sqrt{q_0(t)} r'(t)}$$

$$+ [\frac{q_0'(t)k(t)}{2q_0^{\frac{\alpha+7}{4}}(t)} + \frac{2k(t)}{q_0^{\frac{\alpha+1}{4}} r^{\frac{1}{2}}(t)}] V^{\frac{\alpha+1}{2}}(t) + \frac{(q_0'(t))^2 g^2(t)}{8 q_0^{3}(t) \sum_{i=1}^{n} b_i(t)}. \quad \textbf{(8)}$$

Integrating both sides of (8) from a to t, we obtain

$$V(t)-V(a)\le\int_a^t[\frac{(q_0'(s))^2\sum_{i=1}^n q_i(s)}{4\lambda^2(s)q_0^{\frac{7}{2}}(s)}+\frac{1}{4}\frac{q_0'(s)p(s)}{q_0^{\frac{3}{2}}(s)r^{\frac{1}{2}}(s)}+\frac{3}{2}\sum_{i=1}^n b_i(s)q_0^{-\frac{1}{2}}(s)]V(s)ds$$

$$-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}r(t)x(t)x'(t)+\frac{q_0'(a)}{2q_0^{\frac{3}{2}}(a)}r(a)x(a)x'(a)$$

$$+\frac{1}{2}\int_a^t r(s)x(s)x'(s)\frac{d(q_0'(s)q_0^{-\frac{3}{2}}(s))}{ds}ds$$

$$+\int_a^t[\frac{q_0'(s)k(s)}{2q_0^{\frac{\alpha+7}{4}}(s)}+\frac{2k(s)}{q_0^{\frac{\alpha+1}{4}}r^{\frac{1}{2}}(s)}]V^{\frac{\alpha+1}{2}}(s)ds$$

$$+\int_a^t\frac{(q_0'(s))^2g^2(s)}{8q_0^{3}(s)\sum_{i=1}^n b_i(s)}ds+\int_a^t\frac{2g^2(s)}{\sqrt{q_0(s)}r'(s)}ds.$$

Hence

$$V(t)\le V_0+\int_a^t[\frac{(q_0'(s))^2\sum_{i=1}^n q_i(s)}{4\lambda^2(s)q_0^{\frac{7}{2}}(s)}+\frac{1}{4}\frac{q_0'(s)p(s)}{q_0^{\frac{3}{2}}(s)r^{\frac{1}{2}}(s)}+\frac{3}{2}C(s)q_0^{-\frac{1}{2}}(s)]V(s)ds$$

$$-\frac{q_0'(t)}{2q_0^{\frac{3}{2}}(t)}r(t)x(t)x'(t)+\frac{1}{2}\int_a^t r(s)x(s)x'(s)\frac{d(q_0'(s)q_0^{-\frac{3}{2}}(s))}{ds}ds$$

$$+\int_a^t[\frac{q_0'(s)k(s)}{2q_0^{\frac{\alpha+7}{4}}(s)}+\frac{2k(s)}{q_0^{\frac{\alpha+1}{4}}r^{\frac{1}{2}}(s)}]V^{\frac{\alpha+1}{2}}(s)ds,$$

where

$$V_0=V(a)+\int_a^\infty\frac{(q_0'(t))^2g^2(t)}{8q_0^{3}(t)C(t)}dt+\int_a^\infty\frac{2g^2(t)}{\sqrt{q_0(t)}r'(t)}dt+\frac{q_0'(a)}{2q_0^{\frac{3}{2}}(a)}r(a)x(a)x'(a).$$

By using the estimate $\left|r^{\frac{1}{2}}(t)q_0'(t)q_0^{-\frac{3}{2}}(t)\right|\le\delta<4$, we have

$$V(t)\le V_0+\int_a^t[\frac{(q_0'(s))^2\sum_{i=1}^n q_i(s)}{4\lambda^2(s)q_0^{\frac{7}{2}}(s)}+\frac{\delta}{4}p(s)r^{-1}(s)+\frac{3}{2}C(s)q_0^{-\frac{1}{2}}(s)]V(s)ds$$

$$+\frac{\delta}{4}V(t)+\frac{1}{4}\int_a^t r^{\frac{1}{2}}(s)\frac{d(q_0'(s)q_0^{-\frac{3}{2}}(s))}{ds}V(s)ds$$

$$+\int_a^t[\frac{q_0'(s)k(s)}{2q_0^{\frac{\alpha+7}{4}}(s)}+\frac{2k(s)}{q_0^{\frac{\alpha+1}{4}}r^{\frac{1}{2}}(s)}]V^{\frac{\alpha+1}{2}}(s)ds$$

$$\le\frac{4V_0}{4-\delta}+\frac{4}{4-\delta}\int_a^t[\frac{(q_0'(s))^2\sum_{i=1}^n q_i(s)}{4\lambda^2(s)q_0^{\frac{7}{2}}(s)}+\frac{\delta}{4}p(s)r^{-1}(s)+\frac{3}{2}C(s)q_0^{-\frac{1}{2}}(s)]V(s)ds$$

$$+\frac{1}{4-\delta}\int_a^t r^{\frac{1}{2}}(s)\frac{d(q_0'(s)q_0^{-\frac{3}{2}}(s))}{ds}V(s)ds$$

$$+\frac{4}{4-\delta}\int_a^t[\frac{q_0'(s)k(s)}{2q_0^{\frac{\alpha+7}{4}}(s)}+\frac{2k(s)}{q_0^{\frac{\alpha+1}{4}}r^{\frac{1}{2}}(s)}]V^{\frac{\alpha+1}{2}}(s)ds+\frac{\delta}{4-\delta}V(t).$$

Then, we conclude that

$$V(t)\le\begin{cases}\frac{4V_0}{4-\delta}\exp\{\int_a^t E(s)ds+\frac{4}{4-\delta}\int_a^t[\frac{q_0'(s)k(s)}{2q_0^{\frac{\alpha+7}{4}}(s)}+\frac{2k(s)}{q_0^{\frac{\alpha+7}{4}}(s)r^{\frac{1}{2}}(s)}]ds\},\alpha=1\\ \exp\int_a^t E(s)ds\{V_0^{\frac{1-\alpha}{2}}+\frac{1-\alpha}{2}\int_a^t\frac{4}{4-\delta}[\frac{q_0'(s)k(s)}{2q_0^{\frac{\alpha+7}{4}}(s)}+\frac{2k(s)}{q_0^{\frac{\alpha+1}{4}}r^{\frac{1}{2}}(s)}]\\ \times\exp(\int_a^t\frac{\alpha-1}{2}E(\tau)d\tau)ds\}^{\frac{2}{1-\alpha}},0<\alpha<1,\end{cases}$$

where

$$E(s)=\frac{(q_0'(s))^2\sum_{i=1}^n q_i(s)}{(4-\delta)\lambda^2(s)q_0^{\frac{7}{2}}(s)}+\frac{\delta}{4}p(s)r^{-1}(s)+\frac{6}{4-\delta}C(s)q_0^{-\frac{1}{2}}(s)$$
$$+\frac{1}{4-\delta}r^{\frac{1}{2}}(s)\left|\frac{d(q_0'(s)q_0^{-\frac{3}{2}}(s))}{ds}\right|.$$

In view of the above discussion, it follows that any solution $x(t)$ of Eq. (5) is quadratically integrable and bounded.

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SOME RELATIONS BETWEEN THE q-BERNSTEIN POLYNOMIALS AND TWISTED q-EULER NUMBERS WITH WEIGHT α

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Abstract : Recently, we have studied twisted q-Euler numbers and polynomials with weight α(see [18]). In this paper, by using fermionic p-adic q-integral on $\mathbb{Z}_p$, we give some interesting relationship between the the q-Bernstein polynomials and twisted q-Euler numbers with weight α.

Key words : Euler numbers and polynomials, twisted q-Euler numbers and polynomials with weight α, Bernstein polynomials, q-Bernstein polynomials

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $C_{p^n} = \{w | w^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$T_p = \lim_{n\to\infty} C_{p^n} = C_{p^\infty} = \cup_{n\geq 0} C_{p^n}$$

be the locally constant space. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$. The p-adic absolute value is defined by $|x|_p = \frac{1}{p^r}$, where $x = p^r \frac{s}{t}$($r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s,t) = (p,s) = (p,t) = 1$). In this paper we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ as an indeterminate. The q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \text{ see [1-18].}$$

Note that $\lim_{q\to 1}[x]_q = x$. For

$$f \in UD(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [1-5] .} \tag{1.1}$$

From (1.1), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),$$

where $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, Kim defined q-Bernstein polynomials, which are different q-Bernstein polynomials of Phillips, as follows:

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \text{ see [4].} \tag{1.2}$$

In [9], the p-adic extension of (1.2) is given by

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k}, \text{ where } x \in \mathbb{Z}_p, \text{ and } n,k \in \mathbb{Z}_+. \tag{1.3}$$

For $\alpha \in \mathbb{Z}$, $w \in T_p$, and $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, twisted q-Euler numbers $E_{n,q,w}^{(\alpha)}$ with weight α are defined by

$$E_{n,q,w}^{(\alpha)} = \int_{\mathbb{Z}_p} \phi_w(x)[x]_{q^\alpha}^n d\mu_{-q}(x). \tag{1.4}$$

In the special case, $x = 0$, $E_{n,q,w}^{(\alpha)}(0) = E_{n,q,w}^{(\alpha)}$ are called the n-th twisted q-Euler numbers with weight α.

In this paper we investigate some relations between the q-Bernstein polynomials and the twisted q-Euler numbers with weight α. From these relations, we derive some interesting identities on the twisted q-Euler numbers and polynomials with weight α.

2. On the twisted q-Euler numbers and polynomials with weight α

By using p-adic q-integral and (1.4), we obtain,

$$\begin{aligned}\int_{\mathbb{Z}_p} \phi_w(x)[x]_{q^\alpha}^n d\mu_{-q}(x) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} [x]_{q^\alpha}^n w^x (-q)^x \\ &= [2]_q \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha l+1}}.\end{aligned} \tag{2.1}$$

We set

$$F_{q,w}^{(\alpha)}(t) = \sum_{n=0}^{\infty} E_{n,q,w}^{(\alpha)} \frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}F_{q,w}^{(\alpha)}(t) &= \sum_{n=0}^{\infty} E_{n,q,w}^{(\alpha)} \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \left(\left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha l+1}} \right) \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^m e^{[m]_{q^\alpha} t}.\end{aligned} \tag{2.3}$$

Since $[x+y]_{q^\alpha} = [x]_{q^\alpha} + q^{\alpha x}[y]_{q^\alpha}$, we obtain

$$\begin{aligned}E_{n,q,w}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} \phi_w(y)[y+x]_{q^\alpha} t d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{\alpha x l} [x]_{q^\alpha}^{n-l} \int_{\mathbb{Z}_p} \phi_w(y)[y]_{q^\alpha}^l d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{\alpha x l} [x]_{q^\alpha}^{n-l} E_{l,q,w}^{(\alpha)}.\end{aligned} \tag{2.4}$$

Therefore, we obtain the following theorem.

Theorem 1. For $n \in \mathbb{Z}_+$ and $w \in T_p$, we have

$$E_{n,q,w}^{(\alpha)}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^m [x+m]_{q^\alpha}^n.$$

Furthermore,

$$E_{n,q,w}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha x l} [x]_{q^\alpha}^{n-l} E_{l,q,w}^{(\alpha)} = ([x]_{q^\alpha} + q^{\alpha x} E_{q,w}^{(\alpha)})^n,$$

with usual convention about replacing $(E_{q,w}^{(\alpha)})^n$ by $E_{n,q,w}^{(\alpha)}$.

Let $F_{q,w}^{(\alpha)}(t,x) = \sum_{n=0}^{\infty} E_{n,q,w}^{(\alpha)}(x) \frac{t^n}{n!}$. Then we see that

$$F_{q,w}^{(\alpha)}(t,x) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^m e^{[x+m]_{q^\alpha} t}. \tag{2.5}$$

In the special case, $x = 0$, let $F_{q,w}^{(\alpha)}(t,0) = F_{q,w}^{(\alpha)}(t)$.

By (2.1), we get

$$E_{n,q^{-1},w^{-1}}^{(\alpha)}(1-x) = (-1)^n w q^{\alpha n} E_{n,q,w}^{(\alpha)}(x). \tag{2.6}$$

From (2.3) and (2.5), we note that

$$wqF_{q,w}^{(\alpha)}(t,1) + F_{q,w}^{(\alpha)}(t) = [2]_q. \tag{2.7}$$

By (2.7), we get the following recurrence formula:

$$E_{0,q,w}^{(\alpha)} = \frac{[2]_q}{1+qw}, \text{ and } qwE_{n,q,w}^{(\alpha)}(1) + E_{n,q,w}^{(\alpha)} = 0 \text{ if } n > 0. \tag{2.8}$$

By (2.8) and Theorem 1, we obtain the following theorem.

Theorem 2. For $n \in \mathbb{Z}_+$ and $w \in T_p$, we have

$$E_{0,q,w}^{(\alpha)} = \frac{[2]_q}{1+qw}, \text{ and } qw(q^\alpha E_{q,w}^{(\alpha)} + 1)^n + E_{n,q,w}^{(\alpha)} = 0 \text{ if } n > 0,$$

with usual convention about replacing $(E_{q,w}^{(\alpha)})^n$ by $E_{n,q,w}^{(\alpha)}$.

By (2.4), Theorem 1, and Theorem 2, we have

$$\begin{aligned}
&q^2w^2 E_{n,q,w}^{(\alpha)}(2) - \frac{[2]_q}{1+qw} q^2w^2 - \frac{[2]_q}{1+qw} qw \\
&= q^2w^2 \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l} (q^\alpha E_{q,w}^{(\alpha)} + 1)^l - \frac{[2]_q}{1+qw} q^2w^2 - \frac{[2]_q}{1+qw} qw \\
&= q^2w^2 \sum_{l=1}^{n} \binom{n}{l} q^{\alpha l} (q^\alpha E_{q,w}^{(\alpha)} + 1)^l - \frac{[2]_q}{1+qw} qw \\
&= -qw \sum_{l=1}^{n} \binom{n}{l} q^{\alpha l} E_{l,q,w}^{(\alpha)} - \frac{[2]_q}{1+qw} qw \\
&= -qw \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l} E_{l,q,w}^{(\alpha)} \\
&= -qwE_{n,q,w}^{(\alpha)}(1) = E_{n,q,w}^{(\alpha)} \text{ if } n > 0.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 3. For $n \in \mathbb{N}$, we have

$$E_{n,q,w}^{(\alpha)}(2) = \left(\frac{1}{q^2w^2}\right) E_{n,q,w}^{(\alpha)} + \frac{[2]_q}{1+qw} + \left(\frac{1}{qw}\right) \frac{[2]_q}{1+qw}.$$

By (2.6), we see that

$$\begin{aligned} w\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^n w^x d\mu_{-q}(x) &= (-1)^n q^{\alpha n} w\int_{\mathbb{Z}_p}[x-1]_{q^{\alpha}}^n w^x d\mu_{-q}(x) \\ &= (-1)^n q^{\alpha n} w E_{n,q,w}^{(\alpha)}(-1) = E_{n,q^{-1},w^{-1}}^{(\alpha)}(2). \end{aligned} \tag{2.9}$$

Therefore, we obtain the following theorem.

Theorem 4. For $n \in \mathbb{Z}_+$, we have

$$w\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^n w^x d\mu_{-q}(x) = E_{n,q^{-1},w^{-1}}^{(\alpha)}(2).$$

Let $n \in \mathbb{N}$. By Theorem 3 and Theorem 4, we get

$$\begin{aligned} &w\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^n w^x d\mu_{-q}(x) \\ &= q^2 w^2 E_{n,q^{-1},w^{-1}}^{(\alpha)} + w\left(\frac{[2]_q}{1+qw}\right) + qw^2\left(\frac{[2]_q}{1+qw}\right). \end{aligned} \tag{2.10}$$

From (2.10), we have

$$\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n w^x d\mu_{-q}(x) = q^2 w E_{n,q^{-1},w^{-1}}^{(\alpha)} + \left(\frac{[2]_q}{1+qw}\right) + qw\left(\frac{[2]_q}{1+qw}\right).$$

Therefore, we have the following corollary.

Corollary 5. For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^n w^x d\mu_{-q}(x) = q^2 w E_{n,q^{-1},w^{-1}}^{(\alpha)} + [2]_q.$$

3. Some relations between the q-Bernstein polynomials and twisted q-Euler numbers with weight α

For $x \in \mathbb{Z}_p$, the p-adic q-Bernstein polynomials with weight α of degree n are given by

$$B_{k,n}^{(\alpha)}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{q^{-\alpha}}^{n-k}, \text{ where } n,k \in \mathbb{Z}_+. \tag{3.1}$$

By (3.1), we get the symmetry of q-Bernstein polynomials as follows:

$$B_{k,n}^{(\alpha)}(x,q) = B_{n-k,n}^{(\alpha)}(1-x,q^{-1}), \text{ see [7]}. \tag{3.2}$$

Thus, by Corollary 5, (3.1), and (3.2), we see that

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) w^x d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}^{(\alpha)}(1-x,q^{-1}) w^x d\mu_{-q}(x) \\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n-l} w^x d\mu_{-q}(x) \\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(q^2 w E_{n-l,q^{-1},w^{-1}}^{(\alpha)} + [2]_q\right). \end{aligned}$$

For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) w^x d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(q^2 w E_{n,q^{-1},w^{-1}}^{(\alpha)} + [2]_q \right) \\
&= \begin{cases} q^2 w E_{n,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} E_{n-l,q^{-1},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned} \tag{3.3}$$

Let us take the fermionic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials with weight α of degree n as follows:

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) w^x d\mu_{-q}(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} [x]_q^k [1-x]_{q^{-\alpha}}^{n-k} w^x d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l E_{l+k,q,w}^{(\alpha)}.
\end{aligned} \tag{3.4}$$

Therefore, by (3.3) and (3.4), we have the following theorem.

Theorem 6. Let $n, k \in \mathbb{Z}_+$ with $n > k$. Then we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) w^x d\mu_{-q}(x) \\
&= \begin{cases} q^2 w E_{n,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} E_{n-l,q^{-1},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l E_{l+k,q,w}^{(\alpha)} \\
&= \begin{cases} q^2 w E_{n,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} E_{n-l,q^{-1},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x,q) B_{k,n_2}^{(\alpha)}(x,q) w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n_1+n_2-l} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(q^2 w E_{n_1+n_2-l,q^{-1},w^{-1}}^{(\alpha)} + [2]_q \right) \\
&= \begin{cases} q^2 w E_{n_1+n_2,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left(q^2 w E_{n_1+n_2-l,q^{-1},w^{-1}}^{(\alpha)} + [2]_q \right), & \text{if } k > 0. \end{cases}
\end{aligned} \tag{3.5}$$

Therefore, by (3.5), we obtain the following theorem.

Theorem 7. For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x,q) B_{k,n_2}^{(\alpha)}(x,q) w^x d\mu_{-q}(x) \\
&= \begin{cases} q^2 w E_{n_1+n_2,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E_{n_1+n_2-l,q^{-1},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned}$$

From the binomial theorem, we can derive the following equation.

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x,q) B_{k,n_2}^{(\alpha)}(x,q) w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \int_{\mathbb{Z}_p} [x]_q^{2k+l} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E_{2k+l,q,w}^{(\alpha)}.
\end{aligned} \tag{3.6}$$

Thus, by (3.6) and Theorem 7, we obtain the following corollary.

Corollary 8. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have

$$\begin{aligned}
&\sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E_{2k+l,q}^{(\alpha)} \\
&= \begin{cases} q^2 w E_{n_1+n_2,q^{-\alpha},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E_{n_1+n_2-l,q^{-\alpha},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned}$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials with weight α of degree n as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}^{(\alpha)}(x,q) \cdots B_{k,n_s}^{(\alpha)}(x,q)}_{s-\text{times}} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} [x]_q^{sk} [1-x]_{q^{-\alpha}}^{n_1+\cdots+n_s-sk} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n_1+\cdots+n_s-l} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left(q^2 w E_{n_1+\cdots+n_s-l,q^{-1},w^{-1}}^{(\alpha)} + [2]_q \right) \\
&= \begin{cases} q^2 w E_{n_1+\cdots+n_s,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E_{n_1+\cdots+n_s-l,q^{-1},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned} \tag{3.7}$$

Therefore, by (3.7), we obtain the following theorem.

Theorem 9. For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}^{(\alpha)}(x,q)\cdots B_{k,n_s}^{(\alpha)}(x,q)}_{s-\text{times}} w^x d\mu_{-q}(x) \\
&= \begin{cases} q^2 w E_{n_1+\cdots+n_s,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \binom{n_1}{k}\cdots\binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E_{n_1+\cdots+n_s-l,q^{-1},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned} \tag{3.8}$$

By the definition of q-Bernstein polynomials with weight α and the binomial theorem, we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}^{(\alpha)}(x,q)\cdots B_{k,n_s}^{(\alpha)}(x,q)}_{s-\text{times}} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} \int_{\mathbb{Z}_p} [x]_q^{sk+l} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} E_{sk+l,q,w}^{(\alpha)}.
\end{aligned} \tag{3.9}$$

Therefore, by (3.8) and (3.9), we obtain the following corollary.

Corollary 10. For $w \in T_p, s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we have

$$\begin{aligned}
&\sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} E_{sk+l,q,w}^{(\alpha)} \\
&= \begin{cases} q^2 w E_{n_1+\cdots+n_s,q^{-1},w^{-1}}^{(\alpha)} + [2]_q, & \text{if } k = 0, \\ q^2 w \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E_{n_1+\cdots+n_s-l,q^{-1},w^{-1}}^{(\alpha)}, & \text{if } k > 0. \end{cases}
\end{aligned}$$

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Some formulae for the q-Bernstein polynomials and q-deformed binomial distributions

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Abstract In this paper, we give new identities involving q-Bernstein polynomials and we derive some interesting properties of q-Bernstein polynomials associated with q-deformed binomial distributions.

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1. Introduction

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we usually assume that $|1-q|_p < 1$. Here, the symbol $|\cdot|_p$ stands for the p-adic absolute value on $\mathbb{C}_p$ with $|p|_p \leq 1/p$. For each x, the q-basic numbers are defined by

$$[x]_q = \frac{1-q^x}{1-q}, \text{ and } [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, n \in \mathbb{N}, \text{ (see [1-17])}.$$

Throughout this paper we assume that $q \in \mathbb{C}$ with $|q| < 1$ and we use the notation of Gaussian binomial coefficient in the form

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!}, n, k \in \mathbb{N}.$$

Note that

$$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \text{ (see [4-12])}.$$

The Gaussian binomial coefficient satisfies the following recursion formula:

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q = q^{n-k}\binom{n}{k-1}_q + \binom{n}{k}_q, \text{ (see [11, 12])}. \tag{1}$$

The q-binomial formulae are known as

$$(1-b)_q^n = (b:q)_n = \prod_{i=1}^{n}(1-bq^{i-1}) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-1)^i b^i, \tag{2}$$

and

$$\frac{1}{(1-b)_q^n} = \frac{1}{(b:q)_n} = \frac{1}{\prod_{i=1}^{n}(1-bq^{i-1})} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q b^i, \text{ (see [11, 12, 13])}.$$

Now, we define the q-exponential function as follows:

$$\lim_{n\to\infty}\frac{1}{(x:q)_n} = \lim_{n\to\infty}\sum_{k=0}^{\infty}\binom{n+k-1}{k}_q x^k = \sum_{k=0}^{\infty}\frac{x^k(1-q)^k}{[k]_q!} = e_q(x(1-q)). \tag{3}$$

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well known linear operator in [2]:

$$B_n(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x).$$

Here $B_n(f|x)$ is called the Bernstein operator of order n for f. For $k, n \in \mathbb{Z}_+$, the Bernstein polynomials of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}.$$

By the definition of Bernstein polynomials, we can see that Bernstein basis is the probability mass function of binomial distribution. Based on the

q-integers Phillips introduced the q-analogue of well known Bernstein polynomials (see [16, 17]). For $f \in C[0,1]$, Phillips introduced the q-extension of $\mathbb{B}_n(f|x)$ as follows:

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} B_{k,n}(x,q) f\left(\frac{[k]_q}{[n]_q}\right) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \binom{n}{k}_q x^k (1-x)_q^{n-k}. \quad (4)$$

Here $\mathbb{B}_{n,q}(f \mid x)$ is called the q-Bernstein operator of order n for f. For $k, n \in \mathbb{Z}_+$, the q-Bernstein polynomial of degree n is defined by

$$B_{k,n}(x,q) = \binom{n}{k}_q x^k (1-x)_q^{n-k}, x \in [0,1]. \quad (5)$$

For example, $B_{0,1}(x,q) = 1-x, B_{1,1}(x,q) = x$, and $B_{0,2}(x,q) = 1 - [2]_q x + qx^2, \cdots$. Also $B_{k,n}(x,q) = 0$ for $k > n$, because $\binom{n}{k}_q = 0$.

A Bernoulli trial involves performing an experiment once and noting whether a particular event A occurs. The outcome of Bernoulli trial is said to be "success" if A occurs and a "failure" otherwise. Let k be the number of successes in n independent Bernoulli trials, the probabilities of k are given by the binomial probability law:

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ for } k = 0, 1, \cdots, n,$$

where $p_n(k)$ is the probability of k successes in n trials.

For example, a communication system transmit binary information over channel that introduces random bit errors with probability $\xi = 10^{-3}$. The transmitter transmits each information bit three times, an a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. The receiver can correct a single error, but it will make the wrong decision if the channel introduces two or more errors. If we view each transmission as a Bernoulli trial in which a "success" corresponds to the introduction of an error, then the probability of two or more errors in three Bernoulli trial is

$$p(k \geq 2) = \binom{3}{2}(0.001)^2(0.999) + \binom{3}{3}(0.001)^3 \approx 3(10^{-6}), \text{ see [19]}.$$

The q-binomial distribution associated with the q-boson oscillator are introduced in [19, 20]. For $n, k \in \mathbb{Z}_+$, its probabilities are given by

$$p(X = k) = \binom{n}{k}_q x^k (1-x)_q^{n-k}, \text{ where } x \in [0,1].$$

This distributions are studied by several authors and has applications in physics as well as in approximation theory due to the q-Bernstein polynomials and the q-Bernstein operators (see [13, 14, 19]). From the definition of q-Bernstein polynomials, we note that the q-Bernstein basis is the probability mass function of q-binomial distribution. Recently, several authors have studied the analogs of Bernstein polynomials (see [1, 2, 3, 6, 13, 14, 16, 17, 18]). In [6], Gupta-Kim-Choi-Kim gave the generating function of the q-Bernstein polynomials as follows:

$$\begin{aligned}\frac{x^k t^k}{[k]_q!} e_q((1-x)_q t) &= \frac{x^k t^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{(1-x)_q^n t^n}{[n]_q!} \\ &= \sum_{n=k}^{\infty} \binom{n}{k}_q \frac{x^k (1-x)_q^{n-k}}{[n]_q!} t^n \\ &= \sum_{n=k}^{\infty} B_{k,n}(x,q) \frac{t^n}{[n]_q!}.\end{aligned}$$

Because $B_{k,0}(x,q) = B_{k,1}(x,q) = B_{k,2}(x,q) = \cdots = B_{k,k-1}(x,q) = 0$, we obtain the generating function for $B_{k,n}(x,q)$ as follows:

$$F_q^{(k)}(t,x) = \frac{x^k t^k}{[k]_q!} e_q((1-x)_q t) = \sum_{n=0}^{\infty} B_{k,n}(x,q) \frac{t^n}{[n]_q!}, \text{ see [6]},$$

where $n, k \in \mathbb{Z}_+$ and $x \in [0,1]$.

Notice that

$$B_{k,n}(x,q) = \begin{cases} \binom{n}{k}_q x^k (1-x)_q^{n-k}, & \text{if } n \geq k \\ 0, & \text{if } n < k, \end{cases}$$

for $n, k \in \mathbb{Z}_+$ (see [4, 8, 10, 11, 15, 16]).

In this paper we study the generating function of the q-Bernstein polynomial and give some identities on the q-Bernstein polynomials. From the generating function of q-Bernstein polynomial, we derive recurrence relation and derivative of the q-Bernstein polynomials. Finally, we investigate some interesting properties of q-Bernstein polynomials related to q-deformed binomial distributions and q-Stirling numbers.

2. q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli polynomials

Let

$$F_q^{(k)}(t,x) = \sum_{n=0}^{\infty} B_{k,n}(x,q)\frac{t^n}{[n]_q!}. \tag{6}$$

From (5) and (3), we note that

$$\begin{aligned} F_q^{(k)}(t,x) &= \sum_{n=0}^{\infty} \binom{n}{k}_q x^k (1-x)_q^{n-k} \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k}_q \frac{x^k (1-x)_q^n}{[n+k]_q!} t^{n+k} \\ &= \frac{x^k t^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{(1-x)_q^n}{[n]_q!} t^n \\ &= \frac{x^k t^k}{[k]_q!} e_q((1-x)_q t), \end{aligned}$$

where $n, k \in \mathbb{Z}_+$ and $x \in [0,1]$.

Note that

$$\lim_{q\to 1} F_q^{(k)}(t,x) = \frac{x^k t^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_{k,n}(x)\frac{t^n}{n!},$$

where $B_{k,n}(x)$ are the Bernstein polynomial of degree n.

The q-derivative $D_q f$ of function f is defined by

$$(D_q f)(x) = \frac{df(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}, \text{ (see [4, 6, 7, 8, 9, 10])}. \tag{7}$$

From (7), we note that

$$D_q(fg)(x) = g(x)D_q f(x) + f(qx)D_q g(x), \text{ (see [4, 7, 8, 9, 10])}. \tag{8}$$

The q-Bernstein operator is given by

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} B_{k,n}(x,q) f\left(\frac{[k]_q}{[n]_q}\right), \text{ (see Eq. (4))}.$$

Thus, we have

$$\mathbb{B}_{n,q}(1 \mid x) = \sum_{k=0}^{n} B_{k,n}(x,q) = \sum_{k=0}^{n} \binom{n}{k}_q x^k (1-x)_q^{n-k} = 1,$$

and

$$\mathbb{B}_{n,q}(x \mid x) = \sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q} \right) B_{k,n}(x,q) = x \sum_{k=0}^{n-1} \binom{n-1}{k}_q x^k (1-x)_q^{n-k} = x,$$

where $x \in [0,1]$ and $n, k \in \mathbb{Z}_+$.

For $f \in C[0,1]$, we have

$$\begin{aligned} \mathbb{B}_{n,q}(f \mid x) &= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q} \right) B_{k,n}(x,q) \\ &= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q} \right) \binom{n}{k}_q x^k (1-x)_q^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q} \right) x^k \binom{n}{k}_q \sum_{j=0}^{n-k} \binom{n-k}{j}_q (-1)^j q^{\binom{j}{2}} x^j. \end{aligned} \tag{9}$$

It is easy to show that

$$\binom{n}{k}_q \binom{n-k}{j}_q = \binom{n}{k+j}_q \binom{k+j}{k}_q.$$

Let $k + j = m$. Then we have

$$\binom{n}{k}_q \binom{n-k}{j}_q = \binom{n}{m}_q \binom{m}{k}_q. \tag{10}$$

By (9) and (10), we easily get

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{m=0}^{n} \binom{n}{m}_q x^m \sum_{k=0}^{m} \binom{m}{k}_q q^{\binom{m-k}{2}} (-1)^{m-k} f\left(\frac{[k]_q}{[n]_q} \right). \tag{11}$$

Therefore, we obtain the following proposition.

Proposition 1. For $f \in C[0,1]$ and $n \in \mathbb{Z}_+$, we have

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{m=0}^{n} \binom{n}{m}_q x^m \sum_{k=0}^{m} \binom{m}{k}_q q^{\binom{m-k}{2}} (-1)^{m-k} f\left(\frac{[k]_q}{[n]_q} \right).$$

It is well known that the second kind Stirling numbers are defined by

$$\frac{(e^t-1)^k}{k!}=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt}=\sum_{n=0}^{\infty}S(n,k)\frac{t^n}{n!}, \tag{12}$$

where $k\in\mathbb{N}$ (see [9, 11, 12, 14]).

Let Δ be the shift difference operator with $\Delta f(x)=f(x+1)-f(x)$. By iterative process, we see that

$$\Delta^n f(0)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(k),\ \text{for } n\in\mathbb{N}. \tag{13}$$

From (12) and (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty}S(n,k)\frac{t^n}{n!}&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt}=\sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^n\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{\Delta^k 0^n}{k!}\frac{t^n}{n!},\ \text{where } \Delta^k 0^n=\Delta^k x^n|_{x=0}.\end{aligned} \tag{14}$$

By comparing the coefficients on the both sides of (14), we get

$$S(n,k)=\frac{\Delta^k 0^n}{k!},\ \text{for } n,k\in\mathbb{Z}_+. \tag{15}$$

Now, we consider the q-extension of (13). Let $(Eh)(x)=h(x+1)$ be the shift operator. Then the q-difference operator is defined by

$$\Delta_q^n:=(E-I)_q^n=\prod_{i=1}^{n}(E-Iq^{i-1}),\ \text{(see [11])},$$

where I is an identity operator.

For $f\in C[0,1]$ and $n\in\mathbb{N}$, we have

$$\Delta_q^n f(0)=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}f(n-k)=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^{n-k}q^{\binom{n-k}{2}}f(k). \tag{16}$$

Note that (16) is exactly q-extension of (13). That is, $\lim_{q\to 1}\Delta_q^n f(0) = \Delta^n f(0)$.

As the q-extension of (12), the second kind q-Stirling numbers are defined by

$$\frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}_q q^{\binom{k-j}{2}}e^{[j]_q t} = \sum_{n=0}^{\infty} S(n,k:q)\frac{t^n}{n!}, \text{ (see [11, 12, 14])}. \tag{17}$$

By (16), we obtain the following theorem.

Theorem 2. For $f \in C[0,1]$ and $n \in \mathbb{Z}_+$, we have

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n}\binom{n}{k}_q x^k \Delta_q^k f\left(\frac{0}{[n]_q}\right).$$

In the special case, $f(x) = x^m (m \in \mathbb{Z}_+)$, we obtain the following corollary.

Corollary 3. For $x \in [0,1]$ and $m, n \in \mathbb{Z}_+$, we have

$$[n]_q^m \mathbb{B}_{n,q}(x^m \mid x) = \sum_{k=0}^{n}\binom{n}{k}_q x^k \Delta_q^k 0^m.$$

By (17), we easily get

$$\begin{aligned} S(n,k:q) &= \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\binom{k}{j}_q [k-j]_q^n \\ &= \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^{k-j} q^{\binom{k-j}{2}}\binom{k}{j}_q [j]_q^n \\ &= \frac{q^{-\binom{k}{2}}}{[k]_q!}\Delta_q^k 0^n. \end{aligned} \tag{18}$$

The equation (18) seems to be the q-extension of the equation (15). That is, $\lim_{q\to 1} S(n,k:q) = S(n,k)$.

By Corollary 3 and (18), we obtain the following corollary.

Corollary 4. For $x \in [0,1]$ and $m, n \in \mathbb{Z}_+$, we have

$$[n]_q^m \mathbb{B}_{n,q}(x^m \mid x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k [k]_q! q^{\binom{k}{2}} S(m,k:q).$$

From (1) and (5), for $0 \le k \le n$, we have

$$\begin{aligned} & q^k(1-q^{n-k-1}x)B_{k,n-1}(x,q) + xB_{k-1,n-1}(x,q) \\ &= q^k(1-q^{n-k-1}x)\binom{n-1}{k}_q x^k(1-x)_q^{n-1-k} + x\binom{n-1}{k-1}_q x^{k-1}(1-x)_q^{n-k} \\ &= q^k\binom{n-1}{k}_q x^k(1-x)_q^{n-k} + \binom{n-1}{k-1}_q x^k(1-x)_q^{n-k} \\ &= \binom{n}{k}_q x^k(1-x)_q^{n-k}. \end{aligned} \tag{19}$$

By (2), (7) and (8), we get

$$\frac{dB_{k,n}(x,q)}{d_q x} = -\binom{n}{k}_q x^k [n-k]_q (1-qx)_q^{n-k-1} + \binom{n}{k}_q [k]_q x^{k-1}(1-qx)_q^{n-k}. \tag{20}$$

From the definition of Gaussian binomial coefficient (= q-binomials coefficient) and (2), we note that

$$\binom{n}{k}_q [k]_q x^{k-1}(1-qx)_q^{n-k} = q^{-(k-1)}[n]_q B_{k-1,n-1}(qx,q), \tag{21}$$

and

$$\binom{n}{k}_q x^k [n-k]_q (1-qx)_q^{n-k-1} = [n]_q q^{-k} B_{k,n-1}(qx,q).$$

By (20) and (21), we see that

$$\frac{dB_{k,n}(x,q)}{d_q x} = [n]_q q^{-k}(qB_{k-1,n-1}(qx,q) - B_{k,n-1}(qx,q)). \tag{22}$$

Thus, we note that the q-derivative of the q- Bernstein polynomials of degree n are also polynomial of degree $n-1$. Therefore, by (19) and (22), we obtain the following recurrence formulae:

Theorem 5(Recurrence formulae for $B_{k,n}(x,q)$). For $k \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and $x \in [0,1]$, we have

$$q^k(1-q^{n-k-1}x)B_{k,n-1}(x,q) + xB_{k-1,n-1}(x,q) = B_{k,n}(x,q),$$

and

$$\frac{dB_{k,n}(x,q)}{d_q x} = [n]_q q^{-k}(qB_{k-1,n-1}(qx,q) - B_{k,n-1}(qx,q)).$$

We also get from (5) and (6) that

$$\begin{aligned}
&\frac{[n-k]_q}{[n]_q}B_{k,n}(x,q) + \frac{[k+1]_q}{[n]_q}B_{k+1,n}(x,q) \\
&= (1-xq^{n-k-1})\binom{n-1}{k}_q x^k(1-x)_q^{n-k-1} + x\binom{n-1}{k}_q x^k(1-x)_q^{n-k-1} \\
&= (1-xq^{n-k-1})B_{k,n-1}(x,q) + xB_{k,n-1}(x,q) \\
&= B_{k,n-1}(x,q) + x[n-k-1]_q(1-q)B_{k,n-1}(x,q).
\end{aligned} \tag{23}$$

By (23), we obtain the following theorem.

Theorem 6. For $k, \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and $x \in [0,1]$, we have

$$\frac{[n-k]_q}{[n]_q}B_{k,n}(x,q)+\frac{[k+1]_q}{[n]_q}B_{k+1,n}(x,q) = B_{k,n-1}(x,q)+x[n-k-1]_q(1-q)B_{k,n-1}(x,q).$$

It follows from Theorem 6 that q-Bernstein polynomials can be written as a linear combination of polynomials of higher order.

For $k, n \in \mathbb{N}$, we easily get from (5) that q-Bernstein polynomials can be expressed in the form

$$\begin{aligned}
&\frac{[n-k+1]_q}{[k]_q}\left(\frac{x}{1-xq^{n-k}}\right)x^{k-1}(1-x)_q^{n-k+1}\binom{n}{k-1}_q \\
&= \frac{[n]_q!}{[k]_q![n-k]_q!}x^k(1-x)_q^{n-k} \\
&= \binom{n}{k}_q x^k(1-x)_q^{n-k} \\
&= B_{k,n}(x,q).
\end{aligned} \tag{24}$$

By (24), we obtain the following proposition.

Proposition 7. For $n, k \in \mathbb{N}$ and $x \in [0,1]$, we have

$$B_{k,n}(x,q) = \frac{[n-k+1]_q}{[k]_q} \left(\frac{x}{1-xq^{n-k}} \right) B_{k-1,n}(x,q).$$

The q-Bernstein polynomials of degree n can be written in terms of power basis $\{1, x, x^2, \cdots, x^n\}$. By using the definition of q-Bernstein polynomial and q-binomial theorem, we get

$$\begin{aligned} B_{k,n}(x,q) &= \binom{n}{k}_q x^k (1-x)_q^{n-k} = \binom{n}{k}_q x^k \sum_{i=0}^{n-k} \binom{n-k}{i}_q (-1)^i q^{\binom{i}{2}} x^i \\ &= \sum_{i=0}^{n-k} \binom{n-k}{i}_q \binom{n}{k}_q (-1)^i q^{\binom{i}{2}} x^{i+k} \\ &= \sum_{i=k}^{n} \binom{n-k}{i-k}_q \binom{n}{k}_q (-1)^{i-k} q^{\binom{i-k}{2}} x^i. \end{aligned} \tag{25}$$

By simple calculation, we easily see that

$$\binom{n}{k}_q \binom{n-k}{i-k}_q = \binom{n}{i}_q \binom{i}{k}_q. \tag{26}$$

Therefore, by (25) and (26), we obtain the following theorem.

Theorem 8. For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$B_{k,n}(x,q) = \sum_{i=k}^{n} \binom{n}{i}_q \binom{i}{k}_q (-1)^{i-k} q^{\binom{i-k}{2}} x^i.$$

We get from the properties of q-Bernstein polynomials that

$$\begin{aligned}\sum_{k=1}^{n}\frac{\binom{k}{1}_q}{\binom{n}{1}_q}B_{k,n}(x,q) &= \sum_{k=1}^{n}\frac{[k]_q}{[n]_q}\binom{n}{k}_q x^k(1-x)_q^{n-k}\\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}_q x^k(1-x)_q^{n-k}\\ &= x\sum_{k=0}^{n-1}\binom{n-1}{k}_q x^k(1-x)_q^{n-k-1} = x,\end{aligned}$$

and that

$$\begin{aligned}\sum_{k=2}^{n}\frac{\binom{k}{2}_q}{\binom{n}{2}_q}B_{k,n}(x,q) &= \sum_{k=2}^{n}\binom{n-2}{k-2}_q x^k(1-x)_q^{n-k}\\ &= x^2\sum_{k=0}^{n-2}\binom{n-2}{k}_q x^k(1-x)_q^{n-k-2} = x^2.\end{aligned}$$

Continuing this process, we obtain

$$\sum_{k=i}^{n}\frac{\binom{k}{i}_q}{\binom{n}{i}_q}B_{k,n}(x,q) = x^i.$$

Therefore, we obtain the following theorem.

Theorem 9. For $k, i \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$\sum_{k=i}^{n}\frac{\binom{k}{i}_q}{\binom{n}{i}_q}B_{k,n}(x,q) = x^i.$$

Now we define q-Bernoulli polynomials of order k as follows:

$$\left(\frac{z}{e^z-1}\right)^k e_q(zx) = \sum_{n=0}^{\infty}\beta_n^{(k)}(x,q)\frac{z^n}{[n]_q!}, \quad k \in \mathbb{N}. \tag{27}$$

From the generating function (27) of q-Bernoulli polynomials and (3), we derive

$$\begin{aligned}\left(\frac{z}{e^z-1}\right)^k e_q(zx) &= \left(\sum_{m=0}^{\infty} B_m^{(k)}\frac{z^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{x^l z^l}{[l]_q!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{B_m^{(k)}x^{n-m}[n]_q!}{m![n-m]_q!}\right)\frac{z^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{[m]_q!}{m!}B_m^{(k)}\binom{n}{m}_q x^{n-m}\right)\frac{z^n}{[n]_q!},\end{aligned} \tag{28}$$

where $B_m^{(k)}$ are the n-th Bernoulli numbers of order k(see [6]).

From (27) and (28), we easily get

$$\beta_n^{(k)}(x,q) = \sum_{m=0}^{n}\binom{n}{m}_q \frac{[m]_q!}{m!}x^{n-m}B_m^{(k)}, \tag{29}$$

where $B_m^{(k)}$ are the m-th ordinary Bernoulli numbers of order k.

From (26) and (27), we note that

$$\begin{aligned}\frac{(tx)^k}{[k]_q!}e_q((1-x)_q t) &= \frac{x^k(e^t-1)^k}{[k]_q!}\left(\frac{t}{e^t-1}\right)^k e_q((1-x)_q t)\\ &= \frac{k!}{[k]_q!}x^k\left(\sum_{m=0}^{\infty}S(m,k)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\beta_n^{(k)}((1-x)_q,q)\frac{t^n}{[n]_q!}\right)\\ &= \frac{k!}{[k]_q!}x^k\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}\frac{[m]_q!}{m!}S(m,k)\binom{l}{m}_q\beta_{l-m}^{(k)}((1-x)_q,q)\right)\frac{t^l}{[l]_q!}.\end{aligned} \tag{30}$$

Therefore, by (6) and (30), we obtain the following theorem,

Theorem 10. For $k, l \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$B_{k,l}(x,q) = \frac{k!}{[k]_q!}x^k\sum_{m=0}^{l}\frac{[m]_q!}{m!}S(m,k)\beta_{l-m}^{(k)}((1-x)_q,q)\binom{l}{m}_q,$$

where $\beta_l^{(k)}((1-x)_q,q)$ are called the l-th q-Bernoulli polynomials.

From (15) and Theorem 10, we have the following corollary.

Corollary 11. For $k, l \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$B_{k,l}(x,q) = \frac{x^k}{[k]_q!} \sum_{m=0}^{l} \frac{[m]_q!}{m!} \binom{l}{m}_q \beta_{l-m}^{(k)}((1-x)_q, q) \Delta^k 0^m.$$

It is well known that

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(k, n-k), \text{ (see [7])}, \tag{31}$$

where $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$.

By (31) and Theorem 9, we easily see that

$$\sum_{k=0}^{i} \binom{x}{k} k! S(k, i-k) = \sum_{k=i}^{n} \frac{\binom{k}{i}_q}{\binom{n}{i}_q} B_{k,n}(x,q).$$

3. A matrix representation for q-Bernstein polynomials

Given a polynomial is written as a linear combination of q-Bernstein basis functions:

$$B_q(x) = C_0^q B_{0,n}(x,q) + C_1^q B_{1,n}(x,q) + \cdots + C_n^q B_{n,n}(x,q). \tag{32}$$

It is easy to write (32) as a dot product of two vectors:

$$B_q(x) = \begin{pmatrix} B_{0,n}(x,q), B_{1,n}(x,q), \ldots, B_{n,n}(x,q) \end{pmatrix} \begin{pmatrix} C_0^q \\ C_1^q \\ \vdots \\ C_n^q \end{pmatrix}. \tag{33}$$

Now, we can convert (33) to

$$B_q(x) = \begin{pmatrix} 1, x, \ldots, x^n \end{pmatrix} \begin{pmatrix} b_{0,0}^q & 0 & \ldots & 0 \\ b_{1,0}^q & b_{1,1}^q & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0}^q & b_{n,1}^q & \ldots & b_{n,n}^q \end{pmatrix} \begin{pmatrix} C_0^q \\ C_1^q \\ \vdots \\ C_n^q \end{pmatrix},$$

where $b_{i,j}^q$ are the coefficients of the power basis that are used to determine the respective q-Bernstein polynomials.

From (5) and (6), we note that

$$B_{0,2}(x,q) = (1-x)_q^2 = \sum_{l=0}^{2} \binom{2}{l}_q (-1)^l q^{\binom{l}{2}} = 1 - [2]_q x + qx^2$$

$$B_{1,2}(x,q) = \binom{2}{1}_q x(1-x)_q = [2]_q x(1-x) = [2]_q x - [2]_q x^2$$

$$B_{2,2}(x,q) = x^2.$$

In the quadratic case($n = 2$), the matrix can be represented by

$$B_q(x) = \begin{pmatrix} 1, x, x^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -[2]_q & [2]_q & 0 \\ q & -[2]_q & 1 \end{pmatrix} \begin{pmatrix} C_0^q \\ C_1^q \\ C_2^q \end{pmatrix}.$$

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Efficient Pseudospectral Legendre Preconditioner For Minimum Time Orbit Problem

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Abstract. In this paper the relation between the pseudospectral integration and differentiation matrices has been presented. It has been proved that the integration matrix may be used as an optimal preconditioner. In addition, a new integral Legendre preconditioner has been suggested to convert the pseudospectral approximation of the differential equation arise in optimal control problems to integral pseudospectral approximation of its integral form. This preconditioner, in effect, makes the pseudospectral methods insensitive to specific grid points. The numerical results show the efficiency of the method.

Keywords: Legendre pseudospectral approximation methods; preconditioner matrix.

1. Introduction

In the last few years, the pseudospectral methods have proven efficient for obtaining accurate solutions to differential equations, the basic idea being to replace the exact derivatives with interpolating polynomials at the collocation points (see [2, 4 and 8]). For boundary value problems, this approach involves the solution of very ill-conditioned linear systems of equations. Thus, to apply iterative methods for solving such systems, it becomes important to develop efficient preconditioners. For some previous and recent works on preconditioners for pseudospectral methods, see e.g. [3, 6, 7, 13 and 18]. Conventional pseudospectral methods impose rigid requirement on the computational grids used. The grid points are the nodes of Gauss like quadrature formula. These nodes are denser at the boundaries than in the middle of the domain. While this behavior is suitable for boundary layers, it may create difficulties for other types of problems. Gottlieb [11] pointed out it is apparent that pseudospectral methods which are not constrained to specific nodal points would be more flexible than conventional pseudospectral methods. They show how to implement Legendre Galerkin and Laguerre Galerkin method on arbitrary grids.

The aim of this paper is to introduce some ideas to construct suitable preconditioner matrix of for Legendre pseudospectral method in terms of expansion coefficients on Legendre Gauss-Lobatto (LGL) points. These ideas will be illustrated in the case of pseudospectral solutions of the system dynamic optimal control problems (OCPs), but it will be obvious that they can be applied to time orbit transfer problem.

What is done in this work is in some sense close to the idea presented in [13]. Based on pseudospectral differentiation matrices on Gauss–Lobatto nodal values similar matrices are generalized for arbitrary grids. Throughout this work, we discuss two types of well-known matrices. The first is Legendre pseudospectral differentiation matrix [16]. The second is Legendre pseudospectral integration matrix [15]. The integration matrix could be used as optimal preconditioner (exact inverse). Also, we applied the suggested integral preconditioner on the system dynamics.

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This study has been motivated by the computation of an invariant torus of a system of ordinary differential equations. We find the torus by discretizing optimal control problems with a full dimensional pseudospectral method and by applying Newton's method. This leads to large non-symmetric linear algebraic systems.

The main results of this paper is to show that the integral preconditioner convert the pseudospectral approximation of differential operator to integral pseudospectral approximation of its integral form. A well-conditioned Legendre pseudospectral method is developed through the use of integral equation.

Finally, in this work, we use a various preconditioners to the solution of system of linear equations arising from applications in BVPs and optimal control problems. The preconditioners considered here are the Incomplete LU factorization (ILU), the Incomplete LU factorization with threshold (ILUT), the Crout version of ILUT (ILUC) [21]. We are interested in the comparison of these preconditioners based on their memory efficiency and convergence properties, and we show that for the pseudospectral generally performs better.

The remaining part of this paper is organized as follows: In section 2, preliminary formulations and some useful notations are presented. Also, we introduce the Legendre pseudospectral differentiation and integration matrices. In section 3, the pseudospectral Legendre integration matrix is used to define an optimal preconditioner based on the relation between the integration and differentiation matrices. Some of iterative techniques and preconditioning are applied in section 4. In sections 4 and 5, some of numerical experiments on optimal control problem and time orbit transfer problem are solved.

2. Preliminary Considerations

Let $N \geq 1$ be an integer, and let $x_0, x_1, ..., x_N$ be a set of points in [-1, 1], these points considered throughout the work are Gauss-Lobatto points. The Legendre polynomials, $P_n(x), n = 0,1,...,N$ appears as eigensolutions to the singular Sturm-Liouville problem:

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

with $\lambda = n(n+1)$, in the finite domain $x \in [-1,1]$, the first two being $P_0(x) = 1$, $P_1(x) = x$ while the remaining polynomials are given through the recursion formula [1]:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0. \tag{2.1}$$

The Legendre polynomials in terms of power of x are given by:

$$P_n(x) = \sum_{m=0}^{[n/2]} c_m^n x^{n-2m} \tag{2.2}$$

where

$$c_m^n = \frac{(-1)^m (2n-2m)!}{2^n (n-m)!(n-2m)!m!},$$

and,

$$[n/2]=\begin{cases} n/2 & \text{if } n \text{ is even} \\ (n\text{-}1)/2 & \text{if } n \text{ is odd} \end{cases}.$$

We present here the Legendre approximations of a function $f(x)\in C^{\infty}[-1,1]$, using $(N+1)$ Legendre-Gauss-Lobatto (LGL) points $\{x_i : (1-x_i^2)P_N'(x_i)=0,\ i=0,1,...,N\}$, where N *is* the degree of the expansion [15]:

$$f(x)\cong\sum_{j=0}^{N} a_j P_j(x), \tag{2.3}$$

where

$$a_j \cong \frac{2j+1}{N(N-1)}\sum_{k=0}^{N}\frac{P_j(x_k)f(x_k)}{\left[P_{N-1}(x_k)\right]^2}, \quad j=0,1,...,N\ . \tag{2.4}$$

2.1 Legendre pseudospectral differentiation matrices

The derivatives of a function $f(x)$ can be approximated by interpolating at a set of $(N+1)$ Legendre Gauss-Lobatto (LGL) points. Differentiating the polynomial, and then evaluating the polynomial at the same points. The entries ($d_{i,j}^{(m)}$) of the Legendre differentiation matrix ***D*** are given as follows: [16]

$$\left.\frac{d^m}{dx^m}f(x)\right|_{x=x_i}=\sum_{j=0}^{N} d_{i,j}^{(m)} f(x_j) \tag{2.5}$$

where

$$d_{i,l}^{(m)}=\sum_{j=m}^{N}\sum_{k=0}^{[(j-m)/2]}\frac{2j+1}{N(N-1)}\frac{P_j(x_i)}{\left[P_{N-1}(x_i)\right]^2}a_{k,m}^{(j)}\,x_l^{j-2k-m}, \tag{2.6}$$

and

$$a_{k,m}^{(j)}=\frac{(-1)^k(2j-2k)!(j-2k)(j-2k-1)...(j-2k-m+1)}{2^j(j-k)!(j-2k)!k!}.$$

The matrix form is:

$$\left[\frac{d^m}{dx^m}f\right]=D^m[f].$$

Where D^m is known $(N+1)\times(N+1)$ matrix and their elements given by $[d_{i,j}^{(m)}]$, the elements of the column matrix $[f]$ are given by $f(x_i)$, $i=0,1,...,N$. This formula gives us an improvement of the results as shown in the numerical examples.

2.2 Legendre pseudospectral integration matrices

Approximate the integrals of a function $f(x)$ by interpolating the function with a polynomial $\mathrm{P}_N f$ at Legendre-Gauss-Lobatto (LGL) points.

The values of the integrals:

$$\int_{-1}^{t}\int_{-1}^{t}\cdots\int_{-1}^{t}(\mathrm{P_N f})(t)dtdt...dt \quad (n-times)$$

at th $(N+1)$ points, in fact, be expressed as a fixed linear combination of the given function values and the whole relationship may be written in matrix form:

$$\left[\int_{-1}^{t}\int_{-1}^{t}\cdots\int_{-1}^{t}(\mathrm{P}_N f)(t)dtdt...dt\right] = B^{(n)}\left[\mathrm{P}_N f\right] \quad (n-times)$$

Setting $[\mathbf{P}_N f]=[f(t_0), f(t_1),..., f(t_N)]^T$ as the vector consisting of values of $f(x)$ at $(N+1)$ collocation points,

$$\left[\int_{-1}^{t}(\mathbf{P}_N f)(t)dt\right] = \left[\int_{-1}^{t_0}(\mathbf{P}_N f)(t)dt,...,\int_{-1}^{t_N}(\mathbf{P}_N f)(t)dt\right]^T$$

as the values of the integrals at the collocation points and $B^{(n)}=b_{ij}^{(n)}$, $i, j=0,1,...,N$ as the collocation integral matrix: [15]

$$b_{i,k}^{(n)} = \sum_{j=0}^{N}\sum_{m=0}^{[j/2]}\frac{2j+1}{N(N-1)\left[P_{N-1}(t_k)\right]^2}c_m^j P_j(t_k)\left\{\frac{(j-2m)!}{(j-2m+n)!}[t_i^{j-2m+n}-(-1)^{j-2m+n}]\right.$$

$$\left.-\sum_{r=1}^{n-1}\sum_{s=0}^{r}\frac{(-1)^{j-2m+n-r}(j-2n)!t_i^s}{(r-s)!s!(j-2m+n-r)!}\right\}, \qquad (2.7)$$

$0\le i,k\le N$.

and

$$c_k^n = \frac{(-1)^k(2n-2k)!}{2^n(n-k)!(n-2k)!k!}.$$

It should be noticed that, $B^{(n)}\neq(B^{(1)})^n$. In addition, the row sums of the matrix $B^{(n)}$ is given by:

$$\sum_{k=0}^{N} b_{i,k}^{(n)} = \frac{(t_i+1)^n}{n!} \,.\, i = 0,1,...,N \tag{2.8}$$

One should remember that the matrices obtained are in general very ill conditioned for quite modest values of n. If solving a system of equations with these matrices, preconditioning by finite difference matrices for the same problem has been shown to be an effective way to reduce the spectral condition number [5].

3. Exact Integration Preconditioning matrix

The product formulae proposed in this section are based only on LGL points and weights of Legendre polynomials. These choose lead to make the approximation of the solution more precise; we used the integration preconditioning implementation based on Legendre polynomials (IPL). Therefore, the preconditioning matrix introduces an exact integration and it should be noticed that the truncation error approaches zero faster than any negative power of the number of basis functions used in the approximation. Roughly one can say that, from Eq. (2.8), the convergence is better when the eigenvalues are clustered. This is the reason why we try to precondition the matrices under consideration. Furthermore, such precondition is numerically performed through FORTRAN programming.

Theorem 3.1

If $x_i \in$[-1, 1] are the set of LGL points and $D^{(n)}$, $B^{(m)}$ are the n-differentiation and m-integration Legendre pseudospectral matrices, respectively. Then the exact Legendre integration preconditioning matrixes are given by:

$$B^{(m)}D^{(n)} = \begin{cases} B^{(m-n)} - B^{(m-n)}M - \sum_{k=1}^{n-1} B^{(m-n+k)}M\,D^{(k)}; & m > n \\ D^{(n-m)} - MD^{(n-m)} - \sum_{k=1}^{m-1} B^{(k)}M\,D^{(n-m+k)}; & n > m \\ I - M - \sum_{k=1}^{m-1} B^{(k)}M\,D^{(k)}; & n = m \end{cases},$$

where I is the identity matrix and $M = \lfloor m_{i,j} \rfloor$, are given by:

$$m_{i,j} = \begin{cases} 1 & j = 0 \text{ and } i = 0,...,N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof:

It is clear from the construction of $D^{(m)}$ that the spectral differentiation is exact for any smooth function. Then, for fixed N and differentiation of any order, we have,

$$\left[\int_{-1}^{x_i}\frac{d}{dt}u(t)dt\right]=B^{(1)}D^{(1)}\mathbf{u}=\left[\mathbf{u}_{\mathbf{x}=\mathbf{x}_i}\right]-\left[\mathbf{u}_{\mathbf{x}=\mathbf{x}_0}\right],$$

where $\left[\mathbf{u}_{\mathbf{x}=\mathbf{x}_i}\right]=\left[u(x_0),u(x_1),...,u(x_N)\right]^T$. Therefore,

$$B^{(1)}D^{(1)}=I-M\ , \tag{3.1}$$

and

$$B^{(2)}D^{(2)}=I-M-B^{(1)}M\ D^{(1)} \tag{3.2}$$

From equation (2.8), it can be easily seen by induction that:

Case 1: $m>n$

$$B^{(m)}D^{(n)}=B^{(m-n)}-B^{(m-n)}M-\sum_{k=1}^{m-1}B^{(m-n+k)}M\ D^{(k)}\ , \tag{3.3}$$

where

$$B^{(m-n)}M\left[\mathbf{u}\right]=\left[\frac{(x+1)^{m-n}}{(m-n)!}\right]\left[\mathbf{u}_{\mathbf{x}=\mathbf{x}_0}\right],$$

and

$$B^{(m-n+k)}MD^{(k)}\left[\mathbf{u}\right]=\left[\frac{(x+1)^{m-n+k}}{(m-n+k)!}\right]\left[\mathbf{u}^{(k)}_{\mathbf{x}=\mathbf{x}_0}\right].$$

Case 2: $n>m$

$$B^{(m)}D^{(n)}=D^{(n-m)}-MD^{(n-m)}-\sum_{k=1}^{n-1}B^{(k)}M\ D^{(n-m+k)}\ , \tag{3.4}$$

where

$$MD^{(n-m)}\left[\mathbf{u}\right]=\left[\mathbf{u}^{(n-m)}_{\mathbf{x}=\mathbf{x}_0}\right],$$

and

$$B^{(k)}MD^{(n-m+k)}\left[\mathbf{u}\right]=\left[\frac{(x+1)^{k}}{k!}\right]\left[\mathbf{u}^{(n-m+k)}_{\mathbf{x}=\mathbf{x}_0}\right].$$

Case 3: $m=n$

$$B^{(m)}D^{(n)} = I - M - \sum_{k=1}^{n-1} B^{(k)} M\, D^{(k)}\ ,\ n>1\,. \qquad \blacksquare$$

The equations (3.3) and (3.4) are exact for any N .

Generally, in case of $n=m=1$, we have the preconditioner matrix $\mathbf{P}^{(1)} = B^{(1)}D^{(1)}$, i.e.

$$\mathcal{P}_{i,j} = \begin{cases} -1 & i=1,...,N;\ \ j=0 \\ +1 & i=j;\, i,\, j\neq 0 \\ 0 & otherwise \end{cases}$$

For example, when $N=4$, we get:

$$\mathcal{P}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the next section, we are considered the smooth and continues function $u(x)$, and $\mathcal{P}_N$ are the set of Legendre polynomials of degree at most N . We studied the effectiveness of using Legendre pseudospectral approximation with the suggested integration preconditioning to solve the second and fourth order boundary value problems.

3.1 Preconditioning for the second order operator

Consider the second order, linear nonhomogeneous differential equation of the form:

$$\alpha u''(x) + p(x)u'(x) + q(x)u(x) = f(x),\ x \in (-1,1)\,, \qquad (3.9)$$

subject to the following Robin boundary conditions:

$$a_0 u_x(-1) + a_1 u(-1) = a_2, \qquad b_0 u_x(1) + b_1 u(1) = b_2. \qquad (3.10)$$

Seeking of the solutions of polynomial form leads to a discrete form of the equations (3.9) and (3.10) as follows:

$$\alpha D^{(2)}\mathbf{u} + \bar{\mathbf{P}} D^{(1)}\mathbf{u} + \bar{\mathbf{Q}}\mathbf{u} = \mathbf{f}\ , \qquad (3.11)$$

where,

$$\bar{\mathbf{P}} = \begin{bmatrix} p(x_0) & . & . & 0 & 0 \\ . & p(x_1) & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & p(x_{N-1}) & . \\ 0 & 0 & . & . & p(x_N) \end{bmatrix}, \text{ and } \bar{\mathbf{Q}} = \begin{bmatrix} q(x_0) & . & . & 0 & 0 \\ . & q(x_1) & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & q(x_{N-1}) & . \\ 0 & 0 & . & . & q(x_N) \end{bmatrix}$$

with,

$$a_0 u_x(x_0) + a_1 u(x_0) = a_2,\ b_0 u_x(x_N) + b_1 u(x_N) = b_2. \tag{3.12}$$

Let:

$$D^{(2)}\mathbf{u} = \mathbf{\Phi}.$$

On apply the preconditioners $B^{(2)}$ and $B^{(1)}$ on (3.11) and using (3.12), we get:

$$(\alpha I + \bar{\mathbf{P}}B^{(1)} + \bar{\mathbf{Q}}B^{(2)})\mathbf{\Phi} = \mathbf{f} - \bar{\mathbf{P}}MD\mathbf{u} - \bar{\mathbf{Q}}M\mathbf{u} - u'(x_0)\bar{\mathbf{Q}}[x+1],$$

It can be rewritten in compact form as:

$$\bar{\mathrm{B}}^{(2)}\mathbf{\Phi} = \bar{\mathbf{F}} \tag{3.13}$$

Therefore, the Legendre pseudospectral technique is used to accelerate the convergence of an iteration for solving $\mathcal{L}(D^{(2)})u = f$ by changing the iteration matrix to $\mathcal{L}(B^{(2)})u$. The matrix $\mathcal{L}(B)$ is the "preconditioning matrix" and is chosen to be an approximation of $\mathcal{L}(D)$ (in the sense of having approximately the same eigenvalues), but is much less expensive to invert than $\mathcal{L}(B)$.

3.2 Preconditioning for the fourth order operator

We are now interesting in the pseudospectral approximation of the following model problem,

$$u^{(4)}(x) - \alpha u''(x) - \beta u(x) = f(x),\ x \in (-1,1) \tag{3.14}$$

subject to

$$u(\pm 1) = 0 \text{ and } u'(\pm 1) = 0. \tag{3.15}$$

Seeking of the solutions of polynomial form leads to a discrete form of the equations (3.14) and (3.15) as follows,

$$D^{(4)}\mathbf{u} - \alpha D^{(2)}\mathbf{u} - \beta\mathbf{u} = \mathbf{f} \tag{3.16}$$

subject to

$$u(x_0) = 0,\ u(x_N) = 0,\ u'(x_0) = 0 \text{ and } u'(x_N) = 0. \tag{3.17}$$

Let $D^{(4)}\mathbf{u} = \mathbf{\Phi}$, then on apply the preconditioners on (3.16) and using (3.17), we get:

$$D^{(3)}\mathbf{u} = B^{(1)}\mathbf{\Phi} + MD^{(3)}\mathbf{u};$$

$$D^{(2)}\mathbf{u} = B^{(2)}\mathbf{\Phi} + u'''(-1)\mathbf{[}(x+1)\mathbf{]};$$

$$D^{(1)}\mathbf{u} = B^{(3)}\mathbf{\Phi} + \frac{u'''(-1)}{2!}\mathbf{[}(x+1)^2\mathbf{]} + u''(-1)\mathbf{[}(x+1)\mathbf{]};$$

$$[I - \alpha B^{(2)} - \beta B^{(4)}]\mathbf{\Phi} = \mathbf{f} + \frac{\beta u'''(-1)}{3!}[(x+1)^3] + \frac{\beta u''(-1)}{2!}[(x+1)^2] + \alpha u'''(-1)[(x+1)]$$

The compact form is:

$$\bar{\mathrm{B}}^{(4)}\mathbf{\Phi} = \mathbf{F} \tag{3.18}$$

Further, on using the boundary conditions (3.17) and at the point x_N, we obtain the values of $u''(\text{-}1)$ and $u'''(-1)$. However, in the preconditioned algorithm, we turn to change the implementation of the iterative solvers by preconditioning. Of course, it would be possible to simply apply the methods to the transformed system and proceed as before. But we have to pay a lot of unnecessary expenses.

4. Iterative Techniques and Preconditioning

In this section, without loss of generality, we are interested in finding good preconditioners for solving the m^{th} order linear differential equation with constant coefficients:

$$\mathcal{L}_m(u) = \alpha \frac{d^m u(x)}{dx^m} + \beta u(x) = f(x),$$

i.e.,

$$\mathcal{L}_m(D) = f(x) \tag{4.1}$$

Regarding to the results in the next section, we found that the Legendre pseudospectral $\mathcal{L}_m(B)$ is represented a good preconditioner of the iterative method in (4.1), i.e., it enjoys some basic advantages as well as storage requirements are estimated. First, the memory requirements of $\mathcal{L}_m(B)$ has the same order as $\mathcal{L}_m(D)$. Second, the computational cost is comparable to the computation of matrix–vector products involving $\mathcal{L}_m(D)$ for certain iterative technique, e.g., in case of $m=1$, the total number of multiplications are of order $O(n\log_2 n)$. Finally, the convergence rate is improved and justifies the additional costs. The Legendre pseudospectral preconditioner is related to the discretization of the differential operator in (4.1), that is:

$$\lfloor \alpha D^{(m)} + \beta \rfloor [u_N] = [f(x)].$$

To solve this system, we note that the Legendre pseudospectral preconditioner matrix $B^{(m)}$ admits the system:

$$\bar{\mathrm{B}}^{(m)}\mathbf{\Phi} = \mathbf{F},$$

where,

$$\bar{\mathrm{B}}^{(m)} = \left[I + \alpha B^{(m)} + \beta \right],\ \alpha \neq 0.$$

where $\mathbf{\Phi} = [\varphi_0, \varphi_1, ..., \varphi_N]^T$ are the adjoint slack variables. These variables are used to compute the cost of performing the preconditioning, such that the number of arithmetic operations required for one coefficient matrix vector multiplication in the system (4.1) is $N(2N+1)$.However, the number of operations needed to carry out the above preconditioning algorithm is $2N^2$. The total work for the action of the preconditioner is less than *N*–times the work of a coefficient matrix-vector product.

At the end of this section, we implemented the proposed method in section 3 with FORTRAN 90 in a personal computer and we used the following example in our experiments.

Example (1):
Consider the boundary value problem:

$$y''(x) + x e^{-x} y(x) = x + e^{x}, -1 \le x \le 1, \tag{4.2}$$

with the Robin boundary conditions

$$y'(-1) + 2y(-1) = 3e^{-1} \text{ and } y'(1) - y(1) = 0. \tag{4.3}$$

This problem has the exact solution:

$$y(x) = e^{x}.$$

Apply the preconditioner IPL using $B^{(2)}$ and $B^{(1)}$ on (4.2) and using the boundary conditions (4.3), we get the approximated solution of the problem as follows:

$$y(x_i) = \sum_{k=0}^{N} b_{i,k}^{(2)} \psi(x_k) + (2x_i + 1) \sum_{k=0}^{N} b_{N,k}^{(1)} \psi(x_k) - (2x_i + 1) \sum_{k=0}^{N} b_{N,k}^{(2)} \psi(x_k) - 3e^{-1} x_i,$$

for $i = 0,1,...,N$.

The results in Table (4.1) show some comparisons, based on the choice of a specific iterative algorithm (ILU, ILUT and ILUC), showing the computational cost is strongly reduced for the current preconditioner (IPL).

Table (4.1): Comparison of preconditioners on example (1)

Preconditioners	N	No. of Iterations	CPU time /Iteration	Errors in Boundaries
ILU	8	31	1.09 sec.	1.6E-04
	12	47	2.19 sec.	8.0E-06
	16	90	6.27 sec.	4.9E-07
ILUT	8	14	6.1 sec.	3.8E-09
	12	21	8.2 sec.	2.4E-11
	16	34	13.9 sec.	1.1E-12
ILUC	8	8	4.1 sec.	5.1E-09
	12	11	5.4 sec.	6.2E-12
	16	15	7.0 sec.	7.9E-13
IPL	8	4	1.1 sec.	5.2E-11
	12	6	1.8 sec.	0.1E-14
	16	8	3.7 sec.	7.3E-15

The numerical results confirm the theoretical statements in section 3, the number of iterations does not depend on the values of the coefficients, and the condition number of the matrix $\bar{B}^{(1)}$ is proportional to $D^{(1)}$. The experiments in Table (4.1) show that the iterative method with the proposed IPL preconditioner converges 2 – 3 times faster than with ILUC preconditioner. These results are concerning the conditioning and the discrete relative L_2 error obtained by

El-gendi method [8] and the Chebyshev pseudospectral integration method [4] and Legendre preconditioning method, for N = 8, 10 and 16, respectively.

In terms of accuracy, the preconditioning produces more accurate results than the Chebyshev pseudospectral integration method. Furthermore, when N increases from 8 to 32, the condition number increases from $2.17\times10^{+3}$ to $5.79\times10^{+6}$ for the Chebyshev pseudospectral matrix [8], but only from $1.37\times10^{+2}$ to $4.61\times10^{+4}$ for the Legendre pseudospectral matrix.

On the other hand, Table (4.2) shows the absolute error by using El-gendi method [8], pseudospectral integration method [4] and the proposed Legendre preconditioning method, for different values of N.

Table (4.2): The observed absolute error at N=10

N	Ref. [8]	Ref. [4]	ILP method
8	1.6E-12	4.2E-13	1.14E-14
16	4.7E-12	3.9E-13	1.45E-14
32	1.8E-11	4.5E-13	5.65E-14
64	3.6E-11	1.8E-13	5.24E-14
128	5.1E-11	2.6E-13	4.27E-14
256	5.4E-11	7.9E-13	2.22E-16

Example (2):

Find the optimal control $u(t)$ such that: [12]

Minimizes:

$$J=\int_0^1\left(0.625x^2+0.5xu+0.5u^2\right)d\tau,$$

Subject to:

$$x'(\tau)=0.5x(\tau)+u(\tau),$$

with $x(0)=1.$

This problem has an analytic solution given by

$$u^*(\tau)=-(\tanh(1-\tau)+0.5)\cosh(1-\tau)/\cosh(1)\,,\ \tau\in[0,1]$$

with optimal cost:

$$J^*=e^2\sinh(2)/(1+e^2)^2\simeq 0.380797\,.$$

The first step in solving this problem is to transform the time interval into $t\in[-1,1]$. This will lead to the following problem:

Minimizes $$J=\frac{1}{2}\int_{-1}^{1}\left(0.625x^2+0.5xu+0.5u^2\right)dt,$$

(4.3)

Subject to $$2x'(t)=0.5x(t)+u(t),\qquad(4.4)$$

and $$x(-1)=1. \tag{4.5}$$

Apply the Legendre precondition approximation for Eq. (4.4) and using condition Eq. (4.5), we get:

$$2B^{(1)}D^{(1)}x(t_i)=0.5B^{(1)}x(t_i)+B^{(1)}u(t_i),$$

$$2x(t_i)-2x(t_0)=0.5B^{(1)}x(t_i)+B^{(1)}u(t_i),$$

Let us approximate the control variable by Legendre polynomials as follows:

$$u(t_i)=\sum_{k=0}^{M}a_kP_k(t_i),$$

$$x(t_i)=0.25\sum_{j=0}^{N}b_{i,j}^{(1)}x(t_i)+0.5\sum_{j=0}^{N}b_{i,j}^{(1)}\sum_{k=0}^{M}a_kP_k(t_i)+1$$

Then the problem can be converted to the following constrained optimization problem: Minimize

$$J=\frac{1}{2}\sum_{j=0}^{N}b_{N,j}^{(1)}\left\{0.625x^2(t_i)+0.5x(t_i)\sum_{k=0}^{M}a_kP_k(t_i)+0.5\left(\sum_{k=0}^{M}a_kP_k(t_i)\right)^2\right\} \tag{4.6}$$

Since

$$x(t_i)=0.25\sum_{j=0}^{N}b_{i,j}^{(1)}x(t_i)+0.5\sum_{j=0}^{N}b_{i,j}^{(1)}\sum_{k=0}^{M}a_kP_k(t_i)+1,\ \mathrm{i}=0,1,....,\mathrm{N}. \tag{4.7}$$

The parameter optimization problem (4.6), subject to (4.7), can be solved using the penalty solver given in Ref. [9]. The stopping criteria are:

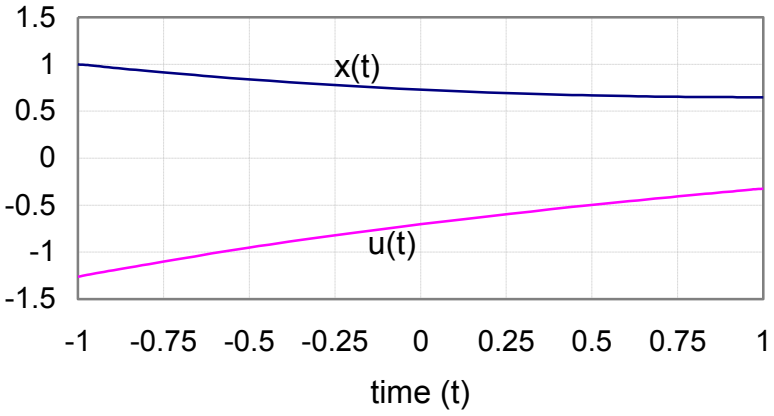
$$|J(\gamma_{N+1},\lambda_{N+1})-J(\gamma_N,\lambda_N)|<\varepsilon$$

to decide whether the computed solution in close enough to the optimal solution (Table 4.4).

Table (4.4): Objectives J^* of Example (2)

N	J^*
4	0.38079160
6	0.38079698
8	0.38079710
10	0.38079700
12	0.38079700
14	0.38079700
Exact solution = 0.380797	

Fig. 6: State and control variables of Example (2)



5. Minimum time orbit problem

One of the best known trajectory optimization examples is the problem of minimizing the transfer time of a constant low-thrust ion rocket between the orbits of Earth and Mars. This involved the determination of the thrust angle history, for which no exact solution is known (El-Gindy et al [10]). The performance index of the problem can be stated as follows:

Minimize $J = T$.

Subject to: $\frac{dx_1}{dt} = \frac{T}{2} x_2$,

$$\frac{dx_2}{dt} = \frac{T}{2}\left[\frac{x_3^2}{x_1} - \frac{\gamma}{x_1^2} + \frac{R_0 Sin\, u}{m_0 + \frac{T}{2} m(t+1)}\right],$$

$$\frac{dx_3}{dt} = \frac{T}{2}\left[-\frac{x_2 x_3}{x_1} + \frac{R_0 \cos u}{m_0 + \frac{T}{2} m(t+1)}\right], \quad -1 \le t \le 1.$$

With the boundary conditions

$$x_1(-1) = 1.0, \quad x_2(-1) = 0.0, \quad x_3(-1) = 1.0$$

$$x_1(1) = 1.525, \quad x_2(1) = 0.0, \quad x_3(1) = 0.8098\,.$$

where T is the unknown final time to be minimized, γ characterizes the gravitational attraction from the sum, R_0 is the constant low-thrust magnitude, u is the control angle measured from the local horizontal, m_0 is the initial mass, m is the constant propellant consumption rate. Using normalized values (Van Dooren et. al, [20]), we have

$\gamma = 1, R_0 = 0.1405, m_0 = 1$ and $m = -0.07487$. Since the state variables x_1, x_2, x_3 are defined in the two boundaries, we shall use the Legendre approximation as follows:
Let

$$x_1'(t) = \Psi_1(t),\ x_2'(t) = \Psi_2(t),\ x_3'(t) = \Psi_3(t)\ ,$$

by integration we have,

$$(I - M_1)x_1(t) = B^{(1)}\Psi_1(t) + C_1, (I - M_2)x_2(t) = B^{(1)}\Psi_2(t) + C_2, \text{ and}$$

$$(I - M_3)x_3(t) = B^{(1)}\Psi_3(t) + C_3.$$

Using boundary conditions $x_1(-1) = 1.0$, $x_2(-1) = 0.0$, $x_3(-1) = 1.0$ and making use of the Legendre integral approximation, we found $C_1 = I$, $C_2 = \mathbf{0}$ and $C_3 = I$ then,

$$x(t_i) = \sum_{j=0}^{N} b_{ij}\ \Psi_1(t_j) + 1,\ i = 0, ..., N$$

$$x_2(t_i) = \sum_{j=0}^{N} b_{ij}\Psi_2(t_j), i = 0, ..., N$$

$$x_3(t_i) = \sum_{j=0}^{N} b_{ij}\Psi_3(t_j) + 1,\ i = 0, ..., N\ .$$

Therefore:

Minimize $J = T$,

Subject to

$$\Psi_1(t_i) - \sum_{j=0}^{N} b_{ij}\Psi_2(t_j) = 0, i = 0, ..., N$$

$$\Psi_2(t_i) - \frac{T}{2}\left[\frac{\left(\sum_{j=0}^{N} b_{ij}\ \Psi_3(t_j) + 1\right)^2}{\left(\sum_{j=0}^{N} b_{ij}\ \Psi_1(t_j) + 1\right)} - \frac{\gamma}{\left(\sum_{j=0}^{N} b_{ij}\ \Psi_1(t_j) + 1\right)^2} + \frac{R_0 Sin\, u(t_i)}{m_0 + \frac{T}{2}m(t_i + 1)}\right] = 0\ ,$$

$$\Psi_3(t_i) - \frac{T}{2}\left[-\frac{\sum_{j=0}^{N} b_{ij}\Psi_2(t_j)\left(\sum_{j=0}^{N} b_{ij}\Psi_3(t_j) + 1\right)}{\left(\sum_{j=0}^{N} b_{ij}\Psi_1(t_j) + 1\right)} + \frac{R_0 \cos u(t_i)}{\frac{T}{2}m(t_i + 1)}\right] = 0,\ -1 \le t \le 1,$$

with the boundary conditions: $x_1(1) = 1.525,\quad x_2(1) = 0.0,\quad x_3(1) = 0.8098$.

Table (4.5): Preconditioner solvers for Min. Time Orbit for N=12

Method	No. of Iterations	Errors in Boundaries	CPU time/ Iteration
ILU	37	5.12E-04	1.1
ILUT	28	1.82E-04	1.1
ILUC	11	5.39E-06	0.9
IPL	5	2.02E-11	0.5

The problem can be converted to the constrained optimization problem and solved by suitable NLP solver. Table (4.5) shows the computational effort with the preconditioner subroutines comparing with other solvers. However, Table (4.6) shows comparative results J^* for solving the time orbit transfer and comparison between the integration preconditioning approach based on Legendre and those obtained by El-Gindy [10] making use of Chebyshev spectral method and other given references.

Table (4.6): Observed objective minimum time J^*

Method		T
Moyer, Pinkham [17]	First	3.317
Gradient	Second	3.317
Hontoir, Cruz [14]		3.3194
Taylor, Constantinides [19]		3.3819
El-Gindy et al [10]		3.3117
Present method (IPL)		3.311

We observe from this table that the IPL implementation have better performance compared with the Chebyshev formulations. This agrees well with the findings in [15].

6. Conclusion

In this paper we calculate the pseudospectral precondition matrices for Legendre polynomials at Legendre Gauss Lobatto (LGL) points. Introduced approximated of any continuous function and an approximation of its finite integrals by Legendre polynomials. The tables and the figure given previously show that the suggested technique is quite reliable. It can be successfully applied to both ordinary boundary value and optimal control problems. Several examples are given and the results obtained using the method introduced in this article show that the newly proposed numerical procedure is efficient and provides very accurate results even with using a small number of collocation points. The method is produced to solve the minimum time orbit transfer problem to get an accurate solution at small number of nodes. Moreover, the results indicate that preconditioning the optimal time control yields better performance than any of the real formulations - a result which concurs with those of [17] and [19]. We see also that the results with the IPL preconditioner are much closer to each other, compared with those of the Chebyshev preconditioner [10].

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TWO FIXED POINT THEOREMS ON A CLASS OF GENERALIZED QUAZI-METRIC SPACES

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Abstract

In this paper we define **generalized quasi-metric space** (gqms)) on the lines of quasi-metric space, where the quasi-triangular inequality has been replaced by quasi-tetrahedral inequality. As such, any generalized metric space is a generalized quasi-metric space but the converse is not true. The well-known fixed point theorems of Banach and of Kannan have been transferred to such a space.

Keywords: Cauchy sequence, generalized quasi-metric spaces, fixed point.

Mathematics Subject Classification: 47H10, 54H25

1. Introduction and Preliminaries

Generalized metric space is treated by Branciari [4] in 2000. Every metric space is a generalized metric space but the converse is not true. Starting with the paper of Branciari [4], some well-known metric fixed point theorems have been transferred to generalized metric spaces (see [1], [5], [6], [7], [9], [13], [14]).

The purpose of this paper is to introduce the concept of *generalized quasi-metric space* and prove two fixed point theorems: the Banach's Contraction principle [2] and the Kannan's Contraction principle [8] in a gqms.

In gqms, differently from metric spaces, the "open" balls $B(a,r)=\{x\in X : d(x,a)<r\}$ are not always open sets and consequently, a generalized quasi-distance is not always continuous of its variables. The gqms is not always a Hausdorff space and the convergent sequence (x_n) in gqms is not always a Cauchy sequence (see Example 2.8**).**

Firstly, we will give some known definitions.
The concept of quasi-metric spaces is treated differently by many authors. In this paper our concept is in line with this treated in [3], [10], [11], [12], [13], [15] etc:

Definition1.1 Let *X* be a set. A nonnegative symmetric function d defined on $X \times X$ is called *a quasi-distance* on X if and only if there exists a constant $k \geq 1$ such that for all x, y and $z \in X$ the following conditions hold:

(1) $d(x,y)=0 \Leftrightarrow x=y$;
(2) $d(x,y)=d(y,x)$;
(3) $d(x,y)\le k[d(x,z)+d(z,y)]$.

Inequality (3) is often called *quasi-triangular inequality* and k is often called the *coefficient* of d. A pair (X,d) is called a *quasi-metric space* if X is a set and d *is* a quasi-distance on X. Metric spaces are a special case of quasi-metric space (for $k=1$).

Definition 1.2 [4] Let X be a set and $d: X^2 \to R^+$ a mapping such that for all $x,y\in X$ and for all distinct points $z,w\in X$, each of them different from x and y, one has
(a) $d(x,y)=0$ if and only if $x=y$,
(b) $d(x,y)=d(y,x)$,
(c) $d(x,y)\le d(x,z)+d(z,w)+d(w,y)$ (Tetrahedral inequality)
Then d is called *a generalized metric* and (X,d) is a *generalized metric space* (or shortly gms).

2. Main results

Definition2.1 Let X be a set. A nonnegative symmetric function d defined on $X \times X$ is called a *generalized quasi-distance* on X if and only if there exists a constant $k\ge 1$ such that for all $x,y\in X$ and for all distinct points $z,w\in X$, each of them different from x and y the following conditions hold:
(i) $d(x,y)=0 \Leftrightarrow x=y$;
(ii) $d(x,y)=d(y,x)$;
(iii) $d(x,y)\le k[d(x,z)+d(z,w)+d(w,y)]$.

Inequality (3) is often called *quasi-tetrahedral inequality* and k is often called the *coefficient* of d. A pair (X,d) is called a *generalized quasi-metric space* if X is a set and *d is* a generalized quasi-distance on X.

The set $B(a,r)=\{x\in X: d(x,a)<r\}$ is called "open" ball with center $a\in X$ and radius $r>0$.

The family $\tau=\{Q\subset X: \forall a\in Q, \exists r>0, B(a,r)\subset Q\}$ is a topology on X and it is called induced topology by the generalized quasi-distance d.

The following example illustrates the existence of the generalized quasi-metric space for an arbitrary constant $k\ge 1$:

Example 2.2 Let $X=R$. Define $d: X\times X \to R$ as follow:

$$d(x,y)=\begin{cases}0 & for\ x=y\\ 3k & for\ x,y\in\{1,2\}, x\ne y\\ 1 & otherwise\end{cases}$$

Then it is easy to see that d is a generalized quasi-distance on X and (X,d) is a generalized quasi-metric space. (X,d) is not a generalized metric space (for $k>1$) because:

$3k = d(1,2) > d(1,3) + d(3,4) + d(4,2) = 1+1+1 = 3$.

The metric spaces are a special case of generalized metric spaces and generalized metric spaces are a special case of generalized quasi-metric spaces (for $k = 1$). Also, every quasi-metric space is a gqms, while the converse is not true (see proposition 2.9).

We give some definitions.

Definition 2.3 A sequence $\{x_n\}$ in a generalized quasi-metric space (X, d) is called Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$

Definition 2.4 Let (X, d) be a generalized quasi-metric space. Then:

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} d(x_n, x) = 0$.

(2) It is called compact if every sequence contains a convergent subsequence.

Definition 2.5 A generalized quasi-metric space (X, d) is called complete, if every Cauchy sequence is convergent.

Definition 2.6 Let (X, d) be a gqms and the *coefficient* of d is k.

A map $T : X \to X$ is called contraction if there exists $0 < c < \frac{1}{k}$ such that

$$d(Tx, Ty) \le cd(x, y) \quad \text{for all } x, y \in X .$$

Definition 2.7 Let $T : X \to X$ be a mapping where X is a gqms. For each $x \in X$, let

$$O(x)=\{x, Tx, T^2x, ...\}$$

which will be called the orbit of T at x. *The space* X is said to be T-orbitally complete if and only if every Cauchy sequence which is contained in $O(x)$ converges to a point in X.

The following example illustrates:

I. The "open" balls $B(a, r) = \{x \in X : d(x, a) < r\}$, in generalized quasi-metric spaces, are not always open sets.

II. The generalized quasi-distance is not always continuous of its variables.

III. The gqms is not always a Hausdorff space and the convergent sequence $\{x_n\}$ in gqms is not always a Cauchy sequence.

Example 2.8 Let $X = \left\{1 - \frac{1}{n} : n = 1, 2, ...\right\} \cup \{1, 2\}$, Define $d : X \times X \to R$ as follow:

$$d(x, y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } x \in \{1,2\} \text{ and } y = 1 - \frac{1}{n} \text{ or } y \in \{1,2\} \text{ and } x = 1 - \frac{1}{n}, x \ne y \\ 3k & \text{for } x, y \in \{1,2\}, x \ne y \\ 1 & \text{otherwise} \end{cases}$$

Then it is easy to see that (X, d) is a generalized quasi-metric space and is not a generalized metric space (for $k > 1$).

Note that the sequence $(x_n) = (1 - \frac{1}{n})$ converges to both 1 and 2 and it is not a Cauchy sequence: $d(x_n, x_m) = d(1 - \frac{1}{n}, 1 - \frac{1}{m}) = 1, \forall n, m \in N$

Observe that $B(1,r) \cap B(2,r) \neq \phi$ for all $r > 0$ and consequently (X, d) is non a Hausdorff generalized quasi-metric space.

The function d is not continuous: $1 = \lim_{n\to\infty} d(1 - \frac{1}{n}, \frac{1}{2}) \neq d(1, \frac{1}{2}) = \frac{1}{2}$.

We have the following proposition.

Proposition 2.9 If (X, d) is a quasi-metric space, then (X, d) is a generalized quasi-metric space.

Proof. Let $x, y, u, v \in X$ be four distinguished point. By the quasi-triangular inequality we have:

$$d(x,y) \leq k[d(x,u) + d(u,y] \leq kd(x,u) + k^2[d(u,v) + d(v,y)] \leq$$
$$\leq k^2[d(x,u) + d(u,v) + d(v,y)]$$

So, (X, d) is a gqms with the coefficient k^2.

The converse proposition doesn't hold true. For example, the gqms taken by the example 2.8, for the case $k = 2$, it is not a quasi-metric space. In contrary, there exists a constant $k' \geq 1$ such that $6 = d(1,2) \leq k'[d(1, 1 - \frac{1}{n}) + d(1 - \frac{1}{n}, 2)] = k'\frac{2}{n}, \forall n \in N$ and under tending in infinity we get $6 \leq 0$. This is a contraction!

We state the following Theorem:

Theorem 2.10 (A generalization of Banach's Contraction principle in a gqms). Let (X, d) be a gqms with coefficient $k \geq 1$ and $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq cd(x, y) \forall x.y \in X \qquad (1)$$

where $0 < c < \frac{1}{k}$. If X is T-orbitally complete then T has a unique fixed point in X.

Proof. Choose any $x_0 \in X$. Define the sequence (x_n) inductively a follows: $x_n = Tx_{n-1}$ for all $n \geq 1$. So by (1) we have,

$$d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n) \leq ... \leq c^n d(x_0, x_1) \qquad (2)$$

And so

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \qquad (3)$$

We divide the proof into two cases:

Case I: Suppose $x_p = x_q$ *for some* $p, q \in N, p \neq q$. Let $p > q$. Then $T^p x_0 = T^{p-q} T^q x_0 = T^q x_0$ i.e. $T^n \alpha = \alpha$ where $n = p - q$ and $T^q x_0 = \alpha$. Now if $n > 1$, by (2), we have

$$d(\alpha, T\alpha) = d(T^n \alpha, T^{n+1} \alpha) \leq c^n d(\alpha, T\alpha)$$

Since $0 < c < 1$, $d(\alpha, T\alpha) = 0$. So $T\alpha = \alpha$ and hence α is a fixed point of *T*.

Case II: Assume that $x_n \neq x_m$ for all $n \neq m$. Then (x_n) is a sequence of distinct points. For $m > n+1$, by *quasi-tetrahedral inequality* and by (2), we have

$$d(x_n, x_m) \leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m)] \leq$$
$$\leq k[c^n d(x_0, x_1) + cd(x_n, x_m) + c^m d(x_1, x_0)] =$$
$$= k(c^n + c^m) d(x_0, x_1) + kcd(x_n, x_m)$$

And so, since $0 < c < \frac{1}{k} \leq 1$, we get $d(x_n, x_m) \leq \frac{k(c^n + c^m)}{1 - kc} d(x_0, x_1)$ and $\lim_{n,m \to \infty} d(x_n, x_m) = 0$.

It implies that $(x_n) = (T^n x_0)$ is a Cauchy sequence. Since X *is T-orbitally complete*, exists a $\alpha \in X$ such that

$$\lim_{n \to \infty} x_n = \alpha \qquad (4)$$

Let us prove $T\alpha = \alpha$. In contrary, if $T\alpha \neq \alpha$, by *quasi-tetrahedral inequality,* we have

$$d(T\alpha, \alpha) \leq k[d(T\alpha, Tx_n) + d(Tx_n, x_n) + d(x_n, \alpha)] \leq$$
$$\leq k[cd(\alpha, x_n) + d(x_{n+1}, x_n) + d(x_n, \alpha)]$$

Letting $n \to \infty$ and using the facts (3) and (4), we have $d(T\alpha, \alpha) = 0$. So α is a fixed point of *T*. If α, α' are fixed points of *T* then $d(\alpha, \alpha') = d(T\alpha, T\alpha') \leq cd(\alpha, \alpha')$. Since $0 < c < 1$ we have $\alpha = \alpha'$. This completes the proof of the theorem.

Corollary 2.11 (Banach's Contraction principle in a gqms).
If (X, d) is a complete gqms with coefficient $k \geq 1$ and $T : X \to X$ is a mapping such that

$$d(Tx, Ty) \leq cd(x, y) \forall x.y \in X$$

where $0 < c < \frac{1}{k}$, then *T* has a unique fixed point in *X*.

The proof follows by Theorem 2.9 since if X is complete then X is T-orbitally complete.

Theorem 2.12 (A generalization of Kannan's Contraction principle in a gqms). Let (X, d) be a gqms with the *coefficient k* and $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \qquad (5)$$

for all $x, y \in X$, where $0 < \beta < \frac{1}{k+1}$. If X is T-orbitally complete, then T has a unique fixed point in X.

Proof. Choose any $x_0 \in X$. Define the sequence (x_n) inductively a follows: $x_n = Tx_{n-1}$ for all $n \geq 1$. So by (5) for $x = x_0$ and $y = x_1$ we have,

$$d(x_1, x_2) \leq \beta[d(x_0, x_1) + d(x_1, x_2)]$$

And so

$$d(x_1, x_2) \leq \frac{\beta}{1 - \beta} d(x_0, x_1)$$

Again

$$d(x_2,x_3)\le\beta[d(x_1,x_2)+d(x_2,x_3)],\text{ i.e., }d(x_2,x_3)\le\frac{\beta}{1-\beta}d(x_1,x_2)\le(\frac{\beta}{1-\beta})^2d(x_0,x_1)$$

Similarly

$$d(x_n,x_{n+1})\le(\frac{\beta}{1-\beta})^n d(x_0,x_1)=r^n d(x_0,x_1) \qquad (6)$$

Where $r=\frac{\beta}{1-\beta}$. Since $0<\beta<\frac{1}{k+1}\le\frac{1}{2}$, clearly $0<r<1$

From (6) it follows

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0 \qquad (7)$$

We divide the proof into two cases:

Case I: Suppose $x_p=x_q$ *for some* $p,q\in N, p\ne q$. Let $p>q$. Then $T^px_0=T^{p-q}T^qx_0=T^qx_0$ i.e. $T^n\alpha=\alpha$ where $n=p-q$ and $T^qx_0=\alpha$. Now if $n>1$, by (6), we have

$$d(\alpha,T\alpha)=d(T^n\alpha,T^{n+1}\alpha)\le r^nd(\alpha,T\alpha)$$

Since $0<r<1$, $d(\alpha,T\alpha)=0$. So $T\alpha=\alpha$ and hence α is a fixed point of *T*.

Case II: Assume that $x_n\ne x_m$ for all $n\ne m$. Then (x_n) is a sequence of distinct points and for $m>n+1$, by *quasi-tetrahedral inequality*, we have

$$\begin{aligned}d(x_n,x_m)&\le k[d(x_n,x_{n+1})+d(x_{n+1},x_{m+1})+d(x_{m+1},x_m)]\le\\&\le k[r^nd(x_0,x_1)+rd(x_n,x_m)+r^md(x_1,x_0)]=\\&=k(r^n+r^m)d(x_0,x_1)+krd(x_n,x_m)\end{aligned}$$

So, since $0<r<1$ and $1-kr>0$ we have $d(x_n,x_m)\le\frac{k(r^n+r^m)}{1-kr}d(x_0,x_1)$.

Therefore $\lim_{n,m\to\infty}d(x_n,x_m)=0$. It implies that $(x_n)=(T^nx_0)$ is a Cauchy sequence. Since X *is T-orbitally complete*, exists a $\alpha\in X$ such that

$$\lim_{n\to\infty}x_n=\alpha \qquad (8)$$

Let us prove $T\alpha=\alpha$. In contrary, if $T\alpha\ne\alpha$, by *quasi-tetrahedral inequality,* we have

$$\begin{aligned}d(T\alpha,\alpha)&\le k[d(T\alpha,Tx_n)+d(Tx_n,x_n)+d(x_n,\alpha)]\le\\&\le k\beta[d(\alpha,T\alpha)+d(x_n,Tx_n)]+kd(Tx_n,x_n)+kd(x_n,\alpha)=\\&=k\beta d(\alpha,T\alpha)+k\beta d(x_n,x_{n+1})+kd(x_{n+1},x_n)+kd(x_n,\alpha)\end{aligned}$$

Letting $n\to\infty$ and using the facts (7) and (8), we have

$d(T\alpha,\alpha)\le k\beta d(T\alpha,\alpha)$. Since $0<k\beta<\frac{k}{k+1}<1$, $d(T\alpha,\alpha)=0$

So α is a fixed point of *T*. If α,α' are fixed points of *T*, then $d(\alpha,\alpha')\le\beta[d(\alpha,T\alpha)+d(\alpha',T\alpha')]=0$ and so $\alpha=\alpha'$. This completes the proof of the theorem.

Corollary 2.13 (Kannan's Contraction principle in a gqms).

If (X,d) is a complete gqms with coefficient $k\ge1$ and $T:X\to X$ is a mapping such

that

$$d(Tx,Ty) \le \beta[d(x,Tx)+d(y,Ty)]$$

where $0 < \beta < \frac{1}{k+1}$, then T has a unique fixed point in X.

Example 2. 14 Let $X = \{1,2,3,4\}$. Define $d : X \times X \to R$ as follow:

$$d(x,y) = \begin{cases} 0 & for\ x = y \\ 3 & for\ x,y \in \{1,2\}, x \ne y \\ 1 & for\ x,y \in \{2,3\}\ or\ x,y \in \{1,3\}, x \ne y \\ 9 & for\ x,y \in \{1,4\}, x \ne y \\ 4 & for\ x,y \in \{2,4\}\ or\ x,y \in \{3,4\}, x \ne y \end{cases}$$

And $Tx = \begin{cases} 1 & if\ x = 4 \\ 3 & if\ x \ne 4 \end{cases}$

Then (X,d) is a T-orbitally complete generalized quasi-metric space with coefficient $k = \frac{3}{2} > 1$ and (X,d) is not a generalized metric space because it lacks two of the tetrahedral inequalities (The fifth and the sixth):

$$3 = d(1,2) < d(1,3)+d(3,4)+d(4,2) = 1+4+4 = 9$$
$$3 = d(1,2) < d(1,4)+d(4,3)+d(3,2) = 9+4+1 = 14$$
$$1 = d(1,3) < d(1,4)+d(4,2)+d(2,3) = 9+4+1 = 14$$
$$1 = d(1,3) < d(1,2)+d(2,4)+d(4,3) = 3+4+4 = 11$$
$$9 = d(1,4) > d(1,2)+d(2,3)+d(3,4) = 3+1+4 = 8$$
$$9 = d(1,4) > d(1,3)+d(3,2)+d(2,4) = 1+1+4 = 6$$
$$1 = d(2,3) < d(2,1)+d(1,4)+d(4,3) = 3+9+4 = 16$$
$$1 = d(2,3) < d(2,4)+d(4,1)+d(1,3) = 4+9+1 = 16$$
$$4 = d(2,4) < d(2,1)+d(1,3)+d(3,4) = 3+1+4 = 8$$
$$4 = d(2,4) < d(2,3)+d(3,1)+d(1,4) = 1+1+9 = 11$$
$$4 = d(3,4) < d(3,1)+d(1,2)+d(2,4) = 1+3+4 = 8$$
$$4 = d(3,4) < d(3,2)+d(2,1)+d(1,4) = 1+3+9 = 13$$

We verify if the inequality $d(Tx,Ty) \le \beta[d(x,Tx)+d(y,Ty)]$ is satisfied for all $x,y \in X$. Note that

$d(T1,T2) = d(T1,T3) = d(T2,T3) = 0$

and in all other cases

$d(Tx,Ty) = 1$, $d(x,Tx)+d(y,Ty) \ge 9$

Hence, for $\beta = \frac{1}{8} < \frac{1}{1+k} = \frac{1}{1+\frac{3}{2}} = \frac{2}{5}$ all conditions of Theorem 2.12 are satisfied to obtain a unique fixed point 3 of X.

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Comment on "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces" [Lakshmikantham and Ćirić, Nonlinear Anal. TMA 70 (2009) 4341–4349]

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Abstract: We prove coupled coincidence and coupled common fixed point theorems for nonlinear contractive strictly mixed g-monotone mappings in partially ordered complete metric spaces. In this paper, we remove the continuity condition of the contractive mapping in the recent fixed point theorems due to Lakshmikantham and Ćirić [V. Lakshmikantham, Lj.B. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. TMA 70 (2009) 4341–4349] when the contractive mapping is strictly mixed g-monotone.

Keywords: Coupled fixed point; coupled coincidence; partially ordered set; mixed strict monotone mapping.

1. Introduction

Existence of a fixed point for contraction type mappings in partially ordered metric spaces and applications have been considered recently by many authors (see, for details, [2]–[18]). The aim of this paper is to extend the results of Lakshmikantham and Ćirić [1] for strictly mixed monotone linear contractive mappings. We prove the main results of Lakshmikantham and Ćirić [1] for strictly mixed monotone contractive mappings without continuity of the contractive mapping.

In this paper, we suppose that X is a nonempty set. Note that an element $(x, y) \in X \times X$ is called a coupled coincidence point of the mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if

$$F(x, y) = g(x), \quad F(y, x) = g(y). \qquad [1]$$

Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ We say F and g are commutative if $g(F(x, y)) = F(g(x), g(y))$ for all $x, y \in X$ (see [1]).

Recently, Lakshmikantham and Ćirić in paper [1] , proved the following theorem:

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Theorem 1.1. *Let $(X, \leq)$ be a partially 0rdered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume there is a function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are such that F has the mixed strict $g-$monotone property and*

$$d(F(x, y), F(u, v)) \leq \varphi(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}) \tag{1}$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \leq g(v)$. Suppose $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F and also suppose either
(a) F is continuous or
(b) X has the following property:
(i) if a non-increasing $\{x_n\} \longrightarrow x$, then $x_n \leq x$ for all n,
(ii) if a non-increasing $\{y_n\} \longrightarrow y$, then $y \leq y_n$ for all n.
If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \quad and \quad g(y_0) \geq F(y_0, x_0),$$

then there exist $x, y \in X$ such that
$g(x) = F(x, y)$ and $\quad g(y) = F(y, x)$,
that is, F and g have a coupled coincidence.

Definition 1.2. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say that F has the mixed strict $g-$monoton property if for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad g(x_1) < g(x_2) \Longrightarrow F(x_1, y) < F(x_2, y)$$
$$y_1, y_2 \in X, \quad g(y_1) < g(y_2) \Longrightarrow F(x, y_1) > F(x, y_2).$$

2. Main Results

We start our work with the following theorem, which remove the conditions (a) and (b) in above theorem for strictly $g-$monotone contractive mapping F.

Theorem 2.1. *Let $(X, \leq)$ be a partially 0rdered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are such that F has the mixed strict $g-$monotone property and*

$$d(F(x, y), F(u, v)) \leq \varphi(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}) \tag{1}$$

for all $x, y, u, v \in X$ for which $g(x) < g(u)$ and $g(y) > g(v)$. Suppose $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that

$$g(x_0) < F(x_0, y_0) \quad and \quad g(y_0) > F(y_0, x_0),$$

then there exist $x, y \in X$ such that
$F(x, y) = g(x)$ and $\quad F(y, x) = g(y)$,
that is, F and g have a coupled coincidence.

Proof. Similarly as in proof of Theorem 2.1 of [1] , we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$g(x_{n+1}) = F(x_n, y_n) \quad$ and $\quad g(y_{n+1}) = F(y_n, x_n) \quad$ for all $n \geq 0$. $\quad (2)$

Also, on the same way from proof of Theorem 2.1 of [1] , we can show that

$$g(x_n) < g(x_{n+1}) \quad for\ all\ n \geq 0 \tag{3}$$

and

$$g(y_n) > g(y_{n+1}) \quad for\ all\ n \geq 0. \tag{4}$$

Therefore, we have

$$g(x_0) < g(x_1) < g(x_2) < \cdots < g(x_n) < g(x_{n+1}) < \cdots \tag{5}$$

and

$$g(y_0) > g(y_1) > g(y_2) > \cdots > g(y_n) > g(y_{n+1}) > \cdots . \tag{6}$$

Again, similarly as in proof of Theorem 2.1 of [1] , we can prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\longrightarrow\infty} g(x_n) = x \ \ and \ \ \lim_{n\longrightarrow\infty} g(y_n) = y. \tag{7}$$

From (5) and (6), $\{g(x_n)\}$ is monotone increasing and $\{g(y_n)\}$ is monotone decreasing therefore, for all $n \geq 0$

$$g(x_n) < x \ \ and \ \ g(y_n) > y. \tag{8}$$

From (7) and continuity of g,

$$\lim_{n\longrightarrow\infty} g(g(x_n)) = g(x) \ \ and \ \ \lim_{n\longrightarrow\infty} g(g(y_n)) = g(y). \tag{9}$$

Thus for all $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$

$$d(g(g(x_n)), g(x)) < \frac{1}{4m} \ \ and \ \ d(g(g(y_n)), g(y)) < \frac{1}{4m}. \tag{10}$$

Hence, from (1), (2), (8), (10), and commutativity of F and g and by triangle inequality, we have

$$\begin{aligned}
d(F(x,y), g(x)) &\leq d(F(x,y), g(g(x_n))) + d(g(g(x_n)), g(x)) \\
&= d(F(x,y), g(F(x_{n-1}, y_{n-1}))) + d(g(g(x_n)), g(x)) \\
&= d(F(x,y), F(g(x_{n-1}), g(y_{n-1}))) + d(g(g(x_n)), g(x)) \\
&\leq \varphi(\frac{d(g(x), g(g(x_{n-1}))) + d(g(y), g(g(y_{n-1})))}{2}) + d(g(g(x_n)), g(x)) \\
&< \frac{d(g(x), g(g(x_{n-1}))) + d(g(y), g(g(y_{n-1})))}{2}) + d(g(g(x_n)), g(x)) \\
&< \frac{1}{8m} + \frac{1}{8m} + \frac{1}{4m} = \frac{1}{2m} \longrightarrow 0.
\end{aligned}$$

Let $m \longrightarrow \infty$ in above inequality, then $F(x, y) = g(x)$. similarly we can show $F(y, x) = g(y)$. This means that F and g have a coupled coincidence point. □

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are such that F has the mixed strict $g-$monotone property and assume that there exists a*

$k \in (0,1)$ *with*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(g(x),g(u)) + d(g(y),g(v))]$$

for all $x,y,u,v \in X$ *for which* $g(x) < g(u)$ *and* $g(y) > g(v)$. *Suppose* $F(X \times X) \subseteq g(X)$ *and* g *is continuous and commutes with* F. *If there exist* $x_0, y_0 \in X$ *such that*
$g(x_0) < F(x_0,y_0)$ *and* $g(y_0) > F(y_0,x_0)$,
then there exist $x,y \in X$ *such that*
$g(x) = F(x,y)$ *and* $g(y) = F(y,x)$,
that is, F *and* g *have a coupled coincidence.*

Proof. By putting $\varphi(t) = kt$ with $k \in (0,1)$ in theorem 2.2, we obtain the result. □

Now, we endow the product space $X \times X$ with the following partial order:
for $(x,y),(u,v) \in X \times X$, $(u,v) < (x,y) \iff u < x, \quad v > y$.

Theorem 2.3. *In addition to the hypotheses of theorem* 2.2, *suppose that for every* $(x,y),(x^*,y^*) \in X \times X$ *there exists a* $(u,v) \in X \times X$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(x^*,y^*),F(y^*,x^*))$. *Then* F *and* g *have a unique coupled fixed point, that is, there exists a unique* $(x,y) \in X \times X$ *such that*
$x = g(x) = F(x,y)$ *and* $y = g(y) = F(y,x)$.

Proof. The proof is similarly to the proof of Theorem 2.2 of [1] . □

Corollary 2.4. *Let* $(X,\leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X,d) *is a complete metric space. Assume there is a function* $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ *with* $\varphi(t) < t$ *and* $\lim_{r \to t^+} \varphi(r) < t$ *for each* $t > 0$ *and also suppose* $F : X \times X \longrightarrow X$ *be a mapping having the mixed strict monotone property on* X *and*

$$d(F(x,y),F(u,v)) \leq \varphi(\frac{d(x,u),d(y,v)}{2}) \tag{11}$$

for each $x < u$ *and* $y > v$.
If there exist $x_0, y_0 \in X$ *such that*

$$x_0 < F(x_0,y_0) \quad \text{and} \quad y_0 > F(y_0,x_0),$$

then there exist $x,y \in X$ *such that*
$x = F(x,y)$ *and* $y = F(y,x)$.
Furthermore, if x_0, y_0 *are comparable, then* $x = y$, *that is,* $x = F(x,x)$.

Proof. Following the proof of Theorem 2.2 with $g = I$ (the identity mapping), we only have to show that $x = F(x,x)$. Let us suppose that $x_0 < y_0$. We shall show that $x_n < y_n$ for all $n \geq 0$, where $x_n = F(x_{n-1},y_{n-1})$, $y_n = F(y_{n-1},x_{n-1})$; $n \in \mathbb{N}$. Suppose $x_n < y_n$ for some fixed $n \geq 0$. Then by mixed strict monotone property of F,

$$x_{n+1} = F(x_n,y_n) < F(y_n,x_n) = y_{n+1}.$$

Thus, we showed that $x_n < y_n$ for all $n \geq 0$. From this, and (11),

$$d(F(x_n,y_n),F(y_n,x_n)) \leq \varphi(d(x_n,y_n)). \tag{12}$$

Now, by the triangle inequality,

$$\begin{aligned} d(x,y) &\leq d(x,x_{n+1}) + d(x_{n+1},y_{n+1}) + d(y_{n+1},y) \\ &= d(F(x_n,y_n),F(y_n,x_n)) + d(x,x_{n+1}) + d(y_{n+1},y) \\ &\leq \varphi(d(x_n,y_n)) + d(x,x_{n+1}) + d(y_{n+1},y) \\ &< d(x_n,y_n) + d(x,x_{n+1}) + d(y_{n+1},y). \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ we get, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, $d(x,y) \leq d(y,x)$. Hence $d(x,y) = 0$. Therefore $x = y$, that is, $x = F(x,x)$. □

Corollary 2.5. *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. $F : X \times X \longrightarrow X$ be a mapping having the mixed strict monotone property on X. Assume that there exists a $k \in (0,1)$ with*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u),d(y,v)]$$

for each $x < u$ and $y > v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 < F(x_0,y_0) \quad and \quad y_0 > F(y_0,x_0),$$

then there exist $x, y \in X$ such that
$x = F(x,y)$ and $y = F(y,x)$.
Furthermore, if x_0, y_0 are comparable, then $x = y$, that is, $x = F(x,x)$.

Proof. Taking $\varphi(t) = kt$ with $k \in (0,1)$ in corollary 2.5 we obtain the result. □

3. **Applications**

In this section, we consider the integral equation
$(x(t),y(t)) = (\int_0^T G(t,s)[(f(s,x(s))+\lambda x(s))-(f(s,y(s))+\lambda y(s))]ds, (\int_0^T G(t,s)[(f(s,y(s))$

$$+\lambda y(s)) - (f(s,x(s)) + \lambda x(s))]ds) \tag{13}$$

where $T > 0$ and $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a integrable function (f is not necessarily continuous) and

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1} & \text{if } 0 \leq s < t \leq T \\ \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1} & \text{if } 0 \leq t < s \leq T. \end{cases} \tag{14}$$

Definition 3.1. *A lower solution for* (13) *is a function $(\alpha,\beta) \in c^1(I,\mathbb{R}) \times c^1(I,\mathbb{R})$ such that*

$$\alpha^{'}(t) + \lambda\beta(t) \leq f(t,\alpha(t)) - f(t,\beta(t)) \ , \ \alpha(0) < \alpha(T)$$

and

$$\beta^{'}(t) + \lambda\alpha(t) \geq f(t,\beta(t)) - f(t,\alpha(t)) \ , \ \beta(0) > \beta(T).$$

Now, we prove the existence of solution for the integral equation (13) in presence of a lower solution. Let $\mathcal{A}$ denote the class of those functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ which satisfies the following conditions:
(i) φ is increasing,
(ii) for all $x \geq 0$, there exists a $k \in [0,1)$ such that $\varphi(x) < \frac{k}{2}x$.

Theorem 3.2. *Consider integral equation* (13) *and suppose that there exists* $\lambda > 0$ *such that for* $x, y \in \mathbb{R}$ *with* $y > x$

$$0 < f(t,y) + \lambda y - [f(t,x) + \lambda x] \leq \lambda\varphi(y-x),$$

where $\varphi \in \mathcal{A}$. *Then the existence of lower solution for* (13) *provides the existence of a solution of* (13).

Proof. Define $F : c(I,\mathbb{R}) \times c(I,\mathbb{R}) \longrightarrow c(I,\mathbb{R})$ by

$$F(x(t),y(t)) = \int_0^T G(t,s)[(f(s,x(s)) + \lambda x(s)) - (f(s,y(s)) + \lambda y(s))]ds.$$

Note that if $(x(t),y(t)) \in c(I,\mathbb{R}) \times c(I,\mathbb{R})$ is a coupled fixed point of F, then $(x(t),y(t))$ is a solution of (13). Now, we will show that the hypotheses in Corollary 2.5 are satisfied. Indeed, $X \times X = c(I,\mathbb{R}) \times c(I,\mathbb{R})$ is a partially ordered set if we define the following order relation in $X \times X$:
$(u(t),v(t)) \leq (x(t),y(t))$ if and only if $u(t) < x(t),\ v(t) \geq y(t)$ for all $(x(t),y(t)),(u(t),v(t)) \in X \times X$ and $t \in I$.
Also, (X,d) is a complete metric space if choose

$$d(x(t),y(t)) = \sup_{t\in I} \mid (x(t) - y(t)) \mid, \quad x(t),y(t) \in X.$$

By hypothesis, for $x_2 > x_1$, we have

$$f(t,x_2) + \lambda x_2 > f(t,x_1) + \lambda x_1,$$

which implies that
$\int_0^T [f(s,x_2(s)) + \lambda x_2(s) - f(s,y(s)) - \lambda y(s)]G(t,s)ds$
$> \int_0^T [f(s,x_1(s)) + \lambda x_1(s) - f(s,y(s)) - \lambda y(s)]G(t,s)ds,$

for all $T \in I$ and all $(t,s) \in I \times I$. This means that

$$F(x_2(t),y(t)) > F(x_1(t),y(t)). \tag{15}$$

On the other hand, for $y_1 < y_2$, we have

$$f(t,y_2) + \lambda y_2 > f(t,y_1) + \lambda y_1,$$

which implies that
$\int_0^T [f(s,x(s)) + \lambda x(s) - f(s,y_2(s)) - \lambda y_2(s)]G(t,s)ds$
$< \int_0^T [f(s,x(s)) + \lambda x(s) - f(s,y_1(s)) - \lambda y_1(s)]G(t,s)ds,$
for all $T \in I$ and all $(t,s) \in I \times I$. This means that

$$F(x(t),y_2(t)) < F(x(t),y_1(t)). \tag{16}$$

It follows from (15) and (16) that F has the mixed strict monotone property. We have

$d(F(x(t),y(t)),F(u(t),v(t))) = \sup_{t\in I} \mid F(x(t),y(t)) - F(u(t),v(t)) \mid$

$= \sup_{t\in I} \mid \int_0^T G(t,s)[f(s,x(s)) + \lambda x(s) - f(s,y(s)) - \lambda y(s)]ds$

$- \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s) - f(s,v(s)) - \lambda v(s)]ds \mid$

$\leq \sup_{t\in I} \int_0^T G(t,s) \mid f(s,x(s)) + \lambda x(s) - f(s,u(s)) - \lambda u(s) + f(s,v(s)) + \lambda v(s)) - f(s,y(s)) - \lambda y(s) \mid ds.$

Since the function $\varphi(x)$ is increasing and $x < u$ and $y > v$, then

$$\varphi(x(s) - u(s)) \leq \varphi(d(x(s),u(s)) \text{ and } \varphi(v(s) - y(s)) \leq \varphi(d(v(s),y(s)),$$

we obtain that

$d(F(x(t),y(t)),F(u(t),v(t))) \leq \sup_{t\in I} \int_0^T G(t,s) \mid \lambda\varphi(x(s) - u(s)) + \lambda\varphi(v(s) - y(s)) \mid ds$

$\leq \lambda \sup_{t\in I} \int_0^T G(t,s) \mid \varphi(d(x(s),u(s))) + \varphi(d(v(s),y(s))) \mid ds$

$= \lambda(\varphi(d(x(s),u(s))) + \varphi(d(v(s),y(s)))). \sup_{t\in I} \int_0^T G(t,s)ds$

$= \lambda(\varphi(d(x(s),u(s))) + \varphi(d(v(s),y(s)))). \sup_{t\in I} \frac{1}{e^{\lambda T}-1}(\frac{1}{\lambda}e^{\lambda(T+s-t)} \mid_0^t + \frac{1}{\lambda}e^{\lambda(s-t)} \mid_t^T)$

$= \lambda(\varphi(d(x(s),u(s))) + \varphi(d(v(s),y(s)))). \frac{1}{\lambda e^{\lambda T}-1}(e^{\lambda T} - 1)$

$= \varphi(d(x(s),u(s))) + \varphi(d(v(s),y(s)))$

$< \frac{k}{2}[d(x(s),u(s)) + d(v(s),y(s))]$

$\leq \frac{k}{2} \sup_{t\in I} \mid x(t) - u(t) \mid + \frac{k}{2} \sup_{t\in I} \mid v(t) - y(t) \mid$

$= \frac{k}{2}[d(x(t),u(t)) + d(y(t),v(t))].$

This means that F satisfies (12). Finally, let $(\alpha(t),\beta(t))$ be a lower (strict) solution for (13) and we will show that $\alpha < F(\alpha,\beta)$ and $\beta > F(\beta,\alpha)$. Indeed,

$$\alpha^{'}(t) + \lambda\beta(t) \leq f(t,\alpha(t)) - f(t,\beta(t)),$$

for all $t \in I$. So we have

$$\alpha^{'}(t) + \lambda\alpha(t) \leq f(t,\alpha(t)) - f(t,\beta(t)) + \lambda\alpha(t) - \lambda\beta(t),$$

for all $t \in I$. Multiplying by $e^{\lambda t}$ both sides of above inequality, we get

$$(\alpha(t)e^{\lambda t})^{'} \leq [(f(t,\alpha(t)) + \lambda\alpha(t)) - (f(t,\beta(t)) + \lambda\beta(t))]e^{\lambda t}$$

for all $t \in I$, and this gives us

$$\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_0^t [(f(s,\alpha(s)) + \lambda\alpha(s)) - f(s,\beta(s)) - \lambda\beta(s)]e^{\lambda s}ds \tag{17}$$

for all $t \in I$. Which implies that

$$\alpha(0)e^{\lambda t} < \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T [f(s,\alpha(s)) + \lambda\alpha(s) - f(s,\beta(s)) - \lambda\beta(s)]e^{\lambda s}ds,$$

and so

$$\alpha(0) < \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[f(s,\alpha(s)) + \lambda\alpha(s) - f(s,\beta(s)) - \lambda\beta(s)]ds.$$

From this inequality and (17) we obtain that

$$\alpha(t)e^{\lambda t} < \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s) - f(s,\beta(s)) - \lambda\beta(s)]ds$$
$$+ \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s) - f(s,\beta(s)) - \lambda\beta(s)]ds$$

and, consequently

$$\alpha(t) < \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s) - f(s,\beta(s)) - \lambda\beta(s)]ds$$
$$+ \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s) - f(s,\beta(s)) - \lambda\beta(s)]ds.$$

Hence,

$$\alpha(t) < \int_0^T G(t,s)[f(s,\alpha(s)) + \lambda\alpha(s) - f(s,\beta(s)) - \lambda\beta(s)]ds = F(\alpha(t),\beta(t)),$$

for all $t \in I$. Similarly, we have $\beta(t) > F(\beta(t),\alpha(t))$. It follows from Corollary 2.5 that F has a coupled fixed point. □

Example 3.3. *Let $x(t) = t$ and $y(t) = 2t$ for $t \in I$, then $x(t), y(t) \in c(I,\mathbb{R})$. Define $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ as follows:*

$$f(p,q) = \begin{cases} p & \text{if } 0 \le p \le \frac{1}{2} \\ \\ p+q & \text{if } \frac{1}{2} < p \le 1. \end{cases}$$

It is easy to see that f is an integrable function. Suppose that $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is an increasing function which satisfies for all $x \ge 0$, there exists a $k \in [0,1)$ such that $\varphi(x) < \frac{k}{2}x$. Also, suppose that there exists $\lambda > 0$ such that for all $x, y \in \mathbb{R}$ with $y > x$,

$$0 < f(t,y) + \lambda y - [f(t,x) + \lambda x] \le \lambda\varphi(y-x).$$

We define $F : c(I,\mathbb{R}) \times c(I,\mathbb{R}) \longrightarrow c(I,\mathbb{R})$ by

$$F(x(t),y(t)) = \int_0^T G(t,s)[(f(s,x(s)) + \lambda x(s)) - (f(s,y(s)) + \lambda y(s))]ds,$$

where $G(t,s)$ defined by (14). Then by Theorem 3.2, the existence of a lower solution for (15), provides the existence of a solution of (15). The example discussed above cannot be covered by Theorem 1.1 of [1], because the function

$$F(x(t),y(t)) = \begin{cases} -\int_0^T G(t,s)\lambda s ds & \text{if } 0 \le s \le \frac{1}{2} \\ \\ -\int_0^T G(t,s)(1+\lambda)s ds & \text{if } \frac{1}{2} < s \le 1, \end{cases}$$

is not continuous. Hence, the condition (a) of Theorem 1.1 does not valid. On the other hand, by choosing $f_n(x) = \frac{sinnx}{\sqrt{n},}$ one can show that condition (b) of Theorem 1.1 does not valid.

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Common fixed point theorems in fuzzy metric spaces using common E.A. property*

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Abstract

Some common fixed points theorems for two pairs of weakly compatible maps in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using common E.A. property are proved. These results generalize some known results in literature.

Keywords: Fuzzy metric spaces; Fixed point theorem; Common E.A. property; Weakly compatible maps

AMS(2000) Subject Classification : 47H10, 54H25, 54A40

1 Introduction

The concept of fuzzy set was introduced by Zadeh[1] in 1965. Later many authors used the concept of fuzziness in metric space. The idea of fuzzy metric space introduced by Kramosil and Michalek[2] was modified by George and Veeramani[3]. Since then, many authors such as Gregori and Ramaguera[4] studied this concept and gave some results on this space.

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. Fixed point theory in fuzzy metric spaces has been developed starting with the paper of Grabiec [5]. Subrahmanyam [6] gave a generalization of Jungk's [7] common fixed point theorem for commuting mappings in the setting of fuzzy metric spaces. Even if in the recent literature weaker conditions of commutativity, as weakly commuting mappings, compatible mappings, R-weakly commuting maps, weakly compatible mappings and several others have been used, the existence of a common fixed point requires some conditions on continuity of the maps, G-completeness of the space, or containment of ranges. Recently, Mihet[8] proved two common fixed point theorems for a pair of weakly compatible maps in fuzzy metric spaces by using E.A. property.

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In this paper, we shall prove some fixed point theorems for two pairs of weakly compatible maps in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using common E.A. property.

2 Preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 [9]A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for all $a \in [0,1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Example 2.1 $a * b = ab, a * b = \min(a, b)$, for all $a, b \in [0,1]$.

Definition 2.2 [2] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space in the sense of Kramosil and Michalek if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X, s, t > 0$,

(FM-1) $M(x, y, 0) = 0$;

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X, s, t > 0$;

(FM-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

Then M is called a fuzzy metric on X. The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t. In what follows, fuzzy metric spaces in the sense of Kramosil and Michalek will be called KM-fuzzy metric spaces.

Lemma 2.1 [5] For all $x, y \in X$, $M(x, y, \cdot)$ is non-decreasing on $(0, \infty)$.

Definition 2.3 [3] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space in the sense of George and Veeramani if X is an arbitrary set, $*$ is a continuous t-norm, and M is fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X, s, t > 0$,

(GV-1) $M(x, y, t) > 0$ for all $t > 0$;

(GV-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(GV-3) $M(x, y, t) = M(y, x, t)$;

(GV-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X, s, t > 0$;

(GV-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

It is worth noticing that, from (GV-2) and (GV-1) it follows that if $x \neq y$, then $0 < M(x, y, t) < 1$ for all $t > 0$. In what follows, fuzzy metric spaces in the sense of Geoge and Veeramani will be called

GV-fuzzy metric spaces. We can find some topological properties and examples of fuzzy metric spaces in [4]. In the following example, we show that every metric induces a fuzzy metric.

Example 2.2 [4] (Induced fuzzy metric) Let (X, d) be a metric space. Denote $a * b = ab$ (or $a * b = \min\{a, b\}$) for all $a, b \in [0, 1]$ and let M_d be the fuzzy set on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a GV-fuzzy metric space, called standard fuzzy metric space induced by (X, d).

Definition 2.4 [2, 3] Let $(X, M, *)$ be a (KM- or GV-)fuzzy metric space:
(a)A sequence $\{x_n\}$ in X converges to x in X (written as $x_n \to x$) if for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1.$$

(b) A sequence $\{x_n\}$ in X is said to be a G-Cauchy sequence if for all $t > 0$ and $p \in N$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1.$$

A fuzzy metric space is said to be G-complete if and only if every G-Cauchy sequence is convergent. It is called compact if every sequence contains a convergent subsequence.

Remark 2.1 Since $*$ is continuous, it follows from (FM-4) and (GV-4) that the limit of the sequence in fuzzy metric space is uniquely determined.

Lemma 2.2 [9] If $(X, M, *)$ is a KM-fuzzy metric space and $\{x_n\}, \{y_n\}$ are sequences in X such that $x_n \to x, y_n \to y$, then $M(x_n, y_n, t) \to M(x, y, t)$ for every continuity point t of $M(x, y, \cdot)$.

The concept of E.A. property in a metric space has been recently introduced by Aamri and Moutawakil[10].

Definition 2.5 [10] Two self-maps A and S of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$$

for some $x \in X$.

Similarly, we can define E.A. property and common E.A. property in fuzzy metric spaces.

Definition 2.6 Let A and S be self maps on a fuzzy metric space $(X, M, *)$. They are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$$

for some $x \in X$.

Definition 2.7 Let A, B, S and T be self maps on a fuzzy metric space $(X, M, *)$. They are said to satisfy common E.A. property if there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = x$$

for some $x \in X$.

For more on (EA) and common (EA) properties, we refer to [10].

The aim of this paper is to emphasize the role of common property E.A. in the existence of a common fixed point for contractive mappings in fuzzy metric spaces. We will work in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani. Our theorems involve the class Φ of all mappings $\varphi : [0,1] \to [0,1]$ satisfying the following properties:

(φ_1) φ is continuous and nondecreasing on $[0,1]$;

(φ_2) $\varphi(x) > x$ for all $x \in (0,1)$.

We note that if $\varphi \in \Phi$, then $\varphi(1) = 1$ and that $\varphi(x) \geq x$ for all $x \in [0,1]$.

3 Main results

Recall that two maps f and g are said to be weakly compatible if they commute at their coincidence points [11].

Theorem 3.1 Let $(X, M, *)$ be a KM-fuzzy metric space satisfying the following property:

$$\forall x, y \in X, x \neq y, \exists t > 0 : 0 < M(x, y, t) < 1$$

and (A, S) and (B, T) be weakly compatible pairs such that there exists $\varphi \in \Phi$ satisfying

$$M(Ax, By, t) \geq \varphi(\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t)\}), \qquad (1)$$

for all $x, y \in X, t > 0$. If A, B, S and T satisfy common E.A. property and the rang of S and T are closed subspaces of X, then A, B, S and T have a unique fixed point in X.

Proof Since A, B, S and T satisfy common E.A. property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some z in X. As $S(X)$ and $T(X)$ are closed subspaces of X, therefore $z = Su = Tv$ for some $u, v \in X$. Now we claim that $Au = z$. Replacing x by u and y by y_n in (1), we obtain

$$M(Au, By_n, t) \geq \varphi(\min\{M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, t), M(Su, By_n, t), M(Au, Ty_n, t)\})$$

which on taking $n \to \infty$ gives

$$M(Au, z, t) \geq \varphi(\min\{M(Su, z, t), M(Au, z, t), M(z, z, t), M(Su, z, t), M(Au, z, t)\})$$

$$= \varphi(\min\{1, M(Au, z, t), 1, 1, M(Au, z, t)\}) = \varphi(M(Au, z, t)).$$

If we had $Au \neq z$, then $0 < M(Au, z, t_0) < 1$ for some $t_0 > 0$. As $M(Au, z, \cdot)$ is left-continuous and the $M(Au, z, \cdot)$ is nondecreasing, that is has only a most countable points of discontinuity, we may suppose that t_0 is a continuity point of $M(Au, z, \cdot)$. Then from condition φ_2 it would follow $\varphi(M(Au, z, t_0)) > M(Au, z, t_0)$, which is a contradiction. Therefore, the equality $Au = z$ holds, that is $Au = z = Su$. Again using (1), we get

$$\begin{aligned} M(Tv, Bv, t) &= M(Au, Bv, t) \\ &\geq \varphi(\min\{M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), M(Su, Bv, t), M(Au, Tv, t)\}) \\ &= \varphi(\min\{1, 1, M(Bv, Tv, t), M(Tv, Bv, t), 1\}) = \varphi(M(Tv, Bv, t)). \end{aligned}$$

As above we obtain $Tv = Bv$ and hence $Au = z = Su = Tv = Bv$.

Now, since (A, S) and (B, T) are weakly compatible pairs, it follows that

$$z = Au = Su \Rightarrow Az = ASu = SAu = Sz \quad \text{and} \quad z = Bv = Tv \Rightarrow Bz = BTv = TBv = Tz.$$

Then, from (1), for all $t > 0$

$$\begin{aligned} M(Az, z, t) &= M(Az, Bv, t) \\ &\geq \varphi(\min\{M(Sz, Tv, t), M(Az, Sz, t), M(Bv, Tv, t), M(Sz, Bv, t), M(Az, Tv, t)\}) \\ &= \varphi(\min\{M(Az, z, t), 1, 1, M(Az, z, t), M(Az, z, t)\}) = \varphi(M(Az, z, t)). \end{aligned}$$

As above we obtain $Az = z = Sz$.

Similarly, we can prove $Bz = z = Tz$. Therefore $Az = Sz = z = Bz = Tz$, that is, z is a common fixed point of A, B, S and T.

For uniqueness, suppose that w is also a common fixed point of A, B, S and T, that is $Aw = Sw = w = Bw = Tw$. If $w \neq z$, then $0 < M(z, w, t) < 1$ for some $t > 0$, hence

$$M(z, w, t) < \varphi(M(z, w, t)).$$

On the other hand, by (φ_1),

$$\begin{aligned} M(z, w, t) &= M(Az, Bw, t) \\ &\geq \varphi(\min\{M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, t), M(Sz, Bw, t), M(Az, Tw, t)\}) \\ &= \varphi(\min\{M(z, w, t), 1, 1, M(z, w, t), M(z, w, t)\}) = \varphi(M(z, w, t)) \end{aligned}$$

for all $t > 0$, which is a contradiction. Therefore, it must be the case that $w = z$. This concludes the proof.

Remark 3.1 Setting $A = B = f$ and $T = S = g$ in Theorem 3.1, one deduces Theorem 2.1 due to Mihet[8].

If $(X, M, *)$ is a fuzzy metric space in the sense of George and Veeramani, then some of the hypotheses in the preceding theorem can be relaxed.

Theorem 3.2 Let $(X, M, *)$ be a GV-fuzzy metric space and (A, S) and (B, T) be weakly compatible pairs such that there exist $\varphi \in \Phi$ and $s > 0$ satisfying

$$M(Ax, By, s) \geq \varphi(\min\{M(Sx, Ty, s), M(Ax, Sx, s), M(By, Ty, s), M(Sx, By, s), M(Ax, Ty, s)\}), \quad (2)$$

for all $x, y \in X$. If A, B, S and T satisfy common E.A. property and the rang of S and T are closed subspaces of X, then A, B, S and T have a unique fixed point in X.

Proof Since A, B, S and T satisfy common E.A. property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some z in X. As $S(X)$ and $T(X)$ are closed subspaces of X, therefore $z = Su = Tv$ for some $u, v \in X$. Now we claim that $Au = z$. Replacing x by u and y by y_n in (2), we obtain

$$M(Au, By_n, s) \geq \varphi(\min\{M(Su, Ty_n, s), M(Au, Su, s), M(By_n, Ty_n, s), M(Su, By_n, s), M(Au, Ty_n, s)\})$$

which on taking $n \to \infty$ gives

$$\begin{aligned} M(Au, z, s) &\geq \varphi(\min\{M(Su, z, s), M(Au, z, s), M(z, z, s), M(Su, z, s), M(Au, z, s)\}) \\ &= \varphi(\min\{1, M(Au, z, s), 1, 1, M(Au, z, s)\}) = \varphi(M(Au, z, s)). \end{aligned}$$

Now, if $Au \neq z$, then $0 < M(Au, z, s) < 1$. Then from condition φ_2 it would follow $\varphi(M(Au, z, t_0)) > M(Au, z, t_0)$, which is a contradiction. This proves that $M(Au, z, s) = 1$, hence, due to (GV-2), the equality $Au = z$ holds, that is $Au = z = Su$. Again using (2), we get

$$\begin{aligned} M(Tv, Bv, s) &= M(Au, Bv, s) \\ &\geq \varphi(\min\{M(Su, Tv, s), M(Au, Su, s), M(Bv, Tv, s), M(Su, Bv, s), M(Au, Tv, s)\}) \\ &= \varphi(\min\{1, 1, M(Bv, Tv, s), M(Tv, Bv, s), 1\}) = \varphi(M(Tv, Bv, s)). \end{aligned}$$

As above we obtain $Tv = Bv$ and hence $Au = z = Su = Tv = Bv$.

Next as in the proof of Theorem 3.1, from weakly compatibility it follows $Az = Sz$ and $Bz = Tz$. Let us prove that $Az = z = Sz = Bz = Tz$. Indeed, we know that

$$\begin{aligned} M(Az, z, s) &= M(Az, Bv, t) \\ &\geq \varphi(\min\{M(Sz, Tv, s), M(Az, Sz, s), M(Bv, Tv, s), M(Sz, Bv, s), M(Az, Tv, s)\}) \\ &= \varphi(\min\{M(Az, z, s), 1, 1, M(Az, z, s), M(Az, z, s)\}) = \varphi(M(Az, z, s)). \end{aligned}$$

If $Az \neq z$, then from (GV-2), $0 < M(Az, z, t) < 1$ for all $t > 0$ and therefore $\varphi(M(Az, z, ts)) > M(Az, z, t)$ $(t > 0)$. In particular,

$$\varphi(M(Az, z, s)) > M(Az, z, s)$$

which contracts the above inequality. Thus, we obtain $Az = z = Sz$.

Similarly, we can prove $Bz = z = Tz$. Therefore $Az = Sz = z = Bz = Tz$, that is, z is a common fixed point of A, B, S and T.

For uniqueness, suppose that w is also a common fixed point of A, B, S and T and $w \neq z$. Then $0 < M(z, w, t) < 1$ for all $t > 0$, and thus

$$\varphi(M(z, w, t)) > M(z, w, t) \text{ for all } t > 0.$$

On the other hand, we know that for some $s > 0$

$$\begin{aligned} M(z, w, s) &= M(Az, Bw, s) \\ &\geq \varphi(\min\{M(Sz, Tw, s), M(Az, Sz, s), M(Bw, Tw, s), M(Sz, Bw, s), M(Az, Tw, s)\}) \\ &= \varphi(\min\{M(z, w, s), 1, 1, M(z, w, s), M(z, w, s)\}) = \varphi(M(z, w, s)) \end{aligned}$$

which is a contradiction. Therefore, it must be the case that $w = z$. This concludes the proof.

Remark 3.2 Setting $A = B = f$ and $T = S = g$ in Theorem 3.2, one deduces Theorem 3.1 due to Mihet[8].

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Some new identities on the twisted (h, q)-Euler numbers and q-Bernstein polynomials

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Abstract In this paper we give some interesting relationship between the q-Bernstein polynomials and the twisted (h, q)-Euler numbers by using fermionic p-adic q-integral on $\mathbb{Z}_p$.

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Key words- Euler numbers and polynomials, twisted (h, q)-Euler numbers and polynomials, Bernstein polynomials, q-Bernstein polynomials

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes

the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $C_{p^n} = \{\zeta | \zeta^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$T_p = \lim_{n\to\infty} C_{p^n} = \cup_{n\geq 1} C_{p^n}.$$

The p-adic absolute value is defined by $|x|_p = \frac{1}{p^r}$, where $x = p^r \frac{s}{t}$($r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s,t) = (p,s) = (p,t) = 1$). In this paper we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ as an indeterminate. The q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \text{ see [1-18].}$$

Note that $\lim_{q\to 1}[x]_q = x$. For

$$f \in UD(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [1-5] .} \tag{1}$$

From (1), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),$$

where $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, Kim defined q-Bernstein polynomials, which are different q-Bernstein polynomials of Phillips, as follows:

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \text{ see [4].} \tag{2}$$

In [9], the p-adic extension of (2) is given by

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \text{ where } x \in \mathbb{Z}_p, \text{ and } n, k \in \mathbb{Z}_+. \tag{3}$$

For $h \in \mathbb{Z}$ and $\zeta \in T_p$, let us consider the twisted (h,q)-Euler polynomials as follows:

$$E_{n,q,\zeta}^{(h)}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n \zeta^y q^{(h-1)y} d\mu_{-q}(y), \text{ for } n \in \mathbb{Z}_+. \tag{4}$$

In the special case, $x = 0$, $E_{n,q,\zeta}^{(h)}(0) = E_{n,q,\zeta}^{(h)}$ are called the n-th twisted (h,q)-Euler numbers.

In this paper we investigate some relations between the q-Bernstein polynomials and the twisted (h,q)-Euler numbers. From these relations, we derive some interesting identities on the the twisted (h,q)-Euler numbers and polynomials.

2. On the twisted (h,q)-Euler numbers and polynomials

From (4), we note that

$$\begin{aligned} E_{n,q,\zeta}^{(h)}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n \zeta^y q^{(h-1)y} d\mu_{-q}(y) \\ &= \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1+q^{h+l}\zeta} \right) \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m \zeta^m q^{hm} [x+m]_q^n, \end{aligned}$$

and

$$\begin{aligned} E_{n,q,\zeta}^{(h)}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n \zeta^y q^{(h-1)y} d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l \zeta^y q^{(h-1)y} d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} E_{l,q,\zeta}^{(h)}. \end{aligned} \tag{5}$$

Therefore, we obtain the following theorem.

Theorem 1. For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, we have

$$E_{n,q,\zeta}^{(h)}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \zeta^m q^{hm} [x+m]_q^n.$$

Furthermore,

$$\begin{aligned} E_{n,q,\zeta}^{(h)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} E_{l,q,\zeta}^{(h)} \\ &= ([x]_q + q^x E_{q,\zeta}^{(h)})^n, \end{aligned}$$

with usual convention about replacing $(E_{q,\zeta}^{(h)})^n$ by $E_{n,q,\zeta}^{(h)}$.

Let

$$F_{q,\zeta}^{(h)}(t,x) = \sum_{n=0}^{\infty} E_{n,q,\zeta}^{(h)}(x) \frac{t^n}{n!}.$$

Then we see that

$$F_{q,\zeta}^{(h)}(t,x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \zeta^m q^{mh} e^{[x+m]_q t}. \tag{6}$$

In the special case, $x = 0$, let $F_{q,\zeta}^{(h)}(t,0) = F_{q,\zeta}^{(h)}(t)$.

By (4), we get

$$\begin{aligned} E_{n,q^{-1},\zeta^{-1}}^{(h)}(1-x) &= \int_{\mathbb{Z}_p} [1-x+y]_{q^{-1}}^n \zeta^{-y} q^{-(h-1)y} d\mu_{-q^{-1}}(y) \\ &= \frac{[2]_q}{(1-q^{-1})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{h-1} \zeta \left(\frac{q^{lx}}{1+q^{h+l}\zeta} \right) \\ &= (-1)^n \zeta q^{n+h-1} \left(\frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{1+q^{h+l}\zeta} \right) \\ &= (-1)^n \zeta q^{n+h-1} E_{n,q,\zeta}^{(h)}(x). \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2. Let $F_{q,\zeta}^{(h)}(t,x) = \sum_{n=0}^{\infty} E_{n,q,\zeta}^{(h)}(x) \frac{t^n}{n!}$. Then we have

$$F_{q^{-1},\zeta^{-1}}^{(h)}(t,1-x) = \zeta q^{h-1} F_{q,\zeta}^{(h)}(-qt,x).$$

Moreover,

$$E_{n,q^{-1},\zeta^{-1}}^{(h)}(1-x) = (-1)^n \zeta q^{n+h-1} E_{n,q,\zeta}^{(h)}(x) \text{ for } n \in \mathbb{Z}_+.$$

From (6), we note that

$$q^h \zeta F_{q,\zeta}^{(h)}(t,1) + F_{q,\zeta}^{(h)}(t) = [2]_q. \tag{7}$$

By (7), we get the following recurrence formula:

$$E_{0,q,\zeta}^{(h)} = \frac{1+q}{1+q^h\zeta}, \text{ and } q^h \zeta E_{n,q,\zeta}^{(h)}(1) + E_{n,q,\zeta}^{(h)} = 0 \text{ if } n > 0. \tag{8}$$

By (8) and Theorem 1, we obtain the following theorem.

Theorem 3. For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, we have

$$E_{0,q,\zeta}^{(h)} = \frac{1+q}{1+q^h\zeta}, \text{ and } q^h \zeta (qE_{q,\zeta}^{(h)} + 1)^n + E_{n,q,\zeta}^{(h)} = 0 \text{ if } n > 0,$$

with usual convention about replacing $(E_{q,\zeta}^{(h)})^n$ by $E_{n,q,\zeta}^{(h)}$.

From Theorem 3, we note that

$$
\begin{aligned}
&q^{2h}\zeta^2 E_{n,q,\zeta}^{(h)}(2) - \frac{1+q}{1+q^h\zeta}q^{2h}\zeta^2 - \frac{1+q}{1+q^h\zeta}q^h\zeta \\
&= q^{2h}\zeta^2 \sum_{l=0}^{n} \binom{n}{l} q^l (qE_{q,\zeta}^{(h)} + 1)^l - \frac{1+q}{1+q^h\zeta}q^{2h}\zeta^2 - \frac{1+q}{1+q^h\zeta}q^h\zeta \\
&= q^{2h}\zeta^2 \sum_{l=1}^{n} \binom{n}{l} q^l (qE_{q,\zeta}^{(h)} + 1)^l - \frac{1+q}{1+q^h\zeta}q^h\zeta \\
&= -q^h\zeta \sum_{l=1}^{n} \binom{n}{l} q^l E_{l,q,\zeta}^{(h)} - \frac{1+q}{1+q^h\zeta}q^h\zeta \\
&= -q^h\zeta \sum_{l=0}^{n} \binom{n}{l} q^l E_{l,q,\zeta}^{(h)} \\
&= -q^h\zeta E_{n,q,\zeta}^{(h)}(1) = E_{n,q,\zeta}^{(h)} \text{ if } n > 0.
\end{aligned}
$$

Therefore, we obtain the following theorem.

Theorem 4. For $n \in \mathbb{N}$, we have

$$
q^{2h}\zeta^2 E_{n,q,\zeta}^{(h)}(2) = E_{n,q,\zeta}^{(h)} + \frac{1+q}{1+q^h\zeta}q^{2h}\zeta^2 + \frac{1+q}{1+q^h\zeta}q^h\zeta.
$$

By Theorem 2, we see that

$$
\begin{aligned}
&q^{h-1}\zeta \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n q^{(h-1)x}\zeta^x d\mu_{-q}(x) \\
&= (-1)^n q^{n+h-1}\zeta \int_{\mathbb{Z}_p} [x-1]_q^n q^{(h-1)x}\zeta^x d\mu_{-q}(x) \\
&= (-1)^n q^{n+h-1}\zeta E_{n,q,\zeta}^{(h)}(-1) = E_{n,q^{-1},\zeta^{-1}}^{(h)}(2).
\end{aligned} \tag{9}
$$

Therefore, we obtain the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$
q^{h-1}\zeta \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n q^{(h-1)x}\zeta^x d\mu_{-q}(x) = E_{n,q^{-1},\zeta^{-1}}^{(h)}(2).
$$

Let $n \in \mathbb{N}$. By Theorem 5 and Theorem 4, we get

$$
\begin{aligned}
&q^{h-1}\zeta \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n q^{(h-1)x}\zeta^x d\mu_{-q}(x) \\
&= q^{2h}\zeta^2 E_{n,q^{-1},\zeta^{-1}}^{(h)} + q^{h-1}\zeta \left(\frac{1+q}{1+q^h\zeta}\right) + q^{2h-1}\zeta^2 \left(\frac{1+q}{1+q^h\zeta}\right).
\end{aligned} \tag{10}
$$

From (10), we have

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n q^{(h-1)x}\zeta^x d\mu_{-q}(x) = q^{h+1}\zeta E_{n,q^{-1},\zeta^{-1}}^{(h)} + \left(\frac{1+q}{1+q^h\zeta}\right) + q^h\zeta\left(\frac{1+q}{1+q^h\zeta}\right).$$

Therefore, we obtain the following corollary.

Corollary 6. For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n q^{(h-1)x}\zeta^x d\mu_{-q}(x) = q^{h+1}\zeta E_{n,q^{-1},\zeta^{-1}}^{(h)} + [2]_q.$$

For $x \in \mathbb{Z}_p$, the p-adic analogues of q-Bernstein polynomials are given by

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k}, \text{ where } n,k \in \mathbb{Z}_+. \tag{11}$$

By (11), we get the symmetry of q-Bernstein polynomials as follows:

$$B_{k,n}(x,q) = B_{n-k,n}(1-x,q^{-1}), \text{ see [7].} \tag{12}$$

Thus, by Corollary 6, (11), and (12), we see that

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x,q)q^{(h-1)x}\zeta^x d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x,q^{-1})q^{(h-1)x}\zeta^x d\mu_{-q}(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n-l}q^{(h-1)x}\zeta^x d\mu_{-q}(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(q^{h+1}\zeta E_{n-l,q^{-1},\zeta^{-1}}^{(h)} + [2]_q\right).\end{aligned}$$

For $n,k \in \mathbb{Z}_+$ with $n > k$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n}(x,q)q^{(h-1)x}\zeta^x d\mu_{-q}(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(q^{h+1}\zeta E_{n,q^{-1},\zeta^{-1}}^{(h)} + [2]_q\right)\\ &= \begin{cases} q^{h+1}\zeta E_{n,q^{-1},\zeta^{-1}}^{(h)} + [2]_q, & \text{if } k=0,\\ q^{h+1}\zeta\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}q^{h+1}\zeta E_{n-l,q^{-1},\zeta^{-1}}^{(h)}, & \text{if } k>0.\end{cases}\end{aligned} \tag{13}$$

Let us take the fermionic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials of degree n as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x,q)q^{(h-1)x}\zeta^x d\mu_{-q}(x) &= \binom{n}{k}\int_{\mathbb{Z}_p}[x]_q^k[1-x]_{q^{-1}}^{n-k}q^{(h-1)x}\zeta^x d\mu_{-q}(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l E_{l+k,q,\zeta}^{(h)}.\end{aligned} \tag{14}$$

Therefore, by (13) and (14), we obtain the following theorem.

Theorem 7. Let $n,k\in\mathbb{Z}_+$ with $n>k$. Then we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n}(x,q)q^{(h-1)x}\zeta^x d\mu_{-q}(x)\\ &=\begin{cases} q^{h+1}\zeta E_{n,q^{-1},\zeta^{-1}}^{(h)}+[2]_q, & \text{if } k=0,\\ q^{h+1}\zeta\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}E_{n-l,q^{-1},\zeta^{-1}}^{(h)}, & \text{if } k>0.\end{cases}\end{aligned}$$

Moreover,

$$\begin{aligned}&\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l E_{l+k,q,\zeta}^{(h)}\\ &=\begin{cases} q^{h+1}\zeta E_{n,q^{-1},\zeta^{-1}}^{(h)}+[2]_q, & \text{if } k=0,\\ q^{h+1}\zeta\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}E_{n-l,q^{-1},\zeta^{-1}}^{(h)}, & \text{if } k>0.\end{cases}\end{aligned}$$

Let $n_1,n_2,k\in\mathbb{Z}_+$ with $n_1+n_2>2k$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)B_{k,n_2}(x,q)q^{(h-1)x}\zeta^x d\mu_{-q}(x)\\ &=\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n_1+n_2-l}q^{(h-1)x}\zeta^x d\mu_{-q}(x)\\ &=\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(q^{h+1}\zeta E_{n_1+n_2-l,q^{-1},\zeta^{-1}}^{(h)}+[2]_q\right)\\ &=\begin{cases} q^{h+1}\zeta E_{n_1+n_2,q^{-1},\zeta^{-1}}^{(h)}+[2]_q, & \text{if } k=0,\\ \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k+l}\left(q^{h+1}\zeta E_{n_1+n_2-l,q^{-1},\zeta^{-1}}^{(h)}+[2]_q\right), & \text{if } k\neq 0.\end{cases}\end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem.

Theorem 8. For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q) B_{k,n_2}(x,q) q^{(h-1)x} \zeta^x d\mu_{-q}(x) \\
&= \begin{cases} q^{h+1}\zeta E^{(h)}_{n_1+n_2,q^{-1},\zeta^{-1}} + [2]_q, & \text{if } k = 0, \\ q^{h+1}\zeta \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E^{(h)}_{n_1+n_2-l,q^{-1},\zeta^{-1}}, & \text{if } k \neq 0. \end{cases}
\end{aligned}$$

From the binomial theorem, we can derive the following equation.

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q) B_{k,n_2}(x,q) q^{(h-1)x} \zeta^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \int_{\mathbb{Z}_p} [x]_q^{2k+l} q^{(h-1)x} \zeta^x d\mu_{-q}(x) \qquad (16) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E^{(h)}_{2k+l,q,\zeta}.
\end{aligned}$$

Thus, by Theorem 8 and (16), we obtain the following corollary.

Corollary 9. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have

$$\begin{aligned}
&\sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E^{(h)}_{2k+l,q} \\
&= \begin{cases} q^{h+1}\zeta E^{(h)}_{n_1+n_2,q^{-1},\zeta^{-1}} + [2]_q, & \text{if } k = 0, \\ q^{h+1}\zeta \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E^{(h)}_{n_1+n_2-l,q^{-1},\zeta^{-1}}, & \text{if } k > 0. \end{cases}
\end{aligned}$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials of

degree n as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q)}_{s-\text{times}} q^{(h-1)x}\zeta^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\int_{\mathbb{Z}_p}[x]_q^{sk}[1-x]_{q^{-1}}^{n_1+\cdots+n_s-sk}q^{(h-1)x}\zeta^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n_1+\cdots+n_s-l}q^{(h-1)x}\zeta^x d\mu_{-q}(x) \qquad (17)\\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\left(q^{h+1}\zeta E^{(h)}_{n_1+\cdots+n_s-l,q^{-1},\zeta^{-1}}+[2]_q\right) \\
&= \begin{cases} q^{h+1}\zeta E^{(h)}_{n_1+\cdots+n_s,q^{-1},\zeta^{-1}}+[2]_q, & \text{if } k=0, \\ q^{h+1}\zeta\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}E^{(h)}_{n_1+\cdots+n_s-l,q^{-1},\zeta^{-1}}, & \text{if } k>0. \end{cases}
\end{aligned}$$

Therefore, by (17), we obtain the following theorem.

Theorem 10. For $s\in\mathbb{N}$ with $s\geq 2$, let $n_1,n_2,\ldots,n_s,k\in\mathbb{Z}_+$ with $n_1+\cdots+n_s>sk$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q)}_{s-\text{times}} q^{(h-1)x}\zeta^x d\mu_{-q}(x) \\
&= \begin{cases} q^{h+1}\zeta E^{(h)}_{n_1+\cdots+n_s,q^{-1},\zeta^{-1}}+[2]_q, & \text{if } k=0, \\ q^{h+1}\zeta\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}E^{(h)}_{n_1+\cdots+n_s-l,q^{-1},\zeta^{-1}}, & \text{if } k>0. \end{cases} \qquad (18)
\end{aligned}$$

By the definition of q-Bernstein polynomials and the binomial theorem, we easily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x,q)\cdots B_{k,n_s}(x,q)}_{s-\text{times}} q^{(h-1)x}\zeta^x d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+\cdots+n_s-sk}(-1)^l\binom{n_1+\cdots+n_s-sk}{l}\int_{\mathbb{Z}_p}[x]_q^{sk+l}q^{(h-1)x}\zeta^z d\mu_{-q}(x) \\
&= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+\cdots+n_s-sk}(-1)^l\binom{n_1+\cdots+n_s-sk}{l}E^{(h)}_{sk+l,q,\zeta}.
\end{aligned}$$

Therefore, we have the following corollary.

Corollary 11. For $\zeta \in T_p, s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we have

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} E^{(h)}_{sk+l,q,\zeta}$$
$$= \begin{cases} q^{h+1}\zeta E^{(h)}_{n_1+\cdots+n_s,q^{-1},\zeta^{-1}} + [2]_q, & \text{if } k = 0, \\ q^{h+1}\zeta \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E^{(h)}_{n_1+\cdots+n_s-l,q^{-1},\zeta^{-1}}, & \text{if } k > 0. \end{cases}$$

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3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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ON THE "ON SOME RESULTS IN FUZZY METRIC SPACES"

REZA SAADATI

ABSTRACT. In this paper, we show that the proof of Theorem 3.9 of the paper George and Veeramani [On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64 (1994), 395-399.] is not true. We also correct the proof in this note.

In Theorem 3.9 of [1] the authors prove that every compact subset A of a fuzzy metric space $(X, M, *)$ is F–bounded. In the proof in [1] with A compact the authors claim that there is a $t' > 0$ and a $0 < s < 1$ such that $M(x, y, t') > 1 - s$ for all $x, y \in A$, hence A is F–bounded. In this paper with an appropriate example we show that their way for finding s is false. We also correct the proof in this note.

Definition 1. [1] The 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times]0, +\infty[$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s \in]0, +\infty[$:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, \cdot) :]0, +\infty[\to [0, 1]$ is continuous.

Note that, if $\lim_{t\to\infty} M(x, y, t) = 1$ then the 3-tuple $(X, M, *)$ is called a *Menger space*.

Lemma 2. *Let $(X, M, *)$ be a Menger space. Define $E_{\lambda,M} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,M}(x, y) = \inf\{t > 0 : M(x, y, t) > 1 - \lambda\} \tag{1}$$

for each $\lambda \in]0, 1[$ and $x, y \in X$. Then, for any $\mu \in]0, 1[$ there exists $\lambda \in]0, 1[$ such that

$$E_{\mu,M}(x, y) \leq E_{\lambda,M}(x, z) + E_{\lambda,M}(z, w) + E_{\lambda,M}(w, y) \tag{2}$$

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for any $x, y, z \in X$.

Proof. For every $\mu \in]0,1[$ we can find a $\lambda \in]0,1[$ such that

$$(1-\lambda) * (1-\lambda) * (1-\lambda) \geq 1-\mu.$$

By the triangular inequality (see Definition 1) we have

$$\begin{aligned} &M(x, y, E_{\lambda,M}(x,z) + E_{\lambda,M}(z,w) + E_{\lambda,M}(w,y) + 3\delta) \\ &\geq M(x, z, E_{\lambda,M}(x,z) + \delta) * M(z, w, E_{\lambda,M}(z,w) + \delta) * M(z, w, E_{\lambda,M}(w,y) + \delta) \\ &\geq (1-\lambda) * (1-\lambda) * (1-\lambda) \geq 1-\mu. \end{aligned}$$

for every $\delta > 0$, which implies that

$$E_{\mu,M}(x,y) \leq E_{\lambda,M}(x,z) + E_{\lambda,M}(z,w) + E_{\lambda,M}(w,y) + 3\delta.$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\mu,M}(x,y) \leq E_{\lambda,M}(x,z) + E_{\lambda,M}(z,w) + E_{\lambda,M}(w,y).$$

□

Let $(X, M, *)$ be a fuzzy metric space. The *open ball* $B(x, r, t)$ with center $x \in X$ and radius r, where $0 < r < 1$ and $t > 0$, is defined in [1, Definition 3.1] as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

In [1, Definition 3.7] a subset A of X is said to be *F–bounded* if and only if there exist $t > 0$ and $r \in]0,1[$ such that

$$M(x, y, t) > 1 - r, \quad \text{for all } x, y \in A,$$

or equivalently, by (1),

$$E_{r,M}(x, y) \leq t, \quad \text{for all } x, y \in A.$$

In the following example we show that using the reasoning of the proof of [1, Theorem 3.9] we can not find a $t > 0$ and a $0 < r < 1$ such that $M(x, y, t) > 1 - r$.

Example 3. Consider the standard fuzzy metric space $(\mathbb{R}, M_0, *)$ in which $M_0(x, y, t) = \frac{t}{t+|x-y|}$ and $a * b = \max\{0, a + b - 1\}$. Let $A = [-4, 13]$. The open cover $\{B(x, 0.8, 1) : x \in A\}$ of A has a finite subcover $\{B(x_i, 0.8, 1) : x_i \in A, 1 \leq i \leq 3\}$ in which

$x_1 = -0.1$, $x_2 = 4$ and $x_3 = 9.1$. Let $x, y \in A$. Then $x \in B(x_i, 0.8, 1)$ and $y \in B(x_j, 0.8, 1)$ for some $1 \leq i, j \leq 3$. Therefore $M_0(x, x_i, 1) > 0.2$ and $M_0(y, x_j, 1) > 0.2$. Now, let

$$\alpha = \min\{M_0(x_i, x_j, 1) : 1 \leq i, j \leq 3\}.$$

Then $\alpha < 0.1$
Now

$$\begin{aligned} M_0(x, y, 3) &\geq M_0(x, x_i, 1) * M_0(x_i, x_j, 1) * M_0(x_j, y, 1) \\ &\geq (0.2) * (0.1) * (0.2) = 0, \end{aligned}$$

and hence from the above we cannot find a $0 < s < 1$ such that $M_0(x, y, 3) \geq (0.2) * (0.1) * (0.2) > 1 - s$, as suggested in the proof of [1, Theorem 3.9].

Now we prove [1, Theorem 3.9].

Theorem 4. *Every compact subset A of a fuzzy metric space $(X, M, *)$ is F–bounded whenever $(X, M, *)$ is Menger space or $a * b > \max\{0, a + b - 1\}$.*

Proof. Let $(X, M, *)$ be Menger space. Given A is a compact subset of X. Fix $t > 0$ and $0 < r < 1$. From Lemma 2 it follows that there exists a $\lambda \in]0, 1[$ such that (2) holds for any $x, y, z \in X$ and for $\mu = r$. Consider an open cover $\{B(x, \lambda, t) : x \in A\}$ of A. Since A is compact there exist $x_1, x_2, ..., x_n \in A$ such that $A \subseteq \cup_{1 \leq i \leq n} B(x_i, \lambda, t)$. Let $x, y \in A$. Then $x \in B(x_i, \lambda, t)$ and $y \in B(x_j, \lambda, t)$ for some $1 \leq i, j \leq n$. Therefore $E_{\lambda,M}(x, x_i) \leq t$ and $E_{\lambda,M}(y, x_j) \leq t$. Now, let

$$\alpha = \max\{E_{\lambda,M}(x_i, x_j) : 1 \leq i, j \leq n\}.$$

Now

$$E_{r,M}(x, y) \leq E_{\lambda,M}(x, x_i) + E_{\lambda,M}(x_i, x_j) + E_{\lambda,M}(x_j, y) \leq 2t + \alpha.$$

Taking $t' = 2t + \alpha$, we have $E_{r,M}(x, y) \leq t'$ for all $x, y \in A$. Hence A is F–bounded.

Now, if in fuzzy metric space $(X, M, *)$, $a * b > \max\{0, a + b - 1\}$, see proof of [1, Theorem 3.9]. □

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[1] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst., 64 (1994) 395–399.

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DIFFERENT TYPES MEIR-KEELER CONTRACTIONS ON PARTIAL METRIC SPACES

İ. M. ERHAN, E. KARAPINAR, D.TÜRKOĞLU

Abstract. In this manuscript, Meir-Keeler contractions on partial metric spaces are introduced. It is shown that if a self-mapping T on a complete partial metric spaces is a Meir-Keeler contraction, then T has a unique fixed point.

1. Introduction and Preliminaries

The theory of fixed points has been a subject of considerable interest since the pioneering work of Banach in 1922 [1]. Fixed points of numerous types of contractions on various spaces has been thoroughly studied from different points of view. In 1992 Matthews [2, 3] introduced a new space called Partial metric space. The main difference between the usual metric spaces and partial metric spaces is that $d(x,x)$ are not necessarily zero. Partial metric spaces have wide application potentials in the research area of computer domains and semantics (see e.g. [4, 5, 6, 7]). Accordingly, the attention paid to these spaces rapidly increases.

In 1994, Matthews [3], generalized the Banach Contraction Principle to the class of complete partial metric spaces as follows: A self mapping T on a complete partial metric space (X,p) has a unique fixed point if there exists $0 \leq k < 1$ such that $p(Tx,Ty) \leq kp(x,y)$, for all $x,y \in X$. After this remarkable contribution of Matthews, many authors have studied partial metric spaces and their topological properties, as well as the fixed points of contractions on partial metric spaces. (See e.g. [8, 9, 10, 11, 12],[14]-[20]).

A partial metric space (See e.g.[2, 3]) is a pair (X,p) of a set X and a function $p : X \times X \to \mathbb{R}^+$ called partial metric, satisfying the following conditions:

(PM1) $p(x,y) = p(y,x)$ (symmetry)
(PM2) If $p(x,x) = p(x,y) = p(y,y)$ then $x = y$ (equality)
(PM3) $p(x,x) \leq p(x,y)$ (small self-distances)
(PM4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$ (triangularity)

for all $x,y,z \in X$. We use the abbreviation PMS for partial metric spaces.

It is easy to see that if p is a partial metric on X, the function $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

is a (usual) metric on X. Observe that each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[x,\varepsilon] = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$

Some basic concepts on partial metric spaces are defined below.

Definition 1. *(See e.g.*[2, 3, 11] *)*

(*i*) *A sequence* $\{x_n\}$ *in a PMS* (X,p) *converges to* $x \in X$ *if and only if* $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(*ii*) *A sequence* $\{x_n\}$ *in a PMS* (X,p) *is called Cauchy sequence if and only if* $\lim_{n,m\to\infty} p(x_n,x_m)$ *exists (and is finite).*
(*iii*) *A PMS* (X,p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges, with respect to* τ_p*, to a point* $x \in X$ *such that* $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(*iv*) *A mapping* $f : X \to X$ *is said to be continuous at* $x_0 \in X$*, if for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

The following lemma is valid on partial metric spaces.

Lemma 2. *(See e.g.*[2, 3, 11] *)*

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(A) *A sequence* $\{x_n\}$ *is Cauchy in a PMS* (X,p) *if and only if* $\{x_n\}$ *is Cauchy in a metric space* (X,d_p).
(B) *A PMS* (X,p) *is complete if and only if a metric space* (X,d_p) *is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m). \tag{1.2}$$

2. Main Results

In this section, we define Meir-Keeler contraction on partial metric spaces and discuss the existence and uniqueness of fixed points of a certain class of Meir-Keeler contractions. First we give two observations which are important in the proofs of fixed point theorems on partial metric spaces in general.

Lemma 3. *(See* [13]*) Let* (X,p) *be a PMS. Then*

(A) *If* $p(x,y)=0$ *then* $x=y$.
(B) *If* $x\neq y$, *then* $p(x,y)>0$.

Proof. Proof of (A). Let $p(x,y)=0$. By (PM3), we have $p(x,x)\leq p(x,y)=0$ and $p(y,y)\leq p(x,y)=0$. Thus, we have

$$p(x,x)=p(x,y)=p(y,y)=0.$$

Hence, by (PM2), we have $x=y$.

Proof of (B). Suppose $x\neq y$. By definition $p(x,y)\geq 0$ for all $x,y\in X$. Assume $p(x,y)=0$. By part (A), $x=y$ which is a contradiction. Hence, $p(x,y)>0$ whenever $x\neq y$.

□

Lemma 4. *(See* [13]*)*

Let $x_n\to z$ *as* $n\to\infty$ *in a PMS* (X,p) *where* $p(z,z)=0$. *Then* $\lim_{n\to\infty} p(x_n,y)=p(z,y)$ *for every* $y\in X$.

Proof. First note that $\lim_{n\to\infty} p(x_n,z)=p(z,z)=0$. By the triangle inequality we have

$$p(x_n,y)\leq p(x_n,z)+p(z,y)-p(z,z)=p(x_n,z)+p(z,y),$$

and

$$p(z,y)\leq p(z,x_n)+p(x_n,y)-p(x_n,x_n)\leq p(x_n,z)+p(x_n,y).$$

Hence

$$0\leq |p(x_n,y)-p(z,y)|\leq p(x_n,z).$$

Letting $n\to\infty$ we obtain the desired result. □

We now give the definition of Meir-Keeler contractions.

Definition 5. *(See* [13]*) Suppose that* $T:X\to X$ *is a self-mapping satisfying the following condition: Given* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$\varepsilon<p(x,y)<\varepsilon+\delta \Rightarrow p(Tx,Ty)\leq\varepsilon. \tag{2.1}$$

Then T *is called Meir-Keeler contractive mapping (Meir-Keeler contraction).*

Remark 6. *Notice that if* $T:X\to X$ *is Meir-Keeler contractive on a PMS* (X,p) *then* T *is contractive, that is,* $p(Tx,Ty)<p(x,y)$, *for all* $x,y\in X$ *with* $x\neq y$.

Inspirit of Hardy-Rogers [22], we state the following definition.

Definition 7. *Suppose that* $T:X\to X$ *is a self-mapping satisfying the following condition: Given* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$\varepsilon\leq H(x,y)<\varepsilon+\delta \Rightarrow p(Tx,Ty)<\varepsilon, \tag{2.2}$$

where

$$H(x,y)=\frac{1}{5}[p(x,y)+p(Tx,x)+p(Ty,y)+p(Tx,y)+p(x,Ty)].$$

Then T *is called Hardy-Rogers type Meir-Keeler contraction.*

Proposition 8. *Let* (X,p) *be a complete partial metric space. Suppose that* $T:X\to X$ *is a Hardy-Rogers type Meir-Keeler contraction. Then, for* $x\in X$ *we have* $p(T^{n+1}x,T^nx)\to 0$, *as* $n\to\infty$.

Proof. Suppose that T is a Hardy-Rogers type Meir-Keeler contraction. Take an arbitrary $x \in X$ and set $x_0 = x$. Define a sequence $\{x_n\}$ as:

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \text{for each } n = 0, 1, 2, \cdots .$$

Due to (2.2), we have

$$\begin{array}{rl} p(Tx, Ty) = 0 & \text{if } H(x, y) = 0, \\ p(Tx, Ty) < H(x, y) & \text{otherwise.} \end{array} \tag{2.3}$$

Then $p(Tx, Ty) \leq H(x, y)$ for every $x, y \in X$. Taking $x = x_{n+1}$ and $y = x_n$ we obtain

$$\begin{array}{rl} p(x_{n+2}, x_{n+1}) & = p(Tx_{n+1}, Tx_n) \leq H(x_{n+1}, x_n) \\ & = \frac{1}{5}\left[p(x_{n+1}, x_n) + p(Tx_{n+1}, x_{n+1}) + p(Tx_n, x_n)\right. \\ & \left. + p(Tx_{n+1}, x_n) + p(x_{n+1}, Tx_n)\right] \\ & = \frac{1}{5}\left[2p(x_{n+1}, x_n) + p(x_{n+2}, x_{n+1}) + p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})\right] \\ & \leq \frac{1}{5}\left[3p(x_{n+1}, x_n) + 2p(x_{n+2}, x_{n+1})\right], \end{array} \tag{2.4}$$

where triangle inequality on partial metric spaces (PM4) has been employed to show that $p(x_{n+2}, x_n) \leq p(x_{n+1}, x_n) + p(x_{n+2}, x_{n+1}) - p(x_{n+1}, x_{n+1})$. Notice that if $H(x_{n+1}, x_n) = 0$, then by definition $p(x_{n+1}, x_n) = p(x_{n+2}, x_{n+1}) = 0$ and hence we get the result. Suppose that $H(x_{n+1}, x_n) > 0$. Due to (2.3) and (2.4) we have

$$p(x_{n+2}, x_{n+1}) < H(x_{n+1}, x_n) \leq \frac{1}{5}\left[3p(x_{n+1}, x_n) + 2p(x_{n+2}, x_{n+1})\right] \tag{2.5}$$

which yields that

$$p(x_{n+2}, x_{n+1}) < p(x_{n+1}, x_n). \tag{2.6}$$

Thus, in either case, we have

$$p(x_{n+2}, x_{n+1}) \leq H(x_{n+1}, x_n) \leq p(x_{n+1}, x_n), \quad \text{for each } n \in \mathbb{N}. \tag{2.7}$$

In other words, the sequence $\{d_n\}_{n=0}^{\infty} = \{p(x_{n+1}, x_n)\}_{n=0}^{\infty}$ is non-increasing and bounded below by 0. Hence, it converges in $[0, \infty)$, to a limit, say L. If $L = 0$, the proof is done. Let $L > 0$. According to (2.2), for this value of L there exists $\delta > 0$ such that

$$L \leq H(x_{m+1}, x_m) \leq L + \delta. \tag{2.8}$$

implies

$$p(Tx_{m+1}, Tx_m) = p(x_{m+2}, x_{m+1}) = d_{m+1} < L, \tag{2.9}$$

which is a contradiction since the sequence $\{d_n\}_{n=0}^{\infty}$ is decreasing. Thus, $L = 0$ which completes the proof. □

The main theorem generalizes the fixed point theorem of Meir and Keeler [21] on partial metric spaces.

Theorem 9. *Let (X, p) be a complete partial metric space. Suppose that $T : X \to X$ is a Hardy-Rogers type Meir-Keeler contraction. Then, T has a unique fixed point, say $z \in X$. Moreover, $\lim_{n\to\infty} p(T^n x, z) = p(z, z)$ for all $x \in X$.*

Proof. Take an arbitrary $x \in X$. We will show that $\{T^n x\}$ is a Cauchy sequence. Assume the contrary. Then there exists an $\varepsilon > 0$ and a subsequence in $\{T^{n(i)}\}$ of $\{T^n x\}$ with

$$p(T^{n(i)}x, T^{n(i+1)}x) > 2\varepsilon. \tag{2.10}$$

For this ε, there exists $\delta > 0$ such that

$$\varepsilon \leq H(x, y) < \varepsilon + \delta \;\text{ implies that }\; p(Tx, Ty) < \varepsilon, \tag{2.11}$$

where

$$H(x, y) = \frac{1}{5}\left[p(x, y) + p(Tx, x) + p(Ty, y) + p(Tx, y) + p(x, Ty)\right].$$

Set $r = \min\{\varepsilon, \delta\}$ and define $d_m = p(T^m x, T^{m+1}x)$. Due to Proposition 8, one can choose $n_0 \in \mathbb{N}$ such that

$$d_m < \frac{r}{4}, \quad \text{for } \; m \geq n_0. \tag{2.12}$$

Let $n(i) \geq n_0$ and suppose $p(T^{n(i)}x, T^{n(i+1)-1}x) \leq \varepsilon + \frac{r}{2}$. Then triangle inequality (PM4) implies

$$\begin{array}{rl} p(T^{n(i)}x, T^{n(i+1)}x) & \leq p(T^{n(i)}x, T^{n(i+1)-1}x) + p(T^{n(i+1)-1}x, T^{n(i+1)}x) \\ & \quad - p(T^{n(i+1)-1}x, T^{n(i+1)-1}x) \\ & \leq p(T^{n(i)}x, T^{n(i+1)-1}x) + p(T^{n(i+1)-1}x, T^{n(i+1)}x) \\ & \leq \varepsilon + \frac{r}{2} + d_{n(i+1)-1} < \varepsilon + \frac{3r}{4} < 2\varepsilon \end{array} \tag{2.13}$$

which contradicts the assumption (2.10). Thus, there are values of k with $n(i) \leq k \leq n(i+1)$ such that $p(T^{n(i)}, T^k x) > \varepsilon + \frac{r}{2}$. Assume that $d_{n(i)} = p(T^{n(i)}x, T^{n(i)+1}x) \geq \varepsilon + \frac{r}{2}$. Then

$$d_{n(i)} \geq \varepsilon + \frac{r}{2} \geq r + \frac{r}{2} > \frac{r}{4}$$

which contradicts (2.12). Hence, there are values of k with $n(i) < k \leq n(i+1)$ such that $p(T^{n(i)}, T^k x) < \varepsilon + \frac{r}{2}$. Choose the smallest integer k with $k \geq n(i)$ such that $p(T^{n(i)}x, T^k x) \geq \varepsilon + \frac{r}{2}$. Then we have,

$$p(T^{n(i)}x, T^{k-1}x) < \varepsilon + \frac{r}{2}. \tag{2.14}$$

From triangle inequality,

$$\begin{aligned} p(T^{n(i)}x, T^k x) &\leq p(T^{n(i)}x, T^{k-1}x) + p(T^{k-1}x, T^k x) - p(T^{k-1}x, T^{k-1}x) \\ &\leq p(T^{n(i)}x, T^{k-1}x) + p(T^{k-1}x, T^k x) \\ &< \varepsilon + \tfrac{r}{2} + \tfrac{r}{4} = \varepsilon + \tfrac{3r}{4}. \end{aligned} \tag{2.15}$$

Then there exists an integer k satisfying $n(i) < k \leq n(i+1)$ such that

$$\varepsilon + \frac{r}{2} \leq p(T^{n(i)}x, T^k x) < \varepsilon + \frac{3r}{4}. \tag{2.16}$$

Employing again (PM4) we obtain

$$\begin{aligned} p(T^{n(i)}x, T^k x) &\leq p(T^{n(i)}x, T^{n(i)+1}x) + p(T^{n(i)+1}x, T^k x) - p(T^{n(i)+1}x, T^{n(i)+1}x) \\ &\leq p(T^{n(i)}x, T^{n(i)+1}x) + p(T^{n(i)+1}x, T^k x) \\ &\leq p(T^{n(i)}x, T^{n(i)+1}x) + p(T^{n(i)+1}x, T^{k+1}x) + p(T^{k+1}x, T^k x). \end{aligned} \tag{2.17}$$

Using the inequalities

$$p(T^{n(i)}x, T^k x) < \varepsilon + \frac{3r}{4},$$
$$p(T^{n(i)}x, T^{n(i)+1}x) = d_{n(i)} < \frac{r}{4},$$
$$p(T^k x, T^{k+1}x) = d_k < \frac{r}{4}$$

we have

$$\begin{aligned} H(T^{n(i)}x, T^k x) &= \tfrac{1}{5}[p(T^{n(i)}x, T^k x) + p(T^{n(i)}x, T^{n(i)+1}x) + p(T^{k+1}x, T^k x) \\ &\quad + p(T^{n(i)}x, T^{k+1}x) + p(T^{n(i)+1}x, T^k x)] \\ &\leq \tfrac{1}{5}[p(T^{n(i)}x, T^k x) + p(T^{n(i)}x, T^{n(i)+1}x) + p(T^{k+1}x, T^k x) \\ &\quad + p(T^{n(i)}x, T^k x) + p(T^k x, T^{k+1}x) - p(T^k x, T^k x) \\ &\quad + p(T^{n(i)+1}x, T^{n(i)}x) + p(T^{n(i)}x, T^k x) - p(T^{n(i)}x, T^{n(i)}x)] \\ &\leq \tfrac{1}{5}[p(T^{n(i)}x, T^k x) + p(T^{n(i)}x, T^{n(i)+1}x) + p(T^{k+1}x, T^k x) \\ &\quad + p(T^{n(i)}x, T^k x) + p(T^k x, T^{k+1}x) \\ &\quad + p(T^{n(i)+1}x, T^{n(i)}x) + p(T^{n(i)}x, T^k x)] \\ &\leq \tfrac{1}{5}[3p(T^{n(i)}x, T^k x) + 2p(T^{n(i)}x, T^{n(i)+1}x) + 2p(T^{k+1}x, T^k x)] \\ &\leq \tfrac{1}{5}[3(\varepsilon + \tfrac{3r}{4}) + 2(\tfrac{r}{4}) + 2(\tfrac{r}{4})] < \varepsilon + r \end{aligned} \tag{2.18}$$

which implies $p(T^{n(i)+1}, T^{k+1}x) < \varepsilon$. But, regarding (2.17) we have

$$\begin{aligned} p(T^{n(i)+1}x, T^{k+1}x) &\geq p(T^{n(i)}x, T^k x) - p(T^{n(i)}x, T^{n(i)+1}x) - p(T^k x, T^{k+1}x) \\ &> \varepsilon + \tfrac{r}{2} - \tfrac{r}{4} - \tfrac{r}{4} = \varepsilon \end{aligned}$$

which contradicts the preceding inequality.

Hence $\{T^n x\}$ is a Cauchy sequence which converges to a limit, say $z \in X$. Moreover,

$$p(z, z) = \lim_{n,m\to\infty} p(T^n x, T^m x) = 0 \tag{2.19}$$

from Proposition 8.

Let us show that $Tz = z$. Assume that $p(Tz, z) > 0$. First we observe that

$$p(T^n x, Tz) < H(T^{n-1}x, z) \text{ if } x_{n-1} \neq z \tag{2.20}$$

where

$$\begin{aligned} H(T^{n-1}x, z) &= \tfrac{1}{5}[p(T^{n-1}x, z) + p(T^n x, T^{n-1}x) + p(Tz, z) \\ &\quad + p(T^n x, z) + p(Tz, T^{n-1}x)]. \end{aligned} \tag{2.21}$$

Taking limit of (2.20) as $n \to \infty$ and making use of Lemma 4, we have

$$p(z, Tz) < \lim_{n\to\infty} H(x_{n-1}, z). \tag{2.22}$$

However, from (2.21),

$$\begin{aligned}\lim_{n\to\infty} H(T^{n-1}x,z) &= \tfrac{1}{5}[p(z,z)+p(z,z)+p(Tz,z)\\ &\quad +p(z,z)+p(Tz,z)]\\ &= \tfrac{1}{5}[3p(z,z)+2p(Tz,z)] = \tfrac{2}{5}p(Tz,z)\end{aligned} \tag{2.23}$$

which contradicts (2.22). Hence $p(Tz,z)=0$. By Lemma 3, $Tz=z$.

Last, we show that z is the unique fixed point of T. Assume that there exists a point $w\in X$ such that $z\neq w$ and $Tw=w$. Due to (2.2),

$$\begin{aligned}p(w,z) &= p(Tw,z)=p(Tz,Tw)<H(z,w)\\ &= \tfrac{1}{5}[p(z,w)+p(Tz,z)+p(Tw,w)+p(Tz,w)+p(z,Tw)]\\ &\leq \tfrac{1}{5}[3p(z,w)+p(w,w)].\end{aligned} \tag{2.24}$$

Therefore,

$$\frac{2}{5}p(z,w)\leq\frac{1}{5}p(w,w), \tag{2.25}$$

which implies $2p(z,w)\leq p(w,w)$. But we always have $p(z,w)\geq p(w,w)$. Hence $2p(w,w)\leq 2p(z,w)\leq p(w,w)$, so $p(w,w)=0$. From (2.25), we deduce $p(z,w)=0$. By Lemma 3, $w=z$. □

Hardy-Rogers type Meir-Keeler contractions form a quite general class of operators. Special type operators from this class can be defined and named after Kannan, Reich and Chattergee respectively.

Definition 10. *Let (X,p) be a partial metric space. Let $T:X\to X$ be a self-mapping satisfying the following:*

(1) *Given $\varepsilon>0$ there exists $\delta>0$ such that*

$$\varepsilon\leq K(x,y)<\varepsilon+\delta\Rightarrow p(Tx,Ty)<\varepsilon, \tag{2.26}$$

where

$$K(x,y)=\frac{1}{2}[p(Tx,x)+p(Ty,y)].$$

Then T is called Kannan type Meir-Keeler contraction.

(2) *Given $\varepsilon>0$ there exists $\delta>0$ such that*

$$\varepsilon\leq R(x,y)<\varepsilon+\delta\Rightarrow p(Tx,Ty)<\varepsilon, \tag{2.27}$$

where

$$R(x,y)=\frac{1}{3}[p(x,y)+p(Tx,x)+p(Ty,y)].$$

Then T is called Reich type Meir-Keeler contraction.

(3) *Given $\varepsilon>0$ there exists $\delta>0$ such that*

$$\varepsilon\leq C(x,y)<\varepsilon+\delta\Rightarrow p(Tx,Ty)<\varepsilon, \tag{2.28}$$

where

$$C(x,y)=\frac{1}{2}[p(x,Ty)+p(y,Tx)].$$

Then T is called Chattergee type Meir-Keeler contraction.

The fixed point theorem 9 is valid also for the operators in Definition 10.

Corollary 11. *Let (X,p) be a complete partial metric space. Suppose that $T:X\to X$ is Meir-Keeler contraction of Kannan type or Riech type or Chattergee type. Then, T has a unique fixed point, say $z\in X$. Moreover, $\lim_{n\to\infty}p(T^nx,z)=p(z,z)$ for all $x\in X$.*

Recently, interest in partial metric spaces increases rapidly because of their extensive applications in computer science. This fact encourages many authors to continue the research in this area.

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q-BERNSTEIN POLYNOMIALS ASSOCIATED WITH q-GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we give the fermionic p-adic integral representation of q-Bernstein polynomials associated with q-Genocchi numbers and polynomials. And we construct some interesting properties of q-Bernstein polynomials associated with q-Genocchi numbers and polynomials.

Key words and Phrases. q-Bernstein polynomials , q-Genocchi numbers and polynomials , p-adic q-integral.

1. Introduction

Let C[0,1] denote the set of continuous functions on $[0,1]$. For$f \in C[0,1]$, Bernstein introduced the following well-known linear positive operator in the field of real numbers $\mathbb{R}$:

(1.1)

$$\mathbb{B}_n(f:x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x), (\text{see [9-11]}).$$

Here, $\mathbb{B}_n(f:x)$ is called the Bernstein operator of order n for f. For $k,n \in \mathbb{Z}_+$, the Bernstein polynomial of degree n is the defined by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \quad (\text{see [9-11]}). \tag{1.2}$$

Let p be a fixed odd prime number.The symbol, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. It is well known that the Genocchi polynomials are defined by

$$t\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad (\text{see [1-8]}). \tag{1.3}$$

In the special case, $x = 0$, and $G_n(0) = G_n$ are called the n-th Genocchi numbers.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < 1$. As the q-extension of (1.3), the q-Genocchi polynomials are defined by

$$t\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}, \quad \text{(see [1-8])}. \tag{1.4}$$

In the special case, $x = 0$, and $G_{n,q}(0) = G_{n,q}$ are called the n-th q-Genocchi numbers.
Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

(1.5)

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad \text{(see [2,6])}.$$

In the special case q=1, the integral

$$I_{-1}(f_1) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x), \quad \text{(see [2,6])}. \tag{1.6}$$

Then, we have the integral equation

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \ \textit{where } f_1(x) = f(x+1).$$

Moreover, for $n \in \mathbb{N}$,let $f_n(x) = f(x+n)$. Then we note that

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{\ell=0}^{n-1}(-1)^{n-1-\ell} f(\ell), \quad \text{(see [2,6])}. \tag{1.7}$$

By using fermionic p-adic integral on $\mathbb{Z}_p$, we see that
(1.8)

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} &= t\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) \\ &= \sum_{m=0}^{\infty}\Big(\frac{2}{(1-q^n)}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{q^{lx}}{1+q^l}\Big)\frac{t^n}{n!}, \quad \text{(see [5,7,8])}.\end{aligned}$$

For $k, n \in \mathbb{Z}_+$,$0 < q < 1$, and $x \in [0,1]$, consider the following q-Bernstein polynomials defined by Kim [11].

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k [1-x]_{\frac{1}{q}}^{n-k}, \quad \text{(see [11])}. \tag{1.9}$$

For example $B_{k,0}(x,q) = B_{k,1}(x,q) =, \ldots = B_{k,k-1}(x,q) = 0$. Thus, for $k, n \in \mathbb{Z}_+$, we note that

$$B_{k,n}(x,q) = \begin{cases} \binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k} & \text{if } n \geq k, \\ 0 & \text{if } n < k. \end{cases} \quad \text{(see [11])} \tag{1.10}$$

By(1.10), we obtain $\lim_{q\longrightarrow 1} B_{k,n}(x,q) = B_{k,n}(x)$.
In this paper, we give the fermionic p-adic integral representation of q-Bernstein polynomials , which are defined by Kim [11], associated with q-Genocchi numbers and polynomials. And we construct some interesting propertics of q-Bernstein polynomials associated with q-Genocchi numbers and polynomials.

2. On the q-Bernstein Polynomials Associated with q-Genocchi Numbers and Polynomials

By (1.4) and (1.8), we see that

$$t\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) = t\sum_{n=0}^{\infty} \frac{G_{n+1,q}(x)}{n+1}\frac{t^n}{n!}. \tag{2.1}$$

We also have

$$t\int_{\mathbb{Z}_p} e^{[1-x+y]_{q^{-1}} t} d\mu_{-1}(y) = t\sum_{n=0}^{\infty} \frac{G_{n+1,q^{-1}}(1-x)}{n+1}\frac{t^n}{n!}. \tag{2.2}$$

From (2.1),(2.2) and (1.8) , we note that $\frac{G_{n+1,q^{-1}}(1-x)}{n+1} = (-1)^n q^n \frac{G_{n+1,q}(x)}{n+1}$.
By(1.7), we have

$$I_n(f(x+n)) = (-1)^n I_n(f(n)) + 2\sum_{\ell=0}^{n-1}(-1)^{n-1+\ell} f(\ell). \tag{2.3}$$

this is well known in [9].

Let $f(x) = [x]_{q^{-1}}^m$, $f_n(x) = f(x+n) = [x+n]_{q^{-1}}^m$, we have

$$\frac{G_{m+1,q^{-1}}(n)}{m+1} = (-1)^n \frac{G_{m+1,q^{-1}}}{m+1} + 2\sum_{\ell=0}^{n-1}(-1)^{n-1-\ell}[\ell]_{q^{-1}}^m. \tag{2.4}$$

In the special case , $n = 2$

$$\frac{G_{m+1,q^{-1}}(2)}{m+1} = \begin{cases} \frac{G_{m+1,q^{-1}}}{m+1} & \text{if } m = 0, \\ 2 + \frac{G_{m+1,q^{-1}}}{m+1} & \text{if } m > 0. \end{cases} \tag{2.5}$$

Theorem 2.1. For $n \in \mathbb{N}$, we have

$$\frac{G_{n+1,q^{-1}}(2)}{n+1} = 2 + \frac{G_{n+1,q^{-1}}}{n+1}. \tag{2.6}$$

By (2.1), (2.2) and (2.6), we see that for $n > 0$,

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n d\mu_{-1}(x) &= (-1)^n q^n \int_{\mathbb{Z}_p} [x-1]_q^n d\mu_{-1}(x) \\ &= \int_{\mathbb{Z}_p} [x+2]_{q^{-1}}^n d\mu_{-1}(x) \\ &= 2 + \int_{\mathbb{Z}_p} [x]_{q^{-1}}^n d\mu_{-1}(x). \end{aligned} \tag{2.7}$$

Therefore, we obtain the following theorem :

Theorem 2.2. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n d\mu_{-1}(x) &= 2 + \frac{G_{n+1,q^{-1}}}{n+1} \\ &= \frac{G_{n+1,q^{-1}}(2)}{n+1}. \end{aligned} \tag{2.8}$$

Theorem 2.2 is important to derive our main result in this paper.
By the the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ the q- Bernstein polynomials in (1.9), we obtain

(2.9)

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x,q) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} \int_{\mathbb{Z}_p} [x]_q^{n-j} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} \frac{G_{n+1-j,q}}{n+1-j}. \end{aligned}$$

Therefore, we obtain the following proposition.

Proposition 2.3 For $k, n \in \mathbb{Z}_+$, we obtain

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q) d\mu_{-1}(x) = \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \frac{G_{k+1+j,q}}{k+1+j}. \tag{2.10}$$

By (1.9), we have this

(2.11)
$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1-x,q^{-1})d\mu_{-1}(x)$$
$$= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n-j}d\mu_{-1}(x).$$

By (2.11) and Theorem 2.2, we see that for $n > k$,

(2.12)
$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_{-1}(x) = \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n-j}d\mu_{-1}(x)$$
$$= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\{2+\int_{\mathbb{Z}_p}[x]_{q^{-1}}^{n-j}d\mu_{-1}(x)\}$$
$$= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\{2+\frac{G_{n+1-j,q^{-1}}}{n+1-j}\}$$
$$= \begin{cases} 2+\frac{G_{n+1,q^{-1}}}{n+1} & \text{if } k=0, \\ \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{G_{n+1-j,q^{-1}}(2)}{n+1-j} & \text{if } k>0. \end{cases}$$

Thus we give the fermionic p-adic integral representation of q-Bernstein polynomials associated with q-Genocchi numbers and polynomials as follow :

Theorem 2.4 For $k, n \in \mathbb{Z}_+$, we obtain

(2.13)
$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_{-1}(x) = \begin{cases} 2+\frac{G_{n+1,q^{-1}}}{n+1} & \text{if } k=0, \\ \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{G_{n+1-j,q^{-1}}(2)}{n+1-j} & \text{if } k>0. \end{cases}$$

By Proposition 2.3 and Theorem 2.4, we obtain the corollary.

Corollary 2.5 For $k, n \in \mathbb{Z}_+$,and $n > k$, we have

$$\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\frac{G_{k+1+j,q}}{k+1+j} = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{G_{n+1-j,q^{-1}}(2)}{n+1-j}. \tag{2.14}$$

For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, fermionic p-adic invariant integral for multiplication of two q-Bernstein polynomials on $\mathbb{Z}_p$ can be given by the following

(2.15)
$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x,q)B_{k,m}(x,q)d\mu_{-1}(x) &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_q^{2k}[1-x]_{q^{-1}}^{n+m-2k}d\mu_{-1}(x)\\ &= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{2k-j}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n+m-j}d\mu_{-1}(x)\\ &= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{2k-j}\{2+\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n+m-j}d\mu_{-1}(x)\}\\ &= \begin{cases} 2+\frac{G_{n+m+1,q^{-1}}}{n+m+1} & \text{if } k=0,\\ \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{2k-j}\frac{G_{n+m-j+1,q^{-1}}(2)}{n+m-j+1} & \text{if } k>0. \end{cases}\end{aligned}$$

Theorem 2.6 For $k, n, m \in \mathbb{Z}_+$, with $m + n > 2k$, we obtain

(2.16)
$$\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_{-1}(x) = \begin{cases} 2+\frac{G_{n+m+1,q^{-1}}}{n+m+1} & \text{if } k=0,\\ \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{2k-j}\frac{G_{n+m-j+1,q^{-1}}(2)}{n+m-j+1} & \text{if } k>0. \end{cases}$$

For $m, n \in \mathbb{Z}_+$, we have

(2.17)
$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_{-1}(x) &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_q^{2k}[1-x]_{q^{-1}}^{n+m-2k}d\mu_{-1}(x)\\ &= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{n+m-2k}\binom{n+m-2k}{j}(-1)^j\int_{\mathbb{Z}_p}[x]_q^{j+2k}d\mu_{-1}(x)\\ &= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{n+m-2k}\binom{n+m-2k}{j}(-1)^j\frac{G_{j+2k+1,q}}{j+2k+1}.\end{aligned}$$

Proposition 2.7 For $k, n \in \mathbb{Z}_+$, we have

(2.18)
$$\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_{-1}(x) = \binom{n}{k}\binom{m}{k}\sum_{j=0}^{n+m-2k}\binom{n+m-2k}{j}(-1)^j\frac{G_{j+2k+1,q}}{j+2k+1}.$$

By Throerm 2.6 and Proposition 2.7 , we obtain the following corollary .

Corollary 2.8 For $k, n \in \mathbb{Z}_+$, with $n+m > 2k$, we obtain

(2.19)
$$\sum_{j=0}^{n+m-2k}\binom{n+m-2k}{j}(-1)^j\frac{G_{j+2k+1,q}}{j+2k+1} = \sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j+2k}\frac{G_{n+m+1-j,q^{-1}}(2)}{n+m+1-j}.$$

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ORTHOGONAL STABILITY OF AN ADDITIVE-QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN SPACES

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of the orthogonally additive-quadratic functional equation

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=\frac{3}{2}f(x)-\frac{1}{2}f(-x)+\frac{1}{2}f(y)+\frac{1}{2}f(-y) \tag{0.1}$$

for all x, y with $x \perp y$, in non-Archimedean Banach spaces. Here $\perp$ is the orthogonality in the sense of Rätz.

1. Introduction and preliminaries

In 1897, Hensel [19] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [12, 27, 28, 35]).

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\leq|r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. Throughout this paper, we assume that the base field is a valued field, hence call it simply a field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\leq\max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

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Definition 1.1. ([34]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x) + f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

G. Pinsker [40] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [51] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \qquad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [18]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [47] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [48] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [47].

Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;

(O_2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;

(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O_4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[3], [7, 14, 24]).

The stability problem of functional equations originated from the following question of Ulam [53]: *Under what condition does there exist an additive mapping near an approximately additive mapping?* In 1941, Hyers [20] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [42] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, $(\varepsilon > 0, p \in [0, 1))$. The result of Rassias has provided a lot of influence in the development of what we now call *generalized Hyers-Ulam stability* or *Hyers-Ulam stability* of functional equations. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [11, 21, 25, 46] and references therein for detailed information on stability of functional equations.

R. Ger and J. Sikorska [17] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all

$x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was F. Skof [50] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. P.W. Cholewa [8] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by S. Czerwik [9] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [10, 39], [43]–[45], [49]).

The orthogonally quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \ x \perp y$$

was first investigated by F. Vajzović [54] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [15], M. Fochi [16], M.S. Moslehian [31, 32] and Gy. Szabó [52] generalized this result. See also [33, 36].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.3. [4, 13] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [22] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 26, 30, 37, 38, 41]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the orthogonally additive-quadratic functional equation (0.1) in non-Archimedean orthogonality spaces for an odd mapping.

In Section 3, we prove the Hyers-Ulam stability of the orthogonally additive-quadratic functional equation (0.1) in non-Archimedean orthogonality spaces for an even mapping.

Throughout this paper, assume that $(X, \perp)$ is a non-Archimedean orthogonality space and that $(Y, \|.\|_Y)$ is a real non-Archimedean Banach space. Assume that $|2| \neq 1$.

2. Stability of the orthogonally additive-quadratic functional equation: an odd mapping case

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally additive-quadratic functional equation

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=\frac{3}{2}f(x)-\frac{1}{2}f(-x)+\frac{1}{2}f(y)+\frac{1}{2}f(-y)$$

for all $x, y \in X$ with $x \perp y$ in non-Archimedean Banach spaces: an odd mapping case.

Definition 2.1. A mapping $f : X \to Y$ is called an *orthogonally additive mapping* if

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq |2|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.1}$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-\frac{3}{2}f(x)+\frac{1}{2}f(-x)-\frac{1}{2}f(y)-\frac{1}{2}f(-y)\right\|_Y \\ \leq \varphi(x, y)\end{aligned} \tag{2.2}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\alpha}{|2| - |2|\alpha}\varphi(x, 0) \tag{2.3}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.2), we get

$$\left\|4f\left(\frac{x}{2}\right)-2f(x)\right\|_Y \leq \varphi(x, 0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|_Y \leq \frac{1}{|4|}\varphi(2x, 0) \leq \frac{\alpha}{|2|} \cdot \varphi(x, 0) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq \mu\varphi(x,0),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [29]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \varphi(x,0)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\|_Y \leq \alpha\varphi(x,0)$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that $d(f, Jf) \leq \frac{\alpha}{|2|}$.

By Theorem 1.3, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$L(2x) = 2L(x) \tag{2.6}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set

$$M = \{g \in S : d(h,g) < \infty\}.$$

This implies that L is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - L(x)\|_Y \quad \leq \quad \mu\varphi(x,0)$$

for all $x \in X$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = L(x)$$

for all $x \in X$;

(3) $d(f,L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f,L) \leq \frac{\alpha}{|2| - |2|\alpha}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\left\|2L\left(\frac{x+y}{2}\right)+2L\left(\frac{x-y}{2}\right)-\frac{3}{2}L(x)+\frac{1}{2}L(-x)-\frac{1}{2}L(y)-\frac{1}{2}L(-y)\right\|_Y\\
&=\lim_{n\to\infty}\frac{1}{|2|^n}\Big\|2f(2^{n-1}(x+y))+2f(2^{n-1}(x-y))\\
&\qquad\qquad-\frac{3}{2}f(2^nx)+\frac{1}{2}f(-2^nx)-\frac{1}{2}f(2^ny)-\frac{1}{2}f(-2^ny)\Big\|_Y\\
&\leq\lim_{n\to\infty}\frac{1}{|2|^n}\varphi(2^nx,2^ny)\leq\lim_{n\to\infty}\frac{|2|^n\alpha^n}{|2|^n}\varphi(x,y)=0
\end{aligned}$$

for all $x,y\in X$ with $x\perp y$. So

$$2L\left(\frac{x+y}{2}\right)+2L\left(\frac{x-y}{2}\right)-\frac{3}{2}L(x)+\frac{1}{2}L(-x)-\frac{1}{2}L(y)-\frac{1}{2}L(-y)=0$$

for all $x,y\in X$ with $x\perp y$. Since f is odd, L is odd. Hence $L:X\to Y$ is an orthogonally additive mapping, i.e.,

$$2L\left(\frac{x+y}{2}\right)+2L\left(\frac{x-y}{2}\right)=2L(x)$$

for all $x,y\in X$ with $x\perp y$. Thus $L:X\to Y$ is a unique orthogonally additive mapping satisfying (2.3), as desired. □

From now on, in corollaries, assume that $(X,\perp)$ is a non-Archimedean orthogonality normed space.

Corollary 2.3. *Let θ be a positive real number and p a real number with $0<p<1$. Let $f:X\to Y$ be an odd mapping satisfying*

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-\frac{3}{2}f(x)+\frac{1}{2}f(-x)-\frac{1}{2}f(y)-\frac{1}{2}f(-y)\right\|_Y\\
&\qquad\leq\theta(\|x\|^p+\|y\|^p)
\end{aligned}\tag{2.7}$$

for all $x,y\in X$ with $x\perp y$. Then there exists a unique orthogonally additive mapping $L:X\to Y$ such that

$$\|f(x)-L(x)\|_Y\leq\frac{\theta}{|2|^p-|2|}\|x\|^p$$

for all $x\in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y)=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$ with $x\perp y$. Then we can choose $\alpha=|2|^{1-p}$ and we get the desired result. □

Theorem 2.4. *Let $f:X\to Y$ be an odd mapping satisfying* (2.2) *for which there exists a function $\varphi:X^2\to[0,\infty)$ such that*

$$\varphi(x,y)\leq\frac{\alpha}{|2|}\varphi\left(2x,2y\right)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{1}{|2| - |2|\alpha}\varphi(x, 0) \tag{2.8}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $d(f, Jf) \leq \frac{1}{|2|}$. So

$$d(f, L) \leq \frac{1}{|2| - |2|\alpha}.$$

Thus we obtain the inequality (2.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.7)*. Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{\theta}{|2| - |2|^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{p-1}$ and we get the desired result. □

3. Stability of the orthogonally additive-quadratic functional equation: an even mapping case

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally additive-quadratic functional equation given in the previous section: an even mapping case.

Definition 3.1. A mapping $f : X \to Y$ is called an *orthogonally quadratic mapping* if

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq |4|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{\alpha}{1-\alpha}\varphi(x, 0) \tag{3.1}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.2), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \varphi(x, 0) \tag{3.2}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x) - \frac{1}{4}f(2x)\right\|_Y \leq \frac{1}{|4|}\varphi(2x, 0) \leq \alpha \cdot \varphi(x, 0) \tag{3.3}$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can obtain an orthogonally quadratic mapping $Q : X \to Y$ defined by

$$\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right) = Q(x)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g\left(2x\right)$$

for all $x \in X$.

It follows from (3.3) that $d(f, Jf) \leq \alpha$. So

$$d(f, Q) \leq \frac{\alpha}{1-\alpha}.$$

So we obtain the inequality (3.1). Thus $Q : X \to Y$ is a unique orthogonally quadratic mapping satisfying (3.1), as desired. □

Corollary 3.3. *Let θ be a positive real number and p a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|4|\theta}{|2|^p - |4|}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{2-p}$ and we get the desired result. □

Theorem 3.4. *Let $f : X \to Y$ be an even mapping satisfying (2.2) and $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\varphi(x, y) \leq \frac{\alpha}{|4|}\varphi\left(2x, 2y\right)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{1-\alpha}\varphi(x, 0) \tag{3.4}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.2) that $d(f, Jf) \leq 1$. So we obtain the inequality (3.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Corollary 3.5. *Let θ be a positive real number and p a real number with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|4|\theta}{|4| - |2|^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{p-2}$ and we get the desired result. □

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_o is an odd mapping and f_e is an even mapping such that $f = f_o + f_e$.

The above corollaries can be summarized as follows:

Theorem 3.6. *Assume that $(X, \perp)$ is a non-Archimedean orthogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 1$ (resp. $p > 2$). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exist an orthogonally additive mapping $L : X \to Y$ and an orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\begin{aligned} \|f(x) - L(x) - Q(x)\|_Y &\leq \left(\frac{1}{|2|^p - |2|} + \frac{|4|}{|2|^p - |4|}\right)\theta\|x\|^p \\ (\textit{resp.} \quad \|f(x) - L(x) - Q(x)\|_Y &\leq \left(\frac{1}{|2| - |2|^p} + \frac{|4|}{|4| - |2|^p}\right)\theta\|x\|^p) \end{aligned}$$

for all $x \in X$.

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Stability of a general mixed additive-cubic equation in F-spaces

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Abstract In this paper, we determine some stability results concerning the general mixed additive-cubic equation

$$f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2f(kx)-2kf(x)$$

in F-spaces, and generalized related results in reference in different aspects.

Keywords F-space, Hyers-Ulam stability, Additive function, Cubic function, Mixed additive-cubic equation.
MR(2000) Subject Classification 39B82, 44B52

1. Introduction and preliminaries

The study of stability problems for functional equations is related to a question of Ulam [33] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [12]. The result of Hyers was generalized by Aoki [2] for approximate additive mappings and by Rassias [19] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x,y)=f(x+y)-[f(x)+f(y)]$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [6], who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$. During the last decades several stability problems for various functional equations have been investigated by many mathematicians, we refer the reader to the monographs [1, 3, 4, 7-11, 13-18, 20-32].

Rassias [20](in 2001) introduced the pioneering cubic functional equation:

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y) \tag{1.1}$$

and established the solution of the Ulam stability problem for this cubic functional equation. The function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function.

Jun and Kim [13](in 2002) introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

and they established the general solution and the generalized Ulam-Hyers stability for the functional equation (1.2). They proved that a function f between real vector spaces X and Y is a solution of (1.2) if and only if

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there exits a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables.

Sadeghi and Moslehian [25] proved the generalized stability of the cubic type functional equation (1.2) and another functional equation

$$f(kx + y) + f(x + ky) = (k + 1)(k - 1)^2[f(x) + f(y)] + k(k + 1)f(x + y), \tag{1.3}$$

where k is an integer with $k \neq 0, \pm 1$ in the framework of non-Archimedean normed spaces.

Gordji and Khodaei [8] achieved the general solution and the generalized Hyers-Ulam stability of the following generalized mixed type cubic, quadratic and additive functional equation functional equation

$$f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(x) \tag{1.4}$$

for fixed integers k with $k \neq 0, \pm 1$ in the quasi-Banach spaces. Gordji [7] proved the Hyers-Ulam stability of an additive-quadratic functional equation of two variables in F-spaces.

In this paper, we consider a mapping $f : X \to Y$ with $f(0) = 0$ and satisfies the following functional equation, which is introduced by the authors,

$$f(kx + y) + f(kx - y) = kf(x + y) + kf(x - y) + 2f(kx) - 2kf(x), \tag{1.5}$$

for a fixed integer with $k \geq 2$ (see [32]). It is easy to see that the function $f(x) = ax^3 + bx$ is a solution of the functional equation (1.5), which is called a general mixed additive-cubic functional equation. We observe that in case $k = 2$, Eq.(1.5) yields mixed additive-cubic equation $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 2f(2x) - 4f(x)$ and there are many interesting results concerning the stability problems of the mixed additive-cubic equation (see [16]). Therefore, Eq.(1.5) is a generalized form of the mixed additive-cubic equation.

The main purpose of this paper is to establish the generalized Ulam-Hyers stability of the general mixed additive-cubic equation (1.5) in F-spaces.

First we recall the following well known concept concerning F-spaces and Fréchet spaces.

Definition 1.1. A linear space $(X, +, \cdot)$ over $\mathbb{R}$ or $\mathbb{C}$ is called F-space if X is a metrizable, complete, topological vector space; and a locally convex F-space is called Fréchet space.

Every Banach space is an F-space as the norm induces an invariant metric and the space is complete with respect to this metric. The L^p spaces are F-spaces for all $p > 0$ and for $p \geq 1$ they are locally convex and thus Fréchet spaces and even Banach spaces. So for instance, $L^p([0, 1])$ for $0 < p < 1$ equipped with the metric $d(f, g) = \int |f(x) - g(x)|^p dx$ is an F-space, which is not a Banach space.

Now suppose that X is an F-space with an invariant metric d, and write $\triangle(x)$ in place of $d(x, 0)$, then it is easy to see that (1) $\triangle(x + y) \leq \triangle(x) + \triangle(y)$, (2) $\triangle(-x) = \triangle(x)$, (3) $\triangle(x) = 0$ if and only if $x = 0$, (4) if $\{x_n\}$ is a sequence in X with $\triangle(x_n - x) \to 0$ and if $\{\alpha_n\}$ is a sequence of scalars with $\alpha_n \to \alpha$, then $\triangle(\alpha_n x_n - \alpha x) \to 0$. There is nothing to guarantee that $\triangle(\alpha x) = |\alpha|\triangle(x)$, and so $\triangle(x)$ may not be a norm. Let $k \geq 2$ be a fixed integer and denote $\triangle_k(x) = \triangle((k - 1)(k^3 - k)x)$ for all $x \in X$. One can easily verify that $x \to \triangle_k(x)$ is a real-valued function on X with properties (1) to (4). Set $d_k(x, y) = \triangle_k(x - y)$ for all $x, y \in X$. Then d_k is an invariant metric on X and, with this metric, X is an F-space.

Lemma 1.2. *Let X be an F-space with a metric d, then $d(nx, ny) \leq nd(x, y)$ for all $x, y \in X$ and all $n \in \mathbb{N}$.*

Proof. For $x, y \in X$, we have

$$d(2x, 2y) = d(x, 2y - x) \leq d(x, y) + d(y, 2y - x) = 2d(x, y).$$

The remaining assertion goes through by mathematical induction. □

Remark 1.3. It is a classical fact that the Fréchet spaces are characterized by the existence of a countable, sufficient and increasing family of semi-norms $\{p_i\}_{i\in\mathbb{N}}$ (that is $p_i(x) = 0$ implies $x = 0$ and $p_i(x) \leq p_{i+1}(x)$ for all $x \in X$ and $i \in \mathbb{N}$), which define the pseudo-norm

$$\triangle(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{p_i(x)}{1 + p_i(x)}$$

and the metric $d(x,y) = \triangle(x-y)$ invariant with respect to translations, such that d generates a complete topology equivalent to that of locally convex space. Also, notice that since $\frac{p_i(x)}{1+p_i(x)} \leq 1$ and $\sum_{i=0}^{\infty} \frac{1}{2^i} = 1$, it follows that $\triangle(x) \leq 1$ for all $x \in X$ (see [5]).

Moreover, d has the properties given by the following.

Theorem 1.4([5]). *Let $(X, \{p_i\}_{i\in\mathbb{N}}, d)$ be a Fréchet space, then*
(1) $d(cx,cy) \leq d(x,y)$ *for* $x,y \in X$, $|c| < 1$;
(2) $d(x+u, y+v) \leq d(x,y) + d(u,v)$ *for* $x,y,u,v \in X$;
(3) $d(kx,ky) \leq d(rx,ry)$ *if* $k,r \in \mathbb{R}$, $0 < k \leq r$;
(4) $d(kx,ky) \leq kd(x,y)$ *for* $x,y \in X$, $k \in \mathbb{N}, k \geq 2$;
(5) $d(cx,cy) \leq (|c|+1)d(x,y)$ *for all* $x,y \in X$ *and* $c \in \mathbb{R}$.

These facts are useful in Section 2. Everywhere in the rest of the paper, $(X, (p_i)_{i\in\mathbb{N}}, d)$ will be a Fréchet space with $(p_i)_{i\in\mathbb{N}}$ and d as in Remark 1.3.

2. Generalized Ulam-Hyers stability of general mixed additive-cubic equations

We now investigate the generalized Ulam-Hyers stability problem for functional equations (1.5). From now on, let X be a real vector space, let Y be a real F-space with a metric d, and let $k \geq 2$ be a fixed integer. Denote $\triangle(u) = d(u,0)$ and $\triangle_k(u) = \triangle((k-1)(k^3-k)u)$ for all $u \in Y$. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$Df(x,y) := f(kx+y) + f(kx-y) - kf(x+y) - kf(x-y) - 2f(kx) + 2kf(x)$$

for all $x,y \in X$.

We start with the following lemmas.

Lemma 2.1. *Let $f : X \to Y$ be a function with $f(0) = 0$ for which there is a function $\varphi : X \times X \to [0,\infty)$ such that*

$$\triangle(Df(x,y)) \leq \varphi(x,y) \tag{2.1}$$

for all $x,y \in X$, then

$$\triangle_k(f(4x) - 10f(2x) + 16f(x)) \leq \tilde{\varphi}(x) \tag{2.2}$$

for all $x \in X$, where

$$\begin{aligned}\tilde{\varphi}(x) := (k-1)\{&(k+1)[\varphi(x,(2k+1)x) + \varphi(x,(2k-1)x)] + \varphi(3x,x) + (1+8k^2)\varphi(x,x) + \\ &\varphi(x,3kx) + \varphi(x,kx) + k^2\varphi(2x,2x) + \varphi(2x,2kx) + 2\varphi(x,(k+1)x) + 2\varphi(x,(k-1)x) + \\ &2\varphi(2x,x) + 2\varphi(2x,kx) + 8\varphi(\tfrac{x}{2},\tfrac{kx}{2}) + 8k\varphi(\tfrac{x}{2},\tfrac{(2k-1)x}{2}) + 8k\varphi(\tfrac{x}{2},\tfrac{(2k+1)x}{2}) + 8\varphi(\tfrac{x}{2},\tfrac{3kx}{2})\} + \\ &(k+1)\varphi(0,(k+1)x) + (8k^2+1)\varphi(0,(k-1)x) + 2\varphi(0,x) + k\varphi(0,(3k-1)x) + \\ &k^2\varphi(0,2(k-1)x) + (k^2+k-1)\varphi(0,2kx) + 8k\varphi(0,\tfrac{(3k-1)x}{2}) + 8k\varphi(0,\tfrac{(k+1)x}{2}) \\ &+(8k^2+2k-8)\varphi(0,kx).\end{aligned} \tag{2.3}$$

Proof. Letting $x = 0$ in (2.1), we get

$$\triangle((k-1)[f(y) + f(-y)]) \leq \varphi(0,y) \tag{2.4}$$

for all $y \in X$. Putting $y = x$ in (2.1), we have

$$\triangle(f((k+1)x) + f((k-1)x) - kf(2x) - 2f(kx) + 2kf(x)) \leq \varphi(x,x) \tag{2.5}$$

for all $x \in X$. Hence

$$\triangle(f(2(k+1)x) + f(2(k-1)x) - kf(4x) - 2f(2kx) + 2kf(2x)) \leq \varphi(2x,2x) \tag{2.6}$$

for all $x \in X$. Letting $y = kx$ in (2.1), we get

$$\triangle(f(2kx) - kf((k+1)x) - kf(-(k-1)x) - 2f(kx) + 2kf(x)) \leq \varphi(x,kx) \tag{2.7}$$

for all $x \in X$. Letting $y = (k+1)x$ in (2.1), we have

$$\triangle(f((2k+1)x) + f(-x) - kf((k+2)x) - kf(-kx) - 2f(kx) + 2kf(x)) \leq \varphi(x,(k+1)x) \tag{2.8}$$

for all $x \in X$. Letting $y = (k-1)x$ in (2.1), we have

$$\triangle(f((2k-1)x) - (k+2)f(kx) - kf(-(k-2)x) + (2k+1)f(x)) \leq \varphi(x,(k-1)x) \tag{2.9}$$

for all $x \in X$. Replacing x and y by $2x$ and x in (2.1), respectively, we get

$$\triangle(f((2k+1)x) + f((2k-1)x) - 2f(2kx) - kf(3x) + 2kf(2x) - kf(x)) \leq \varphi(2x,x) \tag{2.10}$$

for all $x \in X$. Replacing x and y by $3x$ and x in (2.1), respectively, we get

$$\triangle(f((3k+1)x) + f((3k-1)x) - 2f(3kx) - kf(4x) - kf(2x) + 2kf(3x)) \leq \varphi(3x,x) \tag{2.11}$$

for all $x \in X$. Replacing x and y by $2x$ and kx in (2.1), respectively, we have

$$\triangle(f(3kx) + f(kx) - kf((k+2)x) - kf(-(k-2)x) - 2f(2kx) + 2kf(2x)) \leq \varphi(2x,kx) \tag{2.12}$$

for all $x \in X$. Setting $y = (2k+1)x$ in (2.1), we have

$$\triangle(f((3k+1)x) + f(-(k+1)x) - kf(2(k+1)x) - kf(-2kx) - 2f(kx) + 2kf(x)) \leq \varphi(x,(2k+1)x) \tag{2.13}$$

for all $x \in X$. Letting $y = (2k-1)x$ in (2.1), we have

$$\triangle(f((3k-1)x) + f(-(k-1)x) - kf(-2(k-1)x) - kf(2kx) - 2f(kx) + 2kf(x)) \leq \varphi(x,(2k-1)x) \tag{2.14}$$

for all $x \in X$. Letting $y = 3kx$ in (2.1), we have

$$\triangle(f(4kx) + f(-2kx) - kf((3k+1)x) - kf(-(3k-1)x) - 2f(kx) + 2kf(x)) \leq \varphi(x,3kx) \tag{2.15}$$

for all $x \in X$. By (2.4), (2.5), (2.11), (2.13) and (2.14), we get

$$\begin{aligned}&\triangle((k-1)[kf(2(k+1)x) + kf(-2(k-1)x) + 6f(kx) - 2f(3kx) - kf(4x) + 2kf(3x) - 6kf(x)])\\ &\quad\leq (k-1)[\varphi(x,(2k+1)x) + \varphi(x,(2k-1)x) + \varphi(3x,x) + \varphi(x,x)]\\ &\qquad+\varphi(0,(k+1)x) + \varphi(0,(k-1)x) + k\varphi(0,2kx)\end{aligned} \tag{2.16}$$

for all $x \in X$. By (2.4), (2.8) and (2.9), we have

$$\begin{aligned}&\triangle((k-1)[f((2k+1)x) + f((2k-1)x) - kf((k+2)x) - kf(-(k-2)x) - 4f(kx) + 4kf(x)])\\ &\quad\leq (k-1)[\varphi(x,(k+1)x) + \varphi(x,(k-1)x)] + \varphi(0,x) + k\varphi(0,kx)\end{aligned} \tag{2.17}$$

for all $x \in X$. It follows from (2.10) and (2.17) that

$$\begin{aligned}&\triangle((k-1)[kf((k+2)x) + kf(-(k-2)x) - 2f(2kx) + 4f(kx) - kf(3x) + 2kf(2x) - 5kf(x)])\\ &\leq (k-1)[\varphi(x,(k+1)x) + \varphi(x,(k-1)x) + \varphi(2x,x)] + \varphi(0,x) + k\varphi(0,kx)\end{aligned} \tag{2.18}$$

for all $x \in X$. By (2.12) and (2.18), we have

$$\begin{aligned}&\triangle((k-1)[f(3kx) - 4f(2kx) + 5f(kx) - kf(3x) + 4kf(2x) - 5kf(x)])\\ &\quad\leq (k-1)[\varphi(x,(k+1)x) + \varphi(x,(k-1)x) + \varphi(2x,x) + \varphi(2x,kx)]\\ &\qquad+\varphi(0,x) + k\varphi(0,kx)\end{aligned} \tag{2.19}$$

for all $x \in X$. By (2.4), (2.13), (2.14) and (2.15), we have

$$\begin{aligned}&\triangle((k-1)[kf(-(k+1)x) - kf(-(k-1)x) - k^2f(2(k+1)x) + k^2f(-2(k-1)x)\\ &+k^2f(2kx) - (k^2-1)f(-2kx) + f(4kx) - 2f(kx) + 2kf(x)])\\ &\leq (k-1)[k\varphi(x,(2k+1)x) + k\varphi(x,(2k-1)x) + \varphi(x,3kx)] + k\varphi(0,(3k-1)x)\end{aligned} \tag{2.20}$$

for all $x \in X$. It follows from (2.4), (2.6), (2.7) and (2.20) that

$$\begin{aligned}&\triangle((k-1)[f(4kx)-2f(2kx)-k^3f(4x)+2k^3f(2x)])\\&\leq (k-1)[k\varphi(x,(2k+1)x)+k\varphi(x,(2k-1)x)+\varphi(x,3kx)+\varphi(x,kx)+k^2\varphi(2x,2x)]\\&+k\varphi(0,(3k-1)x)+k\varphi(0,(k+1)x)+k^2\varphi(0,2(k-1)x)+(k^2-1)\varphi(0,2kx)\end{aligned} \tag{2.21}$$

for all $x \in X$. Hence

$$\begin{aligned}&\triangle((k-1)[f(2kx)-2f(kx)-k^3f(2x)+2k^3f(x)])\\&\leq (k-1)[k\varphi(\tfrac{x}{2},\tfrac{(2k+1)x}{2})+k\varphi(\tfrac{x}{2},\tfrac{(2k-1)x}{2})+\varphi(\tfrac{x}{2},\tfrac{3kx}{2})+\varphi(\tfrac{x}{2},\tfrac{kx}{2})+k^2\varphi(x,x)]\\&+k\varphi(0,\tfrac{(3k-1)x}{2})+k\varphi(0,\tfrac{(k+1)x}{2})+k^2\varphi(0,(k-1)x)+(k^2-1)\varphi(0,kx)\end{aligned} \tag{2.22}$$

for all $x \in X$. By (2.7), we have

$$\triangle(f(4kx)-kf(2(k+1)x)-kf(-2(k-1)x)-2f(2kx)+2kf(2x)) \leq \varphi(2x,2kx) \tag{2.23}$$

for all $x \in X$. From (2.21) and (2.23), we have

$$\begin{aligned}&\triangle((k-1)[kf(2(k+1)x)+kf(-2(k-1)x)-k^3f(4x)+(2k^3-2k)f(2x)])\\&\leq (k-1)[k\varphi(x,(2k+1)x)+k\varphi(x,(2k-1)x)+\varphi(x,3kx)\\&+\varphi(x,kx)+k^2\varphi(2x,2x)+\varphi(2x,2kx)]+k\varphi(0,(3k-1)x)\\&+k\varphi(0,(k+1)x)+k^2\varphi(0,2(k-1)x)+(k^2-1)\varphi(0,2kx)\end{aligned} \tag{2.24}$$

for all $x \in X$. Also, from (2.16) and (2.24), we get

$$\begin{aligned}&\triangle((k-1)[2f(3kx)-6f(kx)+(k-k^3)f(4x)-2kf(3x)+(2k^3-2k)f(2x)+6kf(x)])\\&\leq (k-1)\{(1+k)[\varphi(x,(2k+1)x)+\varphi(x,(2k-1)x)]+\varphi(3x,x)\\&+\varphi(x,x)+\varphi(x,3kx)+\varphi(x,kx)+k^2\varphi(2x,2x)+\varphi(2x,2kx)\}\\&+(k+1)\varphi(0,(k+1)x)+\varphi(0,(k-1)x)+(k^2+k-1)\varphi(0,2kx)\\&+k\varphi(0,(3k-1)x)+k^2\varphi(0,2(k-1)x)\end{aligned} \tag{2.25}$$

for all $x \in X$.

On the other hand, it follows from (2.19) and (2.25) that

$$\begin{aligned}&\triangle((k-1)[8f(2kx)-16f(kx)+(k-k^3)f(4x)+(2k^3-10k)f(2x)+16kf(x)])\\&\leq (k-1)\{(1+k)[\varphi(x,(2k+1)x)+\varphi(x,(2k-1)x)]+\varphi(3x,x)\\&+\varphi(x,x)+\varphi(x,3kx)+\varphi(x,kx)+k^2\varphi(2x,2x)+\varphi(2x,2kx)\\&+2\varphi(x,(k+1)x)+2\varphi(x,(k-1)x)+2\varphi(2x,x)+2\varphi(2x,kx)\}\\&+2\varphi(0,x)+2k\varphi(0,kx)+(k+1)\varphi(0,(k+1)x)+\varphi(0,(k-1)x)\\&+(k^2+k-1)\varphi(0,2kx)+k\varphi(0,(3k-1)x)+k^2\varphi(0,2(k-1)x)\end{aligned} \tag{2.26}$$

for all $x \in X$. Therefore by (2.22) and (2.26), we get (2.2). □

Theorem 2.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X \times X \to [0,\infty)$ such that*

$$\sum_{i=1}^{\infty} 2^i \varphi(\frac{x}{2^i},\frac{y}{2^i}) < \infty \tag{2.27}$$

and

$$\triangle(Df(x,y)) \leq \varphi(x,y) \tag{2.28}$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\triangle_k(f(2x)-8f(x)-A(x)) \leq \frac{1}{2}\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.29}$$

for all $x \in X$, where $\tilde{\varphi}(x)$ is defined as in Lemma 2.2.

Proof. It follows from (2.27) that

$$\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) < \infty \tag{2.30}$$

for all $x \in X$. Let $g : X \to Y$ be the mapping defined by $g(x) := f(2x) - 8f(x)$. From Lemma 1.2 and the inequality (2.2) we use iterative methods and induction on n to prove the following relation:

$$\triangle_k(2^n g(\frac{x}{2^n}) - g(x)) \leq \frac{1}{2} \sum_{i=1}^{n} 2^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.31}$$

for all $x \in X$. By Lemma 1.2 and replacing x by $\frac{x}{2^m}$ to obtain that

$$\triangle_k(2^{n+m} g(\frac{x}{2^{n+m}}) - 2^m g(\frac{x}{2^m})) \leq \frac{1}{2} \sum_{i=m+1}^{m+n} 2^i \tilde{\varphi}(\frac{x}{2^i}). \tag{2.32}$$

This shows that the sequence $\{2^n g(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x \in X$ by taking the limit $m \to \infty$. Since Y is complete, the sequence $\{2^n g(\frac{x}{2^n})\}$ converges for all $x \in X$. Therefore we see that a mapping $A : X \to Y$ defined by

$$A(x) := \lim_{n\to\infty} 2^n g(\frac{x}{2^n}) \tag{2.33}$$

is well defined for all $x \in X$. Taking the limit as $n \to \infty$ in (2.31), we get

$$\triangle_k(g(x) - A(x)) \leq \frac{1}{2} \sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.34}$$

for all $x \in X$. This means that (2.29) holds. It is easy to see that $A(2x) = 2A(x)$ for all $x \in X$. Also, by (2.28), (2.30) and (2.33), we get

$$\begin{aligned} \triangle_k(DA(x,y)) &= \lim_{n\to\infty} \triangle_k(2^n Dg(\tfrac{x}{2^n}, \tfrac{y}{2^n})) \\ &= \lim_{n\to\infty} \triangle_k(2^n [Df(\tfrac{x}{2^{n-1}}, \tfrac{y}{2^{n-1}}) - 8Df(\tfrac{x}{2^n}, \tfrac{y}{2^n})]) \\ &\leq \lim_{n\to\infty} 2^n [\triangle_k(Df(\tfrac{x}{2^{n-1}}, \tfrac{y}{2^{n-1}})) + 8\triangle_k(Df(\tfrac{x}{2^n}, \tfrac{y}{2^n}))] \\ &\leq \lim_{n\to\infty} (k-1)(k^3-k) 2^n [\varphi(\tfrac{x}{2^{n-1}}, \tfrac{y}{2^{n-1}}) + 8\varphi(\tfrac{x}{2^n}, \tfrac{y}{2^n})] = 0 \end{aligned} \tag{2.35}$$

for all $x, y \in X$. Hence the mapping A satisfies (1.5). By [32, Lemma 2.3], the mapping $x \to A(2x) - 8A(x)$ is additive. Therefore $A(2x) = 2A(x)$ implies that the mapping A is additive. It remains to show that A is unique. Let $T : X \to Y$ be another additive mapping satisfying (2.29). Since $2^n A(\frac{x}{2^n}) = A(x)$ and $2^n T(\frac{x}{2^n}) = T(x)$ for all $x \in X$, we conclude that

$$\begin{aligned} \triangle_k(A(x) - T(x)) &= \triangle_k(2^n A(\tfrac{x}{2^n}) - 2^n T(\tfrac{x}{2^n})) \\ &= \triangle_k(2^n A(\tfrac{x}{2^n}) - 2^n g(\tfrac{x}{2^n}) + 2^n g(\tfrac{x}{2^n}) - 2^n T(\tfrac{x}{2^n})) \\ &\leq 2^n \triangle_k(A(\tfrac{x}{2^n}) - g(\tfrac{x}{2^n})) + 2^n \triangle_k(g(\tfrac{x}{2^n}) - T(\tfrac{x}{2^n})) \\ &\leq \sum_{i=n+1}^{\infty} 2^i \tilde{\varphi}(\tfrac{x}{2^i}) \end{aligned} \tag{2.36}$$

for all $x \in X$. By letting $n \to \infty$ in (2.36), it follows that $A(x) = T(x)$ for all $x \in X$. So $A = T$. □

Theorem 2.3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X \times X \to [0, \infty)$ such that*

$$\sum_{i=1}^{\infty} 8^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \tag{2.37}$$

and

$$\triangle(Df(x,y)) \leq \varphi(x,y) \tag{2.38}$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \to Y$ such that

$$\triangle_k(f(2x) - 2f(x) - C(x)) \leq \frac{1}{8}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.39}$$

for all $x \in X$, where $\tilde{\varphi}(x)$ is defined as in Lemma 2.1.

Proof. The techniques are similar to that of Theorem 2.2. Hence we present a sketch of the proof. It follows from (2.37) that

$$\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i}) < \infty \tag{2.40}$$

for all $x \in X$. Let $h : X \to Y$ be the mapping defined by $h(x) := f(2x) - 2f(x)$. For each $x \in X$, $n \geq 0$ and $m \geq 0$, we can deduce

$$\triangle_k(8^{n+m}h(\frac{x}{2^{n+m}}) - 8^m h(\frac{x}{2^m})) \leq \frac{1}{8}\sum_{i=m+1}^{m+n} 8^i \tilde{\varphi}(\frac{x}{2^i}). \tag{2.41}$$

Thus, $\{8^n h(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x \in X$. Therefore, there is a function $C : X \to Y$ defined by

$$C(x) := \lim_{n\to\infty} 8^n h(\frac{x}{2^n}). \tag{2.42}$$

(2.41) with $m = 0$ implies

$$\triangle_k(f(2x) - 2f(x) - C(x)) \leq \frac{1}{8}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.43}$$

for all $x \in X$.

The remaining assertion goes through by the similar way to corresponding part of Theorem 2.2. □

Corollary 2.4. *Let $r > 3, \theta > 0$, and let X, Y be a normed space and a Banach space, respectively. If $f : X \to Y$ is a function with $f(0) = 0$ such that*

$$\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r) \tag{2.44}$$

for all $x, y \in X$, then there exists a unique cubic function $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{\theta\mu}{6(k-1)(k^3-k)(2^r-8)}\|x\|^r \tag{2.45}$$

for all $x \in X$, where

$$\begin{aligned}\mu = {} & (k-1)\{(1+k+2^{3-r}k)[2+(2k+1)^r+(2k-1)^r]+9+3^r+3k^r+16k^2+3^r k^r+2^{r+1}k^2 \\ & +2^r(5+k^r)+2(k+1)^r+2(k-1)^r+2^{3-r}(2+k^r+3^r k^r)\}+(k+1)^{r+1}+2^{3-r}k(k+1)^r \\ & +(1+8k^2+2^r k^2)(k-1)^r+2^r k^r(k^2+k-1)+2+k(2^{3-r}+1)(3k-1)^r+[2k+8(k^2-1)]k^r.\end{aligned} \tag{2.46}$$

Proof. The result follows from Theorem 2.3 by taking $\triangle'_k(u) = \|6(k-1)(k^3-k)u\|$ for all $u \in Y$, and $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. □

The functional equation (1.5) is not stable for $r = 3, k = 2$ in (2.44) as we shall see in the following example.

Example 2.5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} \alpha x^3, & |x| < 1, \\ \alpha, & \text{otherwise}, \end{cases} \tag{2.47}$$

where $\mu > 0$ is a constant, and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{8^n} \tag{2.48}$$

for all $x \in \mathbb{R}$. Then f satisfies the functional inequality

$$|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-2f(2x)+4f(x)| \leq \frac{12\cdot 8^3\alpha}{7}(|x|^3+|y|^3) \tag{2.49}$$

for all $x, y \in \mathbb{R}$, but there do not exists a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(2x)-2f(x)-C(x)| \leq \beta|x|^3 \quad \text{for all } x \in \mathbb{R}. \tag{2.50}$$

Proof. Evidently, $|\phi(x)| \leq \alpha$ for all $x \in \mathbb{R}$. Therefore, f is bounded by $\frac{8\alpha}{7}$ on $\mathbb{R}$. If $|x|^3+|y|^3 \geq \frac{1}{8}$ or 0, then the left side of (2.49) is less than $\frac{96\alpha}{7}$, and hence (2.49) is true. Now suppose that $0 < |x|^3+|y|^3 < \frac{1}{8}$. Then there exists a positive integer m such that

$$\frac{1}{8^{m+2}} \leq |x|^3+|y|^3 < \frac{1}{8^{m+1}}, \tag{2.51}$$

so that $8^m|x|^3 < \frac{1}{8}, 8^m|y|^3 < \frac{1}{8}$ and $2^{m-1}(2x+y), 2^{m-1}(2x-y), 2^{m-1}(x+y), 2^{m-1}(x-y), 2^{m-1}x$ all belong to the interval $(-1,1)$. Thus, for $n = 0, 1, \cdots, m-1$,

$$\phi(2^n(2x+y))+\phi(2^n(2x-y))-2\phi(2^n(x+y))-2\phi(2^n(x-y))-2\phi(2^n\cdot 2x)+4\phi(2^nx) = 0. \tag{2.52}$$

By the definition of f and (2.51), we obtain that

$$\begin{aligned}
&|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-2f(2x)+4f(x)| \\
&\leq \sum_{n=0}^{\infty} \tfrac{1}{8^n}|\phi(2^n(2x+y))+\phi(2^n(2x-y))-2\phi(2^n(x+y))-2\phi(2^n(x-y))-2\phi(2^n\cdot 2x)+4\phi(2^nx)| \\
&\leq \sum_{n=m}^{\infty} \tfrac{12\alpha}{8^n} = \tfrac{12\alpha 8^{(1-m)}}{7} \leq \tfrac{12\cdot 8^3\alpha}{7}(|x|^3+|y|^3)
\end{aligned} \tag{2.53}$$

for all $x, y \in \mathbb{R}$ with $0 < |x|^3+|y|^3 < \frac{1}{8}$. Hence f satisfies (2.49) for all $x, y \in \mathbb{R}$.

We claim that Eq.(1.5) is not stable for $r = 3, k = 2$ in Corollary 2.4. Suppose on the contrary that there exists a cubic function $C : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (2.50). Since f is bounded and continuous for all $x \in \mathbb{R}$, C is bounded on any open interval containing the origin and continuous at the origin. Then C must have the form $C(x) = \gamma x^3$ for any $x \in \mathbb{R}$ (see [14, Theorem 2.3]), where γ is a real constant. Hence we obtain that

$$|f(2x)-2f(x)| \leq (\beta+|\gamma|)|x|^3, \quad \text{for all } x \in \mathbb{R}. \tag{2.54}$$

On the other hand, we can choose a positive integer $m > 2$ with $m\alpha > \beta+|\gamma|$. If $x \in (0, \frac{1}{2^{m-1}})$, then $2^nx \in (0,1)$ for all $n = 0, 1, \cdots, m-1$, and for such an x we have

$$\begin{aligned}
f(2x)-2f(x) &= \sum_{n=0}^{\infty} \tfrac{\phi(2^n\cdot 2x)}{8^n} - 2\sum_{n=0}^{\infty} \tfrac{\phi(2^nx)}{8^n} \\
&= 8\sum_{n=1}^{\infty} \tfrac{\phi(2^nx)}{8^n} - 2\sum_{n=0}^{\infty} \tfrac{\phi(2^nx)}{8^n} \\
&= 6\sum_{n=1}^{\infty} \tfrac{\phi(2^nx)}{8^n} - 2\phi(x) \\
&\geq 6\sum_{n=1}^{m-1} \tfrac{\phi(2^nx)}{8^n} - 2\phi(x) \\
&= (6m-8)\alpha x^3 > m\alpha x^3 > (\beta+|\gamma|)x^3,
\end{aligned} \tag{2.55}$$

which yields a contradiction. Therefore Eq. (1.5) is not stable if $r = 3$ and $k = 2$ in the inequality (2.44).

Theorem 2.6. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X \times X \to [0, \infty)$ such that*

$$\sum_{i=1}^{\infty} 8^i\varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \tag{2.56}$$

and

$$\triangle(Df(x,y)) \leq \varphi(x,y) \tag{2.57}$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\triangle'_k(f(x) - A(x) - C(x)) \leq \frac{1}{2}\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) + \frac{1}{8}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.58}$$

for all $x \in X$, where $\tilde{\varphi}(x)$ is defined as in Lemma 2.1 and $\triangle'_k(u) := \triangle_k(6u)$ for all $u \in Y$.

Proof. By Theorems 2.2 and 2.3, there exist an additive function $A_0 : X \to Y$ and a cubic function $C_0 : X \to Y$ such that

$$\triangle_k(f(2x) - 8f(x) - A_0(x)) \leq \frac{1}{2}\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.59}$$

and

$$\triangle_k(f(2x) - 2f(x) - C_0(x)) \leq \frac{1}{2}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.60}$$

for all $x \in X$. Hence

$$\triangle_k(6[f(x) + \frac{1}{6}A_0(x) - \frac{1}{6}C_0(x)]) \leq \frac{1}{2}\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) + \frac{1}{8}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i}) \tag{2.61}$$

for all $x \in X$. So we obtain (2.58) by letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$ for all $x \in X$.

To prove the uniqueness of A and C, let $A_1, C_1 : X \to Y$ be different additive and cubic function satisfying (2.58). Let $A' = A - A_1$ and $C' = C - C_1$. So

$$\begin{aligned} \triangle'_k(A'(x) + C'(x)) &\leq \triangle'_k(f(x) - A(x) - C(x)) + \triangle'_k(f(x) - A_1(x) - C_1(x)) \\ &\leq \sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\tfrac{x}{2^i}) + \tfrac{1}{4}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\tfrac{x}{2^i}) \end{aligned} \tag{2.62}$$

for all $x \in X$. Then Lemma 1.2, (2.56) and (2.62) implies that

$$\lim_{n\to\infty} \triangle'_k(2^n[A'(\frac{x}{2^n}) + C'(\frac{x}{2^n})]) = 0$$

for all $x \in X$. Therefore $A' = 0$ and then $C' = 0$. □

Example 2.7. Fix $p \in (0, 1)$ and consider $L^p(0, 1)$. The elements of $L^p(0, 1)$ are those Lebesgue measurable functions x on $[0, 1]$ for which

$$\triangle(x) = \int_0^1 |x(t)|^p dt < \infty,$$

with the usual identification of functions that coincide almost everywhere. Since $0 < p < 1$, the inequality $(a+b)^p \leq a^p + b^p$ holds when $a \geq 0$ and $b \geq 0$. This gives $\triangle(x+y) \leq \triangle(x) + \triangle(y)$, so that $\triangle(x)$ determines an invariant metric on $L^p(0, 1)$. With this metric, $L^p(0, 1)$ is a complete metric linear space, which is not a Banach space.

Let $r > 3$ and $\varphi : L^p(0, 1) \times L^p(0, 1) \to [0, \infty)$ be defined by $\varphi(x, y) = [M\triangle(x) + N\triangle(y)]^{rp^{-1}}$, where $M = 2^{r^{-1}p}k + 2^{r^{-1}p}k^{r^{-1}p^2} + 2^{r^{-1}p^2}k + (2k)^{r^{-1}p^2}$ and $N = 2^{r^{-1}p} + 2^{r^{-1}p}k^{r^{-1}p^2}$. Define $f : L^p(0, 1) \to L^p(0, 1)$ by $f(x) = \triangle(x)^{rp^{-2}}x_0$, where x_0 is an element in $L^p(0, 1)$ such that $\triangle(x_0) = 1$. If $s \geq 1$, then the inequality $(a + b)^s \geq a^s + b^s$ holds when $a \geq 0$ and $b \geq 0$. A straightforward computation shows that

$$\begin{aligned} &\triangle(f(kx + y) + f(kx - y) - kf(x + y) - kf(x - y) - 2f(kx) + 2kf(x)) \\ &\leq [\triangle(kx + y)^{rp^{-2}} + \triangle(kx - y)^{rp^{-2}} + k\triangle(x + y)^{rp^{-2}} + k\triangle(x - y)^{rp^{-2}} + 2\triangle(kx)^{rp^{-2}} + 2k\triangle(x)^{rp^{-2}}]^p \\ &\leq \triangle(kx + y)^{rp^{-1}} + \triangle(kx - y)^{rp^{-1}} + k^p\triangle(x + y)^{rp^{-1}} + k^p\triangle(x - y)^{rp^{-1}} + 2^p\triangle(kx)^{rp^{-1}} + 2^pk^p\triangle(x)^{rp^{-1}} \\ &\leq 2[k\triangle(x) + \triangle(y)]^{rp^{-1}} + 2k^p[\triangle(x) + \triangle(y)]^{rp^{-1}} + 2^p[k\triangle(x)]^{rp^{-1}} + 2^pk^p\triangle(x)^{rp^{-1}} \\ &\leq \{[2^{r^{-1}p}k + 2^{r^{-1}p}k^{r^{-1}p^2} + 2^{r^{-1}p^2}k + (2k)^{r^{-1}p^2}]\triangle(x) + [2^{r^{-1}p} + 2^{r^{-1}p}k^{r^{-1}p^2}]\triangle(y)\}^{rp^{-1}} \\ &= \varphi(x, y) \end{aligned}$$

and

$$\sum_{i=1}^{\infty} 8^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty$$

for all $x, y \in L^p(0,1)$. Therefore, all the conditions of Theorem 2.6 hold and there exist a unique additive mapping $A : L^p(0,1) \to L^p(0,1)$ and a unique cubic mapping $C : L^p(0,1) \to L^p(0,1)$ such that

$$\triangle(f(x) - A(x) - C(x)) \leq \frac{1}{6^p(k-1)^p(k^3-k)^p}[\frac{1}{2}\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) + \frac{1}{8}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i})]$$

for all $x \in X$, where $\tilde{\varphi}(x)$ is defined as in Lemma 2.1.

Corollary 2.8. *Let X be a real vector space, let $(Y, \{p_i\}_{i\in\mathbb{N}}, d)$ be a Fréchet space and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X \times X \to [0, \infty)$ such that*

$$\sum_{i=1}^{\infty} 8^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty$$

and

$$\triangle(Df(x, y)) \leq \varphi(x, y)$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\triangle(f(x) - A(x) - C(x)) \leq \frac{1}{2}\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) + \frac{1}{8}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i})$$

for all $x \in X$, where $\tilde{\varphi}(x)$ is defined as in Lemma 2.1.

Proof. The result follows from Theorem 2.6 and Theorem 1.4. □

Corollary 2.9. *Let X be a real vector space and Y be a Banach space with norm $\|\cdot\|$. Suppose the mapping $\varphi : X \times X \to [0, \infty)$ satisfying*

$$\sum_{i=1}^{\infty} 8^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty$$

for all $x, y \in X$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that

$$\|Df(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in X$, then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\| \leq \frac{1}{12(k-1)(k^3-k)}[\sum_{i=1}^{\infty} 2^i \tilde{\varphi}(\frac{x}{2^i}) + \frac{1}{4}\sum_{i=1}^{\infty} 8^i \tilde{\varphi}(\frac{x}{2^i})]$$

for all $x \in X$, where $\tilde{\varphi}(x)$ is defined as in Lemma 2.1.

Proof. The result follows from Theorem 2.6 by taking $\triangle'_k(u) = \|6(k-1)(k^3-k)u\|$ for all $u \in Y$. □

Corollary 2.10. *Let $r > 3, \theta > 0$, and let X, Y be a normed space and a Banach space, respectively. If $f : X \to Y$ is a function with $f(0) = 0$ such that*

$$\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$, then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\| \leq \frac{(2^r - 5)\theta\mu}{6(k-1)(k^3-k)(2^r-2)(2^r-8)}\|x\|^r$$

for all $x \in X$, where μ is defined as in Corollary 2.4.

Proof. The result follows from Theorem 2.6 by taking $\triangle'_k(u) = \|6(k-1)(k^3-k)u\|$ for all $u \in Y$, and $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. □

Corollary 2.11. *Let $\theta \geq 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s > 3$. Let X, Y be a normed space and a Banach space, respectively. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y)\| \leq \theta[\|x\|^r\|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]$$

for all $x, y \in X$, then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\| \leq \frac{(2^\lambda - 5)\theta\varepsilon}{6(k-1)(k^3-k)(2^\lambda-2)(2^\lambda-8)}\|x\|^\lambda$$

for all $x \in X$, where

$$\begin{aligned}\varepsilon = {} & (k-1)\{(1+k)[(2k+1)^s + (2k-1)^s] + 3^r + 8k^2 + 3^s k^s + k^s + 3k^2 2^\lambda + 2^\lambda k^s + 2(k+1)^s \\ & +2(k-1)^s + 2^{r+1}(1+k^s) + 2^{3-\lambda}k^s + 2^{3-\lambda}k(2k-1)^s + 2^{3-\lambda}k(2k+1)^s + 2^{3-\lambda}3^s k^s \\ & +14 + 2k + 3^\lambda + 16k^2 + 5\cdot 2^\lambda + 2^{4-\lambda}(1+k) + (1+k+2^{3-\lambda}k)[(2k+1)^\lambda + (2k-1)^\lambda] \\ & +(3^\lambda + 3 + 2^\lambda + 2^{3-\lambda})k^\lambda + 2(k+1)^\lambda + 2(k-1)^\lambda + 2^{3-\lambda}3^\lambda k^\lambda\} + (1 + 8k^2 + 2^\lambda k^2)(k-1)^\lambda \\ & +2 + (k+1)^{\lambda+1} + (k + 2^{3-\lambda}k)(3k-1)^\lambda + (k^2 + k - 1)2^\lambda k^\lambda + 2^{3-\lambda}k(k+1)^\lambda + (8k^2 - 6k)k^\lambda.\end{aligned}$$

Proof. The result follows from Theorem 2.6 by taking $\triangle'_k(u) = \|6(k-1)(k^3-k)u\|$ for all $u \in Y$, and $\varphi(x,y) = \theta[\|x\|^r\|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]$ for all $x, y \in X$. □

Corollary 2.12. *Let $\theta, r, s \geq 0$ be non-negative real numbers such that $\lambda := r + s > 3$. Let X, Y be a normed space and a Banach space, respectively. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y)\| \leq \theta\|x\|^r\|y\|^s$$

for all $x, y \in X$, then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\| \leq \frac{(2^\lambda - 5)\theta\eta}{3(k-1)(k^3-k)(2^\lambda-2)(2^\lambda-8)}\|x\|^\lambda$$

for all $x \in X$, where

$$\begin{aligned}\eta = {} & (k-1)\{(1+k)[(2k+1)^s + (2k-1)^s] + 3^r + 1 + 8k^2 + 3^s k^s + k^s + k^2 2^\lambda + 2^\lambda k^s + 2(k+1)^s \\ & +2(k-1)^s + 2^{r+1}(1+k^s) + 2^{3-\lambda}k^s + 2^{3-\lambda}k(2k-1)^s + 2^{3-\lambda}k(2k+1)^s + 2^{3-\lambda}3^s k^s\}.\end{aligned}$$

Proof. The result follows from Theorem 2.6 by taking $\triangle'_k(u) = \|6(k-1)(k^3-k)u\|$ for all $u \in Y$, and $\varphi(x,y) = \theta\|x\|^r\|y\|^s$ for all $x, y \in X$. □

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The Finite Difference Methods and Their Extrapolation for Solving Biharmonic Equations*

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Abstract

In this paper, we research the standard 13−point difference scheme for solving the biharmonic equation. Existence and convergence of finite difference methods(FDM) solutions are obtained by estimating the lower bounds of the minimum eigenvalues of the discrete matrix and making use of the Taylor series expansions, respectively. The accuracy are proved to be $O(h^2)$. Moreover, the extrapolation techniques are used to improve the high accuracy of the solutions. For the given examples, numerical results show that the errors $O(h^4)$ is obtained from the first level extrapolation. The very accurate solutions with the errors $O(10^{-8})$ are obtained by the third level extrapolation techniques under the homogeneous essential boundary conditions.

Keyword: Difference operator; Finite difference method; Biharmonic equation; Extrapolation method; Asymptotic expansion

1 Introduction

Consider the following biharmonic Poisson equation

$$\Delta^2 u(x,y) \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f(x,y), \text{ in } S, \tag{1.1a}$$

$$u = g, \text{ on } \Gamma, \tag{1.1b}$$

$$\frac{\partial u}{\partial n} = g^* \text{ on } \Gamma, \frac{\partial^2 u}{\partial n^2} = g^{**} \text{ on } \Gamma, \tag{1.1c}$$

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where S is a bounded domain with the boundary Γ, and $\partial/(\partial n)$ is the outward normal derivative on Γ. This problem arises from many applications, e.g., the theory of elasticity, mechanics of elastic plates and the theory of flows of viscous fluids. In general, the above problem can be solved by either a finite difference method or a finite element method. The finite difference method always seems to be the best choice not only for regular regions but also for more complex regions when appropriate extrapolation techniques are applied to the boundaries.

Using the FDM to solving biharmonic equation and other Partial Differential Equations(PDEs), the existence and convergence analysis of the former is more difficult than the latter. Because biharmonic equation is no longer within the framework of energy inequality theory. Therefore, many researchers are attracted to discuss this problem. In [2], Zláml has dealt the normal derivative condition $\partial u/\partial n|_{\partial S} = g^*$ by extrapolation of the third degree and the optimal convergence of discrete errors $O(h^2)$ in H^2 norm achieved. For the standard 13-point schemes, the convergence was first proved in Courant et al. [5]. Besides, the discrete error $O(h^2)$ in H^1 norm and the maximal error $O(h^2\sqrt{|\ln h|})$ were proved in Bramble [3]. Moreover, we also mention a few other important references of the FDM for biharmonic equations, such as Lu, et al. [12], Thomee and Westergren [15] and Stys [14].

In this paper, for biharmonic equations, the finite difference methods(FDM) are chosen to seek their approximate solutions. Existence and convergence of finite difference methods(FDM) solutions are obtained by estimate the lower bounds of the minimum eigenvalues of the discrete matrix and make use of the Taylor series expansions, respectively. The accuracy are proved to be $O(h^2)$. Moreover, the extrapolation techniques are used to improve the high accuracy of the solutions.

This paper is organized as follows: In the next section, the standard 13-point difference equations are used for biharmonic equations in 2D. In Section 3, a lower bound of the minimum eigenvalues of the discrete matrix and the convergence of FDM solution are derived. In Section 4, the accuracy order of the approximations by extrapolation techniques are improved. In the last section, some numerical examples are provided to verify the features of the methods discussed in this paper.

2 Finite Difference Method

In this section, we consider the biharmonic equation on the unit square $[0,1]^2$ with the mixed type of the clamped and the support boundary conditions, that is, $u = g,$ on Γ, $\frac{\partial u}{\partial n} = g^*$ on Γ_D, $\frac{\partial^2 u}{\partial n^2} = g^{**}$ on Γ_N, where $\Gamma = \partial S = AB \cup BC \cup CD \cup DA$, $\Gamma_D = AB \cup CD \cup DA$ and $\Gamma_N = BC$. see Figure 1. The functions in (1.1a)-(1.1c) are supposed to be bounded

$$||f||_{0,S} \le C,\ ||g||_{0,\Gamma} \le C,\ ||g^*||_{0,\Gamma_D} \le C,\ ||g^{**}||_{0,\Gamma_N} \le C, \tag{2.1}$$

where C is a bounded constant, and $||f||_{0,S} = \sqrt{\int\int_s f^2 ds},\ ||g||_{0,\Gamma} = \sqrt{\int_\Gamma g^2 dl}$.

Divide S by the uniform difference grids $x_i = ih, y_j = jh$ with $h = 1/N$. Denote $u_{i,j} = u(x_i, y_j) = u(ih, jh)$. We use the standard 13-point difference equation (DE) to

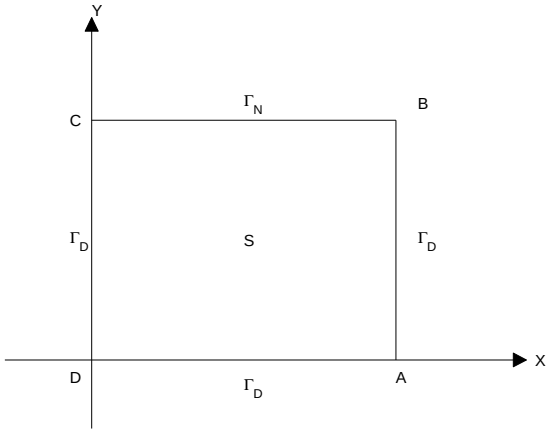


Figure 1: The unit square domain with the mixed boundary conditions.

solve (1.1a-1.1c). This approach yields the following linear system

$$Ax = b, \tag{2.2}$$

where $A \in R^{(N-1)^2 \times (N-1)^2}$ is a block pentadiagonal symmetric matrix and the unknown vector $x = (u_{1,1}, \cdots, u_{1,N-1}, \cdots\cdots, u_{N-1,1}, \cdots, u_{N-1,N-1})^T$. The vector b has been known.

3 Existence and convergence of FDM solutions

3.1 Existence of FDM solution

Below, we will discuss the existence of FDM solution by estimate the lower bound of the minimum eigenvalues of the discrete matrix.

Theorem 3.1[13] *For the stiffness matrix* **A** *in (2.2), there exists the bound*

$$\min_{p,\ q=1,\cdots,N-1} |\lambda_{p,\ q}(A)| \geq ch^4, \tag{3.1}$$

where c is the constant independent of h.

Proof. First, consider the simple support boundary condition on the entire boundary ∂S, and denote A^* the corresponding stiffness matrix. For a matrix A, if it is a block tridiagonal matrix , we denote it by Matlab notation for convenience,

$$A = diag(a, 0) + diag(b, -1) + diag(c, 1).$$

Let A_1 is an $(N-1)^2 \times (N-1)^2$ block tridiagonal matrix.

$$A_1 = diag(a_1, 0) + diag(b_1, -1) + diag(c_1, 1),$$

where $a_1 = B * ones(N-1,1),\ b_1 = I * ones(N-2,1)$ and $c_1 = I * ones(N-2,1)$, and B is an $(N-1)\times(N-1)$ tridiagonal matrix,

$$B = diag(a_2, 0) + diag(b_2, -1) + diag(c_2, 1)$$

with $a_2 = -4 * ones(N-1,1),\ b_2 = ones(N-2,1)$ and $c_2 = ones(N-2,1)$.

A_1 can be expressed in Kronecker product

$$\begin{aligned} A_1 =& diag(a_3, 0) + [diag(a_4, 0) + diag(b_1, -1) + diag(c_1, 1)] \\ =& I \otimes T + T \otimes I, \end{aligned}$$

with $a_3 = T * ones(N-1,1),\ a_4 = -2I * ones(N-1,1)$ and $T = B + 2I$ is an $(N-1)\times(N-1)$ tridiagonal matrix,

$$T = diag(a_5, 0) + diag(b_5, -1) + diag(c_5, 1),$$

where $a_5 = -2 * ones(N-1,1),\ b_5 = ones(N-2,1)$ and $c_5 = ones(N-2,1)$. It is known that the eigenvalues and corresponding eigenvectors of T are[19]

$$\lambda_p(T) = -4\sin^2(\frac{p\pi h}{2}) \text{ and } w_p(T) = (\sin(p\pi h),\ \sin(2p\pi h), \cdots,\ \sin((N-1)p\pi h))^T,$$

Therefore, we can obtain the eigenvalues and corresponding eigenvectors of A_1, which are

$$\lambda_{p,\ q}(A_1) = (\lambda_q(T) + \lambda_p(T)) = -4(\sin^2(\frac{q\pi h}{2}) + \sin^2(\frac{p\pi h}{2}))$$

and

$$w_{p,\ q}(A_1) = \{(w_p(T))^T \sin(q\pi h),\ (w_p(T))^T \sin(2q\pi h), \cdots,\ (w_p(T))^T \sin((N-1)q\pi h)\}^T.$$

Since $A^* = (A_1)^2$, the eigenvalues and corresponding eigenvectors of A^* are

$$\lambda_{p,\ q}(A^*) = \lambda_{p,q}(A_1^2) = \{\lambda_{p,q}(A_1)\}^2 = 16\{\sin^2(\frac{p\pi h}{2}) + \sin^2(\frac{q\pi h}{2})\}^2$$

and

$$w_{p,\ q}(A^*) = \{(w_p(T))^T \sin(q\pi h),\ (w_p(T))^T \sin(2q\pi h), \cdots,\ (w_p(T))^T \sin((N-1)q\pi h)\}^T,$$

$p, q = 1,\ 2, \cdots, N-1$.

Hence the minimal eigenvalue of A^* is

$$\min_{p,\ q=1,\cdots,N-1} |\lambda_{p,\ q}(A^*)| = \lambda_{1,1}(A^*) = 64\sin^4(\pi h/2) \simeq 4\pi^4 h^4. \tag{3.2}$$

Next, consider the mixed type of the clamped and the simple support boundary conditions in (1.1b)-(1.1c). The key difference is that the first coefficient in the FDEs increases from 19 to 21. Hence, we conclude that the only difference between matrices

A and A^* lies in that the diagonal entries of A are larger than those of A^*. Then we have

$$A = A^* + D,$$

where D is a diagonal matrix with D=diag$\{d_{1,1}, ..., d_{n,n}\}$ with $d_{p,p} \geq 0$, and $n = (N-1)^2$. Hence we obtain

$$\begin{aligned}\min_{p,\ q=1,\cdots,N-1} |\lambda_{p,\ q}(A)| &= \min_{(x,x)\neq 0} \frac{(Ax,x)}{(x,x)} = \min_{(x,x)\neq 0} \frac{(A^*x,x)+(Dx,x)}{(x,x)} \\ &\geq \min_{(x,x)\neq 0} \frac{(A^*x,x)}{(x,x)} = \min_{p,\ q=1,\cdots,N-1} |\lambda_{p,\ q}(A^*)|. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we have

$$\min_{p,\ q=1,\cdots,N-1} |\lambda_{p,\ q}(A)| \geq \min_{p,\ q=1,\cdots,N-1} |\lambda_{p,\ q}(A^*)| \geq ch^4. \tag{3.4}$$

This is the desired result (3.1). The proof of Theorem 3.1 is completed. □

3.2 Convergence of FDM solution

In the section we discuss convergence of FDM solution by Taylor series expansion, and the biharmonic equation can be equivalent to the following difference operator equation

$$\Delta_h^2 u(x_i, y_j) = \frac{1}{h^4}(\delta_x^4 u_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j} + \delta_y^4 u_{i,j}) \tag{3.5}$$

where $\delta_x^4 = \delta_x^2(\delta_x^2)$, $\delta_y^4 = \delta_y^2(\delta_y^2)$, $\delta_x^2 = u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)$, $\delta_y^2 = u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})$.

Theorem 3.2 *Let $u(x, y)$ be the exact solution of (1.1a) and $u \in C^{7+\sigma}(S)$, $0 < \sigma < 1$, also let the uniform rectangles $\square_{ij}$ are chosen with $h = 1/N$. Then*

$$\Delta^2 u(x_i, y_j) - \Delta_h^2 u(x_i, y_j) = O(h^2), \tag{3.6}$$

$$\max_{0\leq i,j\leq N} |u(x_i, y_j) - u_{i,j}| \leq Ch^2, \tag{3.7}$$

where $u_{i,j}$ be the numerical solution of (2.2), N is the division number along AB (see Figure 1) and C is the constant independent of h.

Proof. We first prove (3.6). According to Taylor′s expansion, we obtain

$$\begin{aligned} u(x_i \pm h, y_j) =& u(x_i, y_j) \pm h\frac{\partial}{\partial x}u(x_i, y_j) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}u(x_i, y_j) \pm \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}u(x_i, y_j) \\ &+ \frac{h^4}{4!}\frac{\partial^4}{\partial x^4}u(x_i + \theta_\pm h, y_j), \end{aligned} \tag{3.8}$$

where $|\theta_{\pm}| \leq 1$. Therefore,

$$\frac{u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)}{h^2}-\frac{\partial^2}{\partial x^2}u(x_i,y_j)=\frac{h^2}{12}\frac{\partial^4}{\partial x^4}u(x_i+\theta_1 h,y_j), \quad (3.9)$$

where $|\theta_1| \leq 1$. Analogously, we have

$$\frac{u(x_i,y_{j+1})-2u(x_i,y_j)+u(x_i,y_{j-1})}{h^2}-\frac{\partial^2}{\partial y^2}u(x_i,y_j)=\frac{h^2}{12}\frac{\partial^4}{\partial y^4}u(x_i,y_j+\theta_2 h), \quad (3.10)$$

where $|\theta_2| \leq 1$. Since $\delta_x^2 = u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)$, $\delta_y^2 = u(x_i,y_{j+1})-2u(x_i,y_j)+u(x_i,y_{j-1})$, $\delta_x^4 = \delta_x^2(\delta_x^2)$, $\delta_y^4 = \delta_y^2(\delta_y^2)$, take (2.5)-(2.7) into account, we obtain

$$\begin{aligned}\Delta^2 u(x_i,y_j)-\Delta_h^2 u(x_i,y_j) =&(\frac{\partial^4 u}{\partial x^4}-\frac{1}{h^4}\delta_x^4 u_{i,j})+(\frac{\partial^4 u}{\partial y^4}-\frac{1}{h^4}\delta_y^4 u_{i,j})\\ &+2(\frac{\partial^4 u}{\partial x^2\partial y^2}-\frac{1}{h^4}\delta_x^2\delta_y^2 u_{i,j})\\ =&O(h^2). \end{aligned} \quad (3.11)$$

This is the desired result (3.6).

By using the result of (3.6), it is easy to know

$$\max_{0\leq i,j\leq N}|u(x_i,y_j)-u_{i,j}| \leq Ch^2,$$

Hence, the proof of Theorem 3.2 is completed. □

4 Asymptotic expansion of errors and extrapolation

In order to improve the accuracy order of approximation solutions, we give the asymptotic expansions of the errors, which are the foundation of extrapolation.

Theorem 4.1 *Under the hypotheses of Theorem 3.2. Then there exists a function $d_i(x,y)$ independent of h such that the following asymptotic expansions hold*

$$u^h(x,y)-u(x,y)=\sum_{i=1}^{m}h^{2i}d_i(x,y)+O(h^{2m+2}). \quad (4.1)$$

where $u^h(x,y)$ be the numerical solution of (2.2).

Proof. Under the hypotheses of Theorem 3.2, we know $u \in C^{7+\sigma}(S)$ and

$$\begin{aligned}\Delta^2 u(x,y) - \Delta_h^2 u(x,y) =&(\frac{\partial^4 u}{\partial x^4} - \frac{1}{h^4}\delta_x^4 u_{i,j}) + (\frac{\partial^4 u}{\partial y^4} - \frac{1}{h^4}\delta_y^4 u_{i,j}) \\ &+ 2(\frac{\partial^4 u}{\partial x^2 \partial y^2} - \frac{1}{h^4}\delta_x^2\delta_y^2 u_{i,j}) \\ =&\frac{h^2}{3}[\frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial x^4 \partial y^2}] + O(h^2). \end{aligned} \tag{4.2}$$

Let $\frac{h^2}{3}[\frac{\partial^6 u}{\partial x^2\partial y^4} + \frac{\partial^6 u}{\partial x^4\partial y^2}] = w_1(x,y) \in C^{1+\sigma}(S)$, and construct auxiliary problem

$$\begin{cases} \Delta^2 d_1(x,y) = w_1(x,y), \text{ in } S, \\ d_1(x,y) = 0, \text{ on } \Gamma, \end{cases} \tag{4.3}$$

where $d_1(x,y) \in C^{5+\sigma}$. According to (4.3) we obtain

$$\Delta^2 u(x,y) - \Delta_h^2 u(x,y) = O(h^2). \tag{4.4}$$

Therefore,

$$\Delta_h^2(u^h(x,y) - u(x,y) - h^2 d_1(x,y)) = \Delta^2 u(x,y) - \Delta_h^2 u(x,y) - h^2 w_1(x,y). \tag{4.5}$$

$$(\Delta^2 d_1(x,y) - \Delta_h^2 d_1(x,y)) \cdot h^2 = O(h^4), \text{ in } S \tag{4.6}$$

and

$$u^h(x,y) - u(x,y) - h^2 d_1(x,y) = 0, \quad \text{on } \Gamma. \tag{4.7}$$

Hence,

$$u^h(x,y) - u(x,y) = h^2 d_1(x,y) + O(h^4). \tag{4.8}$$

By mathematical induction, we can obtain the result of Theorem 4.1. Hence, the proof of Theorem 4.1 is completed. □

Remark 4.2 We can improve the accuracy order of approximation solutions by Richardson extrapolation. This method just needs to compute little work, but the precision is much higher than non-extrapolation.

For example, We may have the multi-level extrapolations by[1]

$$u_{h,i} = \frac{2^{2i} u_{h/2,i-1} - u_{h,i-1}}{2^{2i} - 1}, i = 1, 2, 3, \tag{4.9}$$

where $u_{h,0} = u_h$, and $u_{h,i}$ is called the $i-th$ level extrapolation solution. Hence, the maximal nodal errors $O(h^4)$ is obtained from the first level extrapolation. That is

$$||u - u_{h,1}||_\infty = O(h^4). \tag{4.10}$$

Moreover, when $h = 1/64$, the very accurate solutions with the maximal nodal errors $O(10^{-8})$ are obtained by the third level extrapolation techniques for the homogeneous essential boundary conditions.

[1]Eq. (4.10) is based on the assumption that the error expansions hold as$||u-u_h||_\infty = d(x,y)h^2 + \sum_{i=1}^{2} d_i(x,y)h^{2(i+1)} + O(h^7)$, where $d_i(x,y)$ are the functions independent of h.

5 Numerical experiments

The uniform rectangles $\square_{ij}$ are chosen with $h = 1/N$, where N is the division number along AB (see Figure 1), $S = [0,1] \times [0,1]$ and $\Gamma = \partial S = AB \cup BC \cup CD \cup DA$. We choose the biharmonic equation with three different boundary conditions, which are called Models I, II and III, respectively.

Model I[13]. Choose the exact solution

$$u(x,y) = [x(1-x)\sin(\pi x)][y(1-y)^2\sin(\pi y)]$$

to satisfy

$$\Delta^2 u = f, \text{ in } s,$$
$$u|_\Gamma = 0, \partial u/\partial n|_{AB\cup AD\cup CD} = 0, \partial^2 u/\partial n^2|_{BC} = 0.$$

Model II[13]. Choose the exact solution

$$u(x,y) = [x(1-x)\sin(\pi x)][y(1-y)\sin(\pi y)]$$

to satisfy

$$\Delta^2 u = f, \text{ in } s,$$
$$u|_\Gamma = 0, \partial u/\partial n|_{AB\cup AD\cup CD} = 0, \partial^2 u/\partial n^2|_{BC} = 2\pi x(1-x)\sin(\pi x).$$

Model III[13]. Choose the exact solution

$$u(x,y) = [x(1-x)\sin\frac{\pi x}{2}][y(1-y)^2\sin(\pi y)]$$

to satisfy

$$\Delta^2 u = f, \text{ in } s,$$
$$u|_\Gamma = 0, \partial u/\partial n|_{AB\cup CD} = 0, \partial u/\partial n|_{AB} = -y(1-y)^2\sin(\pi y), \partial^2 u/\partial n^2|_{BC} = 0.$$

The difference equations are established as in Section 2, and the numerical solutions are obtained from the Gaussian elimination. The numerical results are listed in Tables 1,2,3 for Models I, II and III, respectively. From Tables 1-3, we can see the following asymptotes,

$$||u - u_h||_\infty = \max_{0\le i,j\le N} |u(x_i,y_j) - u_h(x_i,y_j)| = O(h^2), \tag{5.1}$$

Eq. (5.1) coincides with the results in Zlamal [1, 2].

Below, we intend to improve the accuracy of the solutions by multi-level extrapolation techniques. Suppose that the solutions by the 13-points difference equation have the following errors

$$||u - u_h||_\infty = d(x,y)h^2 + O(h^{2+p}), p = 1, 2, \tag{5.2}$$

where $d(x,y) > 0$ is a function independent of h. Hence, for the first level extrapolation solution, we have from (5.2)

$$||u - u_{h,1}||_\infty = O(h^k), k = 3 \text{ or } 4. \tag{5.3}$$

Table 1: The maximal errors for Model I

N	8	16	32	64
$\|\|u-u_h\|\|_\infty$	0.491(-2)	0.120(-2)	0.300(-3)	0.750(-4)
$\|\|u-u_{h,1}\|\|_\infty$	0.410(-4)	0.230(-5)	0.138(-6)	–
$\|\|u-u_{h,2}\|\|_\infty$	0.348(-6)	0.748(-8)	–	–
$\|\|u-u_{h,3}\|\|_\infty$	0.306(-8)	–	–	–
$\lambda_{max}(A)$	59.38	62.79	63.69	63.92
$\lambda_{min}(A)$	0.1841	0.124(-1)	0.800(-3)	0.603(-4)

Table 2: The maximal errors for Model II

N	8	16	32	64
$\|\|u-u_h\|\|_\infty$	0.727(-2)	0.177(-2)	0.438(-3)	0.109(-3)
$\|\|u-u_{h,1}\|\|_\infty$	0.705(-4)	0.419(-5)	0.259(-6)	–
$\|\|u-u_{h,2}\|\|_\infty$	0.270(-6)	0.113(-7)	–	–
$\|\|u-u_{h,3}\|\|_\infty$	0.451(-8)	–	–	–
$\lambda_{max}(A)$	59.38	62.79	63.69	63.92
$\lambda_{min}(A)$	0.1841	0.124(-1)	0.800(-3)	0.603(-4)

Table 3: The maximal errors for Model III

N	8	16	32	64
$\|\|u-u_h\|\|_\infty$	0.261(-2)	0.704(-3)	0.176(-3)	0.440(-4)
$\|\|u-u_{h,1}\|\|_\infty$	0.137(-3)	0.221(-4)	0.302(-5)	–
$\|\|u-u_{h,2}\|\|_\infty$	0.510(-5)	0.526(-6)	–	–
$\|\|u-u_{h,3}\|\|_\infty$	0.395(-6)	–	–	–
$\lambda_{max}(A)$	59.38	62.79	63.69	63.92
$\lambda_{min}(A)$	0.1841	0.124(-1)	0.800(-3)	0.603(-4)

The numerical solutions are listed in Tables 1-3. From Tables 1 and 2 with the homogeneous essential boundary conditions (e.g., $g = g^* = 0$), we can see

$$||u - u_{h,1}||_\infty = O(h^4), \tag{5.4}$$

to agree with (5.3). Note that the third level extrapolation solutions are very accurate. Then the ratios are obtained from Tables 1 and 2

$$\frac{||u - u_{h,0}||_\infty|_{N=64}}{||u - u_{h,3}||_\infty} = \frac{0.750 \times 10^{-4}}{0.306 \times 10^{-8}} = 2.45 \times 10^4,$$

$$\frac{||u - u_{h,0}||_\infty|_{N=64}}{||u - u_{h,3}||_\infty} = \frac{0.109 \times 10^{-3}}{0.451 \times 10^{-8}} = 2.41 \times 10^4, \tag{5.5}$$

respectively. Eqs.(5.5) display significance of the extrapolation techniques to improve accuracy of the solutions. However, from Table 3 with the non-homogeneous clamped boundary condition, we can only see

$$||u - u_{h,1}||_\infty = O(h^3), \tag{5.6}$$

which is less efficient, compared with (5.4). Still, the ratio from Table 3

$$\frac{||u - u_{h,0}||_\infty|_{N=64}}{||u - u_{h,3}||_\infty} = \frac{0.440 \times 10^{-4}}{0.395 \times 10^{-6}} = 111, \tag{5.7}$$

yields an evident improvement of the solution accuracy. Although only the maximal nodal errors are provided in Tables 1-3, the improvements of the maximal nodal derivatives by the extrapolation techniques are similar. Since the improvements of the solution accuracy are significant and the algorithms are simple, the extrapolation techniques are strongly recommended for numerical biharmonic equations.

To close this paper, let us make a few concluding remark.

The improvement of accuracy by the extrapolation techniques is significant, especially for the homogeneous essential boundary conditions, see Tables 1 and 2 for Models I and II. When $N = 64$, after the third level extrapolation techniques have been applied, the obtained solutions are very accurate with the errors $O(10^{-8})$. Interestingly, when the boundary conditions are homogeneous, the excellent stability and the very high accuracy can also be reached simultaneously.

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$\mathcal{I}$–Summability and $\mathcal{I}$-approximation through invariant mean

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Abstract. The concept I_u convergence was introduced by Baláz and Šalát [Uniform density u and corresponding I_u-convergence, *Math. Comm.*, 11 (2006), 1-7]. Recently, Mursaleen and Edely [On the invariant mean and statistical convergence, *Appl. Math. Letters,* 22 (2009) 1700-1704] used the concept of invariant mean and density to study some new type of convergence methods. In this paper we generalize these two methods by using the concept of ideal convergence and invariant mean. Further we apply our method to prove a Korovkin type approximation theorem along with an example in support of our result.

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1. Introduction and preliminaries

The idea of statistical convergence was introduced by Fast [6] and since then several generalizations and application of this concept have been investigated by various authors. One of its generalizations is the ideal convergence or I-convergence which was introduced by Kastyrko et al [11]. I-convergence and I-Cauchy in 2-normed spaces have been studied in [10] and [19]. Recently the concept of ideal convergence for double sequences has bee studied in [5], [15] and [18]. In this paper we define and study ideal convergence through the notion of invariant mean. Further we apply our method to prove a Korovkin type approximation theorem. We also give an example of a sequence of positive linear operators satisfying the conditions of our theorem but does not satisfy the conditions of the classical Korovkin approximation theorem.

First we recall some basic concepts.

Definition 1.1. Let c and l_∞ denote the space of all convergent and bounded sequences, respectively and note that $c \subset l_\infty$. Let σ be a mapping of the set of positive

integers $\mathbb{N}$ in to itself. A continuous linear functional φ defined on the space l_∞ of all bounded sequences is called an *invariant mean* (or a *σ-mean*) if it is nonnegative, normal and $\varphi(x) = \varphi((x_{\sigma(n)}))$.

A sequence $x = (x_k)$ is said to be *σ-convergent* to the number L if and only if all of its σ-means coincide to L, i.e. $\varphi(x) = L$ for all φ. A bounded sequence $x = (x_k)$ is σ-convergent (cf. [21]) to the number L if and only if $\lim_{p\longrightarrow\infty} t_{pm} = L$ uniformly in m, where

$$t_{pm} = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} \cdots\cdots + x_{\sigma^p(m)}}{p+1}.$$

We denote the set of all σ-convergent sequences by V_σ and in this case we write $x_k \longrightarrow L(V_\sigma)$ and L is called the σ-limit of x. Note that a σ-mean extends the limit functional on c in the sense that $\varphi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits (cf. [15]) and $c \subset V_\sigma \subset l_\infty$.

If σ is a translation then σ-mean is called a *Banach limit* and σ-convergence is reduced to the concept of *almost convergence* introduced by Lorentz [11].

Definition 1.2. Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$.Then the *natural density* of K is defined (cf. [4]) by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

Definition 1.3. A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to L provided that for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero (cf. [6], [9], [20]). In this case L is called the δ-limit of x.

Recently the following concepts were given in [17].

Definition 1.4. Let $K_\sigma(m+1, m+p)$ denote the cardinality of the set $\{\sigma(m) \leq k \leq \sigma^p(m) : k \in K\}$, and write $N_p = \min_m K_\sigma(m+1, m+p)$, $N^p = \max_m K_\sigma(m+1, m+p)$. It is easy to see that the limits $\underline{\delta}_\sigma(K) = \lim_{p\longrightarrow\infty} \frac{N_p}{p}$ and $\bar{\delta}_\sigma(K) = \lim_{p\longrightarrow\infty} \frac{N^p}{p}$ exist. These are called respectively the *lower* and *upper* *σ-density* of the set K . If $\underline{\delta}_\sigma(K) = \bar{\delta}_\sigma(K)$, then the common value $\delta_\sigma(K)$ is called the σ-density of the set K. It is clear that for $K \subseteq \mathbb{N}$, $\underline{\delta}_\sigma(K) \leq \underline{\delta}(K) \leq \bar{\delta}(K) \leq \bar{\delta}_\sigma(K)$. For $\sigma(n) = n+1$, σ-density is reduced to the uniform density [3].

Definition 1.5. A sequence $x = (x_k)$ is said to be *σ-statistically convergent* to L if for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has σ-density zero, i.e. $\delta_\sigma(K_\epsilon) = 0$. In this case we write $\sigma(\delta)\text{-}\lim x = L$. That is,

$$\lim_p \frac{1}{p} |\{\sigma(m) \leq k \leq \sigma^p(m) : |x_k - L| \geq \epsilon\}| = 0, \text{uniformly in } m.$$

Definition 1.6. A sequence $x = (x_k)$ is said to be *statistically σ- convergent* to L if for every $\epsilon > 0$ the set $K_\epsilon(\varphi) := \{k \in \mathbb{N} : \varphi(|x_k - L|) \geq \epsilon\}$ has natural density zero, i.e. $\delta(K_\epsilon(\varphi)) = 0$. In this case we write $\delta(\sigma)\text{-lim}\, x = L$. That is,

$$\lim_n \frac{1}{n} |\{p \leq n : |t_{pm} - L| \geq \epsilon\}| = 0, \text{uniformly in } m.$$

Definition 1.7. A sequence $x = (x_k)$ is said to be *strongly σ_q-converget* ($0 < q < \infty$) to the limit L if $\lim_p \frac{1}{p} \sum_{j=1}^{p} |x_{\sigma^j(m)} - L|^q = 0$,uniformly in m, and we write it as $x_k \longrightarrow L[V_\sigma]_q$. In this case L is called the $[V_\sigma]_q$-limit of x. Note that, for $q = 1$, $[V_\sigma]_q \cap l_\infty = [V_\sigma]$ (cf. [16]).

We recall the notion of ideal convergence which is a generalization of statistical convergence introduced by Kastyrko et al [11].

Definition 1.8. If X is a non-empty set then a family of subsets of X is called an *ideal* in X if and only if

(a) $\emptyset \in I$,

(b) $A, B \in I$ implies $A \cup B \in I$,

(c) For each $A \in I$ and $B \subset A$ we have $B \in I$,

where $P(X)$ is the power set of X. I is called *nontrivial ideal* if $X \in I \neq \emptyset$.

Definition 1.9. Let X be a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a *filter* on X if and only if

(a) $\emptyset \notin F$,

(b) $A, B \in F$ implies $A \cap B \in F$,

(c) For each $A \in F$ and $B \supset A$ we have $B \in F$.

Definition 1.10. A non-trivial ideal I in X is called an *admissible ideal* if it is different from $P(\mathbb{N})$ and it contains all singletons i.e., $\{x\} \in I$ for each $x \in X$.

Definition 1.11. Let $I \subset P(X)$ be a non-trivial ideal. Then a class $F(I) = \{M \subset X : M = X \setminus A, \text{ for some } A \in I\}$ is a filter on X, called the *filter associated with the ideal I*.

Definition 1.12. An admissible ideal $I \subset P(\mathbb{N})$ is said to satisfy the *condition* (AP) if for every sequence $(A_n)_{n\in}$ of pairwise disjoint sets from I there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every n and $\bigcup_{n\in} B_n \in I$.

Definition 1.13. Let I be a non trivial ideal of $\mathbb{N}$. A sequence $x = (x_k)$ is said to be *I-convergent* to $L \in X$ if for each $\epsilon > 0$,

$$\{k \in \mathbb{N} : |x_k - \ell| \geq \epsilon\} \in I.$$

In this case we write $I\text{-}\lim x = L$.

2. $\mathcal{I}$-summability methods

In this paper, we generalize the definitions of σ-statistical convergence and statistical σ-convergence through the concept of ideal and establish some results.

Throughout this paper we consider I to be an admissible ideal.

Definition 2.1. A sequence x is said to be $I(\sigma)$-*convergent* to L provided that for every $\varepsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ belongs to $I(\sigma)$; where $I(\sigma) = \{K \subset \mathbb{N} : \delta_\sigma(K) = 0\}$. In this case we write $I(\sigma)\text{-}\lim x = L$.

Definition 2.2. A sequence x is said to be $\sigma(I)$-*convergent* to L provided that for every $\varepsilon > 0$ the set $\{p \in \mathbb{N} : |t_{pm} - L| \geq \epsilon\} \in I$ uniformly in m; where t_{pm} is defined as above. In this case we write $\sigma(I)\text{-}\lim x = L$.

We prove the following relation between $I(\sigma)$-convergence and $\sigma(I)$-convergence:

Theorem 2.3. If a sequence $x = (x_k)$ is bounded and $I(\sigma)$-convergent to L then it is $\sigma(I)$-convergent to L but not conversely.

Proof. Let $x = (x_k)$ be bounded and σ-statistically convergent to L. Write $K_\sigma(\epsilon) := \{\sigma(m) \leq k \leq \sigma^p(m) : |x_k - L| \geq \epsilon\}$. Then

$$|t_{pm} - L| = |\frac{1}{p}\sum_{j=1}^{p} x_{\sigma^j(m)} - L| = |\frac{1}{p}\sum_{k=\sigma(m)}^{\sigma^p(m)} x_k - L| = |\frac{1}{p}\sum_{k=\sigma(m)}^{\sigma^p(m)} (x_k - L)|,$$

$$\leq |\frac{1}{p}\sum_{k\in K_\sigma(\epsilon)} (x_k - L)| \leq \frac{1}{p}(\sup_k |x_k - L|)(K_\sigma(m+1, m+p)),$$

$$\leq \frac{1}{p}(\sup_k |x_k - L|)(\max_m K_\sigma(m+1, m+p)),$$

$$= \frac{1}{p}(\sup_k |x_k - L|)n^p \to 0 \text{ as } p \to \infty,$$

which implies that $t_{pm} \to L$ as $p \to \infty$, uniformly in m. That is, x is $\sigma(I)$-convergent to L.

For converse, consider the case $\sigma(n) = n + 1$ and the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} 1 \ ; & \text{if } k \text{ is odd,} \\ -1 \ ; & \text{if } k \text{ is even.} \end{cases} \tag{2.0}$$

Of course this sequence is not convergent or $I(\sigma)$-convergent. On the other hand x is σ-convergent to 0 and hence $\sigma(I)$-convergent to 0.

This completes the proof of the theorem.

Theorem 2.4. A sequence $x = (x_k)$ is $\sigma(I)$-convergent to L if and only if there exists a set $K = \{k_1 < k_2 < ... < k_n < ..\} \in F(I)$ and $\sigma\text{-lim}\, x_{k_n} = L$.

Proof. Suppose that there exists a set $K = \{k_1 < k_2 < ... < k_n < ..\} \in F(I)$ and $\sigma\text{-lim}\, x_{k_n} = L$. Then there is a positive integer N such that for $n > N$,

$$\varphi(|x_{k_n} - L|) < \epsilon. \tag{2.1}$$

Put $K_\epsilon(\varphi) := \{n \in \mathbb{N} : \varphi(|x_{k_n} - L|) \geq \epsilon\}$ and $K' = \{k_{N+1}, k_{N+2}, ...\}$. Then $K' \in F(I)$ and $K_\epsilon(\varphi) \in I$ which implies that $x = (x_k)$ is $\sigma(I)$-convergent to L .

Conversely, let $x = (x_k)$ be $\sigma(I)$-convergent to L. For $r = 1, 2, 3, ...$, put $K_r(\varphi) := \{j \in \mathbb{N} : \varphi(|x_{k_j} - L|) \geq 1/r\}$ and $M_r(\varphi) := \{j \in \mathbb{N} : \varphi(|x_{k_j} - L|) < 1/r\}$.Then $\delta(K_r(\varphi)) = 0$ and

$$M_1(\varphi) \supset M_2(\varphi) \supset M_i(\varphi) \supset M_{i+1}(\varphi) \supset \tag{2.2}$$

and

$$\delta(M_r(\varphi)) = 1, \ r = 1, 2, 3, ... \tag{2.3}$$

Now we have to show that for $j \in M_r(\varphi), (x_{k_j})$ is σ-convergent to L. Suppose that (x_{k_j}) is not σ-convergent to L.Therefore there is $\epsilon > 0$ such that $\varphi(|x_{k_j} - L|) \geq \epsilon$ for infinitely many terms. Let $M_\epsilon(\varphi) := \{j \in \mathbb{N} : \varphi(|x_{k_j} - L|) < \epsilon\}$ and $\epsilon > 1/r$ $(r = 1, 2, 3, ...)$. Then

$$\delta(M_\epsilon(\varphi)) = 0, \tag{2.4}$$

and by (2.2), $M_r(\varphi) \subset M_\epsilon(\varphi)$. Hence $\delta(M_r(\varphi)) = 0$, which contradicts (2.3) and therefore (x_{k_j}) is σ-convergent to L.

This completes the proof of the theorem.

Similarly on the same lines we can prove the following dual statement:

Theorem 2.5. A sequence $x = (x_k)$ is $I(\sigma)$-statistically convergent to L if and only if there exists a set $K = \{k_1 < k_2 < ... < k_n < ..\} \in F(I(\sigma))$ and $\lim x_{k_n} = L$.

Theorem 2.6. (*a*) If $0 < q < \infty$ and a sequence $x = (x_k)$ is strongly σ_q-convergent to the limit L, then it is $I(\sigma)$-convergent to L.

(*b*) If $x = (x_k)$ is bounded and $I(\sigma)$-statistically convergent to L then $x_k \longrightarrow L[V_\sigma]_q$.

Proof. (*a*) If $0 < q < \infty$ and $x_k \longrightarrow L[V_\sigma]_q$, then as $p \to \infty$,

$$0 \leftarrow \frac{1}{p}\sum_{j=1}^{p} |x_{\sigma^j(m)} - L|^q \geq \frac{1}{p} \sum_{\substack{j=1 \\ |x_{\sigma^j(m)}-L|\geq\epsilon}}^{p} |x_{\sigma^j(m)} - L|^q$$

$$\geq \frac{\epsilon^q}{p} K_\sigma(m+1, m+p) \geq \frac{\epsilon^q}{p} N^p.$$

That is, $\lim_{p\to\infty} N^p/p = 0$ and so $\delta_\sigma(K_\epsilon) = 0$, where $K_\epsilon := \{k \leq n : |x_k - L| \geq \epsilon\}$. Hence $x = (x_k)$ is $I(\sigma)$-converget to L.

(*b*) Suppose that $x = (x_k)$ is bounded and $I(\sigma)$-onvergent to L. Then for $\epsilon > 0$, we have $\delta_\sigma(K_\epsilon) = 0$. Since $x \in l_\infty$, there exists $M > 0$ such that $|x_k - L| \leq M$ $(k = 1, 2, ...)$. For every $m \in \mathbb{N}$, we have

$$\frac{1}{p}\sum_{j=1}^{p} |x_{\sigma^j(m)} - L|^q = \frac{1}{p} \sum_{\substack{k=\sigma(m) \\ k\notin K_\epsilon}}^{\sigma^p(m)} |x_k - L|^q + \frac{1}{p} \sum_{\substack{k=\sigma(m) \\ k\in K_\epsilon}}^{\sigma^p(m)} |x_k - L|^q,$$

$$= S_1(m,p) + S_2(m,p),$$

where

$$S_1(m,p) = \frac{1}{p} \sum_{\substack{k=\sigma(m) \\ k\notin K_\epsilon}}^{\sigma^p(m)} |x_k - L|^q,$$

and

$$S_2(m,p) = \frac{1}{p} \sum_{k=\sigma(m)\ k\in K_\epsilon}^{\sigma^p(m)} |x_k - L|^q.$$

Now if $k \notin K_\epsilon$ then $S_1(m,p) < \epsilon^q$. For $k \in K_\epsilon$, we have

$$S_2(m,p) \leq (\sup |x_k - L|)(\max_{m\geq 0} K_\sigma(m+1, m+p)/p) \leq M\frac{N^p}{p} \to 0,$$

as $p \to \infty$, since $\delta_\sigma(K_\epsilon) = 0$. Hence $x_k \longrightarrow L[V_\sigma]_q$.

This completes the proof of the theorem.

3. Application: A Korovkin type approximation theorem

Let $C_M[a,b]$ be the space of all continuous functions on $[a,b]$ and bounded on the entire line, that is $|f(x)| \leq K_f, -\infty < x < \infty$, where K_f is a constant depending on f. Also we denote the space of all bounded functions on $[a,b]$ by $B[a,b]$ which is equipped

with the supremum norm $\| f \|_{B[a,b]} = \sup_{x\in[a,b]} |f(x)|$, $(f \in B[a,b])$. Let L be a linear operator from $C_M[a,b]$ into $B[a,b]$.Then, we say that L is a positive linear operator if $f \geq 0$ implies $L(f) \geq 0$. Also, the value of $L(f)$ at a point $x \in [a,b]$ is denoted by $L(f(u);x)$ or, briefly by $L(f;x)$.

Throughout the paper, we also use the following test functions

$$f_0(u) = 1,\ f_1(u) = u, \text{and } f_2(u) = u^2.$$

The following is the classical Korovkin approximation theorem (cf. [12]).

Theorem A. Let $\{L_m\}$ be a sequence of positive linear operators acting from $C_M[a,b]$ into $B[a,b]$. Then, for all $f \in C_M[a,b]$,

$$\lim \|L_m(f) - f\|_{B[a,b]} = 0 \text{ if and only if}$$
$$\lim \|L_m(f_j) - f_j\|_{B[a,b]} = 0 \ (j = 0,1,2), \text{ where } f_j(u) = u^j,\ j = 0,1,2.$$

For the last few years several variations and generalizations of this theorem have been appeared. Recently in [5], the $\delta(\sigma)$-version of the above theorem has been proved and $\mathcal{I}$-version is proved in [1] and [7]. Quite recently, Korovkin type approximation theorem for a function of two variables by using almost convergence of double sequences is proved in [2]. Now we prove the following.

Theorem 3.1. Let I be an admissible ideal in $\mathbb{N}$. Let $\{L_m\}$ be a sequence of positive linear operators from $C_M[a,b]$ into $B[a,b]$. Then, for all $f \in C_M[a,b]$,

$$\sigma(I) - \lim \|L_m(f) - f\|_{B[a,b]} = 0 \tag{3.1}$$

if and only if

$$\sigma(I) - \lim \|L_m(f_j) - f_j\|_{B[a,b]} = 0 \ (j = 0,1,2), \tag{3.2}$$

where $f_j(u) = u^j$, $j = 0,1,2$.

Proof. Since $f \in C_M[a,b]$ and f is bounded on the whole real line, we have

$$|f(x)| \leq K, \quad -\infty < x < \infty.$$

Also, since f is continuous on $[a,b]$, we write that for every $\varepsilon > 0$, there exists a number $\delta_1 > 0$ such that for all $x \in [a,b]$,

$$|f(u) - f(x)| < \varepsilon, \quad \forall\ |u - x| < \delta_1.$$

Hence we get

$$|f(u) - f(x)| < \varepsilon + \frac{2K}{\delta_1^2}(t - x)^2 \tag{3.3}$$

Using the properties of positivity and monotonicity of L_m, we obtain, from (3.3),

$$\begin{aligned}\left|t_{pm}(L_m(f;x))-f(x)\right| &= \left|\frac{L_m(f;x)+L_{\sigma(m)}(f;x)+\cdots+L_{\sigma^p(m)}(f;x)}{p+1}-f(x)\right.\\ &\quad -\frac{[L_m(f_0;x)+L_{\sigma(m)}(f_0;x)+\cdots+L_{\sigma^p(m)}(f_0;x)]f(x)}{p+1}\\ &\quad \left.+\frac{[L_m(f_0;x)+L_{\sigma(m)}(f_0;x)+\cdots+L_{\sigma^p(m)}(f_0;x)]f(x)}{p+1}\right|\\ &\leq \frac{L_m(|f(u)-f(x)|;x)+L_{\sigma(m)}(|f(u)-f(x)|;x)+\cdots+L_{\sigma^p(m)}(|f(u)-f(x)|;x)}{p+1}\\ &\quad +|f(x)|\left|t_{pm}(L_m(f_0;x))-f_0(x)\right|\\ &\leq \varepsilon+(\varepsilon+K+\frac{2K}{\delta_1^2}c^2)\left|t_{pm}(L_m(f_0;x))-f_0(x)\right|\\ &\quad +\frac{4K}{\delta_1^2}c\left|t_{pm}(L_m(f_1;x))-f_1(x)\right|+\frac{2K}{\delta_1^2}\left|t_{pm}(L_m(f_2;x))-f_2(x)\right|,\end{aligned}$$

where $c := \max|x|$. Then taking the supremum we get

$$\begin{aligned}\|t_{pm}(L_m(f))-f\|_{B[a,b]} &\leq \varepsilon+N\Big\{\|t_{pm}(L_m(f_0))-f_0\|_{B[a,b]}\\ &\quad +\|t_{pm}(L_m(f_1))-f_1\|_{B[a,b]}+\|t_{pm}(L_m(f_2))-f_2\|_{B[a,b]}\Big\}\end{aligned}$$

where

$$N := \max\{\varepsilon+K+\frac{2K}{\delta_1^2}c^2,\frac{4K}{\delta_1^2}c,\frac{2K}{\delta_1^2}\}.$$

Now, take

$$\begin{aligned}D &= \left\{p\in\mathbb{N}:\|t_{pm}(L_m(f))-f\|_{B[a,b]}\geq\varepsilon\right\},\\ D_0 &= \left\{p\in\mathbb{N}:\|t_{pm}(L_m(f_0))-f_0\|_{B[a,b]}\geq\varepsilon\right\}.\\ D_1 &= \left\{p\in\mathbb{N}:\|t_{pm}(L_m(f_1))-f_1\|_{B[a,b]}\geq\varepsilon\right\},\\ D_2 &= \left\{p\in\mathbb{N}:\|t_{pm}(L_m(f_2))-f_2\|_{B[a,b]}\geq\varepsilon\right\}.\end{aligned}$$

Then $D\subseteq D_0\cup D_1\cup D_2$ and $D\in I$ if and only if $D_j\in I$ $(j=0,1,2)$. That is (3.1) holds if and only if (3.2) holds.

This completes the proof of the theorem.

In the following we show that our theorem is stronger than the classical Korovkin theorem.

Example 3.2. Let $\sigma(n) = n+1$ and $I = I_\delta = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$. Consider the sequence of classical Bernstein polynomials

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k};\ 0 \le x \le 1.$$

Let the sequence (P_n) be defined by $P_n : C_M[0,1] \to B[0,1]$ with $P_n(f,x) = (1+z_n)B_n(f,x)$, where z_n is defined as above by (2.0). Then

$$B_n(f_0,x) = 1,\ B_n(f_1,x) = x,\ B_n(f_2,x) = x^2 + \frac{x-x^2}{n},$$

and the sequence (P_n) satisfies the conditions (3.2) and hence we have

$$\sigma(I)\text{-}\lim \|P_n(f,x) - f(x)\|_{B[0,1]} = 0.$$

On the other hand, we get $P_n(f,0) = (1+ z_n)f(0)$, since $B_n(f,0) = f(0)$, and hence

$$\|P_n(f,x) - f(x)\|_{B[0,1]} \geq \mid P_n(f,0) - f(0) \mid = z_n \mid f(0) \mid .$$

We see that (P_n) does not satisfy the conditions of Theorem A, since $\lim_{n\to\infty} z_n$ does not exists.

Therefore, we conclude that our Theorem 3.1 works for our new method while the classical Korovkin approximation theorem does not work.

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STABILITY OF HIGHER RING DERIVATIONS IN FUZZY BANACH ALGEBRAS

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Abstract. In this paper, we take account of the stability for higher ring derivations in fuzzy Banach algebras.

1. Introduction and preliminaries

The stability problem of functional equations has originally been formulated by S.M. Ulam [26] : *Under what condition does there exists a homomorphism near an approximate homomorphism*? As an answer to the problem of Ulam, D.H. Hyers has proved the stability of the additive functional equation [13], which states that *if* $\varepsilon > 0$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is mapping with* $\mathcal{X}$ *a normed space,* $\mathcal{Y}$ *a Banach space, such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{1.1}$$

for all $x, y \in \mathcal{X}$, *then there exists a unique additive mapping* $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - \mathcal{L}(x)\| \leq \varepsilon$$

for all $x \in \mathcal{X}$. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $f(x+y) = f(x) + f(y)$.

A generalized version of the theorem of Hyers for additive mappings was given by T. Aoki [1] (cf. also [7]) and for linear mappings was presented by Th.M. Rassias [23] by considering the case when the inequality (1.1) is unbounded. Due to the fact, the additive functional equation is said to have the *generalized Hyers-Ulam stability* (the *Hyers-Ulam-Rassias stability*) property. Since then, a great deal of work has been done by a number of authors and the problems concerned with the generalizations and the applications of the stability to functional equations have been developed as well (for instant, [8, 11, 12, 14, 22]). In particular, the stability concerning derivations between operator algebras was first obtained by P. Šemrl [25]. Recently, R. Badora [2] gave a generalization of the Bourgin's result [6]. He also dealt with the Hyers-Ulam stability and the Bourgin-type superstability of derivations in [3].

We recall the following definition (See [14]).

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the real or complex field $\mathbb{F}$. Let $\mathbb{N}$ be the set of the natural numbers. From $m \in \mathbb{N} \cup \{0\}$, a sequence $H = \{h_0, h_1, \cdots, h_m\}$

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(resp. $H = \{h_0, h_1, \cdots, h_k, \cdots\}$) of additive operators from $\mathcal{A}$ into $\mathcal{B}$ is called a *higher ring derivation* of rank m (resp. infinite rank) if the functional equation

$$h_k(xy) = \sum_{i=0}^{k} h_i(x)h_{k-i}(y).$$

holds for each $k = 0, 1, \cdots, m$ (resp. $k = 0, 1, \cdots$) and for all $x, y \in \mathcal{A}$. A higher ring derivation H of additive operators on $\mathcal{A}$ is *strong* if h_0 is an identity operator.

Of course, a higher ring derivation of rank 0 from $\mathcal{A}$ into $\mathcal{B}$ (resp. a strong higher ring derivation of rank 1 on $\mathcal{A}$) is a ring homomorphism (resp. a ring derivation). Note that a higher ring derivation is a generalization of both a ring homomorphism and a ring derivation.

A.K. Katsaras [15] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some authors have defined fuzzy norms on a vector space from various points of view [10, 17, 27]. In particular, T. Bag and S.K. Samanta [4], following S.C. Cheng and J.N. Mordeson [9], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [16]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [5].

Following [4], we give the notation of fuzzy norm as follows (cf. [18, 19, 20, 21]).

Definition 1.2. Let $\mathcal{X}$ be a real vector space. A function $N : \mathcal{X} \times \mathbb{R} \to [0, 1]$ is said to be a *fuzzy norm* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and $a, b \in \mathbb{R}$,

(N1) $N(x, a) = 0$ if $a \leq 0$;
(N2) $x = 0$ if and only if $N(x, a) = 1$ for all $a > 0$;
(N3) $N(ax, b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
(N4) $N(x + y, a + b) \geq \min\{N(x, a), N(y, b)\}$
(N5) $N(x, .)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x, a) = 1$;
(N6) For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

In this case, the pair $(\mathcal{X}, N)$ is called a *fuzzy normed vector space*.

The examples of fuzzy norms and the properties of fuzzy normed vector spaces are given in [18, 19, 20, 21].

Definition 1.3. Let $(\mathcal{X}, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be *convergent* if there exists $x \in \mathcal{X}$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In this case, x is the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 1.4. Let $(\mathcal{X}, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $\mathcal{X}$ is called *Cauchy* if for each ε and each $a > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

Definition 1.5. [24] Let $\mathcal{X}$ be an algebra and $(\mathcal{X}, N)$ be a fuzzy Banach space. The pair $(\mathcal{X}, N)$ is said to be a *fuzzy Banach algebra* if for every $x, y \in \mathcal{X}$, $a, b \in \mathbb{R}$,

$$N(xy, ab) \geq \min\{N(x, a), N(y, b)\}.$$

Example 1.6. Let $(\mathcal{X}, \|.\|)$ be a Banach algebra. Define

$$N(x,a) = \begin{cases} 0 & \text{if } a \leq \|x\| \\ 1 & \text{if } a > \|x\|. \end{cases}$$

Then $(\mathcal{X}, N)$ is a fuzzy Banach algebra (See [24]).

In the present paper, we use the definition of fuzzy normed spaces given in [18, 19, 20, 21] to establish the stability of higher ring derivations in fuzzy Banach algebras.

2. Main results

Theorem 2.1. *Let $\mathcal{A}$ be an algebra and let $(\mathcal{C}, N')$ be a fuzzy normed space. Assume that $\varphi : \mathcal{A}^2 \to \mathcal{C}$ and $\phi : \mathcal{A}^2 \to \mathcal{C}$ are functions such that for some $0 < \alpha < 1$ and some $0 < \beta < 2$,*

$$N'(\varphi(2x, 2y), t) \geq N'(\alpha\varphi(x,y), t) \quad \text{and} \quad N'(\phi(2x,y), t) \geq N'(\beta\varphi(x,y), t) \tag{2.1}$$

for all $x, y \in \mathcal{A}$ and all $t > 0$. Suppose that $F = \{f_0, f_1, \cdots, f_k, \cdots\}$ is a sequence of mappings from $\mathcal{A}$ to a fuzzy Banach algebra $(\mathcal{B}, N)$ such that for each $k = 0, 1, \cdots$,

$$N(f_k(x+y) - f_k(x) - f_k(y), t) \geq N'(\varphi(x,y), t) \tag{2.2}$$

and

$$N(f_k(xy) - \sum_{i=0}^{k} f_i(x) f_{k-i}(y), s) \geq N'(\phi(x,y), s) \tag{2.3}$$

for all $x, y \in \mathcal{A}$ and all $t, s > 0$. Then there exists a unique higher ring derivation $H = \{h_0, h_1, \cdots\}$ of any rank such that for each $k = 0, 1, \cdots$,

$$N(f_k(x) - h_k(x), t) \geq N'\Big(\frac{2\varphi(x,x)}{2-\alpha}, t\Big) \tag{2.4}$$

for all $x \in \mathcal{A}$ and all all $t > 0$. In addition,

$$\sum_{i=0}^{k} h_i(x)\{f_{k-i}(y) - h_{k-i}(y)\} = 0 \tag{2.5}$$

for all $x \in \mathcal{A}$.

Proof. From (2.2) and Theorem 3.1 [19], we note that for each $k = 0, 1, \cdots$, there exists a unique additive operator $h_k : \mathcal{A} \to \mathcal{B}$ satisfying (2.4), where

$$h_k(x) = N - \lim_{n\to\infty} \frac{f_k(2^n x)}{2^n} \tag{2.6}$$

for all $x \in \mathcal{A}$.

Now we are in the position to verify that the sequence $H = \{h_0, h_1, \cdots, h_k, \cdots\}$ satisfies the identity

$$h_k(xy) = \sum_{i=0}^{k} h_i(x) h_{k-i}(y)$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Let us assume that $\Delta_k : \mathcal{A}^2 \to \mathcal{B}$ is a function defined by

$$\Delta_k(x, y) = f_k(xy) - \sum_{i=0}^{k} f_i(x) f_{k-i}(y)$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Hence we see that

$$N - \lim_{n\to\infty} \frac{\Delta_k(2^n x, y)}{2^n} = 0 \tag{2.7}$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Indeed, by letting $x := 2^n x$ and $s := 2^n s$ in (2.3), then we get

$$N\Big(\frac{f_k(2^n xy)}{2^n} - \sum_{i=0}^{k} \frac{f_i(2^n x) f_{k-i}(y)}{2^n}, s\Big) \geq N'\Big(\phi(x, y), \frac{2^n}{\beta^n} s\Big),$$

which implies that the condition (2.7) holds as $n \to \infty$. Now, using (2.6), we have

$$h_k(xy) = N - \lim_{n\to\infty} \frac{f_k(2^n x \cdot y)}{2^n} \tag{2.8}$$

$$= N - \lim_{n\to\infty} \sum_{i=0}^{k} \Big[\frac{f_i(2^n x) f_{k-i}(y)}{2^n} + \frac{\Delta_k(2^n x, y)}{2^n}\Big]. \tag{2.9}$$

Note that

$$N\Big(\sum_{i=0}^{k} \Big[\frac{f_i(2^n x) f_{k-i}(y)}{2^n} - \sum_{i=0}^{k} h_i(x) f_{k-i}(y) + \frac{\Delta_k(2^n x, y)}{2^n}\Big], s\Big) \tag{2.10}$$

$$\geq \min\Big\{N\Big(\sum_{i=0}^{k} \frac{f_i(2^n x) f_{k-i}(y)}{2^n} - \sum_{i=0}^{k} h_i(x) f_{k-i}(y), \frac{s}{2}\Big), N\Big(\sum_{i=0}^{k} \frac{\Delta_k(2^n x, y)}{2^n}, \frac{s}{2}\Big)\Big\}$$

for all $s > 0$. In particular, by the additivity of each h_k, $k = 0, 1, \cdots$, we obtain

$$N\Big(\sum_{i=0}^{k} \frac{f_i(2^n x) f_{k-i}(y)}{2^n} - \sum_{i=0}^{k} h_i(x) f_{k-i}(y), \frac{s}{2}\Big) \tag{2.11}$$

$$\geq \min_{0\leq i\leq k} \Big\{N\Big(\frac{f_i(2^n x) f_{k-i}(y)}{2^n} - h_i(x) f_{k-i}(y), \frac{s}{2(k+1)}\Big)\Big\}$$

$$= \min_{0\leq i\leq k} \Big\{N\Big(f_i(2^n x) f_{k-i}(y) - h_i(2^n x) f_{k-i}(y), \frac{2^{n-1} s}{k+1}\Big)\Big\}$$

for all $s > 0$. Since $(\mathcal{B}, N)$ is a fuzzy Banach algebra, we see that

$$N\Big(f_i(2^n x) f_{k-i}(y) - h_i(2^n x) f_{k-i}(y), \frac{2^{n-1} s}{k+1}\Big) \tag{2.12}$$

$$\geq \min\Big\{N(f_i(2^n x) - h_i(2^n x), \sqrt{s}), N\Big(f_{k-i}(y), \frac{2^{n-1}\sqrt{s}}{k+1}\Big)\Big\}$$

for all $s > 0$. On the other hand, replacing x by $2^n x$ and s by $\sqrt{s}$ in (2.4), we have by (2.1)

$$N(f_i(2^n x) - h_i(2^n x), \sqrt{s}) \geq N'\Big(\varphi(x, x), \frac{2-\alpha}{2} \cdot \frac{\sqrt{s}}{\alpha^n}\Big)$$

for all $s > 0$. The right-hand side of the above inequality tends to 1 as $n \to \infty$. Hence

$$\lim_{n\to\infty} N(f_i(2^n x) - h_i(2^n x), \sqrt{s}) = 1 \tag{2.13}$$

for all $s > 0$. By virtue of (2.12) and (2.13), we deduce that

$$\lim_{n\to\infty} N\Big(f_i(2^n x)f_{k-i}(y) - h_i(2^n x)f_{k-i}(y), \frac{2^{n-1}s}{k+1}\Big) = 1$$

for all $s > 0$. Therefore, in view of (2.11), we have

$$\lim_{n\to\infty} N\Big(\sum_{i=0}^{k} \frac{f_i(2^n x)f_{k-i}(y)}{2^n} - \sum_{i=0}^{k} h_i(x)f_{k-i}(y), \frac{s}{2}\Big) = 1. \tag{2.14}$$

Sending $n \to \infty$ in (2.10), we have by (2.14) and (2.7)

$$N - \lim_{n\to\infty} \sum_{i=0}^{k} \Big[\frac{f_i(2^n x)f_{k-i}(y)}{2^n} + \frac{\Delta_k(2^n x, y)}{2^n}\Big] = \sum_{i=0}^{k} h_i(x)h_{k-i}(y).$$

Thus the previous result and (2.8) imply that

$$h_k(xy) = \sum_{i=0}^{k} h_i(x)f_{k-i}(y) \tag{2.15}$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Let $n \in \mathbb{N}$ be fixed. Then, applying (2.15) and the additivity of each h_k, $k = 0, 1, \cdots$, we get

$$\begin{aligned}\sum_{i=0}^{k} h_i(x)f_{k-i}(2^n y) &= h_k(x \cdot 2^n y) = h_k(2^n x \cdot y)\\ &= \sum_{i=0}^{k} h_i(2^n x)f_{k-i}(y) = 2^n \sum_{i=0}^{k} h_i(x)f_{k-i}(y)\end{aligned}$$

Hence we have

$$\sum_{i=0}^{k} h_i(x)f_{k-i}(y) = \sum_{i=0}^{k} h_i(x)\frac{f_{k-i}(2^n y)}{2^n} \tag{2.16}$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. As in the above proof with (2.16), we see that

$$\begin{aligned}\sum_{i=0}^{k} h_i(x)h_{k-i}(y) &= N - \lim_{n\to\infty} \sum_{i=0}^{k} h_i(x)\frac{f_{k-i}(2^n y)}{2^n} \qquad (2.17)\\ &= \sum_{i=0}^{k} h_i(x)f_{k-i}(y)\end{aligned}$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$, which means (2.5). Combining (2.16) with (2.17), it follows that H satisfies the relation

$$h_k(xy) = \sum_{i=0}^{k} h_i(x)h_{k-i}(y).$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. This completes the proof of the theorem. □

Corollary 2.2. *Let $(\mathcal{A}, N)$ be a fuzzy Banach algebra with unit and let $(\mathcal{C}, N')$ be a fuzzy normed space. Assume that $\varphi : \mathcal{A}^2 \to \mathcal{C}$ and $\phi : \mathcal{A}^2 \to \mathcal{C}$ are functions satisfying* (2.1). *Suppose that $F = \{f_0, f_1, \cdots, f_k, \cdots\}$ is a sequence of operators on $\mathcal{A}$ satisfying* (2.2) *and* (2.3)*, where f_0 is an identity operator. Then F is a strong higher ring derivation of any rank on $\mathcal{A}$.*

Proof. By induction, we lead to the conclusion. From (2.6), we see that for all $x \in A$,

$$h_0(x) = N - \lim_{n\to\infty} \frac{f_0(2^n x)}{2^n} = x$$

and so $h_0(= f_0)$ is an identity operator. If $k = 1$, it follows from (2.5) that $f_1(x) = h_1(x)$ for all $x \in \mathcal{A}$, since $\mathcal{A}$ contains the unit. Let us assume that $f_m(x) = h_m(x)$ is valid for all $x \in \mathcal{A}$ and $m < k$. Then (2.5) implies that

$$x(f_k(y) - h_k(y)) = 0.$$

Since $\mathcal{A}$ contains the unit, $f_k(y) = h_k(y)$ for all $y \in \mathcal{A}$. Hence we conclude that $f_k(x) = h_k(x)$ for all $k = 0, 1, \cdots$ and all $x \in \mathcal{A}$ and Theorem 2.1 tell us that F is a strong higher ring derivation of any rank on $\mathcal{A}$, which completes the proof.

Theorem 2.3. *Let $\mathcal{A}$ be an algebra. Suppose that $F = \{f_0, f_1, \cdots, f_k, \cdots\}$ is a sequence of mappings from $\mathcal{A}$ to a fuzzy Banach algebra $(\mathcal{B}, N)$ such that for each $k = 0, 1, \cdots$, $f_k(x) = 0$. Assume that $\varphi : \mathcal{A} \to \mathcal{C}$ and $\phi : \mathcal{A} \to \mathcal{C}$ are functions from $\mathcal{A}$ to a fuzzy normed space $(\mathcal{C}, N')$ such that for each $k = 0, 1, \cdots$,*

$$N\Big(2f_k\Big(\frac{x+y}{2}\Big) - f_k(x) - f_k(y), t+s\Big) \geq \min\{N'(\varphi(x), t), N'(\varphi(y), s)\} \tag{2.18}$$

and

$$N(f_k(xy) - \sum_{i=0}^{k} f_i(x) f_{k-i}(y), u+v) \geq \min\{N'(\phi(x), u), N'(\phi(y), v)\} \tag{2.19}$$

for all $x, y \in \mathcal{A}\backslash\{0\}$ and all $t, s, u, v > 0$. If $\varphi(3x) = \alpha\varphi(x)$ and $\phi(3x) = \beta\phi(x)$ for some $0 < \alpha < 1$ and some $0 < \beta < 3$, then there exists a unique higher ring derivation $H = \{h_0, h_1, \cdots, h_k, \cdots\}$ of any rank such that for each $k = 0, 1, \cdots$,

$$N(f_k(x) - h_k(x), t) \geq M\Big(x, \frac{(3-\alpha)t}{6}\Big) \tag{2.20}$$

for all $x \in \mathcal{A}$ and all $t > 0$, where

$$M(x,t) := \min\Big\{N'\Big(\varphi(x), \frac{3}{4}t\Big), N'\Big(\varphi(-x), \frac{3}{4}t\Big), N'\Big(\varphi(3x), \frac{3}{4}t\Big)\Big\}.$$

Moreover,

$$\sum_{i=0}^{k} h_i(x)\{f_{k-i}(y) - h_{k-i}(y)\} = 0 \tag{2.21}$$

for all $x \in \mathcal{A}$.

Proof. It follows by (2.18) and Theorem 2.1 in [18] that for each $k = 0, 1, \cdots$, there exists a unique additive mapping $h_k : \mathcal{A} \to \mathcal{B}$ satisfying (2.20), where

$$h_k(x) = N - \lim_{n\to\infty} \frac{f_k(3^n x)}{3^n} \tag{2.22}$$

for all $x \in \mathcal{A}$

Next we want to show that the sequence $H = \{h_0, h_1, \cdots, h_k, \cdots\}$ satisfies the relation

$$h_k(xy) = \sum_{i=0}^{k} h_i(x)h_{k-i}(y)$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. We now define a function $\Delta_k : \mathcal{A}^2 \to \mathcal{B}$ by

$$\Delta_k(x, y) = f_k(xy) - \sum_{i=0}^{k} f_i(x)f_{k-i}(y)$$

for each $k = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$. Letting $x := 3^n x, u := \frac{3^n}{2}t$ and $v := \frac{3^n}{2}t$ in (2.19), we obtain that for each $k = 0, 1, \cdots$,

$$N\Big(\frac{f_k(3^n xy)}{3^n} - \sum_{i=0}^{k} \frac{f_i(3^n x)}{3^n} f_{k-i}(y), t\Big) \geq \min\Big\{N'\Big(\phi(x), \frac{3^n}{2\alpha^n}t\Big), N'\Big(\phi(y), \frac{3^n}{2\alpha^n}t\Big)\Big\}$$

for all $x, y \in \mathcal{A}$ and all $t > 0$. Hence we deduce that

$$N - \lim_{n\to\infty} \frac{\Delta_k(3^n x, y)}{3^n} = 0 \tag{2.23}$$

as $n \to \infty$.

The remaining part of the proof can be carried out similarly as in that of the previous theorem. □

Similarly, as we did in the proof of Corollary 2.2, we also apply theorem 2.3 and prove the following.

Corollary 2.4. *Let $\mathcal{A}$ be a fuzzy Banach algebra with unit. If $F = \{f_0, f_1, \cdots, f_k, \cdots\}$ is a sequence of operators on $\mathcal{A}$ such that for each $k = 0, 1, \cdots, f_k(0) = 0$. Assume that $\varphi : \mathcal{A} \to \mathcal{C}$ and $\phi : \mathcal{A} \to \mathcal{C}$ are functions from $\mathcal{A}$ to a fuzzy normed space $(\mathcal{C}, N')$ satisfying* (2.18) *and* (2.19)*, where f_0 is an identity operator. If $\varphi(3x) = \alpha\varphi(x)$ and $\phi(3x) = \beta\phi(x)$ for some $0 < \alpha < 1$ and some $0 < \beta < 3$, then F is a strong higher ring derivation of any rank on $\mathcal{A}$.*

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L_p Convergence With Rates of General Singular Integral Operators

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Abstract. In this article we study the general singular integral operators on the line regarding their convergence to the unit operator with rates in the L_p norm, $p \geq 1$. The related established inequalities involve the higher order L_p modulus of smoothness of the engaged function or its higher order derivative. We give particular applications to trigonometric singular integral operators.

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1 Introduction

The rate of convergence of singular integrals has been studied in [12], [13], [14], [7], [8], [3], [4], [5], [6] and these articles motivate our work. Here we study the L_p, $p \geq 1$, convergence of general singular integral operators on $\mathbb{R}$ to the unit operator with rates over smooth functions with higher order derivatives in $L_p(\mathbb{R})$. We establish related Jackson type inequalities involving the higher L_p modulus of smoothness of the engaged function or its higher order derivative. The discussed operators are not in general positive. Other motivation derives from [1], [2]. Especially we give applications to trigonometric singular integral operator.

2 Results

In the next we introduce and deal with the *smooth general singular integral operators* $\Theta_{r,\xi}(f;x)$ defined as follows.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Let $\xi > 0$ and let μ_ξ be Borel probability measures on $\mathbb{R}$.

Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$, we define for $x \in \mathbb{R}$, $\xi > 0$ the integral

$$\Theta_{r,\xi}(f;x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) d\mu_\xi(t). \tag{2}$$

$\Theta_{r,\xi}$ operators are not positive operators, see [9].

We notice that $\Theta_{r,\xi}(c,x) = c$, c constant, and

$$\Theta_{r,\xi}(f;x) - f(x) = \sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x+jt) - f(x))\, d\mu_\xi(t). \tag{3}$$

We need the rth L_p-modulus of smoothness

$$\omega_r(f^{(n)}, h)_p := \sup_{|t| \leq h} \|\Delta_t^r f^{(n)}(x)\|_{p,x}, \quad h > 0, \tag{4}$$

where

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x+jt), \tag{5}$$

see [10], p. 44. Here we have that $\omega_r(f^{(n)}, h)_p < \infty$, $h > 0$.

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}. \tag{6}$$

Call

$$\tau(w,x) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x+jw) - \delta_n f^{(n)}(x). \tag{7}$$

Notice also that

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{8}$$

According to [3], p. 306, [1], we get

$$\tau(w,x) = \Delta_w^r f^{(n)}(x). \tag{9}$$

Thus

$$\|\tau(w,x)\|_{p,x} \leq \omega_r(f^{(n)}, |w|)_p, \quad w \in \mathbb{R}. \tag{10}$$

Using Taylor's formula one has

$$\sum_{j=0}^{r} \alpha_j [f(x+jt) - f(x)] = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k t^k + \mathcal{R}_n(0,t,x), \tag{11}$$

where

$$\mathcal{R}_n(0,t,x) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!} \tau(w,x) dw, \quad n \in \mathbb{N}. \tag{12}$$

Assume

$$c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t) \in \mathbb{R}, \ k = 1, \ldots, n. \tag{13}$$

Using the above terminology we derive

$$\Delta(x) := \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k c_{k,\xi} = \mathcal{R}_n^*(x), \tag{14}$$

where

$$\mathcal{R}_n^*(x) := \int_{-\infty}^{\infty} \mathcal{R}_n(0,t,x)d\mu_\xi(t), \quad n \in \mathbb{N}. \tag{15}$$

We present the first result.

Theorem 1. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ and the rest as above. Furthermore assume that*

$$M_\xi := \int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1 \right) |t|^{np-1} \, d\mu_\xi(t) < \infty.$$

Then

$$\|\Delta(x)\|_p \le \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}\,(rp+1)^{\frac{1}{p}}} \cdot \left(\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1 \right) |t|^{np-1} \, d\mu_\xi(t) \right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)}, \xi)_p. \tag{16}$$

If $M_\xi \le \lambda$, $\forall \xi > 0, \lambda > 0$, and as $\xi \to 0$ we get that $\|\Delta(x)\|_p \to 0$.

Proof. We observe that

$$\begin{aligned} |\Delta(x)|^p = |\mathcal{R}_n^*(x)|^p &= \left| \int_{-\infty}^{\infty} \mathcal{R}_n(0,t,x)d\mu_\xi(t) \right|^p \\ &\le \left(\int_{-\infty}^{\infty} |\mathcal{R}_n(0,t,x)| \, d\mu_\xi(t) \right)^p \\ &\le \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!} |\tau(sign(t)w,x)| dw \right) d\mu_\xi(t) \right)^p. \end{aligned} \tag{17}$$

Hence we have

$$I_1 := \int_{-\infty}^{\infty} |\Delta(x)|^p dx \le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \gamma(t,x) d\mu_\xi(t) \right)^p dx = I_2, \tag{18}$$

where

$$\gamma(t,x) := \int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!} |\tau(sign(t)w,x)| dw \ge 0. \tag{19}$$

Hence

$$I_2 \le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \gamma^p(t,x) d\mu_\xi(t) \right) dx = I_3. \tag{20}$$

But we need to treat

$$\gamma(t,x) \le \frac{\left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw \right)^{\frac{1}{p}}}{(n-1)!} \frac{|t|^{\left(n-1+\frac{1}{q}\right)}}{(q(n-1)+1)^{1/q}} \tag{21}$$

That is

$$\gamma^p(t,x) \le \frac{\left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw \right)}{((n-1)!)^p} \frac{|t|^{np-1}}{(q(n-1)+1)^{p/q}}. \tag{22}$$

Consequently we have

$$I_3 \leq \frac{1}{((n-1)!)^p(q(n-1)+1)^{p/q}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw \right) |t|^{np-1} \right] d\mu_\xi(t) \right) dx \tag{23}$$

(calling

$$c_1 := \frac{1}{((n-1)!)^p(q(n-1)+1)^{p/q}})$$

$$= c_1 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} |\Delta^r_{sign(t)w} f^{(n)}(x))|^p dw \right) dx \right) |t|^{np-1} d\mu_\xi(t) \tag{24}$$

$$= c_1 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} |\Delta^r_{sign(t)w} f^{(n)}(x))|^p dx \right) dw \right) |t|^{np-1} d\mu_\xi(t) \tag{25}$$

$$\leq c_1 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_r(f^{(n)}, w)_p^p \, dw \right) |t|^{np-1} d\mu_\xi(t).$$

So far we have proved

$$I_1 \leq c_1 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_r(f^{(n)}, w)_p^p dw \right) |t|^{np-1} d\mu_\xi(t) = I_4. \tag{26}$$

By [10], p. 45 we have

$$I_4 = c_1 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_r(f^{(n)}, \xi \frac{w}{\xi})_p^p dw \right) |t|^{np-1} d\mu_\xi(t)$$

$$\leq c_1 \omega_r(f^{(n)}, \xi)_p^p \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{w}{\xi} \right)^{rp} dw \right) |t|^{np-1} d\mu_\xi(t) \right)$$

$$= \frac{\xi c_1}{(rp+1)} \omega_r(f^{(n)}, \xi)_p^p \left(\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{rp+1} - 1 \right) |t|^{np-1} d\mu_\xi(t) \right). \tag{27}$$

We have proved that

$$I_1 \leq \frac{\xi \omega_r(f^{(n)}, \xi)_p^p}{((n-1)!)^p(q(n-1)+1)^{p/q}\,(rp+1)} \left(\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{rp+1} - 1 \right) |t|^{np-1} d\mu_\xi(t) \right), \tag{28}$$

proving the claim. ■

The counterpart of Theorem 1 follows, case of $p = 1$.

Theorem 2. *Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R}), n \in \mathbb{N}$. Assume that*

$$\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right) |t|^{n-1} \, d\mu_\xi(t) < \infty.$$

Then

$$\|\Delta(x)\|_1 \leq \frac{1}{(r+1)\,(n-1)!} \left(\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right) |t|^{n-1} d\mu_\xi(t) \right) \xi \omega_r(f^{(n)}, \xi)_1. \tag{29}$$

Additionally assume that

$$\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\,d\mu_{\xi}(t)\leq\lambda,\lambda>0,$$

$\forall\xi>0$. *Hence as* $\xi\to 0$ *we obtain* $\|\Delta(x)\|_1\to 0$.

Proof. It follows

$$\begin{aligned}|\Delta(x)| &= \left|\int_{-\infty}^{\infty}\mathcal{R}_n(0,t,x)d\mu_{\xi}(t)\right| \\ &\leq \int_{-\infty}^{\infty}|\mathcal{R}_n(0,t,x)|\,d\mu_{\xi}(t) \\ &\leq \int_{-\infty}^{\infty}\left(\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)\cdot w,x)|dw\right)d\mu_{\xi}(t).\end{aligned}\tag{30}$$

Thus

$$\begin{aligned}\|\Delta(x)\|_1 &= \int_{-\infty}^{\infty}|\Delta(x)|dx \\ &\leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w,x)|dw\right)d\mu_{\xi}(t)\right)dx =: (*)\end{aligned}\tag{31}$$

But we see that

$$\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w,x)|dw\leq\frac{|t|^{n-1}}{(n-1)!}\int_0^{|t|}|\tau(sign(t)w,x)|dw.\tag{32}$$

Therefore it holds

$$\begin{aligned}(*) &\leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\frac{|t|^{n-1}}{(n-1)!}\int_0^{|t|}|\tau(sign(t)w,x)|dw\right)d\mu_{\xi}(t)\right)dx \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\frac{|t|^{n-1}}{(n-1)!}\int_0^{|t|}|\tau(sign(t)w,x)|dw\right)dx\right)d\mu_{\xi}(t) \\ &= \frac{1}{(n-1)!}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}|\tau(sign(t)w,x)|dw\right)dx\right)|t|^{n-1}d\mu_{\xi}(t) \\ &= \frac{1}{(n-1)!}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty}|\tau(sign(t)w,x)|dx\right)dw\right)|t|^{n-1}d\mu_{\xi}(t)\right) \\ &\leq \frac{1}{(n-1)!}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_r(f^{(n)},w)_1dw\right)|t|^{n-1}d\mu_{\xi}(t)\right).\end{aligned}\tag{33}$$

That is we get

$$\|\Delta(x)\|_1\leq\frac{1}{(n-1)!}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_r(f^{(n)},w)_1dw\right)|t|^{n-1}d\mu_{\xi}(t)\right).\tag{34}$$

Consequently we have

$$\begin{aligned}\|\Delta(x)\|_1 &\leq \frac{1}{(n-1)!}\omega_r(f^{(n)},\xi)_1\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{w}{\xi}\right)^r dw\right)|t|^{n-1}d\mu_{\xi}(t)\right) \\ &\leq \frac{\xi\omega_r(f^{(n)},\xi)_1}{(n-1)!\,(r+1)}\left(\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\,d\mu_{\xi}(t)\right),\end{aligned}\tag{35}$$

proving the claim. ■

The case $n = 0$ is met next.

Proposition 3. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the rest as above. Assume that*

$$\rho_\xi := \int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t) < \infty.$$

Then

$$\|\Theta_{r,\xi}(f) - f\|_p \leq \omega_r(f, \xi)_p \left(\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t)\right)^{\frac{1}{p}}. \tag{36}$$

Additionally assume that $\rho_\xi \leq \lambda, \lambda > 0$, $\forall \xi > 0$, then as $\xi \to 0$ we obtain $\Theta_{r,\xi} \to$ unit operator I in the L_p norm, $p > 1$.

Proof. We notice that

$$\Theta_{r,\xi}(f; x) - f(x) = \int_{-\infty}^{\infty} \left(\Delta_t^r f\right)(x) d\mu_\xi(t). \tag{37}$$

Hence

$$|\Theta_{r,\xi}(f; x) - f(x)| \leq \int_{-\infty}^{\infty} \left|\left(\Delta_t^r f\right)(x)\right| d\mu_\xi(t). \tag{38}$$

We next estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\Theta_{r,\xi}(f; x) - f(x)|^p dx \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left|\left(\Delta_t^r f\right)(x)\right| d\mu_\xi(t)\right)^p dx \\
&\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left|\left(\Delta_t^r f\right)(x)\right|^p d\mu_\xi(t)\right) dx \qquad (39) \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left|\left(\Delta_t^r f\right)(x)\right|^p dx\right) d\mu_\xi(t) \\
&\leq \int_{-\infty}^{\infty} \left(\omega_r(f, |t|)_p^p\right) d\mu_\xi(t) \\
&= \int_{-\infty}^{\infty} \left(\omega_r\left(f, \xi \frac{|t|}{\xi}\right)_p^p\right) d\mu_\xi(t) \\
&\leq \omega_r(f, \xi)_p^p \int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t), \qquad (40)
\end{aligned}$$

proving the claim. ■

We also give

Proposition 4. *Assume*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{r} d\mu_\xi(t) < \infty.$$

Then

$$\|\Theta_{r,\xi}(f) - f\|_1 \leq \omega_r(f, \xi)_1 \left(\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{r} d\mu_\xi(t)\right). \tag{41}$$

Additionally assuming that

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{r} d\mu_\xi(t) \leq \lambda, \lambda > 0,$$

$\forall \xi > 0$, we get as $\xi \to 0$ that $\Theta_{r,\xi} \to I$ in the L_1 norm.

Proof. We do have again

$$|\Theta_{r,\xi}(f;x)-f(x)| \leq \int_{-\infty}^{\infty} |\Delta_t^r f(x)|\, d\mu_\xi(t). \tag{42}$$

We estimate

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Theta_{r,\xi}(f;x)-f(x)|\, dx \leq & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\Delta_t^r f(x)|\, d\mu_\xi(t) \right) dx \qquad (43)\\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\Delta_t^r f(x)|\, dx \right) d\mu_\xi(t)\\
&\leq \int_{-\infty}^{\infty} \omega_r(f,|t|)_1 d\mu_\xi(t)\\
&= \int_{-\infty}^{\infty} \omega_r(f,\xi\tfrac{|t|}{\xi})_1 d\mu_\xi(t) \qquad (44)\\
&\leq \omega_r(f,\xi)_1 \left(\int_{-\infty}^{\infty} \left(1+\frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right), \qquad (45)
\end{aligned}$$

proving the claim. ■

3 Applications to General Trigonometric Singular Operators

We make

Remark 5. We need the following preliminary result.

Let p and m be integers with $1 \leq p \leq m$. We define the integral

$$I(m;p) := \int_{-\infty}^{\infty} \frac{(\sin x)^{2m}}{x^{2p}} dx = 2\int_0^{\infty} \frac{(\sin x)^{2m}}{x^{2p}} dx. \tag{46}$$

This is an (absolutely) convergent integral.

According to [11], page 210, item 1033, we obtain

$$I(m;p) = \pi \frac{(-1)^p (2m)!}{4^{m-p}(2p-1)!} \sum_{k=1}^{m} (-1)^k \frac{k^{2p-1}}{(m-k)!(m+k)!}. \tag{47}$$

In particular, for $p=m$ the above formula becomes

$$\int_0^{\infty} \frac{(\sin x)^{2m}}{x^{2m}} dx = \pi(-1)^m m \sum_{k=1}^{m} (-1)^k \frac{k^{2m-1}}{(m-k)!(m+k)!}. \tag{48}$$

In this section we apply the general theory of this article to the trigonometric smooth general singular integral operators $T_{r,\xi}(f,x)$ defined as follows. Let $\xi > 0$.

Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$, $\beta \in \mathbb{N}$, we define for $x \in \mathbb{R}$ the integral

$$T_{r,\xi}(f;x) := \frac{1}{W} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) \left(\frac{\sin(t/\xi)}{t} \right)^{2\beta} dt, \tag{49}$$

where

$$\begin{aligned} W &= \int_{-\infty}^{\infty} \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt \\ &= 2\xi^{1-2\beta} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^{2\beta} dt. \qquad (50) \\ &\overset{(48)}{=} 2\xi^{1-2\beta}\pi(-1)^{\beta}\beta \sum_{k=1}^{\beta}(-1)^k \frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}. \end{aligned}$$

$T_{r,\xi}$ operators are not positive operators, see [9].

Let $\lceil\cdot\rceil$ denote the ceiling of a real number. We present our first result of this section

Theorem 6. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n, \beta \in \mathbb{N}$, $\beta > \frac{\lceil rp\rceil + np + 1}{2}$ and the rest as above. Then*

$$\|\Delta(x)\|_p \leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \qquad (51)$$
$$\cdot \left[\frac{1}{\int_0^{\infty}\left(\frac{\sin t}{t}\right)^{2\beta} dt} \sum_{j=1}^{\lceil rp\rceil+1} \int_0^{\infty} \left[t^{np-1+j}\left(\frac{\sin t}{t}\right)^{2\beta}\right] dt\right]^{\frac{1}{p}} \xi^n \omega_r(f^{(n)}, \xi)_p.$$

Moreover, as $\xi \to 0$ we get that $\|\Delta(x)\|_p \to 0$.

Proof. It is well known that $\lim_{t\to 0} \frac{\sin t}{t} = 1$. Therefore, $\lim_{t\to 0}\left(\frac{\sin t}{t}\right)^{2\beta} = 1$. Hence, on $(0,1]$, the following function $t^j\left(\frac{\sin t}{t}\right)^{2\beta}$ is continuous and bounded. Therefore

$$\int_0^1 t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt = \int_{0+}^1 t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt < \infty. \qquad (52)$$

For $j \in \mathbb{R}$ such that $2\beta - j > 1$ we have

$$\begin{aligned} \int_1^{\infty} t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt &\leq \int_1^{\infty} t^j \left(\frac{1}{t}\right)^{2\beta} dt \\ &= \int_1^{\infty} t^{j-2\beta} dt \\ &= \frac{1}{2\beta - j - 1} \\ &< \infty. \qquad (53) \end{aligned}$$

Relations (52) and (53) imply

$$\begin{aligned} \int_0^{\infty} t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt &= \int_0^1 t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt + \int_1^{\infty} t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt \\ &\leq \int_{0+}^1 t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt + \int_1^{\infty} t^j \left(\frac{1}{t}\right)^{2\beta} dt \\ &< \infty. \qquad (54) \end{aligned}$$

By (54) we obtain, for $\beta > \frac{\lceil rp \rceil + np + 1}{2}$,

$$\begin{aligned} M_\xi &:= \frac{1}{W} \int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{rp+1} - 1 \right) |t|^{np-1} \left(\frac{\sin(t/\xi)}{t} \right)^{2\beta} dt \\ &= \frac{2\xi^{np-2\beta}}{W} \int_0^{\infty} \left((1+t)^{rp+1} - 1 \right) t^{np-1} \left(\frac{\sin t}{t} \right)^{2\beta} dt \\ &\leq \frac{2\xi^{np-2\beta}}{W} \int_0^{\infty} \left((1+t)^{\lceil rp \rceil + 1} - 1 \right) t^{np-1} \left(\frac{\sin t}{t} \right)^{2\beta} dt \\ &= \frac{2\xi^{np-2\beta}}{W} \int_0^{\infty} \left[\left(\sum_{j=1}^{\lceil rp \rceil + 1} t^j \right) t^{np-1} \left(\frac{\sin t}{t} \right)^{2\beta} \right] dt \\ &= \frac{2\xi^{np-2\beta}}{W} \sum_{j=1}^{\lceil rp \rceil + 1} \int_0^{\infty} \left[t^{np-1+j} \left(\frac{\sin t}{t} \right)^{2\beta} \right] dt \qquad (55) \\ &< \infty. \end{aligned}$$

Using Theorem 1 we obtain

$$\|\Delta(x)\|_p \leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} M_\xi^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)}, \xi)_p. \qquad (56)$$

Using (55) we get

$$M_\xi \leq \frac{2\xi^{np-2\beta}}{W} \sum_{j=1}^{\lceil rp \rceil + 1} \int_0^{\infty} \left[t^{np-1+j} \left(\frac{\sin t}{t} \right)^{2\beta} \right] dt \qquad (57)$$

therefore,

$$\begin{aligned} \|\Delta(x)\|_p \leq & \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \\ & \cdot \left[\frac{1}{\int_0^{\infty} \left(\frac{\sin t}{t} \right)^{2\beta} dt} \sum_{j=1}^{\lceil rp \rceil + 1} \int_0^{\infty} \left[t^{np-1+j} \left(\frac{\sin t}{t} \right)^{2\beta} \right] dt \right]^{\frac{1}{p}} \xi^n \omega_r(f^{(n)}, \xi)_p \qquad (58) \end{aligned}$$

proving the claim of the theorem. ■

The counterpart of Theorem 6 follows, case of $p = 1$.

Theorem 7. *Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R}), n \in \mathbb{N}$, $\beta \in \mathbb{N}$, $\beta > \frac{r+1+n}{2}$. Then*

$$\|\Delta(x)\|_1 \leq \frac{1}{(r+1)(n-1)! \left[\int_0^{\infty} \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]} \sum_{j=1}^{r+1} \left(\int_0^{\infty} \left[t^{n-1+j} \left(\frac{\sin t}{t} \right)^{2\beta} \right] dt \right) \xi^n \omega_r(f^{(n)}, \xi)_1. \qquad (59)$$

Hence as $\xi \to 0$ we obtain $\|\Delta(x)\|_1 \to 0$.

Proof. As in the proof of Theorem 6, inequality (54), for $\beta > \frac{r+1+n}{2}$ we have,

$$\begin{aligned}
&\frac{1}{W}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\left(\frac{\sin\left(t/\xi\right)}{t}\right)^{2\beta}dt\\
&=\frac{2\xi^{n-2\beta}}{W}\int_{0}^{\infty}\left(\left(1+t\right)^{r+1}-1\right)t^{n-1}\left(\frac{\sin t}{t}\right)^{2\beta}dt\\
&=\frac{2\xi^{n-2\beta}}{W}\sum_{j=1}^{r+1}\int_{0}^{\infty}\left[t^{n-1+j}\left(\frac{\sin t}{t}\right)^{2\beta}\right]dt \qquad (60)\\
&<\infty.
\end{aligned}$$

Then, by Theorem 2, we obtain

$$\begin{aligned}
\|\Delta(x)\|_1 &\leq \frac{1}{(r+1)(n-1)!}\left(\frac{1}{W}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\left(\frac{\sin\left(t/\xi\right)}{t}\right)^{2\beta}dt\right)\xi\omega_r(f^{(n)},\xi)_1\\
&= \frac{1}{(r+1)(n-1)!}\left(\frac{2\xi^{n-2\beta}}{W}\sum_{j=1}^{r+1}\int_{0}^{\infty}\left[t^{n-1+j}\left(\frac{\sin t}{t}\right)^{2\beta}\right]dt\right)\xi\omega_r(f^{(n)},\xi)_1\\
&\overset{(50)}{=} \frac{1}{(r+1)(n-1)!\left[\int_0^{\infty}\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]}\sum_{j=1}^{r+1}\left(\int_{0}^{\infty}\left[t^{n-1+j}\left(\frac{\sin t}{t}\right)^{2\beta}\right]dt\right)\xi^n\omega_r(f^{(n)},\xi)_1,
\end{aligned}$$

poving the claim of the theorem. ■

The case $n=0$ is met next.

Proposition 8. *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $\beta\in\mathbb{N}$, $\beta>\frac{\lceil rp\rceil+1}{2}$ and the rest as above.* Then

$$\|T_{r,\xi}(f)-f\|_p \leq \omega_r(f,\xi)_p\left(\frac{1}{\left[\int_0^{\infty}\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]}\sum_{j=0}^{\lceil rp\rceil}\left[\int_0^{\infty}t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]\right)^{\frac{1}{p}}. \qquad (61)$$

Also as $\xi\to 0$ we obtain $T_{r,\xi}\to$ unit operator I in the L_p norm, $p>1$.

Proof. As in the proof of Theorem 6, inequality (54), for $\beta>\frac{\lceil rp\rceil+1}{2}$ we have

$$\begin{aligned}
\rho_\xi &:= \frac{1}{W}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp}\left(\frac{\sin\left(t/\xi\right)}{t}\right)^{2\beta}dt\\
&=\frac{2\xi^{1-2\beta}}{W}\int_0^{\infty}(1+t)^{rp}\left(\frac{\sin t}{t}\right)^{2\beta}dt\\
&\leq\frac{2\xi^{1-2\beta}}{W}\int_0^{\infty}(1+t)^{\lceil rp\rceil}\left(\frac{\sin t}{t}\right)^{2\beta}dt\\
&=\frac{2\xi^{1-2\beta}}{W}\sum_{j=0}^{\lceil rp\rceil}\left[\int_0^{\infty}t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]\\
&<\infty.
\end{aligned}$$

By use of Proposition 3 we obtain

$$\|T_{r,\xi}(f)-f\|_p \leq \omega_r(f,\xi)_p \left(\frac{2\xi^{1-2\beta}}{W}\sum_{j=0}^{\lceil rp\rceil}\left[\int_0^\infty t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]\right)^{\frac{1}{p}}$$
$$\overset{(50)}{=} \omega_r(f,\xi)_p \left(\frac{1}{\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta}dt\right]}\sum_{j=0}^{\lceil rp\rceil}\left[\int_0^\infty t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]\right)^{\frac{1}{p}}$$

proving the claim of the proposition. ■

We also give

Proposition 9. *For $\beta\in\mathbb{N}$, $\beta>\frac{r+1}{2}$, we have*

$$\|T_{r,\xi}(f)-f\|_1 \leq \frac{\omega_r(f,\xi)_1}{\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta}dt\right]}\sum_{j=0}^{r}\left[\int_0^\infty t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]. \tag{62}$$

Moreover *as $\xi\to 0$ we get that $T_{r,\xi}\to I$ in the L_1 norm.*

Proof. As in the proof of Theorem 6, inequality (54), for $\beta>\frac{r+1}{2}$ we have

$$\frac{1}{W}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^r\left(\frac{\sin(t/\xi)}{t}\right)^{2\beta}dt$$
$$=\frac{2\xi^{1-2\beta}}{W}\int_0^\infty (1+t)^r\left(\frac{\sin t}{t}\right)^{2\beta}dt$$
$$=\frac{2\xi^{1-2\beta}}{W}\sum_{j=0}^{r}\left[\int_0^\infty t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]$$
$$<\infty.$$

By Proposition 4 we obtain

$$\|T_{r,\xi}(f)-f\|_1 \leq \omega_r(f,\xi)_1\left(\frac{1}{W}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^r\left(\frac{\sin(t/\xi)}{t}\right)^{2\beta}dt\right)$$
$$\overset{(50)}{=} \frac{\omega_r(f,\xi)_1}{\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta}dt\right]}\sum_{j=0}^{r}\left[\int_0^\infty t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right],$$

which proves the claim of the proposition.

■

4 Applications to Particular Trigonometric Singular Operators

In this section we work on the approximation results given in the previous section, for some particular values of r, n, p and β.

Case $\beta=2$. We have the following results

Corollary 10. *Let $f\in C^1(\mathbb{R})$ and $f'\in L_1(\mathbb{R})$. Then*

$$\|T_{1,\xi}(f;x)-f(x)\|_1 \leq \frac{3}{2\pi}\left(\ln 2+\frac{\pi}{4}\right)\xi\omega_1(f',\xi)_1. \tag{63}$$

Hence as $\xi \to 0$ we obtain $\|T_{1,\xi}(f;x) - f(x)\|_1 \to 0$.
Proof. By Theorem 7, with $r = n = 1, \beta = 2$. ■

Corollary 11. *It holds*

$$\|T_{1,\xi}(f) - f\|_{1.5} \leq \omega_1(f,\xi)_{1.5} \left(\frac{7}{4} + \frac{3}{\pi}\ln 2\right)^{\frac{2}{3}}. \tag{64}$$

Also as $\xi \to 0$ we obtain $T_{1,\xi} \to$ unit operator I in the $L_{1.5}$ norm.
Proof. By Proposition 8, with $p = 1.5, r = 1, \beta = 2$. ■

Corollary 12. *We have*

$$\|T_{1,\xi}(f) - f\|_2 \leq \omega_1(f,\xi)_2 \sqrt{\frac{7}{4} + \frac{3}{\pi}\ln 2}. \tag{65}$$

Also as $\xi \to 0$ we obtain $T_{1,\xi} \to$ unit operator I in the L_2 norm.
Proof. By Proposition 8, with $p = 2, r = 1, \beta = 2$. ■

Corollary 13. *It holds*

$$\|T_{1,\xi}(f) - f\|_1 \leq \omega_1(f,\xi)_1 \left(\frac{3\ln 2}{\pi} + 1\right). \tag{66}$$

Moreover *as $\xi \to 0$ we get that $T_{1,\xi} \to I$ in the L_1 norm.*
Proof. By Proposition 9, with $r = 1, \beta = 2$. ■

Corollary 14. *We have*

$$\|T_{2,\xi}(f) - f\|_1 \leq \omega_2(f,\xi)_1 \left(\frac{7}{4} + \frac{3}{\pi}\ln 2\right). \tag{67}$$

Moreover *as $\xi \to 0$ we get that $T_{2,\xi} \to I$ in the L_1 norm.*
Proof. By Proposition 9, with $r = 2, \beta = 2$. ■

Case $\beta = 3$. We have the following results
Corollary 15. *It holds*

$$\|T_{1,\xi}(f;x) - f(x)\|_2 \leq \left(\sqrt{\frac{5}{22\pi}\ln\frac{256}{27} + \frac{25}{66}}\right) \xi\omega_1(f',\xi)_p. \tag{68}$$

Moreover, as $\xi \to 0$ we get that $\|T_{1,\xi}(f;x) - f(x)\|_2 \to 0$.
Proof. By Theorem 6, with $p = 2, r = 1, n = 1, \beta = 3$. ■

Corollary 16. *Let $f \in C^1(\mathbb{R})$ and $f' \in L_1(\mathbb{R})$. Then*

$$\|T_{1,\xi}(f;x) - f(x)\|_1 \leq \left(\frac{20}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{5}{22}\right) \xi\omega_1(f',\xi)_1. \tag{69}$$

Hence as $\xi \to 0$ we obtain $\|T_{1,\xi}(f;x) - f(x)\|_1 \to 0$.
Proof. By Theorem 7, with $r = 1, n = 1, \beta = 3$. ■

Corollary 17. *Let $f \in C^1(\mathbb{R})$ and $f' \in L_1(\mathbb{R})$. Then*

$$\|T_{2,\xi}(f;x) - f(x)\|_1 \leq \left(\frac{40}{33\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{5}{33} + \frac{5}{22\pi}\ln\frac{256}{27}\right) \xi\omega_2(f',\xi)_1. \tag{70}$$

Hence as $\xi \to 0$ *we obtain* $\|T_{2,\xi}(f;x) - f(x)\|_1 \to 0$.
Proof. By Theorem 7, with $r = 2, n = 1, \beta = 3$. ■

Corollary 18. *Let* $f \in C^2(\mathbb{R})$ and $f'' \in L_1(\mathbb{R})$. *Then*

$$\|T_{2,\xi}(f;x) - f(x) - \frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_1 \leq \left(\frac{25}{66} + \frac{5}{22\pi}\ln\frac{256}{27}\right)\xi^2\omega_2(f'',\xi)_1. \tag{71}$$

Hence as $\xi \to 0$ *we obtain* $\|T_{2,\xi}(f;x) - f(x) - \frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_1 \to 0$.
Proof. By Theorem 7, with $r = 2, n = 2, \beta = 3$. ■

Corollary 19. *It holds*

$$\|T_{2,\xi}(f) - f\|_2 \leq \omega_2(f,\xi)_2\sqrt{\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{15}{22\pi}\ln\frac{256}{27} + \frac{47}{22}}. \tag{72}$$

Also as $\xi \to 0$ *we obtain* $T_{r,\xi} \to$ *unit operator* I *in the* L_2 *norm.*
Proof. By Proposition 8, with $p = r = 2, \beta = 3$. ■

Corollary 20. *It holds*

$$\|T_{1,\xi}(f) - f\|_4 \leq \omega_1(f,\xi)_4\sqrt[4]{\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{15}{22\pi}\ln\frac{256}{27} + \frac{47}{22}}. \tag{73}$$

Also as $\xi \to 0$ *we obtain* $T_{r,\xi} \to$ *unit operator* I *in the* L_p *norm,* $p > 1$.
Proof. By Proposition 8, with $p = 4, r = 1, \beta = 3$. ■

Corollary 21. *It holds*

$$\|T_{1,\xi}(f) - f\|_3 \leq \omega_1(f,\xi)_3\sqrt[3]{\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{16}{11} + \frac{15}{22\pi}\ln\frac{256}{27}}. \tag{74}$$

Also as $\xi \to 0$ *we obtain* $T_{r,\xi} \to$ *unit operator* I *in the* L_p *norm,* $p > 1$.
Proof. By Proposition 8, with $p = 3, r = 1, \beta = 3$. ■

Corollary 22. *We have*

$$\|T_{1,\xi}(f) - f\|_1 \leq \omega_1(f,\xi)_1\left(1 + \frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right)\right). \tag{75}$$

Moreover *as* $\xi \to 0$ *we get that* $T_{1,\xi} \to I$ *in the* L_1 *norm.*
Proof. By Proposition 9, with $r = 1, \beta = 3$. ■

Corollary 23. *We have*

$$\|T_{2,\xi}(f) - f\|_1 \leq \omega_2(f,\xi)_1\left(\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{16}{11}\right). \tag{76}$$

Moreover *as* $\xi \to 0$ *we get that* $T_{2,\xi} \to I$ *in the* L_1 *norm.*
Proof. By Proposition 9, with $r = 2, \beta = 3$. ■

Corollary 24. *We have*

$$\|T_{3,\xi}(f)-f\|_1 \leq \omega_3(f,\xi)_1\left(\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right)+\frac{16}{11}+\frac{15}{22\pi}\ln\frac{256}{27}\right). \tag{77}$$

Moreover *as* $\xi\to 0$ *we get that* $T_{3,\xi}\to I$ *in the* L_1 *norm.*
Proof. By Proposition 9, with $r=3,\beta=3$. ■

Case $\beta=4$. We have the following results
Corollary 25. *It holds*

$$\|T_{2,\xi}(f;x)-f(x)\|_2 \leq \left(\sqrt{\frac{126}{151\pi}\ln\left(\frac{3^{27/8}}{32}\right)+\frac{651}{2416}+\frac{567}{604\pi}\ln\frac{4}{3}}\right)\xi\omega_2(f',\xi)_2. \tag{78}$$

Moreover, as $\xi\to 0$ *we get that* $\|T_{2,\xi}(f;x)-f(x))\|_2\to 0$.
Proof. By Theorem 6, with $p=r=2,n=1,\beta=4$. ■

Corollary 26. *It holds*

$$\|T_{1,\xi}(f;x)-f(x)\|_2 \leq \left(\sqrt{\frac{35}{151}+\frac{210}{151\pi}\ln\left(\frac{3^{27/8}}{32}\right)}\right)\xi\omega_1(f',\xi)_2. \tag{79}$$

Moreover, as $\xi\to 0$ *we get that* $\|T_{1,\xi}(f;x)-f(x)\|_2\to 0$.
Proof. By Theorem 6, with $p=2,r=n=1,\beta=4$. ■

Corollary 27. *It holds*

$$\|T_{2,\xi}(f;x)-f(x)-\frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_{1.5} \leq \frac{1}{4}\left[\frac{630}{151\pi}\ln\frac{3^{9/4}}{2^{11/4}}+\frac{2415}{2416}\right]^{\frac{2}{3}}\xi^2\omega_2(f'',\xi)_{1.5}. \tag{80}$$

Moreover, as $\xi\to 0$ *we get that* $\|T_{2,\xi}(f;x)-f(x)-\frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_{1.5}\to 0$.
Proof. By Theorem 6, with $p=1.5,r=2,n=2,\beta=4$. ■

Corollary 28. *Let* $f\in C^1(\mathbb{R})$ and $f'\in L_1(\mathbb{R})$. *Then*

$$\|T_{2,\xi}(f;x)-f(x)\|_1 \leq \left(\frac{210}{151\pi}\ln\left(\frac{2^{104/15}}{3^{81/20}}\right)+\frac{35}{302}+\frac{210}{151\pi}\ln\left(\frac{3^{27/8}}{32}\right)\right)\xi\omega_2(f',\xi)_1. \tag{81}$$

Hence as $\xi\to 0$ *we obtain* $\|T_{2,\xi}(f;x)-f(x)\|_1\to 0$.
Proof. By Theorem 7, with $r=2,n=1,\beta=4$. ■

Corollary 29. *Let* $f\in C^1(\mathbb{R})$ and $f'\in L_1(\mathbb{R})$. *Then*

$$\|T_{5,\xi}(f;x)-f(x)\|_1 \leq \left(\frac{105}{151\pi}\ln\frac{2^{29/15}}{3^{27/40}}+\frac{945}{1208\pi}\ln\frac{4}{3}+\frac{1085}{4832}\right)\xi\omega_5(f',\xi)_1. \tag{82}$$

Hence as $\xi\to 0$ *we obtain* $\|T_{5,\xi}(f;x)-f(x)\|_1\to 0$.
Proof. By Theorem 7, with $r=5,n=1,\beta=4$. ■

Corollary 30. *Let* $f\in C^n(\mathbb{R})$ and $f^{(n)}\in L_1(\mathbb{R})$. *Then*

$$\|T_{2,\xi}(f;x)-f(x)-\frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_1 \leq \left(\frac{105}{151\pi}\ln\frac{3^{9/4}}{2^{11/4}}+\frac{35}{604}\right)\xi^3\omega_2(f''',\xi)_1. \tag{83}$$

Hence as $\xi \to 0$ *we obtain* $\|T_{2,\xi}(f;x) - f(x) - \frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_1 \to 0$.
Proof. By Theorem 7, with $r = 2, n = 3, \beta = 4$. ■

Corollary 31. *Let* $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R})$. *Then*

$$\|T_{3,\xi}(f;x) - f(x) - \frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_1 \leq \left(\frac{315}{604\pi}\ln\frac{3^{9/4}}{2^{11/4}} + \frac{2415}{19328}\right)\xi^3\omega_3(f''',\xi)_1. \tag{84}$$

Hence as $\xi \to 0$ *we obtain* $\|T_{3,\xi}(f;x) - f(x) - \frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_1 \to 0$.
Proof. By Theorem 7, with $r = 3, n = 3, \beta = 4$. ■

Corollary 32. *It holds*

$$\|T_{2,\xi}(f) - f\|_2 \leq \omega_2(f,\xi)_2\sqrt{\frac{630}{151\pi}\ln\frac{2^{29/15}}{3^{27/40}} + \frac{256}{151}}. \tag{85}$$

Also as $\xi \to 0$ *we obtain* $T_{2,\xi} \to$ *unit operator* I *in the* L_2 *norm.*
Proof. By Proposition 8, with $p = r = 2, \beta = 4$. ■

Corollary 33. *It holds*

$$\|T_{2,\xi}(f) - f\|_3 \leq \omega_2(f,\xi)_3\sqrt[3]{\frac{630}{151\pi}\ln\frac{2^{251/60}}{3^{9/5}} + \frac{5671}{2416}}. \tag{86}$$

Also as $\xi \to 0$ *we obtain* $T_{2,\xi} \to$ *unit operator* I *in the* L_3 *norm.*
Proof. By Proposition 8, with $p = 3, r = 2, \beta = 4$. ■

Corollary 34. *It holds*

$$\|T_{3,\xi}(f) - f\|_2 \leq \omega_3(f,\xi)_2\sqrt{\frac{630}{151\pi}\ln\frac{2^{251/60}}{3^{9/5}} + \frac{5671}{2416}}. \tag{87}$$

Also as $\xi \to 0$ *we obtain* $T_{3,\xi} \to$ *unit operator* I *in the* L_2 *norm.*
Proof. By Proposition 8, with $p = 2, r = 3, \beta = 4$. ■

Corollary 35. *It holds*

$$\|T_{1,\xi}(f) - f\|_2 \leq \omega_1(f,\xi)_2\sqrt{\frac{630}{151\pi}\ln\frac{2^{104/15}}{3^{81/20}} + \frac{407}{302}}. \tag{88}$$

Also as $\xi \to 0$ *we obtain* $T_{1,\xi} \to$ *unit operator* I *in the* L_2 *norm.*
Proof. By Proposition 8, with $p = 2, r = 1, \beta = 4$. ■

Corollary 36. *We have*

$$\|T_{1,\xi}(f) - f\|_1 \leq \omega_1(f,\xi)_1\left(1 + \frac{630}{151\pi}\ln\frac{2^{104/15}}{3^{81/20}}\right). \tag{89}$$

Moreover *as* $\xi \to 0$ *we get that* $T_{1,\xi} \to I$ *in the* L_1 *norm.*
Proof. By Proposition 9, with $r = 1, \beta = 4$. ■

Corollary 37. *We have*

$$\|T_{2,\xi}(f) - f\|_1 \leq \omega_2(f,\xi)_1\left(\frac{630}{151\pi}\ln\frac{2^{104/15}}{3^{81/20}} + \frac{407}{302}\right). \tag{90}$$

Moreover *as* $\xi \to 0$ *we get that* $T_{2,\xi} \to I$ *in the* L_1 *norm.*
Proof. By Proposition 9, with $r = 2, \beta = 4$. ■

Corollary 38. *We have*

$$\|T_{3,\xi}(f) - f\|_1 \leq \omega_3(f,\xi)_1 \left(\frac{630}{151\pi} \ln \frac{2^{29/15}}{3^{27/40}} + \frac{407}{302} \right). \tag{91}$$

Moreover *as* $\xi \to 0$ *we get that* $T_{3,\xi} \to I$ *in the* L_1 *norm.*
Proof. By Proposition 9, with $r = 3, \beta = 4$. ■

Corollary 39. *We have*

$$\|T_{6,\xi}(f) - f\|_1 \leq \omega_6(f,\xi)_1 \left(\frac{630}{151\pi} \ln \frac{2^{251/60}}{3^{9/5}} + \frac{5671}{2416} \right). \tag{92}$$

Moreover *as* $\xi \to 0$ *we get that* $T_{6,\xi} \to I$ *in the* L_1 *norm.*
Proof. By Proposition 9, with $r = 6, \beta = 4$. ■

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CONVERGENCE THEOREMS OF ITERATIVE SCHEMES FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN METRIC SPACES

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Abstract. The purpose of this paper is to introduce and study the convergence problem of an implicit iterative scheme with errors for a finite family of asymptotically quasi-nonexpansive type mappings in convex metric spaces.

1. Introduction

In 1953, Mann [20] introduced the general iterative schemes and proved the following convergence theorem of the iterative schemes:

Let K be a nonempty subset of a Banach space E and $T : K \to K$ be a mapping. For any given $x_0 \in K$, the scheme $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \quad (n \geq 0) \tag{1.1}$$

is called the *Mann iterative scheme*, where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying appropriate conditions.

Further, in 1974, Ishikawa [12] proved the following theorem by introducing the *Ishikawa iterative scheme*, that is, for any given $x_0 \in K$, the scheme $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \quad (n \geq 0), \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying appropriate conditions. In particular, if $\beta_n = 0$ for all $n \geq 0$ in (1.2), then the Ishikawa iterative scheme becomes the Mann iterative scheme (1.1).

These two iterative schemes described above have been studied extensively by many authors for approximating either fixed points of nonlinear operators or solutions of nonlinear operator equations in Banach spaces. Concerning the convergence problems of Mann and Ishikawa iterative schemes have been studied extensively by many authors, see [11], [12], [20], [30] and [31].

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In 2001, Xu-Ori [29] have introduces an *implicit iterative scheme* for a finite family of nonexpansive mappings. Let $T_1, T_2, \cdots, T_N$ be N self-mappings of C and suppose that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, the set of common fixed points of T_i, $i = 1, 2, \cdots, N$. Hereafter, we will denote the index set $\{1, 2, \cdots, N\}$ by I. An implicit iterative scheme for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$\begin{aligned} x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\ x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\ &\cdots \\ x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\ x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\ &\cdots \end{aligned}$$

which can be written in the following compact form :

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1, \tag{1.3}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in I.) And they proved the weak convergence of the process (1.3) to a common fixed point in the setting of a Hilbert space.

In 2003, Sun [23] extend the schemes (1.3) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in C$, which is defined as follows :

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 \\ &\cdots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1}, \\ &\cdots \\ x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N}, \\ x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1}, \\ &\cdots \end{aligned}$$

which can be written in the following compact form :

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1, \tag{1.4}$$

where $n = (k - 1)N + i$, $i \in I$.

And also, Sun [23] proved that the *modified implicit iterative scheme* (1.4) for a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the family in a uniformly convex Banach space, requiring one member T in the family to be semi-compact. The result presented Sun [23] generalized and extended the corresponding main results of Wittmann [28], Xu-Ori [29].

It is very interesting to consider the convergence theorems of the iterative schemes for nonlinear operators in metric spaces.

In 1970, Takahashi [24] introduced the concept of convexity in a metric space:

Let (X,d) be a metric space and $I=[0,1]$. A mapping $W: X\times X\times I\to X$ is said to be a *convex structure* on X if for each $(x,y,\lambda)\in X\times X\times I$ and $u\in X$,

$$d(u,W(x,y,\lambda))\leq \lambda d(u,x)+(1-\lambda)d(u,y).$$

X together with a convex structure W is called a *convex metric space*, denoted it by (X,d,W). A nonempty subset K of (X,d,W) is said to be *convex* if $W(x,y,\lambda)\in K$ for all $(x,y,\lambda)\in K\times K\times I$.

Recall that a function $\Phi:[0,\infty)\to[0,\infty)$ is said to *satisfy the condition* (C_Φ) if it is nondecreasing continuous from right, $\Phi(t)<t$ for all $t>0$ and $\Phi(0)=0$.

Let (X,d) be a complete metric space, $T: X\to X$ be a mapping.

(1) T is said to be *quasi-contractive* if there exists a constant $k\in[0,1)$ such that

$$\begin{aligned} &d(Tx,Ty) \\ &\leq k\max\Big\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big\}, \end{aligned} \tag{1.5}$$

for all $x,y\in X$.

(2) T is said to be *generalized quasi-contractive* if there exists a function $\Phi:[0,\infty)\to[0,\infty)$ satisfying the condition (C_Φ) such that

$$\begin{aligned} &d(Tx,Ty) \\ &\leq \Phi\Big(\max\Big\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big\}\Big), \end{aligned} \tag{1.6}$$

for all $x,y\in X$.

In 1988, Ding [8] established the following result for Ishikawa type iterative schemes of a quasi-contractive mappings in convex metric spaces.

Theorem 1.1. *Let K be a nonempty closed convex subset of a complete convex metric space (X,d,W) and let $T: K\to K$ be a quasi-contractive mapping* (1.5)*. Suppose that $\{x_n\}$ is the Ishikawa type iterative scheme defined by*

$$\begin{cases} x_0\in K, \\ x_{n+1}=W(Ty_n,x_n,\alpha_n), \\ y_n=W(Tx_n,x_n,\beta_n), \quad n\geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0\leq\alpha_n,\beta_n\leq 1$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then $\{x_n\}$ converges to a unique fixed point of T in K.

Recently, in 2003, Chang-Kim [3] proved convergence theorems of the Ishikawa type iterative schemes with errors for generalized quasi-contractive mapping (1.6) in convex metric spaces, which is improved and extended the corresponding results in [4], [8] and others.

Theorem 1.2. *Let (X, d, W) be a complete convex metric space with a convex structure $W : X^3 \times I^3 \to X$ of X, T be a generalized quasi-contractive mapping satisfying condition* (1.6) *and $\{x_n\}$ be the Ishikawa type iterative scheme with errors of T defined by*

$$\begin{cases} x_{n+1} = W(x_n, Ty_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(x_n, Tx_n, v_n; \xi_n, \eta_n, \delta_n), \quad \forall\, n \geq 0, \end{cases}$$

where $\{u_n\}, \{v_n\}$ are two sequences in X. Then the scheme $\{x_n\}$ converges to a unique fixed point of T in X.

And, Chang-Kim-Jin [2] proved convergence theorems of the Ishikawa type iterative schemes with errors for asymptotically quasi-nonexpansive type mappings in convex metric spaces.

Theorem 1.3. *Let (X, d, W) be a complete convex metric space and $T : X \to X$ be an asymptotically quasi-nonexpansive type mapping satisfying the following conditions : there exist constants $L > 0$ and $\alpha > 0$ such that*

$$d(Tx, p) \leq L \cdot d(x, p)^{\alpha}, \quad \forall\, x \in X, \ \ \forall\, p \in F(T).$$

For any given $x_0 \in X$, let $\{x_n\}$ be the iterative scheme with errors defined by

$$\begin{cases} x_{n+1} = W(x_n, T^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(x_n, T^n x_n, v_n; \alpha'_n, \beta'_n, \gamma'_n), \quad \forall\, n \geq 0, \end{cases}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in X. If the sequences $\{\beta_n\}$ and $\{\gamma_n\}$ satisfying the following conditions :

$$\sum_{n=0}^{\infty} \beta_n < \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty.$$

Then $\{x_n\}$ converges strongly to a fixed point of T in X if and only if

$$\liminf_{n \to \infty} D_d(x_n, F(T)) = 0,$$

where, $D_d(x, A) = \inf_{a \in A} d(x, a)$ for any set A.

Very recently, in 2006, Kim-Kim-Kim [15] proved convergence theorems of an implicit iterative schemes with errors for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces.

Theorem 1.4. *Let (X, d, W) be a complete convex metric space. Let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive mappings from X into X, that is,*

$$d(T_i^n x, p_i) \leq (1 + h_{n(i)}) d(x, p_i)$$

for all $x \in X$, $p_i \in F(T_i)$, $i \in I$. Suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset [s, 1-s]$ for some $s \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} h_{n(i)} < \infty \ \ (i \in I)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{u_n\}$ is arbitrary bounded sequence in X. Then the implicit iterative scheme with error $\{x_n\}$ generated by (2.2) *converges to a common fixed point in F if and only if*

$$\liminf_{n \to \infty} D_d(x_n, F) = 0,$$

where $D_d(x,F)$ *denotes the distance from* x *to the set* F*, that is,* $D_d(x,F) = \inf_{y\in F} d(x,y)$.

The purpose of this paper is to introduce and study the convergence problem of an implicit iterative sequences with errors for a finite family of asymptotically quasi-nonexpansive type mappings in convex metric spaces. And also we point out that the results of this paper not only generalize and improve the corresponding results of Chang et al. [1], Liu [17], [18], Petryshyn-Williamson [21], Tan-Xu [25], Ghosh-Debnath [10] and Chang [4]-[6] but also contain the corresponding results in [4]-[6], [10], [15]-[19], [21], [24] as its special cases.

2. Preliminaries

Throughout this paper, we assume that $F(T_i)$ $(i \in I)$ is the set of fixed points of T_i respectively, that is, $F(T_i) = \{x \in X : T_i x = x\}$. The set of common fixed points of T_i $(i \in I)$ denotes by F, that is, $F = \bigcap_{i=1}^{N} F(T_i)$.

Definition 2.1. *Let* E *be a metric space and* $T : D(T) \subset E \to E$ *be a mapping.*

(1) T *is said to be nonexpansive if*

$$d(Tx,Ty) \le d(x,y)$$

for all $x, y \in D(T)$.

(2) T *is said to be quasi-nonexpansive if* $F(T) \neq \emptyset$ *and*

$$d(Tx,p) \le d(x,p)$$

for all $x \in D(T)$ *and* $p \in F(T)$.

(3) T *is said to be asymptotically nonexpansive if there exists a sequence* $h_n \in [1,\infty)$ *with* $\lim_{n\to\infty} h_n = 1$ *such that*

$$d(T^n x, T^n y) \le h_n d(x,y)$$

for all $x, y \in D(T)$ *and* $n \ge 1$.

(4) T *is said to be asymptotically quasi-nonexpansive if* $F(T) \neq \emptyset$ *and there exists a sequence* $h_n \in [1,\infty)$ *with* $\lim_{n\to\infty} h_n = 1$ *such that*

$$d(T^n x, p) \le h_n d(x,p)$$

for all $x \in D(T)$, $p \in F(T)$ *and* $n \ge 1$.

(5) T *is said to be asymptotically nonexpansive type if*

$$\limsup_{n\to\infty}\left\{\sup_{x,y\in D(T)}\left\{d(T^n x, T^n y) - d(x,y)\right\}\right\} \le 0.$$

(6) T *is said to be asymptotically quasi- nonexpansive type if* $F(T) \neq \emptyset$ *and*

$$\limsup_{n\to\infty}\left\{\sup_{x\in D(T)}\left\{d(T^n x, p) - d(x,p)\right\}\right\} \le 0 \tag{2.1}$$

for all $p \in F(T)$.

Remark 2.2. *We know that the following implications hold :*

$$\begin{array}{ccccc} (1) & \Longrightarrow & (3) & \Longrightarrow & (5) \\ \Downarrow F(T)\neq\emptyset & & \Downarrow F(T)\neq\emptyset & & \Downarrow F(T)\neq\emptyset \\ (2) & \Longrightarrow & (4) & \Longrightarrow & (6) \end{array}$$

Definition 2.3. *([2]) Let (X,d) be a metric space and $I=[0,1]$. A mapping $W: X^3\times I^3\to X$ is said to be a convex structure on X if it satisfies the following conditions : for all $u,x,y,z\in X$ and for all $\alpha,\beta,\gamma\in I$ with $\alpha+\beta+\gamma=1$,*

(1) $W(x,y,z;\alpha,0,0)=x$,
(2) $d(u,W(x,y,z;\alpha,\beta,\gamma))\leq \alpha d(u,x)+\beta d(u,y)+\gamma d(u,z)$.

If (X,d) is a metric space with a convex structure W, then (X,d) is called a convex metric space and denotes it by (X,d,W).

Remark 2.4. *Every linear normed space is a convex metric space, where a convex structure $W(x,y,z;\alpha,\beta,\gamma)=\alpha x+\beta y+\gamma z$, for all $x,y,z\in X$ and $\alpha,\beta,\gamma\in I$ with $\alpha+\beta+\gamma=1$. In fact,*

$$\begin{aligned} d(u,W(x,y,z;\alpha,\beta,\gamma)) &= \|u-(\alpha x+\beta y+\gamma z)\| \\ &\leq \alpha\|u-x\|+\beta\|u-y\|+\gamma\|u-z\| \\ &= \alpha d(u,x)+\beta d(u,y)+\gamma d(u,z), \quad \forall\, u\in X. \end{aligned}$$

But there exists some convex metric spaces which can not be embedded into any linear normed space.

Example 2.5. *Let $X=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1>0,\ x_2>0,\ x_3>0\}$. For $x=(x_1,\ x_2,\ x_3)$, $y=(y_1,y_2,y_3)\in X$ and $\alpha,\beta,\gamma\in I$ with $\alpha+\beta+\gamma=1$, we define a mapping $W: X^3\times I^3\to X$ by*

$$W(x,y,z;\alpha,\beta,\gamma)=(\alpha x_1+\beta y_1+\gamma z_1,\ \alpha x_2+\beta y_2+\gamma z_2,\ \alpha x_3+\beta y_3+\gamma z_3)$$

and define a metric $d: X\times X\to[0,\infty)$ by

$$d(x,y)=\Big|x_1y_1+x_2y_2+x_3y_3\Big|.$$

Then we can show that (X,d,W) is a convex metric space, but it is not linear normed space.

Definition 2.6. *Let (X,d,W) be a convex metric space with a convex structure W and let $T_i: X\to X$ $(i\in I)$ be asymptotically quasi-nonexpansive type*

mappings. For any given $x_0 \in X$, *the following iterative scheme* $\{x_n\}$ *defined by*

$$\begin{aligned} x_1 &= W(x_0, T_1x_1, u_1; \alpha_1, \beta_1, \gamma_1), \\ &\cdots \\ x_N &= W(x_{N-1}, T_Nx_N, u_N; \alpha_N, \beta_N, \gamma_N), \\ x_{N+1} &= W(x_N, T_1^2x_{N+1}, u_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}), \\ &\cdots \\ x_{2N} &= W(x_{2N-1}, T_N^2x_{2N}, u_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}), \\ x_{2N+1} &= W(x_{2N}, T_1^3x_{2N+1}, u_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}) \\ &\cdots \end{aligned}$$

which can be written in the following compact form :

$$x_n = W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 1 \tag{2.2}$$

where $n = (k-1)N + i$, $i \in I$, $\{u_n\}$ *is bounded scheme in* X, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *be three sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *for* $n = 1, 2, 3, \cdots$. *Schemes (2.2) is called the implicit iterative scheme with error for a finite family of mappings* T_i $(i = 1, 2, \cdots, N)$.

If $u_n = 0$ *in (2.2) then,*

$$x_n = W(x_{n-1}, T_i^k x_n; \alpha_n, \beta_n), \quad n \geq 1, \tag{2.3}$$

where $n = (k-1)N + i$, $i \in I$, $\{\alpha_n\}$, $\{\beta_n\}$ *be two sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n = 1$ *for* $n = 1, 2, 3, \cdots$. *Scheme (2.3) is called implicit iterative for a finite family of mappings* T_i $(i = 1, 2, \cdots, N)$.

3. Main Results

In order to prove the main theorems of this paper, we need the following lemma:

Lemma 3.1. ([26]) *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two nonnegative sequences satisfying*

$$a_{n+1} \leq a_n + b_n \quad (n \geq n_0),$$

where $\sum_{n=1}^{\infty} b_n < \infty$ *and* n_0 *is some positive integer. Then* $\lim_{n\to\infty} a_n$ *exists.*

Theorem 3.2. *Let (E,d,W) be a complete convex metric space. Let* $\{T_i : i \in I\}$ *be a finite family of asymptotically quasi-nonexpansive type mappings from* E *into* E *satisfying the following conditions: there exist constants* $L_i > 0$ *and* $a_i > 0$ *such that*

$$d(T_ix, p_i) \leq L_i d(x, p_i)^{a_i} \tag{3.1}$$

for all $x \in E$, $p_i \in F(T_i)$, $i \in I$. *Suppose that* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and let* $x_0 \in E$ *be any given point,* $\{x_n\}$ *defined by the implicit iterative scheme with error* (2.2) *satisfying the following conditions:*

(i) $s \leq \beta_n \leq 1 - s$, *for some* $s \in (0, \frac{1}{2})$,
(ii) $\sum_{n=0}^{\infty} \beta_n < \infty$,
(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\{x_n\}$ strongly converges to a common fixed point in F if and only if

$$\liminf_{n\to\infty} D_d(x_n, F) = 0, \tag{3.2}$$

where $D_d(x, F) = \inf_{y\in F} d(x, y)$.

In order to prove Theorem 3.2, we first prove the following important lemma.

Lemma 3.3. *Suppose that all assumptions in Theorem 3.2 are satisfied. Then for any given $\varepsilon > 0$, there exist a positive integer n_0 and $M > 0$ such that*

(1) $d(x_n, p) \le d(x_{n-1}, p) + M\left(\beta_n + \frac{\gamma_n}{s}\right)$,

(2) $d(x_{n+m}, p) \le d(x_n, p) + M\left(\sum_{l=n+1}^{n+m} \beta_l + \frac{1}{s}\sum_{l=n+1}^{n+m} \gamma_l\right)$,

(3) $\lim_{n\to\infty} D_d(x_n, F)$ *exists,*

for any $p \in F$ and $n \ge n_0$, where $M = \max\left\{\frac{\varepsilon}{5s},\ \sup_{n\ge 0} d(u_n, p)\right\}$.

Proof. (1) For any $p \in F$, from (2.2), where $n = (k-1)N+i$, $T_n = T_{n(modN)} = T_i$, $i \in I$,

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), p) \\ &\le \alpha_n d(x_{n-1}, p) + \beta_n d(T_i^k x_n, p) + \gamma_n d(u_n, p) \\ &\le \alpha_n d(x_{n-1}, p) + \beta_n\{d(T_i^k x_n, p) - d(x_n, p)\} \\ &\quad + \beta_n d(x_n, p) + \gamma_n d(u_n, p) \end{aligned} \tag{3.3}$$

From (2.1), for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for $n \ge n_0$ with $k > \frac{n_0}{N} + 1$, we have

$$\sup_{x\in E}\{d(T_i^n x, p) - d(x, p)\} \le \frac{\varepsilon}{6}, \quad \forall\, p \in F. \tag{3.4}$$

Since $\{x_n\} \subset E$,

$$d(T_i^n x_n, p) - d(x_n, p) \le \frac{\varepsilon}{5} \quad (n \ge n_0)\ (i \in I).$$

Substituting (3.4) into (3.3) and simplifying, we have

$$d(x_n, p) \le \alpha_n d(x_{n-1}, p) + \frac{\varepsilon}{5}\cdot\beta_n + \beta_n d(x_n, p) + \gamma_n d(u_n, p),$$

for all $n \ge n_0$, $p \in F$. Then we have

$$\begin{aligned} d(x_n, p) &\le \frac{\alpha_n}{1-\beta_n} d(x_{n-1}, p) + \frac{\varepsilon}{5}\cdot\frac{\beta_n}{1-\beta_n} + \frac{\gamma_n}{1-\beta_n} d(u_n, p) \\ &\le d(x_{n-1}, p) + \frac{\varepsilon}{5s}\cdot\beta_n + \frac{\gamma_n}{s} d(u_n, p), \end{aligned} \tag{3.5}$$

for $p \in F$ and for any $n \geq n_0$. Here, let $M = \max\left\{\frac{\varepsilon}{5s},\ \sup_{n\geq 0} d(u_n, p)\right\}$. Then we have

$$d(x_n, p) \leq d(x_{n-1}, p) + M\left(\beta_n + \frac{\gamma_n}{s}\right).$$

(2) It follows from conclusion (1) that for any $m \geq 1$,

$$\begin{aligned} d(x_{n+m}, p) \leq\ & d(x_{n+m-1}, p) + M\left(\beta_{n+m} + \frac{\gamma_{n+m}}{s}\right) \\ \leq\ & d(x_{n+m-2}, p) + M\left(\beta_{n+m-1} + \frac{\gamma_{n+m-1}}{s}\right) + M\left(\beta_{n+m} + \frac{\gamma_{n+m}}{s}\right) \\ & \cdots \\ \leq\ & d(x_n, p) + M\left(\sum_{l=n+1}^{n+m} \beta_l + \frac{1}{s}\sum_{l=n+1}^{n+m} \gamma_l\right), \end{aligned}$$

for any $p \in F$ and $n \geq n_0$.

(3) Again, from the conclusion (1), it follows that

$$\inf_{q\in F} d(x_n, q) \leq \inf_{q\in F} d(x_{n-1}, q) + M\beta_n + \frac{\gamma_n}{s} d(u_n, p)$$

for any $p \in F$ and $n \geq n_0$. Therefore we have

$$D_d(x_n, F) \leq D_d(x_{n-1}, F) + M\beta_n + \frac{\gamma_n}{s} d(u_n, p).$$

for any $p \in F$ and $n \geq n_0$. Since $\{u_n\}$ is a bounded sequence in E, $\{d(u_n, p)\}$ is also bounded in $[0, \infty)$ and by conditions (i) and (ii) in Theorem 3.2,

$$\sum_{n=0}^{\infty}\left(M\beta_n + \frac{\gamma_n}{s} d(u_n, p)\right) < \infty.$$

Hence from Lemma 3.1 we know that the $\lim_{n\to\infty} D_d(x_n, F)$ exists. □

Proof of Theorem 3.2: The necessity is obvious. Now, we prove the sufficiency. Suppose that the condition (3.1) is satisfied. Then, from Lemma 3.3 (3), we have

$$\lim_{n\to\infty} D_d(x_n, F) = 0. \tag{3.6}$$

First, we prove that $\{x_n\}$ is Cauchy in E. In fact, by the condition (ii), (iii) and (3.6), for given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ (where n_0 is the positive integer appeared in Lemma 3.3) such that, for any $n \geq n_1$,

$$D_d(x_n, F) < \frac{\varepsilon}{7}, \tag{3.7}$$

$$\sum_{i=1}^{N}\sum_{k=[\frac{n_1+1}{N}]}^{\infty} \beta_{ki} < \frac{\varepsilon}{3M} \quad \text{and} \quad \sum_{i=1}^{N}\sum_{k=[\frac{n_1+1}{N}]}^{\infty} \gamma_{ki} < \frac{\varepsilon s}{3M}. \tag{3.8}$$

By the definition of infimum, it follows from (3.7) that for any given $n \geq n_1$, there exists a $p^* \in F$ such that

$$d(x_n, p^*) < \frac{\varepsilon}{6}. \tag{3.9}$$

On the other hand, it follows from Lemma 3.3 (2) that for the given $\varepsilon > 0$ and for any $n \geq n_1 \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq 2d(x_n, p^*) + M \sum_{i=1}^{N} \sum_{k=[\frac{n_1+1}{N}]}^{\infty} \beta_{ki} + \frac{M}{s} \sum_{i=1}^{N} \sum_{k=[\frac{n_1+1}{N}]}^{\infty} \gamma_{ki}, \end{aligned} \tag{3.10}$$

for any $m \geq 1$, where M is the positive constant appeared in Lemma 3.3. Therefore from (3.7)-(3.9), for any $n \geq n_1 \geq n_0$, we have

$$d(x_{n+m}, x_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad (m \geq 1).$$

This implies that $\{x_n\}$ is Cauchy in E. Since E is a complete metric space, there exists $q \in E$ such that $x_n \to q$ as $n \to \infty$.

Now, we prove that q is a common fixed point of T_i in E.

Since $\lim_{n\to\infty} x_n = q$ and $\lim_{n\to\infty} D_d(x_n, F) = 0$, for given $\varepsilon > 0$, there exists a positive integer n_2 such that for $n_2 \geq n_1 \geq n_0$, for $n \geq n_2$,

$$d(x_n, q) < \frac{\varepsilon}{5} \quad \text{and} \quad D_d(x_n, F) < \frac{\varepsilon}{7}. \tag{3.11}$$

It follow from the second inequality in (3.11) that there exists $q^* \in F$ such that

$$d(x_{n_2}, q^*) < \frac{\varepsilon}{5}. \tag{3.12}$$

Moreover, it follow from (3.4) that for any $n \geq n_2$, we have

$$d(T_i^n q, q^*) - d(q, q^*) \leq \frac{\varepsilon}{5}. \tag{3.13}$$

Therefore, from (3.11)-(3.13), for any $n \geq n_2$,

$$\begin{aligned} d(T_i^n q, q) &\leq d(T_i^n q, q^*) + d(q, q^*) + d(q, q^*) - d(q, q^*) \\ &\leq d(T_i^n q, q^*) - d(q, q^*) + 2\{d(q, x_{n_2}) + d(x_{n_2}, q^*)\} \\ &\leq \frac{\varepsilon}{5} + 2(\frac{\varepsilon}{5} + \frac{\varepsilon}{5}) = \varepsilon. \end{aligned}$$

This implies that $T_i^n q \to q \quad (\text{as } \ n \to \infty)$. Again since

$$d(T_i^n q, T_i q) \leq (d(T_i^n q, q^*) - d(q, q^*)) + d(q, q^*) + d(T_i q, q^*),$$

for any $n \geq n_2$, by the condition (3.1) and (3.11)-(3.13), we have

$$\begin{aligned} d(T_i^n q, T_i q) &\leq \frac{\varepsilon}{5} + d(q, q^*) + L_i d(q, q^*)^{a_i} \\ &\leq \frac{\varepsilon}{5} + d(q, x_{n_2}) + d(x_{n_2}, q^*) + L_i(d(q, x_{n_2}) + d(x_{n_2}, q^*))^{a_i} \\ &< \frac{3\varepsilon}{5} + L_i\left(\frac{2\varepsilon}{5}\right)^{a_i}, \quad \forall i \in I. \end{aligned}$$

This shows that $T_i^n q \to T_i q$ as $n \to \infty$. From the uniqueness of limit, we have $q = T_i q$, that is, q is a common fixed point of T_i, for all $i \in I$. □

If $u_n = 0$, in Theorem 3.2, we can easily obtain the following theorem.

Theorem 3.4. *Let (X, d, W) be a complete convex metric space. Let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive type mappings from E into E satisfying the following conditions: there exist constants $L_i > 0$ and $a_i > 0$ such that*

$$d(T_i x, p_i) \leq L_i d(x, p_i)^{a_i}$$

for all $x \in E$, $p_i \in F(T_i)$, $i \in I$. Suppose that $F \neq \emptyset$ and that $x_0 \in E$, $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$ and $\sum_{n=0}^{\infty} \beta_n < \infty$. Then the implicit iterative scheme $\{x_n\}$ generated by (2.3) converges to a common fixed point in F if and only if

$$\liminf_{n\to\infty} D_d(x_n, F) = 0.$$

Remark 3.5. We note the followings:

(1) The Theorem 3.2 is the extension of the main theorem of Kim-Kim-Kim [15] for the case of any quasi-nonexpansive type mappings.

(2) We point out that Theorem 3.2 not only generalize and improve the corresponding results of Chang et al. [1], Liu [17]-[18], Petryshyn-Williamson [21], Tan-Xu [25], Ghosh-Debnath [10] and Chang [4]-[6] but also contain the corresponding results in [4]-[6], [10], [15]-[19], [21], [24] as its special cases.

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Operational Formula For Jacobi-Pineiro Polynomials

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Abstract

The main purpose of this paper is to develop certain operational formula for Jacobi-Pineiro polynomials and to find some properties of these polynomials with the help of operational formulas.

Key words : Jacobi-Pineiro polynomials, Generating functions, Operational formula, Recurrence relation.

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1 Introduction

The Jacobi-Pineiro polynomials $P_{\mathbf{n}}^{(\alpha_0,\boldsymbol{\alpha})}$ of degree $\mathbf{n}$ for the multi-index $\mathbf{n}=n_1+...+n_r \in \mathbb{N}^r$ and $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_r)$, are the classical multiple orthogonal polynomials associated with an AT system consisting Jacobi weights on [0,1]. The polynomials $P_{\mathbf{n}}^{(\alpha_0,\boldsymbol{\alpha})}$ are defined by the orthogonality conditions [1]

$$\int_0^1 P_{\mathbf{n}}^{(\alpha_0,\boldsymbol{\alpha})} x^{\alpha_i}(1-x)^{\alpha_0}x^k dx=0,\ k=0,1,...,n_i-1,\ i=1,2,...,r,$$

and the Rodrigues formula for the Jacobi-Pineiro polynomials $P_{\mathbf{n}}^{(\alpha_0,\boldsymbol{\alpha})}$ is given by

$$(-1)^{|\mathbf{n}|}\prod_{j=1}^{r}(|\mathbf{n}|+\alpha_0+\alpha_j+1)_{n_j}P_{\mathbf{n}}^{(\alpha_0,\boldsymbol{\alpha})}(x)=(1-x)^{-\alpha_0}\prod_{j=1}^{r}\left[x^{-\alpha_j}D^{n_j}x^{n_j+\alpha_j}\right](1-x)^{\alpha_0+|\mathbf{n}|}.$$

In [1], various properties of these polynomials have been investigated via the Rodrigues formula.

There are several studies related with the generalization of the known special functions via the Rodrigues formula [2], [3], [4], [5], [8], [9]. In the recent papers [6] and [7], the Rodrigues type extensions of multiple Laguerre polynomials and multiple Hermite polynomials were introduced and operational formulas, generating functions and some interesting recurrence relations have been proved.

The main purpose of this paper is to develop certain operational formula and generating function for Jacobi-Pineiro polynomials $P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x)$ and to find some properties of these polynomials with the help of operational formulas.

The following definition provides a Rodrigues type generalization of Jacobi-Pineiro polynomials $P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x)$:

$$\begin{aligned}&(-1)^{n+m}(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p) \qquad (1)\\ &=(1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}D^n x^{\alpha_1-\alpha_2+n}D^m x^{\alpha_2+m}(1-x)^{\alpha_0+n+m}e^{px},\end{aligned}$$

where $\alpha_0,\alpha_1,\alpha_2>-1$, $\alpha_1-\alpha_2\notin\mathbb{Z}$ and $D=\frac{d}{dx}$. It should be noticed that, in particular case $p=0$ the equation (1) gives the Rodrigues formula of the Jacobi-Pineiro polynomials $P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x)$ of degree $n+m$.

2 Main Results

In this section we derive the explicit formula and some operational formulas for the extended Jacobi-Pineiro polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$.

From the Rodrigues formula (1) and Leibniz' formula we get

$$\begin{aligned}
&(-1)^{n+m}(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)\\
&= (1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}D^n x^{\alpha_1-\alpha_2+n}D^m x^{\alpha_2+m}(1-x)^{\alpha_0+n+m}e^{px}\\
&= (1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}D^n x^{\alpha_1-\alpha_2+n}\sum_{k=0}^{m}\binom{m}{k}D^k(x^{\alpha_2+m}e^{px})D^{m-k}\left((1-x)^{\alpha_0+n+m}\right)\\
&= m!(1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{\alpha_2+m}{j}\binom{\alpha_0+n+m}{m-k}\frac{p^{k-j}}{(k-j)!}(-1)^{m-k}\\
&\times D^n\left(x^{\alpha_1+m+n-j}e^{px}(1-x)^{\alpha_0+n+k}\right).
\end{aligned}$$

Applying the Leibniz' formula one more time to the last part of the above equation gives the explicit formula for the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$ as follows:

$$\begin{aligned}
&(-1)^{n+m}(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p) \qquad (2)\\
&= m!n!\sum_{k=0}^{m}\sum_{j=0}^{k}\sum_{i=0}^{n}\sum_{l=0}^{i}\binom{\alpha_2+m}{j}\binom{\alpha_0+n+m}{m-k}\binom{\alpha_0+n+k}{n-i}\binom{\alpha_1+n+m-j}{l}\\
&\times\frac{p^{k+i-j-l}}{(k-j)!(i-l)!}(-1)^{m-k+n-i}x^{m+n-j-l}(1-x)^{k+i}.
\end{aligned}$$

Remark 1 *It should be noticed that, since $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,0)=P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x)$, in particular case $p=0$ the explicit form (2) represents the explicit representation of Jacobi-Pineiro polynomials [1].*

In order to find an operational formula for the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$, we will prove the following operational formula.

Lemma 2 *For the sufficiently differentiable function Y of x, we have*

$$\prod_{j=1}^{n}[(x-x^2)D-(\alpha+\beta+2j)x+p(x-x^2)+\beta+j]Y=(1-x)^{-\alpha}x^{-\beta}e^{-px}D^n[x^{\beta+n}(1-x)^{\alpha+n}e^{px}Y]. \quad (3)$$

Proof. To prove this identity we use the method of induction. It is obvious that for $n=1$ the identity (3) is correct. Let for n it is true. If we consider

$$Y=[(x-x^2)D-(\alpha+\beta+2n+2)x+p(x-x^2)+\beta+n+1]Y$$

in equation (3), we get

$$\begin{aligned}
&\prod_{j=1}^{n}[(x-x^2)D-(\alpha+\beta+2j)x+p(x-x^2)+\beta+j] \qquad (4)\\
&\times[(x-x^2)D-(\alpha+\beta+2n+2)x+p(x-x^2)+\beta+n+1]Y\\
&=(1-x)^{-\alpha}x^{-\beta}e^{-px}D^n[x^{\beta+n}(1-x)^{\alpha+n}e^{px}[(x-x^2)D-(\alpha+\beta+2n+2)x+p(x-x^2)+\beta+n+1]Y].
\end{aligned}$$

Using the identity

$$D[x^{\beta+n}(1-x)^{\alpha+n}e^{px}Y]=(1-x)^{\alpha+n-1}x^{\beta+n-1}e^{px}[(x-x^2)D-(\alpha+\beta+2n)x+p(x-x^2)+\beta+n]Y$$

(4) becomes

$$\prod_{j=1}^{n+1}[(x-x^2)D-(\alpha+\beta+2j)x+p(x-x^2)+\beta+j]Y=(1-x)^{-\alpha}x^{-\beta}e^{-px}D^{n+1}[x^{\beta+n+1}(1-x)^{\alpha+n+1}e^{px}Y].$$

Then the proof is completed. ■

Now, we can find an operational formula for the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$ by using the identity (3). For convenience, we will use the notation

$$\Omega(x)=(1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}D^n x^{\alpha_1-\alpha_2+n}D^m x^{\alpha_2+m}(1-x)^{\alpha_0+n+m}e^{px}Y.$$

Taking (3) into the consideration with $n=m, \alpha=\alpha_0+n,\ \beta=\alpha_2$ we get

$$\begin{aligned}\Omega(x)&=(1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}D^n[x^{\alpha_1+n}(1-x)^{\alpha_0+n}e^{px}\\&\quad\times\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+p(x-x^2)+\alpha_2+j]Y]\end{aligned}\tag{5}$$

and then applying (3) in (5) one more time we get first equality as follows:

$$\begin{aligned}\Omega(x)&=\prod_{k=1}^{n}[(x-x^2)D-(\alpha_0+\alpha_1+2k)x+p(x-x^2)+\alpha_1+k]\\&\quad\times\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+p(x-x^2)+\alpha_2+j]Y.\end{aligned}\tag{6}$$

On the other hand, with the help of Leibniz' formula the second equality can be found in the following form:

$$\begin{aligned}&\Omega(x)\\&=m!n!\sum_{r_1=0}^{m}\sum_{k=0}^{m-r_1}\sum_{j=0}^{k}\sum_{r_2=0}^{n}\sum_{i=0}^{n-r_2}\sum_{l=0}^{i}\binom{\alpha_2+m}{j}\binom{\alpha_0+n+m}{m-r_1-k}\binom{\alpha_0+n+r_1+k}{n-r_2-i}\binom{\alpha_1+n+m-j}{l}\\&\times\frac{p^{k+i-j-l}}{(k-j)!(i-l)!}(-1)^{m-r_1-k+n-r_2-i}x^{m+n-j-l}(1-x)^{k+i}\frac{D^{r_1+r_2}Y}{r_1!r_2!}.\end{aligned}\tag{7}$$

The general operational formula for the extended Jacobi-Pineiro polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$ is obtained by combining (2), (6) and (7).

Theorem 3 *For the extended Jacobi-Pineiro polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$ we have the following general operational formula:*

$$\begin{aligned}&\sum_{r_1=0}^{m}\sum_{r_2=0}^{n}\binom{m}{r_1}\binom{n}{r_2}(-1)^{n+m-r_1-r_2}x^{r_1+r_2}(n+m+\alpha_0+\alpha_1+r_1+r_2+1)_{n-r_2}\\&\times(n+m+\alpha_0+\alpha_2+r_1+1)_{m-r_1}R_{n-r_2,m-r_1}^{(\alpha_0+r_1+r_2,\alpha_1+r_1+r_2,\alpha_2+r_1)}(x,p)D^{r_1+r_2}(Y)\\&=\prod_{k=1}^{n}[(x-x^2)D-(\alpha_0+\alpha_1+2k)x+p(x-x^2)+\alpha_1+k]\\&\times\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+p(x-x^2)+\alpha_2+j]\ Y,\end{aligned}\tag{8}$$

where Y is sufficiently differentiable function with respect to x.

Setting $p = 0$ in Theorem 3 we get:

Theorem 4 *For the Jacobi-Pineiro polynomial $P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}$ we have the following general operational formula:*

$$\begin{aligned}
&\sum_{r_1=0}^{m}\sum_{r_2=0}^{n}\binom{m}{r_1}\binom{n}{r_2}(-1)^{n+m-r_1-r_2}x^{r_1+r_2}(n+m+\alpha_0+\alpha_1+r_1+r_2+1)_{n-r_2} \qquad (9)\\
&\times(n+m+\alpha_0+\alpha_2+r_1+1)_{m-r_1}P_{n-r_2,m-r_1}^{(\alpha_0+r_1+r_2,\alpha_1+r_1+r_2,\alpha_2+r_1)}(x)D^{r_1+r_2}(Y)\\
&=\prod_{k=1}^{n}[(x-x^2)D-(\alpha_0+\alpha_1+2k)x+\alpha_1+k]\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+\alpha_2+j]\ Y,
\end{aligned}$$

where Y is sufficiently differentiable function with respect to x.

Setting $Y = 1$ in (8), we obtain the following theorem.

Theorem 5 *For the extended Jacobi-Pineiro polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$ we have the following operational formula:*

$$\begin{aligned}
&(-1)^{n+m}(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p) \qquad (10)\\
&=\prod_{k=1}^{n}[(x-x^2)D-(\alpha_0+\alpha_1+2k)x+p(x-x^2)+\alpha_1+k]\\
&\times\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+p(x-x^2)+\alpha_2+j].1\ .
\end{aligned}$$

Moreover for each suitable choice of the sufficiently differentiable function Y, various operational formulas can be proven for the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$.

3 Applications of Operational Formula

In this section we will derive some recurrence relations for the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$.

If we consider the operational formula (8) we get

$$\begin{aligned}
&\prod_{k=1}^{n}[(x-x^2)D-(\alpha_0+\alpha_1+2k)x+p(x-x^2)+\alpha_1+k] \qquad (11)\\
&\times\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+p(x-x^2)+\alpha_2+j]\ x\\
&=(-1)^{n+m}x(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)\\
&+(-1)^{n+m-1}m!x(n+m+\alpha_0+\alpha_1+2)_n(n+m+\alpha_0+\alpha_2+2)_{m-1}R_{n,m-1}^{(\alpha_0+1,\alpha_1+1,\alpha_2+1)}(x,p)\\
&+(-1)^{n+m-1}n!x(n+m+\alpha_0+\alpha_1+2)_{n-1}(n+m+\alpha_0+\alpha_2+1)_m R_{n-1,m}^{(\alpha_0+1,\alpha_1+1,\alpha_2)}(x,p).
\end{aligned}$$

By the equation (6) we can easily obtain the following two relations:

$$\begin{aligned}
&\prod_{k=1}^{n}[(x-x^2)D-(\alpha_0+\alpha_1+2k)x+p(x-x^2)+\alpha_1+k] \qquad (12)\\
&\times\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+p(x-x^2)+\alpha_2+j]\ x\\
&=(-1)^{n+m}x(n+m+\alpha_0+\alpha_1+2)_n(n+m+\alpha_0+\alpha_2+2)_m R_{n,m}^{(\alpha_0,\alpha_1+1,\alpha_2+1)}(x,p),
\end{aligned}$$

and

$$\prod_{k=1}^{n}[(x-x^2)D-(\alpha_0+\alpha_1+2k)x+p(x-x^2)+\alpha_1+k] \tag{13}$$
$$\times\prod_{j=1}^{m}[(x-x^2)D-(\alpha_0+n+\alpha_2+2j)x+p(x-x^2)+\alpha_2+j]\ (1-x)$$
$$=(-1)^{n+m}(1-x)(n+m+\alpha_0+\alpha_1+2)_n(n+m+\alpha_0+\alpha_2+2)_m R_{n,m}^{(\alpha_0+1,\alpha_1,\alpha_2)}(x,p),$$

respectively.

Equating (11) and (12) we have the following relation:

$$R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$$
$$=\frac{(n+m+\alpha_0+\alpha_1+n+1)}{(n+m+\alpha_0+\alpha_1+1)(n+m+\alpha_0+\alpha_2+1)}[(n+m+\alpha_0+\alpha_2+m+1)R_{n,m}^{(\alpha_0,\alpha_1+1,\alpha_2+1)}(x,p)$$
$$+m!R_{n,m-1}^{(\alpha_0+1,\alpha_1+1,\alpha_2+1)}(x,p)+\frac{(n+m+\alpha_0+\alpha_2+1)}{(n+m+\alpha_0+\alpha_1+n+1)}n!R_{n-1,m}^{(\alpha_0+1,\alpha_1+1,\alpha_2)}(x,p)].$$

Also, adding (11) and (13), we have the following recurrence relation

$$(1-x)R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$$
$$=\frac{(n+m+\alpha_0+\alpha_1+n+1)}{(n+m+\alpha_0+\alpha_1+1)(n+m+\alpha_0+\alpha_2+1)}[(n+m+\alpha_0+\alpha_2+m+1)(1-x)R_{n,m}^{(\alpha_0+1,\alpha_1,\alpha_2)}(x,p)$$
$$-m!xR_{n,m-1}^{(\alpha_0+1,\alpha_1+1,\alpha_2+1)}(x,p)-\frac{(n+m+\alpha_0+\alpha_2+1)}{(n+m+\alpha_0+\alpha_1+n+1)}n!xR_{n-1,m}^{(\alpha_0+1,\alpha_1+1,\alpha_2)}(x,p)].$$

Furthermore, combining (12) and (13) gives

$$R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)=\frac{(n+m+\alpha_0+\alpha_1+n+1)(n+m+\alpha_0+\alpha_2+m+1)}{(n+m+\alpha_0+\alpha_1+1)(n+m+\alpha_0+\alpha_2+1)}$$
$$\times\left(xR_{n,m}^{(\alpha_0,\alpha_1+1,\alpha_2+1)}(x,p)+(1-x)R_{n,m}^{(\alpha_0+1,\alpha_1,\alpha_2)}(x,p)\right).$$

Remark 6 *Setting $p=0$ in the above recurrence relations we can get the corresponding relations for the Jacobi-Pineiro polynomial $P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}$.*

4 Generating Function

In this section we obtain generating function for the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$. We start with the following theorem.

Theorem 7 *Let the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$ be defined by (1). Then we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}(\alpha_0+\alpha_1+1)_n(\alpha_0+\alpha_2+1)_m R_{n,m}^{(\alpha_0-n-m,\alpha_1,\alpha_2)}(x,p)\frac{\left(\frac{t_1}{1-x}\right)^n}{n!}\frac{\left(\frac{t_2}{1-x}\right)^m}{m!} \tag{14}$$
$$=\frac{(1-x)^{-\alpha_0}}{(1-t_1)^{1+\alpha_1}(1-t_2)^{1+\alpha_2}}\left(1-\frac{x}{(1-t_1)(1-t_2)}\right)^{\alpha_0}e^{\frac{px(1-(1-t_1)(1-t_2))}{(1-t_1)(1-t_2)}}.$$

Proof. Considering

$$\Psi=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}(\alpha_0+\alpha_1+1)_n(\alpha_0+\alpha_2+1)_m R_{n,m}^{(\alpha_0-n-m,\alpha_1,\alpha_2)}(x,p)\frac{\left(\frac{t_1}{1-x}\right)^n}{n!}\frac{\left(\frac{t_2}{1-x}\right)^m}{m!}$$

then

$$\begin{aligned}\Psi &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}D^n x^{\alpha_1-\alpha_2+n}D^m x^{\alpha_2+m}(1-x)^{\alpha_0}e^{px}\frac{t_1^n}{n!}\frac{t_2^m}{m!}\\ &= (1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}\sum_{n=0}^{\infty}D^n x^{\alpha_1-\alpha_2+n}\left(\frac{m!}{2\pi i}\sum_{m=0}^{\infty}\int_{C_1}\frac{z^{\alpha_2+m}(1-z)^{\alpha_0}e^{pz}}{(z-x)^{m+1}}dz\frac{t_2^m}{m!}\right)\frac{t_1^n}{n!}\\ &= (1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}\sum_{n=0}^{\infty}D^n x^{\alpha_1-\alpha_2+n}\left(\frac{1}{2\pi i}\oint_{C_1}\frac{z^{\alpha_2}(1-z)^{\alpha_0}e^{pz}}{z-x}\sum_{m=0}^{\infty}\left(\frac{zt_2}{z-x}\right)^m dz\right)\frac{t_1^n}{n!}\\ &= \frac{(1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}}{1-t_2}\sum_{n=0}^{\infty}D^n x^{\alpha_1-\alpha_2+n}(\frac{x}{1-t_2})^{\alpha_2}(1-\frac{x}{1-t_2})^{\alpha_0}e^{\frac{px}{1-t_2}}\frac{t_1^n}{n!}.\end{aligned}$$

Applying Cauchy integral gives

$$\begin{aligned}\Psi &= \frac{(1-x)^{-\alpha_0}x^{-\alpha_1}e^{-px}}{1-t_2}\frac{1}{2\pi i}\oint_{C_2}\frac{z^{\alpha_1-\alpha_2}(\frac{z}{1-t_2})^{\alpha_2}(1-\frac{z}{1-t_2})^{\alpha_0}e^{\frac{pz}{1-t_2}}}{z-x}\sum_{n=0}^{\infty}\left(\frac{zt_1}{z-x}\right)^n dz\\ &= \frac{(1-x)^{-\alpha_0}}{(1-t_1)^{1+\alpha_1}(1-t_2)^{1+\alpha_2}}\left(1-\frac{x}{(1-t_1)(1-t_2)}\right)^{\alpha_0}e^{\frac{px(1-(1-t_1)(1-t_2))}{(1-t_1)(1-t_2)}}\end{aligned}$$

where C_1 is a circle in the complex $z-$plane along the negative real axis, (centered at $z=\frac{x}{1-t_2}$) with radius $\epsilon>0$, which is described in the positive direction and the closed contour C_2 is a circle in the complex $\eta-$plane, cut along the negative real axis and it is a circle (centered at $\eta=\frac{x}{1-t_1}$) of sufficiently small radius. ■

Remark 8 *Letting $p=0$ in (14), we get the generating function for Jacobi-Pineiro polynomials $P_{n,m}^{(\alpha_0,\boldsymbol{\alpha}_1,\alpha_2)}$.*

Furthermore, using the generating relation (14) gives the following recurrence relation.

Theorem 9 *For the polynomials $R_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x,p)$, we have*

$$\begin{aligned}&(-1)^{n+m}(\alpha_0+\beta_0+\alpha_1+\beta_1+2)_n(\alpha_0+\beta_0+\alpha_2+\beta_2+2)_m R_{n,m}^{(\alpha_0+\beta_0-n-m,\alpha_1+\beta_1+1,\alpha_2+\beta_2+1)}(x,p)\\ &=\sum_{r=0}^{n}\sum_{l=0}^{m}\binom{n}{r}\binom{m}{l}(-1)^{n+m}(\alpha_0+\alpha_1+1)_{n-r}(\alpha_0+\alpha_2+1)_{m-l}(\beta_0+\beta_1+1)_r(\beta_0+\beta_2+1)_l\\ &\times R_{n-r,m-l}^{(\alpha_0-n-r-m-l,\alpha_1,\alpha_2)}(x,\frac{p}{2})R_{r,l}^{(\beta_0-r-l,\beta_1,\beta_2)}(x,\frac{p}{2}).\end{aligned}$$

Proof. Taking (14) into consideration we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}(\alpha_0+\beta_0+\alpha_1+\beta_1+2)_n(\alpha_0+\beta_0+\alpha_2+\beta_2+2)_m$$
$$\times R_{n,m}^{(\alpha_0+\beta_0-n-m,\alpha_1+\beta_1+1,\alpha_2+\beta_2+1)}(x,p)\frac{\left(\frac{t_1}{1-x}\right)^n}{n!}\frac{\left(\frac{t_2}{1-x}\right)^m}{m!}$$
$$=\frac{(1-x)^{-\alpha_0-\beta_0}}{(1-t_1)^{2+\alpha_1+\beta_1}(1-t_2)^{2+\alpha_2+\beta_2}}\left(1-\frac{x}{(1-t_1)(1-t_2)}\right)^{\alpha_0+\beta_0}e^{\frac{px(1-(1-t_1)(1-t_2))}{(1-t_1)(1-t_2)}}$$
$$=\frac{(1-x)^{-\alpha_0}}{(1-t_1)^{1+\alpha_1}(1-t_2)^{1+\alpha_2}}\left(1-\frac{x}{(1-t_1)(1-t_2)}\right)^{\alpha_0}e^{\frac{p}{2}\frac{x(1-(1-t_1)(1-t_2))}{(1-t_1)(1-t_2)}}$$
$$\times\frac{(1-x)^{-\beta_0}}{(1-t_1)^{1+\beta_1}(1-t_2)^{1+\beta_2}}\left(1-\frac{x}{(1-t_1)(1-t_2)}\right)^{\beta_0}e^{\frac{p}{2}\frac{x(1-(1-t_1)(1-t_2))}{(1-t_1)(1-t_2)}}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}(\alpha_0+\alpha_1+1)_n(\alpha_0+\alpha_2+1)_mR_{n,m}^{(\alpha_0-n-m,\alpha_1,\alpha_2)}(x,\frac{p}{2})\frac{\left(\frac{t_1}{1-x}\right)^n}{n!}\frac{\left(\frac{t_2}{1-x}\right)^m}{m!}$$
$$\times\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{r+l}(\beta_0+\beta_1+1)_r(\beta_0+\beta_2+1)_lR_{r,l}^{(\beta_0-r-l,\beta_1,\beta_2)}(x,\frac{p}{2})\frac{\left(\frac{t_1}{1-x}\right)^r}{r!}\frac{\left(\frac{t_2}{1-x}\right)^l}{l!}.$$

Taking $n\longrightarrow n-r$ and $m\to m-l$, we get

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{n}\sum_{l=0}^{m}(-1)^{n+m}(\alpha_0+\alpha_1+1)_{n-r}(\alpha_0+\alpha_2+1)_{m-l}R_{n-r,m-l}^{(\alpha_0-(n-r)-(m-l),\alpha_1,\alpha_2)}(x,\frac{p}{2})$$
$$\times(\beta_0+\beta_1+1)_r(\beta_0+\beta_2+1)_lR_{r,l}^{(\beta_0-r-l,\beta_1,\beta_2)}(x,\frac{p}{2})\frac{\left(\frac{t_1}{1-x}\right)^n}{r!(n-r)!}\frac{\left(\frac{t_2}{1-x}\right)^m}{l!(m-l)!}.$$

Hence the result. ■

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FIXED POINTS AND STABILITY OF ADDITIVE FUNCTIONAL EQUATIONS ON THE BANACH ALGEBRAS

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Abstract. Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras and of derivations on Banach algebras for the additive functional equation

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big)$$

for all $m \in \mathbb{N}$ with $m \geq 2$.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [13] concerning the stability of group homomorphisms:

Let $(G_1, *)$ be a group and $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. For any $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that, if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

If the answer is affirmative, we would say that the equation of homomorphism $H(x*y) = H(x)\diamond H(y)$ is *stable*. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [4] gave a first affirmative answer to the question of Ulam for Banach spaces as follows:

Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$.

Rassias [9] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded in the following:

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Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The above inequality (1.1) has provided a lot of influence in the development of what is now known as a *generalized Hyers-Ulam stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [3] generalized the Rassias' result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [5], [10]–[12]).

Theorem 1.2. [6, 7, 8] *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R} - \{1\}$ such that f satisfies inequality $\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}$ for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying $\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}||x||^p$ for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.*

We recall a fundamental result in fixed point theory for our main results.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.3. [2] *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for any element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

This paper is organized as follows: In Section 2, using the fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the Cauchy functional equation. In Section 3, using the fixed point methods, we prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the Cauchy functional equation.

2. **Stability of homomorphisms in Banach algebras**

Throughout this paper, assume that A is a complex Banach algebra with norm $\|\cdot\|_A$ and B is a complex Banach algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \to B$, we define

$$D_\mu f(x_1, \cdots, x_m) := \sum_{i=1}^{m} \mu f(mx_i + \sum_{j=1, j\neq i}^{m} x_j) + \mu f(\sum_{i=1}^{m} x_i) - 2f(\mu \sum_{i=1}^{m} mx_i)$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and $x_1, \cdots, x_m \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in Banach algebras if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the functional equation $D_\mu f(x_1, \cdots, x_m) = 0$.

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there are functions $\varphi : A^m \to [0, \infty)$ and $\psi : A^2 \to [0, \infty)$ such that*

$$\lim_{j\to\infty} m^{-j}\varphi(m^j x_1, \cdots, m^j x_m) = 0, \tag{2.1}$$

$$\|D_\mu f(x_1, \cdots, x_m)\|_B \leq \varphi(x_1, \cdots, x_m), \tag{2.2}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \psi(x, y), \tag{2.3}$$

$$\lim_{j\to\infty} m^{-2j}\psi(m^j x, m^j y) = 0 \tag{2.4}$$

for all $\mu \in \mathbb{T}^1$ and $x_1, \cdots, x_m, x, y \in A$. If there exists $L < 1$ such that $\varphi(mx, 0, \cdots, 0) \leq mL\varphi(x, 0, \cdots, 0)$ for all $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{m - mL}\varphi(x, 0, \cdots, 0) \tag{2.5}$$

for all $x \in A$.

Proof. Consider the set $X := \{g : A \to B\}$ and introduce the *generalized metric* on X as follows:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \leq C\varphi(x, 0, \cdots, 0),\ \forall x \in A\}.$$

It is easy to show that (X, d) is complete.

Now, we consider the linear mapping $J : X \to X$ such that $Jg(x) := \frac{1}{m}g(mx)$ for all $x \in A$. By Theorem 3.1 of [1], $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in X$. Letting $\mu = 1$,

$x = x_1$ and $x_2 = \cdots = x_m = 0$ in (2.2), we get

$$\|f(mx) - mf(x)\|_B \leq \varphi(x, 0, \cdots, 0) \tag{2.6}$$

for all $x \in A$ and so

$$\|f(x) - \frac{1}{m} f(mx)\|_B \leq \frac{1}{m}\varphi(x, 0, \cdots, 0)$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{1}{m}$. Thus, by Theorem 1.3, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(mx) = mH(x) \tag{2.7}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.6) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x, 0, \cdots, 0)$$

for all $x \in A$;

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(m^n x)}{m^n} = H(x) \tag{2.8}$$

for all $x \in A$;

(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality $d(f, H) \leq \frac{1}{m-mL}$. This implies that the inequality (2.5) holds.

It follows from (2.1), (2.2) and (2.7) that

$$\begin{aligned}
&\Big\| \sum_{i=1}^{m} H(mx_i + \sum_{j=1, j\neq i}^{m} x_j) + H(\sum_{i=1}^{m} x_i) - 2H(\sum_{i=1}^{m} mx_i) \Big\|_B \\
= \ & \lim_{n\to\infty} \frac{1}{m^n} \Big\| \sum_{i=1}^{m} f(m^{n+1}x_i + \sum_{j=1, j\neq i}^{m} m^n x_j) + f(\sum_{i=1}^{m} m^n x_i) - 2f(\sum_{i=1}^{m} m^{n+1} x_i) \Big\|_B \\
\leq \ & \lim_{n\to\infty} \frac{1}{m^n} \varphi(m^n x_1, \cdots, m^n x_m) = 0
\end{aligned}$$

for all $x_1, \cdots, x_m \in A$ and so

$$\sum_{i=1}^{m} H(mx_i + \sum_{j=1, j\neq i}^{m} x_j) + H(\sum_{i=1}^{m} x_i) = 2H(\sum_{i=1}^{m} mx_i) \tag{2.9}$$

for all $x_1, \cdots, x_m \in A$.

By a similar method to above, we get

$$\mu H(mx) = H(m\mu x)$$

for all $\mu \in \mathbb{T}^1$ and $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

Finally, it follows from (2.3) that

$$\begin{aligned}\|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty}\frac{1}{m^n}\|f(m^nxy)-f(m^nx)f(m^ny)\|_B\\ &\le \lim_{n\to\infty}\frac{1}{m^n}\psi(m^nx,m^ny)=0\end{aligned}$$

for all $x,y\in A$ and so $H(xy)=H(x)H(y)$ for all $x,y\in A$. Thus $H:A\to B$ is a homomorphism satisfying (2.5). This completes the proof. □

Corollary 2.2. *Let $r<1$ and θ be nonnegative real numbers and $f:A\to B$ be a mapping such that*

$$\|D_\mu f(x_1,\cdots,x_m)\|_B \le \theta\cdot(\|x_1\|_A^r+\|x_2\|_A^r+\cdots+\|x_m\|_A^r), \tag{2.10}$$

$$\|f(xy)-f(x)f(y)\|_B \le \theta\cdot(\|x\|_A^r\cdot\|y\|_A^r) \tag{2.11}$$

for all $\mu\in\mathbb{T}^1$ and $x_1,\cdots,x_m,x,y\in A$. Then there exists a unique homomorphism $H:A\to B$ such that

$$\|f(x)-H(x)\|_B\le\frac{\theta}{m-m^r}\|x\|_A^r$$

for all $x\in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x_1,\cdots,x_m)=\theta\cdot(\|x_1\|_A^r+\|x_2\|_A^r+\cdots+\|x_m\|_A^r),$$

$$\psi(x,y):=\theta\cdot(\|x\|_A^r\cdot\|y\|_A^r)$$

for all $x_1,\cdots,x_m,x,y\in A$ and $L=m^{r-1}$. □

Theorem 2.3. *Let $f:A\to B$ be a mapping for which there are functions $\varphi:A^m\to[0,\infty)$ and $\psi:A^2\to[0,\infty)$ such that*

$$\lim_{j\to\infty}m^j\varphi(m^{-j}x_1,\cdots,m^{-j}x_m) = 0, \tag{2.12}$$

$$\|D_\mu f(x_1,\cdots,x_m)\|_B \le \varphi(x_1,\cdots,x_m), \tag{2.13}$$

$$\|f(xy)-f(x)f(y)\|_B \le \psi(x,y), \tag{2.14}$$

$$\lim_{j\to\infty}m^{2j}\psi(m^{-j}x,m^{-j}y) = 0 \tag{2.15}$$

for all $\mu\in\mathbb{T}^1$ and $x_1,\cdots,x_m,x,y\in A$. If there exists an $L<1$ such that $\varphi(x,0,\cdots,0)\le\frac{L}{m}\varphi(mx,0,\cdots,0)$ for all $x\in A$, then there exists a unique homomorphism $H:A\to B$ such that

$$\|f(x)-H(x)\|_B\le\frac{L}{m-mL}\varphi(x,0,\cdots,0) \tag{2.16}$$

for all $x\in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := mg\left(\frac{x}{m}\right)$$

for all $x \in A$. It follows from (2.6) that

$$\left\|f(x) - mf(\frac{x}{m})\right\|_B \leq \varphi\left(\frac{x}{m}, 0, \cdots, 0\right) \leq \frac{L}{m}\varphi(x, 0, \cdots, 0)$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{L}{m}$.

By Theorem 1.3, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(mx) = mH(x) \tag{2.17}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.13) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x, 0, \cdots, 0)$$

for all $x \in A$;

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} m^n f(\frac{x}{m^n}) = H(x)$$

for all $x \in A$;

(3) $d(f, H) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{L}{m - mL},$$

which implies that the inequality (2.16) holds.

The rest of the proof is similar to the proof of Theorem 2.1. This completes the proof. □

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers and $f : A \to B$ be a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_B \leq \theta \cdot (\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r), \tag{2.18}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \theta \cdot (\|x\|_A^r \cdot \|y\|_A^r) \tag{2.19}$$

for all $\mu \in \mathbb{T}^1$ and $x_1, \cdots, x_m, x, y \in A$. Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{m^r - m}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x_1, \cdots, x_m) = \theta \cdot (\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r),$$
$$\psi(x, y) := \theta \cdot (\|x\|_A^r \cdot \|y\|_A^r)$$

for all $x_1, \cdots, x_m, x, y \in A$ and $L = m^{1-r}$. □

3. **Stability of derivations on Banach algebras**

Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the functional equation $D_\mu f(x_1, \cdots, x_m) = 0$.

Theorem 3.1. *Let $f : A \to A$ be a mapping for which there are functions $\varphi : A^m \to [0, \infty)$ and $\psi : A^2 \to [0, \infty)$ such that*

$$\lim_{j\to\infty} m^{-j}\varphi(m^j x_1, \cdots, m^j x_m) = 0, \tag{3.1}$$
$$\|D_\mu f(x_1, \cdots, x_m)\|_A \leq \varphi(x_1, \cdots, x_m), \tag{3.2}$$
$$\|f(xy) - f(x)y - xf(y)\|_A \leq \psi(x, y), \tag{3.3}$$
$$\lim_{j\to\infty} m^{-2j}\psi(m^j x, m^j y) = 0 \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$ and $x_1, \cdots, x_m, x, y \in A$. If there exists an $L < 1$ such that $\varphi(mx, 0, ..., 0) \leq mL\varphi(x, 0, \cdots, 0)$ for all $x \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{1}{m - mL}\varphi(x, 0, \cdots, 0) \tag{3.5}$$

for all $x \in A$.

Proof. By the same reasoning in as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^n} \tag{3.6}$$

for all $x \in A$. It follows from (3.3), (3.4) and (3.6) that

$$\begin{aligned}
&\|\delta(xy) - \delta(x)y - x\delta(y)\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{m^{2n}}\|f(m^{2n}xy) - f(m^n x) \cdot m^n y - m^n x f(m^n y)\|_A \\
&\quad \leq \lim_{n\to\infty} \frac{1}{m^{2n}}\psi(m^n x, m^n y) = 0
\end{aligned}$$

for all $x, y \in A$ and so

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.5). This completes the proof. □

Corollary 3.2. *Let $r < 1$ and θ be nonnegative real numbers and $f : A \to A$ be a mapping such that*

$$\begin{aligned} \|D_\mu f(x_1, \cdots, x_m)\|_B &\leq \theta \cdot (\|x_1\|_A^r + \cdots \|x_m\|_A^r), && (3.7)\\ \|f(xy) - f(x)y - xf(y)\|_A &\leq \theta \cdot (\|x\|_A^r \cdot \|y\|_A^r) && (3.8) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and $x_1, \cdots, x_m, x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{\theta}{m - m^r}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x_1, \cdots, x_m) := \theta \cdot (\|x_1\|_A^r + \cdots \|x_m\|_A^r)$$

and

$$\psi(x, y) := \theta \cdot (\|x\|_A^r \cdot \|y\|_A^r),$$

for all $x_1, \cdots, x_m, x, y \in A$ and $L = m^{r-1}$. □

Theorem 3.3. *Let $f : A \to B$ be a mapping for which there are functions $\varphi : A^m \to [0, \infty)$ and $\psi : A^2 \to [0, \infty)$ such that*

$$\begin{aligned} \lim_{j\to\infty} m^j \varphi(m^{-j}x_1, \cdots, m^{-j}x_m) &= 0, && (3.9)\\ \|D_\mu f(x_1, \cdots, x_m)\|_B &\leq \varphi(x_1, \cdots, x_m), && (3.10)\\ \|f(xy) - f(x)y - xf(y)\|_B &\leq \psi(x, y), && (3.11)\\ \lim_{j\to\infty} m^{2j}\psi(m^{-j}x, m^{-j}y) &= 0 && (3.12) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and $x_1, \cdots, x_m, x, y \in A$. If there exists an $L < 1$ such that $\varphi(mx, 0, \cdots, 0) \leq \frac{L}{m}\varphi(x, 0, \cdots, 0)$ for all $x \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{L}{m - mL}\varphi(x, 0, \cdots, 0) \tag{3.13}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. □

Corollary 3.4. *Let $r > 1$, θ be nonnegative real numbers and $f : A \to A$ be a mapping such that*

$$\begin{aligned} \|D_\mu f(x_1, \cdots, x_m)\|_B &\leq \theta \cdot (\|x_1\|_A^r + \cdots \|x_m\|_A^r), && (3.14)\\ \|f(xy) - f(x)y - xf(y)\|_A &\leq \theta \cdot (\|x\|_A^r \cdot \|y\|_A^r) && (3.15) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and $x_1, \cdots, x_m, x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{\theta}{m^r - m}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x_1, \cdots, x_m) := \theta \cdot (\|x_1\|_A^r + \cdots \|x_m\|_A^r)$$

and

$$\psi(x, y) := \theta \cdot (\|x\|_A^r \cdot \|y\|_A^r),$$

for all $x_1, \cdots, x_m, x, y \in A$ and $L = m^{1-r}$. □

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Oscillation of second order nonlinear neutral differential equation

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Abstract

In this article, we present some criteria for the oscillation of the second order nonlinear neutral differential equation

$$(r(t)([(x(t))+p(t)x(\tau(t))]')^{\gamma})'+q(t)x^{\gamma}(\sigma(t))=0.$$

We use the oscillation results of the first order equation to establish that of second order equation.

Keywords: Oscillation, neutral differential equation, second-order.

1 Introduction

In this article, we consider the second order neutral differential equation

$$(r(t)([x(t)+p(t)x(\tau(t))]')^{\gamma})'+q(t)x^{\gamma}(\sigma(t))=0, \tag{1}$$

where $q(t)\in C([t_0,\infty),(0,\infty))$, $r(t),\tau(t)$ and $\sigma(t)\in C^1([t_0,\infty),(0,\infty))$ with $\tau(t),\sigma(t)\to\infty$ as $t\to\infty$, $p(t)\in C^1([t_0,\infty),[0,\infty))$ with $p(t)\leqslant p_0<\infty$, and $\gamma\geqslant 1$ is ratio of odd positive integers. In addition, suppose that the following conditions are also satisfied:

(H_1) $\tau'(t)\geqslant\tau_0>0,\ \tau o\sigma=\sigma o\tau$,

(H_2) $R(t)=\int_{t_0}^{t}\frac{ds}{r^{\frac{1}{\gamma}}(s)}\to\infty$ as $t\to\infty$.

By a solution of (1) we mean a nontrivial reel valued function which has the properties $x(t)+p(t)x(\tau(t))\in C^1([t_1,\infty),\mathbb{R})$, $r(t)([x(t)+p(t)x(\tau(t))]')^{\gamma}\in C^1([t_1,\infty),\mathbb{R})$ and (1) is satisfied for $t\geqslant t_1\geqslant t_0$. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros, otherwise, it is said to be nonoscillatory. The equation (1) is called oscillatory if all of its solutions are oscillatory.

Numerous oscillation results have been obtained for the second order neutral delay differential equations. Grammatikopoulos et al.[1] considered the second order linear neutral delay equation

$$(y(t)+p(t)y(t-\tau))''+q(t)y(t-\delta)=0,\quad t\geqslant t_0 \tag{2}$$

and they proved that if $q(t) > 0$, $0 \leqslant p(t) < 1$, and

$$\int_{t_0}^{\infty} q(s)(1 - p(s - \delta))ds = \infty,$$

then every solution of (2) oscillates. Graef et al.[2] considered the second order nonlinear neutral delay equation

$$(y(t) + p(t)y(t - \tau))'' + q(t)f(y(t - \delta)) = 0, \quad t \geqslant t_0, \tag{3}$$

which is more general than (2), and they proved that if $q(t) > 0$, $0 \leqslant p(t) < 1$, and

$$\int_{t_0}^{\infty} q(s)f((1 - p(s - \delta))c)ds = \infty, \quad c > 0$$

then every solution of (3) oscillates. Later Grace and Lalli [3] studied more general equation of the form

$$(r(t)[x(t) + p(t)x(t - \tau)]')' + q(t)f(x(t - \sigma)) = 0, \tag{4}$$

under the conditions $\frac{f(x)}{x} \geqslant k$, $\int^{\infty} \frac{ds}{r(s)} = \infty$, and

$$\int^{\infty} \rho(s)q(s)(1 - p(s - \sigma)) - \frac{(\rho'(s))^2 r(s - \sigma)}{4k\rho(s)}ds = \infty,$$

they proved that (4) is oscillatory. Baculíková and Džurina [4] studied neutral differential equation

$$(r(t)[x(t) + p(t)x(\tau(t))]')' + q(t)x(\sigma(t)) = 0, \tag{5}$$

under the condition $0 \leqslant p(t) \leqslant p_0 < \infty$, $\tau'(t) \geqslant \tau_0 > 0$, $\tau o \sigma = \sigma o \tau$ and $R(t) = \int_{t_0}^{t} \frac{ds}{r(s)} \to \infty$ as $t \to \infty$. Our results originated from results of Baculíková and Džurina [4]. We note that when $\gamma = 1$ in (1) we get (5) therefore our results improve and generalize that of in [4]. For some other oscillation results we refer the reader to the papers [5-7] and the references cited therein and related books [8-10].

2 Main Results

Let us use the following notations for convenience:
$y(t) = x(t) + p(t)x(\tau(t))$ and $Q(t) = min\{q(t), q(\tau(t))\}$.

Lemma 1. *Assume that* (H_2) *holds. Let* $x(t)$ *be an eventually positive solution of (1), then there exists a* t_1 *such that*

$$y(t) > 0, \quad r(t)(y'(t))^{\gamma} > 0, \quad (r(t)(y'(t))^{\gamma})' < 0 \tag{6}$$

for $t \geqslant t_1$.

Proof. Since $x(t)$ is an eventually positive solution of (1), in view of property of $\sigma(t)$ there exists a $t_1 \geqslant t_0$ such that $x(\sigma(t)) > 0$ for $t \geqslant t_1$. Then it follows from (1) that

$$(r(t)(y'(t))^\gamma)' = -q(t)x^\gamma(\sigma(t)). \tag{7}$$

Because of postitive nature of $q(t)$, from (7) $r(t)(y'(t))^\gamma$ is strictly decreasing for $t \geqslant t_1$. Thus we can have either $y'(t) > 0$ for $t \geqslant t_1$ or $y'(t) < 0$ for $t \geqslant t_2 \geqslant t_1$. Suppose $y'(t) < 0$ for $t \geqslant t_2 \geqslant t_1$, then there exists a $c > 0$ such that

$$r(t)(y'(t))^\gamma < -c < 0. \tag{8}$$

Dividing (8) by $r(t)$, taking γ root of both sides and integrating from t_2 to t, respectively, we obtain

$$y(t) \leqslant y(t_2) - c^{1/\gamma} \int_{t_2}^{t} \frac{ds}{r^{\frac{1}{\gamma}}(s)} \to -\infty \quad as \quad t \to \infty,$$

which contradicts the positive nature of $y(t)$ and therefore $y'(t) > 0$ for $t \geqslant t_1$. Thus the proof is complete. □

Theorem 1. *Assume that the first order neutral differential inequality*

$$\left(w(t) + \frac{{p_0}^\gamma}{\tau_0} w(\tau(t))\right)' + \frac{Q(t)}{2^{\gamma-1}}(R(\sigma(t)) - R(t_1))^\gamma w(\sigma(t)) \leqslant 0 \tag{9}$$

has no positive solution, then (1) is oscillatory.

Proof. Assume that $x(t) > 0$ is a solution of (1). Then $y(t)$ satisfies

$$\begin{aligned} [y(\sigma(t))]^\gamma &= [x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t)))]^\gamma \\ &\leqslant [x(\sigma(t)) + p_0 x(\sigma(\tau(t)))]^\gamma \\ &\leqslant 2^{\gamma-1}[x^\gamma(\sigma(t)) + {p_0}^\gamma x^\gamma(\sigma(\tau(t)))]. \end{aligned} \tag{10}$$

On the other hand, using (1) and (H_1), we have

$$(r(t)(y'(t))^\gamma)' + q(t)x^\gamma(\sigma(t)) = 0, \tag{11}$$

or

$$\begin{aligned} 0 &= \frac{{p_0}^\gamma}{\tau'(t)}(r(\tau(t))(y'(\tau(t)))^\gamma)' + {p_0}^\gamma q(\tau(t))x^\gamma(\sigma(\tau(t))) \\ &\geqslant \frac{{p_0}^\gamma}{\tau_0}(r(\tau(t))(y'(\tau(t)))^\gamma)' + {p_0}^\gamma q(\tau(t))x^\gamma(\sigma(\tau(t))). \end{aligned} \tag{12}$$

Combining (11) and (12), we see that

$$(r(t)(y'(t))^\gamma)' + \frac{{p_0}^\gamma}{\tau_0}(r(\tau(t))(y'(\tau(t)))^\gamma)' + {p_0}^\gamma q(\tau(t))x^\gamma(\sigma(\tau(t))) + q(t)x^\gamma(\sigma(t)) \leqslant 0. \tag{13}$$

Taking (10) into account and using (13), we obtain

$$(r(t)(y'(t))^\gamma)' + \frac{{p_0}^\gamma}{\tau_0}(r(\tau(t))(y'(\tau(t)))^\gamma)' + \frac{Q(t)}{2^{\gamma-1}} y^\gamma(\sigma(t)) \leqslant 0. \tag{14}$$

Moreover, since $w(t) = r(t)(y'(t))^\gamma > 0$ is decreasing from Lemma 1, we obtain the following inequality

$$\begin{aligned} y(t) \geqslant \int_{t_1}^{t} \frac{1}{r^{1/\gamma}(s)}(r^{1/\gamma}(s)y'(s))ds &\geqslant w^{1/\gamma}(t)\int_{t_1}^{t}\frac{1}{r^{1/\gamma}(s)}ds \\ &= w^{1/\gamma}(t)(R(t)-R(t_1)). \end{aligned} \tag{15}$$

Using (15) into (14) we see that $w(t)$ is a positive solution of (9) which contradicts our assumption and completes the proof.

□

Theorem 2. *Assume that $\tau(t) \geqslant t$. If the first order differential inequality*

$$z'(t) + \left(\frac{\tau_0}{\tau_0 + {p_0}^\gamma}\right)\frac{Q(t)}{2^{\gamma-1}}(R(\sigma(t)) - R(t_1))^\gamma z(\sigma(t)) \leqslant 0 \tag{16}$$

has no positive solution, then (1) is oscillatory.

Proof. Assume that $x(t) > 0$ is a solution of (1). We know from Lemma 1 and the proof of Theorem 1 that $w(t) = r(t)(y'(t))^\gamma > 0$ is decreasing and it satisfies (9). Let us denote $z(t) = w(t) + \frac{{p_0}^\gamma}{\tau_0}w(\tau(t))$. Since $\tau(t) \geqslant t$, we have

$$z(t) \leqslant w(t)\left(1 + \frac{{p_0}^\gamma}{\tau_0}\right).$$

Substituting these terms into (9), we see that $z(t)$ is a positive solution of (16) and therefore we have a contradiction which completes the proof. □

Corollary 1. *Assume that $\tau(t) \geqslant t$ and $\sigma(t) \leqslant t$. If*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)R^\gamma(\sigma(s))ds > \frac{2^{\gamma-1}(\tau_0 + {p_0}^\gamma)}{\tau_0 e} \tag{17}$$

then (1) is oscillatory.

Proof. It is well known that when (17) holds (16) has no eventually positive solutions and therefore (1) is oscillatory. □

Theorem 3. *Assume that $\tau(t) \leqslant t$. If the first order differential inequality*

$$z'(t) + \left(\frac{\tau_0}{\tau_0 + {p_0}^\gamma}\right)\frac{Q(t)}{2^{\gamma-1}}(R(\sigma(t)) - R(t_1))^\gamma z(\tau^{-1}(\sigma(t))) \leqslant 0, \tag{18}$$

where $\tau^{-1}(t)$ is inverse function of $\tau(t)$, has no positive solution, then (1) is oscillatory.

Proof. Assume that $x(t) > 0$ is a solution of (1). As in the proof of Theorem 1 we see that $w(t) = r(t)(y'(t))^\gamma > 0$ is decreasing and it satisfies (9). Let us denote $z(t) = w(t) + \frac{{p_0}^\gamma}{\tau_0}w(\tau(t))$. Since $\tau(t) \leqslant t$, we have

$$z(t) \leqslant w(\tau(t))\left(1 + \frac{{p_0}^\gamma}{\tau_0}\right).$$

Using the last inequality in (9), we see that $z(t)$ is a positive solution of (18) and therefore we have a contradiction which completes the proof. □

Corollary 2. *Assume that* $\sigma(t) \leqslant \tau(t) \leqslant t$. *If*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} Q(s)R^{\gamma}(\sigma(s))ds > \frac{2^{\gamma-1}(\tau_0 + p_0{}^{\gamma})}{\tau_0 e}$$

then (1) is oscillatory.

Proof. The proof is similar to that of Corollary 1 therefore it is omitted. □

Example 1. *Consider the nonlinear neutral differential eqaution*

$$\left\{ t^{7/6} \left[\left(x(t) + \frac{1}{t}x(2t) \right)' \right]^{5/3} \right\}' + \frac{2^{29/6}}{t^{3/2}} x^{5/3}(t/2) = 0, \qquad t \geqslant 1, \tag{19}$$

where $\gamma = 5/3$, $\tau(t) = 2t$ *and* $\tau'(t) = 2 \geqslant 1 = (\tau_0)^{1/\gamma}$, $p(t) = 1/t < 1 = p_0$, *for* $t \geqslant 1$. *We will apply Corollary 1 to (19).*

$$Q(t) = min\{q(t), q(\tau(t))\} = q(2t) = \frac{2^{10/3}}{t^{3/2}}.$$

$$R(t) = \int_{t_0}^{t} \frac{ds}{r(s)^{1/\gamma}} = \int_{0}^{t} \frac{ds}{((s)^{7/6})^{3/5}} = \frac{10}{3} t^{3/10}.$$

Then we get

$$\begin{aligned} \liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)R^{\gamma}(\sigma(s))ds &= \liminf_{t\to\infty} \int_{t/2}^{t} \frac{2^{10/3}}{s^{3/2}} \left(\frac{10}{3} (\frac{s}{2})^{3/10} \right)^{5/3} ds \\ &= \liminf_{t\to\infty} \int_{t/2}^{t} 2^{17/6} \left(\frac{10}{3} \right)^{5/3} \frac{1}{s} ds > 2^{\gamma-1} \frac{(\tau_0 + p_0^{\gamma})}{\tau_0 e} = \frac{2^{5/3}}{e}, \end{aligned}$$

which guaranties the oscillation of (19).

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On the fuzzy stability of a generalized Jensen quadratic functional equation

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of the following generalized Jensen quadratic functional equation

$$f\left(\frac{x+y}{r}+sz\right)+f\left(\frac{x+y}{r}-sz\right)+f\left(\frac{x-y}{r}+sz\right)+f\left(\frac{x-y}{r}-sz\right)=\frac{4}{r^2}f(x)+\frac{4}{r^2}f(y)+4s^2f(z)$$

in fuzzy Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Th. M. Rassias [23] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. (Th.M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x)-L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

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is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [27] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [6] proved the Hyers-Ulam stability of the quadratic functional equation.

In this paper, we consider the following *generalized Jensen functional equation*

$$\begin{aligned} f\left(\frac{x+y}{r}+sz\right)+f\left(\frac{x+y}{r}-sz\right) &+ f\left(\frac{x-y}{r}+sz\right) \\ &+ f\left(\frac{x-y}{r}-sz\right)=\frac{4}{r^2}f(x)+\frac{4}{r^2}f(y)+4s^2f(z) \end{aligned} \tag{1.1}$$

and prove the Hyers-Ulam stability of the functional equation (1.1) in fuzzy Banach spaces.

Jun and Cho [11] proved that a mapping $f : X \to Y$ satisfies the functional equation (1.1) if and only if the mapping $f : X \to Y$ is quadratic. Moreover, they proved the Hyers-Ulam stability of the functional equation (1.1) in Banach spaces. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [9, 12], [19]–[21], [24]–[26]).

Katsaras [13] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms a vector space from various points of view (see [8, 15, 22]). In particular, Bag and Samanta [1], following Cheng and Mordeson [4], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [14]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

2. **Preliminaries**

Definition 2.1. (Bag and Samanta [1]) *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
$(N1)$ $N(x,t) = 0$ for $t \leq 0$;
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
$(N3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
$(N4)$ $N(x+y, c+t) \geq min\{N(x,s), N(y,t)\}$;
$(N5)$ $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
$(N6)$ for $x \neq 0$, $N(x,.)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space. The properties of fuzzy normed vector space and examples of fuzzy norms are given in (see [17, 18]).

Example 2.1. *Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 2.2. (Bag and Samanta [1]) *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t\to\infty} x_n = x$.*

Definition 2.3. (Bag and Samanta [1]) *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.
We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [2]).

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Definition 2.4. *Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:*

(1) *$d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;*

(2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

(3) *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

Theorem 2.1. ([3, 7]) *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$*;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

Lemma 2.1. *Let* X *and* Y *be vector spaces. If a mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and (1.1) for all* $x, y, z \in X$*, then the mapping* f *is quadratic.*

Proof. Letting $x = y$ in (1.1), we get

$$f\left(\frac{2x}{r} + sz\right) + f\left(\frac{2x}{r} - sz\right) + f(sz) + f(-sz) = \frac{8}{r^2} f(x) + 4s^2 f(z) \tag{2.1}$$

for all $x, z \in X$. Letting $x = 0$ in (2.1), we get $2f(sz) + 2f(-sz) = 4s^2 f(z)$. Setting $y = -x$ in (1.1), we obtain

$$f\left(\frac{2x}{r} + sz\right) + f\left(\frac{2x}{r} - sz\right) + f(sz) + f(-sz) = \frac{4}{r^2} f(x) + \frac{4}{r^2} f(-x) + 4s^2 f(z). \tag{2.2}$$

By (2.1) and (2.2), we conclude that f is even. And by setting $z = 0$ in (2.1), we get $f\left(\frac{2x}{r}\right) = \frac{4}{r^2} f(x)$ for all $x \in X$. So we get

$$f\left(\frac{2x}{r} + sz\right) + f\left(\frac{2x}{r} - sz\right) + f(sz) + f(-sz) = 2f\left(\frac{2x}{r}\right) + 4f(sz)$$

for all $x, z \in X$. Hence we get

$$f\left(\frac{2x}{r} + sz\right) + f\left(\frac{2x}{r} - sz\right) = 2f\left(\frac{2x}{r}\right) + 2f(sz)$$

for all $x, z \in X$. So f is quadratic. □

The mapping $f : X \to Y$ given in the statement of Lemma 2.1 is called a *generalized Jensen quadratic mapping.* Putting $z = 0$ in (1.1) with $r = 2$, we get the Jensen quadratic mapping, i.e.,

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

and putting $x = y$ in (1.1) with $r = 2$ and $s = 1$, we get the quadratic mapping, i.e.,

$$f(x + z) + f(x - z) = 2f(x) + 2f(z).$$

3. **Fuzzy stability of the functional equation (1.1)**

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y,z) : &= f\left(\frac{x+y}{r}+sz\right)+f\left(\frac{x+y}{r}-sz\right)+f\left(\frac{x-y}{r}+sz\right) \\ &+ f\left(\frac{x-y}{r}-sz\right)-\frac{4}{r^2}f(x)-\frac{4}{r^2}f(y)-4s^2f(z) \end{aligned}$$

for all $x, y, z \in X$.

Theorem 3.1. *Let $\phi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{rx}{2},\frac{ry}{2},\frac{rz}{2}\right) \leq \frac{r^2L}{4}\phi(x,y,z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(Df(x,y,z),t\right) \geq \frac{t}{t+\phi(x,y,z)} \tag{3.1}$$

for all $x, y, z \in X$, all $t > 0$ and $r, s \neq 0$. Then

$$Q(x) := N\text{-}\lim_{n\to\infty}\left(\frac{4}{r^2}\right)^n f\left(\left(\frac{r}{2}\right)^n x\right)$$

exists for each $x \in X$ and defines a unique generalized Jensen quadratic mapping $Q : X \to Y$ such that

$$N(f(x)-Q(x),t) \geq \frac{(8-8L)t}{(8-8L)t+r^2L\phi(x,x,0)}. \tag{3.2}$$

Proof. Putting $x = y$ and $z = 0$ in (3.1), we have

$$N\left(2f\left(\frac{2x}{r}\right)-\frac{8}{r^2}f(x),t\right) \geq \frac{t}{t+\phi(x,x,0)} \tag{3.3}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S := \{g : X \to Y, g(0) = 0\}$$

and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\phi(x,x,0)}, \forall x \in X,\, t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [16, Lemma 2.1]).

Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{4}{r^2}g\left(\frac{rx}{2}\right)$$

for all $x \in X$. Let $g, h \in S$ satisfy $d(g, h) = \epsilon$. Then

$$N(g(x)-h(x),\epsilon t) \geq \frac{t}{t+\phi(x,x,0)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(\frac{4}{r^2}g\left(\frac{rx}{2}\right) - \frac{4}{r^2}h\left(\frac{rx}{2}\right), L\epsilon t\right) \\ &= N\left(g\left(\frac{rx}{2}\right) - h\left(\frac{rx}{2}\right), \frac{r^2}{4}L\epsilon t\right) \\ &\geq \frac{\frac{r^2}{4}Lt}{\frac{r^2}{4}Lt + \phi(\frac{rx}{2}, \frac{rx}{2}, 0)} \\ &\geq \frac{\frac{r^2}{4}Lt}{\frac{r^2}{4}Lt + \frac{r^2}{4}L\phi(x, x, 0)} \\ &= \frac{t}{t + \phi(x, x, 0)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (3.3) that

$$\begin{aligned} N\left(f(x) - \frac{4}{r^2}f\left(\frac{rx}{2}\right), t\right) &\geq \frac{2t}{2t + \phi(\frac{rx}{2}, \frac{rx}{2}, 0)} \\ &\geq \frac{2t}{2t + \frac{r^2L}{4}\phi(x, x, 0)} \\ &= \frac{\frac{8t}{r^2L}}{\frac{8t}{r^2L} + \phi(x, x, 0)}. \end{aligned}$$

Therefore,

$$N\left(f(x) - \frac{4}{r^2}f\left(\frac{rx}{2}\right), \frac{r^2Lt}{8}\right) \geq \frac{t}{t + \phi(x, x, 0)}.$$

This means, $d(f, Jf) \leq \frac{r^2L}{8}$.

By Theorem 2.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q\left(\frac{rx}{2}\right) = \frac{r^2}{4}Q(x) \tag{3.4}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.4) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \phi(x, x, 0)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} N\text{-}\left(\frac{4}{r^2}\right)^n f\left(\left(\frac{r}{2}\right)^n x\right) = Q(x)$$

for all $x \in X$.

(3) $d(f, Q) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, Q) \leq \frac{r^2 L}{8 - 8L}.$$

This implies that the inequality (3.2) holds.

Replacing x, y and z by $(\frac{r}{2})^n x$, $(\frac{r}{2})^n y$ and $(\frac{r}{2})^n z$, respectively, in (3.1), we get

$$\begin{aligned}
&N\bigg(\left(\frac{4}{r^2}\right)^n\bigg[f\left(\left(\frac{r}{2}\right)^n\left[\frac{x+y}{r} \pm sz\right]\right) + f\left(\left(\frac{r}{2}\right)^n\left[\frac{x-y}{r} \pm sz\right]\right)\\
&-\frac{4}{r^2}f\left(\left(\frac{r}{2}\right)^n x\right) - \frac{4}{r^2}f\left(\left(\frac{r}{2}\right)^n y\right) - 4s^2 f\left(\left(\frac{r}{2}\right)^n z\right)\bigg], t\bigg)\\
&\geq \frac{(\frac{4}{r^2})^n t}{(\frac{4}{r^2})^n t + \phi((\frac{r}{2})^n x, (\frac{r}{2})^n y, (\frac{r}{2})^n z)}\\
&\geq \frac{(\frac{4}{r^2})^n t}{(\frac{4}{r^2})^n t + (\frac{r^2}{4})^n L^n \phi(x, y, z)}
\end{aligned}$$

for all $x, y, z \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{(\frac{4}{r^2})^n t}{(\frac{4}{r^2})^n t + (\frac{r^2}{4})^n L^n \phi(x, y, z)} = 1$$

for all $x, y, z \in X$ and all $t > 0$, we deduce that

$$N\left(DQ(x, y, z), t\right) = 1$$

for all $x, y, z \in X$ and all $t > 0$. Thus the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 3.1. *Let $\theta \geq 0$ and let p be a real number with $p > 2$ and $0 < |r| < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(Df(x, y, z), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)} \tag{3.5}$$

for all $x, y, z \in X$ and all $t > 0$. Then

$$Q(x) := N\text{-}\lim_{n\to\infty}\left(\frac{4}{r^2}\right)^n f\left(\left(\frac{r}{2}\right)^n x\right)$$

exists for each $x \in X$ and defines a generalized Jensen quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{4(2^p - |r|^p)t}{4(2^p - |r|^p)t + r^2|r|^p\theta\|x\|^p}.$$

Proof. The proof follows from Theorem 3.1 by taking

$$\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = \left(\frac{|r|}{2}\right)^p$ and we get the desired result. □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$ and $|r| > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.5). Then*

$$Q(x) := N\text{-}\lim_{n\to\infty} \left(\frac{4}{r^2}\right)^n f\left(\left(\frac{r}{2}\right)^n x\right)$$

exists for each $x \in X$ and defines a generalized Jensen quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(|r|^p - 2^p)t}{(|r|^p - 2^p)t + r^2 \cdot 2^{p-2}\theta\|x\|^p}.$$

Proof. The proof follows from Theorem 3.1 by taking

$$\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = \left(\frac{2}{|r|}\right)^p$ and we get the desired result. □

Theorem 3.2. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x, y, z) \leq \frac{4L}{r^2}\phi\left(\frac{rx}{2}, \frac{ry}{2}, \frac{rz}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and satisfying (3.1). Then

$$R(x) := N\text{-}\lim_{n\to\infty} \left(\frac{r^2}{4}\right)^n f\left(\left(\frac{2}{r}\right)^n x\right)$$

exists for each $x \in X$ and defines a generalized Jensen quadratic mapping $R : X \to Y$ such that

$$N(f(x) - R(x), t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + r^2\phi(x, x, 0)} \tag{3.6}$$

Proof. Let (S, d) be the generalized metric space defined as in the proof of Theorem 3.1. Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{r^2}{4} g\left(\frac{2x}{r}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$N(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \phi(x, x, 0)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(\frac{r^2}{4} g\left(\frac{2x}{r}\right) - \frac{r^2}{4} h\left(\frac{2x}{r}\right), L\epsilon t\right) \\ &= N\left(g\left(\frac{2x}{r}\right) - h\left(\frac{2x}{r}\right), \frac{4}{r^2} L\epsilon t\right) \\ &\geq \frac{\frac{4}{r^2} Lt}{\frac{4}{r^2} Lt + \phi(\frac{2x}{r}, \frac{2x}{r}, 0)} \\ &\geq \frac{\frac{4}{r^2} Lt}{\frac{4}{r^2} Lt + \frac{4L}{r^2}\phi(x, x, 0)} \\ &= \frac{t}{t + \phi(x, x, 0)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (3.3) that

$$N\left(\frac{r^2}{4} f\left(\frac{2x}{r}\right) - f(x), \frac{r^2 t}{8}\right) \geq \frac{t}{t + \phi(x, x, 0)}$$

for all $x \in X$ and $t > 0$. So $d(f, Jf) \leq \frac{r^2}{8}$. By Theorem 2.1, there exists a mapping $R : X \to Y$ satisfying the following:

(1) R is a fixed point of J, that is,

$$\frac{4}{r^2} R(x) = R\left(\frac{2x}{r}\right) \tag{3.7}$$

for all $x \in X$. The mapping R is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that R is a unique mapping satisfying (3.7) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - R(x), \mu t) \geq \frac{t}{t + \phi(x, x, 0)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, R) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} N\text{-}\left(\frac{r^2}{4}\right)^n f\left(\left(\frac{2}{r}\right)^n x\right) = R(x)$$

for all $x \in X$.

(3) $d(f, R) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, R) \leq \frac{r^2}{8-8L}.$$

This implies that the inequality (3.6) holds.

The rest of the proof is similar to that of the proof of Theorem 3.1. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$ and $|r| > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (3.5)*. Then*

$$R(x) := N\text{-}\lim_{n\to\infty}\left(\frac{r^2}{4}\right)^n f\left(\left(\frac{2}{r}\right)^n x\right)$$

exists for each $x \in X$ and defines a generalized Jensen quadratic mapping $R : X \to Y$ such that

$$N(f(x) - R(x), t) \geq \frac{(|r|^p - 2^p)t}{(|r|^p - 2^p)t + 4^{-1} \cdot r^2|r|^p\theta\|x\|^p}.$$

Proof. The proof follows from Theorem 3.2 by taking

$$\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = \left(\frac{|r|}{2}\right)^{-p}$ and we get the desired result. □

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$ and $|r| < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (3.5)*. Then*

$$R(x) := N\text{-}\lim_{n\to\infty}\left(\frac{r^2}{4}\right)^n f\left(\left(\frac{2}{r}\right)^n x\right)$$

exists for each $x \in X$ and defines a generalized Jensen quadratic mapping $R : X \to Y$ such that

$$N(f(x) - R(x), t) \geq \frac{(2^p - |r|^p)t}{(2^p - |r|^p)t + r^2.2^{p-2}\theta\|x\|^p}.$$

Proof. The proof follows from Theorem 3.2 by taking

$$\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = \left(\frac{|r|}{2}\right)^p$ and we get the desired result. □

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Landau type inequalities on time scales

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Abstract

Here we prove some new type Landau inequalities on Time Scales. Both delta and nabla cases are presented. We give applications.

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Key Words and Phrases: Time Scale, Landau inequality, delta and nabla derivatives.

1 Introduction

We are motivated by [27], where E. Landau in 1913 proved the following two theorems:

Theorem 1 *Let $f \in C^2(\mathbb{R}_+, \mathbb{R})$ with $\|f\|_{\infty,\mathbb{R}_+}$, $\|f''\|_{\infty,\mathbb{R}_+} < \infty$. Then*

$$\|f'\|_{\infty,\mathbb{R}_+} \leq 2\sqrt{\|f\|_{\infty,\mathbb{R}_+} \|f''\|_{\infty,\mathbb{R}_+}},$$

with 2 the best constant.

Theorem 2 *Let $f \in C^2(\mathbb{R}, \mathbb{R})$ with $\|f\|_{\infty,\mathbb{R}}$, $\|f''\|_{\infty,\mathbb{R}} < \infty$. Then*

$$\|f'\|_{\infty,\mathbb{R}} \leq \sqrt{2\|f\|_{\infty,\mathbb{R}} \|f''\|_{\infty,\mathbb{R}}},$$

with $\sqrt{2}$ the best constant.

In this article we establish a new type of Landau inequalities on time scales. We treat both delta and nabla cases with applications. To keep paper short for basics on time scales we refer the reader to the following sources: [1], [2], [9], [12], [14], [15], [16], [19], [20], [21], [23], [24], [25], [26], [28], [29].

2 Background

Let $f \in C_{rd}^{1}(\mathbb{T})$, $s,t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale, then

$$\int_{s}^{t} f^{\Delta}(\tau)\,\Delta\tau = f(t) - f(s).$$

Here we use $h_0(t,s) = 1$, $\forall\ s,t \in \mathbb{T}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau$, $\forall\ s,t \in \mathbb{T}$.

Notice that

$$h_1(r,s) = \int_{s}^{r} 1\Delta\tau = r - s,\ \ \forall\ r,s \in \mathbb{T},$$

and

$$h_2(r,s) = \int_{s}^{r} (\tau - s)\,\Delta\tau,\ \ \forall\ r,s \in \mathbb{T}.$$

If $r \geq s$, then $0 \leq \tau - s \leq r - s$ and $0 \leq h_2(r,s) \leq (r-s)^2$.

If $r \leq s$, then

$$0 \leq h_2(r,s) = \int_{r}^{s} (s - \tau)\,\Delta\tau \leq (s-r)^2.$$

Hence in general

$$0 \leq h_2(r,s) \leq (r-s)^2,\ \forall\ r,s \in \mathbb{T}.$$

Also $h_2(r,s) \neq h_2(s,r)$.

Similarly we define and use

$$\widehat{h_0}(t,s) = 1,\ \ \forall\ s,t \in \mathbb{T},$$

and

$$\widehat{h_{k+1}}(t,s) = \int_{s}^{t} \widehat{h_k}(\tau,s)\,\nabla\tau,\ \ \forall\ s,t \in \mathbb{T},\ k \in \mathbb{N}_0.$$

We have that

$$\widehat{h_1}(t,s) = t - s,$$

and

$$\widehat{h_2}(t,s) = \int_{s}^{t} (\tau - s)\,\nabla\tau.$$

If $t \geq s$, then

$$0 \leq \widehat{h_2}(t,s) \leq (t-s)^2.$$

If $t \leq s$, then

$$0 \leq \widehat{h_2}(t,s) = \int_{t}^{s} (s - \tau)\,\nabla\tau \leq (s-t)^2.$$

Therefore

$$0 \leq \widehat{h_2}(t,s) \leq (t-s)^2,\ \ \forall\ t,s \in \mathbb{T}.$$

Let $f \in C_{ld}^{1}(\mathbb{T})$, then

$$\int_{s}^{t} f^{\nabla}(\tau)\,\nabla\tau = f(t) - f(s).$$

We need

Theorem 3 *(see [9], [13], p. 634) Here $\mathbb{T} = \mathbb{T}^{k}$. Let $f \in C_{rd}^{1}(\mathbb{T})$, $a, b, c \in \mathbb{T}$: $a \leq c \leq b$. Then*

$$\left| \frac{1}{b-a} \int_{a}^{b} f(t)\,\Delta t - f(c) \right| \leq \left(\frac{h_2(a,c) + h_2(b,c)}{b-a} \right) \left\| f^{\Delta} \right\|_{\infty,[a,b]\cap\mathbb{T}}. \tag{1}$$

The last is a basic Δ-Ostrowski inequality on time scales, see for basics on $\mathbb{R}$ [30].

Remark 4 *We notice that*

$$R.H.S.(1) \leq \left(\frac{(c-a)^2 + (b-c)^2}{b-a} \right) \left\| f^{\Delta} \right\|_{\infty,[a,b]\cap\mathbb{T}}$$

$$\leq \frac{(b-a)^2}{(b-a)} \left\| f^{\Delta} \right\|_{\infty,[a,b]\cap\mathbb{T}} = (b-a) \left\| f^{\Delta} \right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Therefore

$$\left| \frac{1}{b-a} \int_{a}^{b} f(t)\,\Delta t - f(c) \right| \leq (b-a) \left\| f^{\Delta} \right\|_{\infty,[a,b]\cap\mathbb{T}}. \tag{2}$$

We need

Theorem 5 *(see [12], [13], p. 659) Here $\mathbb{T} = \mathbb{T}^{k} = \mathbb{T}_{k}$. Let $f \in C_{ld}^{1}(\mathbb{T})$, $a, b, c \in \mathbb{T} : a \leq c \leq b$. Then*

$$\left| \frac{1}{b-a} \int_{a}^{b} f(t)\,\nabla t - f(c) \right| \leq \left(\frac{\widehat{h_2}(a,c) + \widehat{h_2}(b,c)}{b-a} \right) \left\| f^{\nabla} \right\|_{\infty,[a,b]\cap\mathbb{T}}. \tag{3}$$

The last is a basic ∇-Ostrowski inequality on time scales.

Remark 6 *We notice that*

$$R.H.S.(3) \leq \left(\frac{(c-a)^2 + (b-c)^2}{b-a} \right) \left\| f^{\nabla} \right\|_{\infty,[a,b]\cap\mathbb{T}}$$

$$\leq \frac{(b-a)^2}{(b-a)} \left\| f^{\nabla} \right\|_{\infty,[a,b]\cap\mathbb{T}} = (b-a) \left\| f^{\nabla} \right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Therefore

$$\left| \frac{1}{b-a} \int_{a}^{b} f(t)\,\nabla t - f(c) \right| \leq (b-a) \left\| f^{\nabla} \right\|_{\infty,[a,b]\cap\mathbb{T}}. \tag{4}$$

3 Main Results

We give the following Δ-Landau type inequality.

Theorem 7 *Here $\mathbb{T}=\mathbb{T}^k$. Let $f\in C_{rd}^2(\mathbb{T})$, $a,b\in\mathbb{T}$, $a<b$. Then*

$$\left\|f^{\Delta}\right\|_{\infty,[a,b]\cap\mathbb{T}}\leq\left(\frac{2}{b-a}\right)\|f\|_{\infty,[a,b]\cap\mathbb{T}}+(b-a)\left\|f^{\Delta^2}\right\|_{\infty,[a,b]\cap\mathbb{T}}. \tag{5}$$

Proof. By (2) we have

$$\left|\frac{1}{b-a}\int_a^b f^{\Delta}(t)\,\Delta t-f'(c)\right|\leq(b-a)\left\|f^{\Delta^2}\right\|_{\infty,[a,b]\cap\mathbb{T}},$$

for any $c\in\mathbb{T}:a\leq c\leq b$.

Therefore

$$\left|\frac{1}{b-a}\left(f(b)-f(a)\right)-f^{\Delta}(c)\right|\leq(b-a)\left\|f^{\Delta^2}\right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Hence

$$\left|f^{\Delta}(c)\right|-\frac{1}{b-a}\left|f(b)-f(a)\right|\leq(b-a)\left\|f^{\Delta^2}\right\|_{\infty,[a,b]\cap\mathbb{T}}$$

and

$$\left|f^{\Delta}(c)\right|\leq\left(\frac{2}{b-a}\right)\|f\|_{\infty,[a,b]\cap\mathbb{T}}+(b-a)\left\|f^{\Delta^2}\right\|_{\infty,[a,b]\cap\mathbb{T}},$$

for any $c\in[a,b]\cap\mathbb{T}$.

The last implies (5). ■

Next we present a ∇-Landau type inequality.

Theorem 8 *Here $\mathbb{T}=\mathbb{T}^k=\mathbb{T}_k$. Let $f\in C_{ld}^2(\mathbb{T})$, $a,b\in\mathbb{T}$, $a<b$. Then*

$$\left\|f^{\nabla}\right\|_{\infty,[a,b]\cap\mathbb{T}}\leq\frac{2\|f\|_{\infty,[a,b]\cap\mathbb{T}}}{(b-a)}+(b-a)\left\|f^{\nabla^2}\right\|_{\infty,[a,b]\cap\mathbb{T}}. \tag{6}$$

Proof. By (4) we get

$$\left|\frac{1}{b-a}\int_a^b f^{\nabla}(t)\,\nabla t-f^{\nabla}(c)\right|\leq(b-a)\left\|f^{\nabla^2}\right\|_{\infty,[a,b]\cap\mathbb{T}},$$

for any $c\in\mathbb{T}:a\leq c\leq b$.

That is

$$\left|\left(\frac{f(b)-f(a)}{b-a}\right)-f^{\nabla}(c)\right|\leq(b-a)\left\|f^{\nabla^2}\right\|_{\infty,[a,b]\cap\mathbb{T}},$$

and

$$\left|f^{\nabla}(c)\right|-\frac{1}{b-a}\left|f(b)-f(a)\right|\leq(b-a)\left\|f^{\nabla^2}\right\|_{\infty,[a,b]\cap\mathbb{T}},$$

and

$$\left|f^{\nabla}(c)\right| \leq \frac{1}{b-a}\left|f(b)-f(a)\right| + (b-a)\left\|f^{\nabla^2}\right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Therefore

$$\left|f^{\nabla}(c)\right| \leq \frac{2\left\|f\right\|_{\infty,[a,b]\cap\mathbb{T}}}{(b-a)} + (b-a)\left\|f^{\nabla^2}\right\|_{\infty,[a,b]\cap\mathbb{T}},$$

for any $c \in [a,b]\cap\mathbb{T}$, proving the claim. ■

4 Applications

i) When $\mathbb{T}=\mathbb{R}$, then see [3], [4].

ii) When $\mathbb{T}=\mathbb{Z}$, see that $\mathbb{Z}=\mathbb{Z}^k=\mathbb{Z}_k$, then

$$\begin{aligned} f^{\Delta}(t) &= f(t+1)-f(t), \\ f^{\nabla}(t) &= f(t)-f(t-1), \\ f^{\Delta^2}(t) &= f(t)-2f(t+1)+f(t+2), \end{aligned} \tag{7}$$

and

$$f^{\nabla^2}(t) = f(t)-2f(t-1)+f(t-2),$$

where $f:\mathbb{Z}\to\mathbb{R}$.

Then by (5) we get

$$\left\|f(\cdot+1)-f(\cdot)\right\|_{\infty,[a,b]\cap\mathbb{Z}} \leq$$

$$\left(\frac{2}{b-a}\right)\left\|f\right\|_{\infty,[a,b]\cap\mathbb{Z}} + (b-a)\left\|f(\cdot)-2f(\cdot+1)+f(\cdot+2)\right\|_{\infty,[a,b]\cap\mathbb{Z}}. \tag{8}$$

Also by (6) we have that

$$\left\|f(\cdot)-f(\cdot-1)\right\|_{\infty,[a,b]\cap\mathbb{Z}} \leq$$

$$\frac{2\left\|f\right\|_{\infty,[a,b]\cap\mathbb{Z}}}{(b-a)} + (b-a)\left\|f(\cdot)-2f(\cdot-1)+f(\cdot-2)\right\|_{\infty,[a,b]\cap\mathbb{Z}}. \tag{9}$$

iii) Next let $q>1$, $q^{\mathbb{Z}}=\{q^k : k\in\mathbb{Z}\}$, and we take $\mathbb{T}=q^{\overline{\mathbb{Z}}}=q^{\mathbb{Z}}\cup\{0\}$, which is a very important time scale to q-difference equations. See that $q^{\overline{\mathbb{Z}}}=\left(q^{\overline{\mathbb{Z}}}\right)^k=\left(q^{\overline{\mathbb{Z}}}\right)_k$.

Let $f: q^{\overline{\mathbb{Z}}}\to\mathbb{R}$.

By [23], p. 17 we get that

$$f^{\Delta}(t) = \frac{f(qt)-f(t)}{(q-1)t}, \quad \forall\ t\in q^{\overline{\mathbb{Z}}}-\{0\}$$

and

$$f^{\Delta}(0)=\lim_{s\to 0}\frac{f(s)-f(0)}{s},$$

provided that the limit exists.

For $t\neq 0$ we have that

$$f^{\Delta^2}(t)=\frac{f\left(q^2t\right)-(q+1)f(qt)+qf(t)}{q(q-1)^2t^2},$$

and

$$f^{\Delta^2}(0)=\lim_{s\to 0}\frac{f^{\Delta}(s)-f^{\Delta}(0)}{s},$$

provided that the limit exists.

Here 0 is a right-dense minimum and every other point in $q^{\overline{\mathbb{Z}}}$ is isolated, also $\sigma(t)=qt$ and $\rho(t)=\frac{t}{q}$, $t\in q^{\overline{\mathbb{Z}}}$, see [23], p. 16.

For $t\in q^{\overline{\mathbb{Z}}}-\{0\}$ we have that (see [14])

$$f^{\nabla}(t)=\frac{f(t)-f\left(\frac{t}{q}\right)}{t-\frac{t}{q}}=\frac{f\left(\frac{1}{q}t\right)-f(t)}{\left(\frac{1}{q}-1\right)t},$$

and since zero is a left-dense point of $q^{\overline{\mathbb{Z}}}$ we have

$$f^{\nabla}(0)=\lim_{s\to 0}\frac{f(s)-f(0)}{s},$$

provided that the limit exists.

For $t\neq 0$ we have that

$$f^{\nabla^2}(t)=\frac{f\left(\frac{1}{q^2}t\right)-\left(\frac{1}{q}+1\right)f\left(\frac{1}{q}t\right)+\frac{1}{q}f(t)}{\frac{1}{q}\left(\frac{1}{q}-1\right)^2t^2},$$

and

$$f^{\nabla^2}(0)=\lim_{s\to 0}\frac{f^{\nabla}(s)-f^{\nabla}(0)}{s},$$

provided that the limit exists.

Let $f\in C_{rd}^2\left(q^{\overline{\mathbb{Z}}}\right)$; $a,b\in q^{\overline{\mathbb{Z}}}$, $a<b$, then by (5) we get

$$\left\|f^{\Delta}\right\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}\leq\left(\frac{2}{b-a}\right)\|f\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}+(b-a)\left\|f^{\Delta^2}\right\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}. \tag{10}$$

If $f\in C_{ld}^2\left(q^{\overline{\mathbb{Z}}}\right)$, then by (6) we get

$$\left\|f^{\nabla}\right\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}\leq\frac{2\|f\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}}{(b-a)}+(b-a)\left\|f^{\nabla^2}\right\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}. \tag{11}$$

5 Addendum

Similarly as in Theorem 1.24 (i), [23], pp. 9-10, we have

Theorem 9 *Let α be a constant and $m \in \mathbb{N}$. For $f(t) = (t-\alpha)^m$ we have*

$$f^{\nabla}(t) = \sum_{\nu=0}^{m-1} (\rho(t) - \alpha)^{\nu} (t-\alpha)^{m-1-\nu}. \tag{12}$$

Proof. By mathematical induction:

If $m = 1$, then $f(t) = t - \alpha$, and $f^{\nabla}(t) = 1$, so that (12) is valid.

Now we assume that (12) is true for $m \in \mathbb{N}$, we will prove it correct for $m+1$.

So let $F(t) = (t-\alpha)^{m+1} = (t-\alpha) f(t)$. Hence by ∇-product rule (see p. 331, Theorem 8.41(iii), of [23]) we get:

$$F^{\nabla}(t) = ((t-\alpha) f(t))^{\nabla} = (f(t)(t-\alpha))^{\nabla}$$

$$= f^{\nabla}(t)(t-\alpha) + f(\rho(t))(t-\alpha)^{\nabla} = f^{\nabla}(t)(t-\alpha) + f(\rho(t))$$

$$= f(\rho(t)) + f^{\nabla}(t)(t-\alpha) = f(\rho(t)) + (t-\alpha) f^{\nabla}(t)$$

$$= (\rho(t)-\alpha)^m + (t-\alpha) \sum_{\nu=0}^{m-1} (\rho(t)-\alpha)^{\nu} (t-\alpha)^{m-1-\nu}$$

$$= (\rho(t)-\alpha)^m + \sum_{\nu=0}^{m-1} (\rho(t)-\alpha)^{\nu} (t-\alpha)^{m-\nu} = \sum_{\nu=0}^{m} (\rho(t)-\alpha)^{\nu} (t-\alpha)^{m-\nu}.$$

The claim is proved. ■

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GENERALIZED INTEGRATION OPERATORS FROM THE SPACE OF INTEGRAL TRANSFORMS INTO BLOCH-TYPE SPACES

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Abstract. We consider the generalized integration operator

$$I^{(n)}_{\varphi,g}f(z)=\int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d\zeta$$

where g is a holomorphic function on the unit disk $\mathbb{D}$, φ is a holomorphic self-map of $\mathbb{D}$, and $n\in\mathbb{N}_0$. The boundedness and compactness of the operator from the space of Cauchy transforms to the Bloch-type spaces are characterized.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measure on $\mathbb{D}$ (i.e. $A(\mathbb{D})=1$), $d\lambda(z)=dA(z)/(1-|z|^2)^2$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, $S(\mathbb{D})$ the class of all holomorphic self-maps of $\mathbb{D}$ and $\mathfrak{M}$ the space of all complex Borel measures on $\partial\mathbb{D}$.

A positive continuous function ν on $\mathbb{D}$ is called a (*weight*). A weight ν is called *typical* if it is radial, i.e. $\nu(z)=\nu(|z|)$, $z\in\mathbb{D}$, and $\nu(|z|)$ decreasingly converges to 0 as $|z|\to 1$. A positive continuous function ν on the interval $[0,1)$ is called normal ([16]) if there are $\delta\in[0,1)$ and τ and t, $0<\tau<t$ such that

$$\frac{\nu(r)}{(1-r)^\tau}\ \text{ is decreasing on }\ [\delta,1)\ \text{ and }\ \lim_{r\to 1}\frac{\nu(r)}{(1-r)^\tau}=0;$$

$$\frac{\nu(r)}{(1-r)^t}\ \text{ is increasing on }\ [\delta,1)\ \text{ and }\ \lim_{r\to 1}\frac{\nu(r)}{(1-r)^t}=\infty.$$

If we say that a function $\nu:\mathbb{D}\to[0,\infty)$ is normal we also assume that it is radial.

For a weight ν, the Bloch-type space $\mathcal{B}_\nu(\mathbb{D})=\mathcal{B}_\nu$ consists of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\nu}:=|f(0)|+b_\nu(f)=|f(0)|+\sup_{z\in\mathbb{D}}\nu(z)|f'(z)|<\infty.$$

The little Bloch-type space $\mathcal{B}_{\nu,0}(\mathbb{D})=\mathcal{B}_{\nu,0}$ consists of all $f\in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1}\nu(z)|f'(z)|=0.$$

With the norm $\|\cdot\|_{\mathcal{B}_\nu}$ the Bloch-type space is Banach and the little Bloch-type space is a closed subspace of the Bloch-type space.

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If ν is normal we have that $\nu(a) \asymp \nu(z)$ when $z \in D(a,(1-|a|)/2)$. Using this fact, the subharmonicity of $|f'|^2$, the asymptotic relation $|1-\bar{a}z| \asymp 1-|a|$, $z \in D(a,1-|a|)$, and Proposition 1.4.10 in [13] we have that $f \in \mathcal{B}_\nu$ if and only if

$$\Psi(f) := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty, \tag{1}$$

where $\eta_a(z) = (a-z)/(1-\bar{a}z)$, $a, z \in \mathbb{D}$. Moreover $b^2_\nu(f) \asymp \Psi(f)$, for $f \in H(\mathbb{D})$.

An $f \in H(\mathbb{D})$ belongs to the space of Cauchy transforms $\mathcal{K}$ if

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1-\bar{\zeta}z} \tag{2}$$

for some $\mu \in \mathfrak{M}$. The space $\mathcal{K}$ becomes a Banach space under the norm

$$\|f\|_{\mathcal{K}} = \inf\left\{ \|\mu\| \ : \ f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1-\bar{\zeta}z} \right\}, \tag{3}$$

where $\|\mu\|$ denotes the total variation of measure μ. It is known that $H^1 \subset \mathcal{K} \subset \cap_{0<p<1} H^p$, where H^p is the Hardy space (see, e.g. [2, 4]).

Let $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $n \in \mathbb{N}_0$. We define the generalized integration operator as follows (see also [15])

$$I^{(n)}_{\varphi,g} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) g(\zeta) d\zeta, \quad z \in \mathbb{D}. \tag{4}$$

Operator (4) is an extension of many operators appearing in the literature. For example, if $n = 0$, then is obtained one of the operators defined in [8] (see also [10, 11, 21, 28]), which is a natural extension of the integral operator by Pommerenke ([12]). If $n = 1$ then is obtained, so called *generalized composition operator* which was introduced in [7] (see, also [18, 19]). Some n-dimensional extensions of these operators, of which some are also extensions of the operators introduced in [5] and [6], are given in [17], [22] and [30], and later studied in [1, 9, 23, 24, 25, 26, 27, 29, 31, 32, 33, 34, 36, 37, 38, 39]. If $\varphi(z) = z$, operator (4) will be denoted by $I^{(n)}_g$.

We provide function theoretic characterizations of when g and φ induce bounded or compact operators in (4) from the space of Cauchy transforms into the Bloch-type spaces, continuing the line of research in [2, 3, 4, 35]. Among other results, operator norm of $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is calculated (for related results, see also [20, 21, 29]).

Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. We write $A \asymp B$ if there is a positive constant C such that $CA \le B \le A/C$.

2. Boundedness and compactness of $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ (or $\mathcal{B}_{\nu,0}$)

In this section, we characterize the boundedness and compactness of operators $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ (or $\mathcal{B}_{\nu,0}$).

Theorem 1. *Let ν be a weight, $n \in \mathbb{N}_0$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded if and only if*

$$M_1 := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\bar{\zeta}\varphi(z)|^{n+1}} < \infty. \tag{5}$$

Moreover, if $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded then

$$\|I^{(n)}_{\varphi,g}\|_{\mathcal{K}\to\mathcal{B}_\nu} = n! M_1. \tag{6}$$

Proof. First, suppose that (5) holds. Let $f \in \mathcal{K}$, then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{K}}$ and

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}.$$

We have

$$f^{(n)}(z) = n! \int_{\partial\mathbb{D}} \frac{(\overline{\zeta})^n}{(1-\overline{\zeta}z)^{n+1}} d\mu(\zeta), \quad n \in \mathbb{N}_0. \tag{7}$$

Replacing z in (7) by $\varphi(z)$, using a known inequality and multiplying such obtained inequality by $\nu(z)|g(z)|$, we obtain

$$\nu(z)|g(z)||f^{(n)}(\varphi(z))| \leq n! \int_{\partial\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}} d|\mu|(\zeta), \tag{8}$$

from which it follows that for each $z \in \mathbb{D}$

$$\nu(z)|(I^{(n)}_{\varphi,g}f)'(z)| \leq n! \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}} \|f\|_{\mathcal{K}}. \tag{9}$$

Taking the supremum in (9) over $z \in \mathbb{D}$, we get

$$b_\nu(I^{(n)}_{\varphi,g}f) \leq n! M_1 \|f\|_{\mathcal{K}}.$$

This inequality along with the fact $|(I^{(n)}_{g,\varphi}f)(0)| = 0$, shows that

$$\|I^{(n)}_{\varphi,g}f\|_{\mathcal{B}_\nu} \leq n! M_1 \|f\|_{\mathcal{K}}. \tag{10}$$

Thus $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded.

Now suppose that $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded. Let

$$f_\zeta(z) = \frac{1}{1-\overline{\zeta}z}, \quad \zeta \in \partial\mathbb{D}. \tag{11}$$

Then

$$\|f_\zeta\|_{\mathcal{K}} = 1, \quad \zeta \in \partial\mathbb{D} \quad \text{and} \quad f^{(n)}_\zeta(z) = n! \frac{(\overline{\zeta})^n}{(1-\overline{\zeta}z)^{n+1}}. \tag{12}$$

From this and the boundedness of the operator $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ we have that $\|I^{(n)}_{\varphi,g} f_\zeta\|_{\mathcal{B}_\nu} \leq \|I^{(n)}_{\varphi,g}\|_{\mathcal{K}\to\mathcal{B}_\nu}$, for every $\zeta \in \partial\mathbb{D}$ and so

$$n! \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}} \leq \|I^{(n)}_{\varphi,g}\|_{\mathcal{K}\to\mathcal{B}_\nu}. \tag{13}$$

If $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded, then from (10) and (13), equality (6) follows. □

Corollary 1. *Let ν be a weight, $n \in \mathbb{N}_0$ and $g \in H(\mathbb{D})$. Then $I^{(n)}_g : \mathcal{K} \to \mathcal{B}_\nu$ is bounded if and only if*

$$M_2 := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}z|^{n+1}} < \infty. \tag{14}$$

Moreover, if $I^{(n)}_g : \mathcal{K} \to \mathcal{B}_\nu$ is bounded, then $\|I^{(n)}_g\|_{\mathcal{K}\to\mathcal{B}_\nu} = n! M_2$.

Corollary 2. *Let ν be a normal weight, $n \in \mathbb{N}_0$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded if and only if*

$$L_1 := \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2n+2}} \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty. \tag{15}$$

Moreover, if $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded, then the asymptotic relation $L_1 \asymp M_1^2$ holds.

Proof. First assume that (15) holds. Since ν is normal, $\nu(a) \asymp \nu(z)$ when $z \in D(a,(1-|a|)/2)$. Also it is known that $|1-\bar{a}z| \asymp 1-|a|^2$, for $z \in D(a,(1-|a|)/2)$. Using these two facts and the subharmonicity of the function

$$h(z) = \frac{|g(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2n+2}}$$

we obtain

$$\begin{aligned} L_1 &\geq \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|g(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2n+2}} \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) \\ &\asymp \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \frac{\nu^2(a)|g(a)|^2}{|1-\overline{\zeta}\varphi(a)|^{2n+2}} = M_1^2, \end{aligned} \tag{16}$$

which implies (5). By Theorem 1 the operator $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded.

Now assume that the operator $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded. By Theorem 1 we have that (5) holds. From this and by Proposition 1.4.10 in [13] we have

$$L_1 \leq M_1^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) = M_1^2 C < \infty. \tag{17}$$

The asymptotic relation $L_1 \asymp M_1^2$ follows from (16) and (17). □

Since the unit ball $B_{\mathcal{K}}$ of $\mathcal{K}$ is a normal family, a standard argument from [14] yields the proof of the next lemma.

Lemma 1. *Let ν be a weight, $n \in \mathbb{N}_0$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{K}$ converging to zero on compacts of $\mathbb{D}$, we have $\lim_{n\to\infty} \|I^{(n)}_{\varphi,g} f_n\|_{\mathcal{B}_\nu} = 0$.*

Theorem 2. *Let ν be a normal weight, $n \in \mathbb{N}_0$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent:*

(i) $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ *is compact;*

(ii) $M_3 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \infty$

and

$$\lim_{r\to 1} \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2n+2}} (1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) = 0. \tag{18}$$

Proof. $(i) \Rightarrow (ii)$. Since $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ is bounded and by using the asymptotic relation $b^2_\nu(g) \asymp \Psi(g)$ (see (1)), for the test function $f(z) = z^n/n! \in \mathcal{K}$, we obtain

$$M_3 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \infty.$$

Let $f_m(z) = z^m$, $m \in \mathbb{N}$. It is a norm bounded sequence in $\mathcal{K}$ converging to zero uniformly on compacts of $\mathbb{D}$. Hence by Lemma 1, it follows that $\|I^{(n)}_{\varphi,g} f_m\|_{\mathcal{B}_\nu} \to 0$ as $m \to \infty$. Thus for every $\varepsilon > 0$, there is an $m_0 \in \mathbb{N}$, $m_0 \geq n$, such that for $m \geq m_0$, we have that

$$\Big(\prod_{j=0}^{n-1}(m-j)\Big)^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi(z)|^{2(m-n)} \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \varepsilon. \tag{19}$$

From (19), we have that for each $r \in (0,1)$

$$r^{2(m-n)}\Big(\prod_{j=0}^{n-1}(m-j)\Big)^2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \varepsilon. \tag{20}$$

Hence for $r \in \big[\prod_{j=0}^{n-1}(m_0-j))^{-\frac{1}{m_0-n}}, 1\big)$, we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \varepsilon. \tag{21}$$

Let $f \in B_{\mathcal{K}}$ and $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0<t<1} \|f_t\|_{\mathcal{K}} \leq \|f\|_{\mathcal{K}}$, $f_t \in \mathcal{K}$, $t \in (0,1)$ and $f_t \to f$ uniformly on compacts of $\mathbb{D}$ as $t \to 1$. The compactness of $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$ implies that

$$\lim_{t\to 1} \|I^{(n)}_{\varphi,g} f_t - I^{(n)}_{\varphi,g} f\|_{\mathcal{B}_\nu} = 0.$$

Hence for every $\varepsilon > 0$, there is a $t \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \varepsilon. \tag{22}$$

From inequalities (21) and (22), we have that for $r \in \big[\prod_{j=0}^{n-1}(m_0-j))^{-\frac{1}{m_0-n}}, 1\big)$

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) \\
&\qquad \leq 2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) \\
&\qquad\quad + 2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_t^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) \\
&\qquad \leq 2\varepsilon(1 + \|f_t^{(n)}\|_\infty^2).
\end{aligned}$$

Hence for every $f \in B_{\mathcal{K}}$, there is a $\delta_0 \in (0,1)$, $\delta_0 = \delta_0(f,\varepsilon)$, such that for $r \in (\delta_0, 1)$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \varepsilon. \tag{23}$$

From the compactness of $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_\nu$, we have that for every $\varepsilon > 0$ there is a finite collection of functions $f_1, f_2, \ldots, f_k \in B_{\mathcal{K}}$ such that for each $f \in B_{\mathcal{K}}$, there is a $j \in \{1, 2, \ldots, k\}$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(\varphi(z)) - f_j^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |g(z)|^2 d\lambda(z) < \varepsilon. \tag{24}$$

On the other hand, from (23) it follows that if $\delta := \max_{1\le j\le k}\delta_j(f_j,\varepsilon)$, then for $r\in(\delta,1)$ and all $j\in\{1,2,\dots,k\}$ we have

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f_j^{(n)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|g(z)|^2d\lambda(z)<\varepsilon. \tag{25}$$

From (24) and (25) we have that for $r\in(\delta,1)$ and every $f\in B_{\mathcal{K}}$

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f^{(n)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|g(z)|^2d\lambda(z)<4\varepsilon. \tag{26}$$

Applying (26) to the functions $f_\zeta(z)=1/(1-\overline{\zeta}z)$, $\zeta\in\partial\mathbb{D}$, we obtain

$$\sup_{\zeta\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2n+2}}(1-|\eta_a(z)|^2)^2|g(z)|^2d\lambda(z)<4\varepsilon/(n!)^2,$$

from which (18) follows.

(ii) $\Rightarrow$ *(i)*. Assume that $(f_m)_{m\in\mathbb{N}}$ is a bounded sequence in $\mathcal{K}$, say by L, converging to 0 uniformly on compacts of $\mathbb{D}$ as $m\to\infty$. Then by the Weierstrass theorem, $f_m^{(k)}$ also converges to 0 uniformly on compacts of $\mathbb{D}$, as $m\to\infty$, for each $k\in\mathbb{N}$. We show that $\|I_{\varphi,g}^{(n)}f_m\|_{\mathcal{B}_\nu}\to 0$ as $m\to\infty$, and then apply Lemma 1.

For each $m\in\mathbb{N}$, we can find a $\mu_m\in\mathfrak{M}$ with $\|\mu_m\|=\|f_m\|_{\mathcal{K}}$ such that

$$f_m(z)=\int_{\partial\mathbb{D}}\frac{d\mu_m(\zeta)}{1-\overline{\zeta}z}. \tag{27}$$

Differentiating (27) n times, composing such obtained equation by φ, applying Jensen's inequality, as well as the boundedness of the sequence $(f_m)_{m\in\mathbb{N}}$, we obtain

$$|f_m^{(n)}(\varphi(w))|^2\le L(n!)^2\int_{\partial\mathbb{D}}\frac{d|\mu_m|(\zeta)}{|1-\overline{\zeta}\varphi(w)|^{2n+2}}. \tag{28}$$

By the second condition in *(ii)*, we have that for every $\varepsilon>0$, there is an $r_1\in(0,1)$ such that for $r\in(r_1,1)$, we have

$$\sup_{\zeta\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2n+2}}(1-|\eta_a(z)|^2)^2|g(z)|^2d\lambda(z)<\varepsilon. \tag{29}$$

By (1), we have

$$\|I_{\varphi,g}^{(n)}f_m\|_{\mathcal{B}_\nu}^2\asymp\sup_{a\in\mathbb{D}}\Big(\int_{|\varphi(z)|\le r}+\int_{|\varphi(z)|>r}\Big)|f_m^{(n)}(\varphi(z))|^2(1-|\eta_a(z)|^2)^2|g(z)|^2\nu^2(z)d\lambda(z).$$

Using (28), (29), Fubini's theorem and the fact that $\sup_{|w|\le r}|f_m^{(n)}(w)|^2<\varepsilon$, for sufficiently large m, say $m\ge m_1$, we have that for $m\ge m_1$

$$\begin{aligned}\|I_{\varphi,g}^{(n)}f_m\|_{\mathcal{B}_\nu}^2&\le C\sup_{|\varphi(z)|\le r}|f_m^{(n)}(\varphi(z))|^2\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\le r}(1-|\eta_a(z)|^2)^2|g(z)|^2\nu^2(z)d\lambda(z)\\&\quad+C\sup_{a\in\mathbb{D}}\int_{\partial\mathbb{D}}\int_{|\varphi(z)|>r}\frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2n+2}}(1-|\eta_a(z)|^2)^2|g(z)|^2d\lambda(z)d|\mu_m|(\zeta)\\&\le C\Big(M_3+\int_{\partial\mathbb{D}}d|\mu_m|(\zeta)\Big)\varepsilon\\&\le C(M_3+L)\varepsilon.\end{aligned}$$

From this and since ε is an arbitrary positive number the claim follows. □

Theorem 3. *Let ν be a weight, $n \in \mathbb{N}_0$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is bounded if and only if condition* (5) *holds and*

$$\lim_{|z|\to 1} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}} = 0 \tag{30}$$

for every $\zeta \in \partial\mathbb{D}$.

Proof. First suppose that (5) and (30) hold. Note that (8) holds. From (5) and (30), the integrand in (8) tends to zero for every $\zeta \in \partial\mathbb{D}$, as $|z| \to 1$, and is dominated by the function $f(z) = M_1$. Thus by the Lebesgue convergence theorem, the integral in (8) tends to zero as $|z| \to 1$, implying

$$\lim_{|z|\to 1} \nu(z)|(I^{(n)}_{g,\varphi} f)'(z)| = 0.$$

Hence, for every $f \in \mathcal{K}$ we have that $I^{(n)}_{\varphi,g} f \in \mathcal{B}_{\nu,0}$, from which along with Theorem 1, the boundedness of $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ follows.

Now suppose that $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is bounded. Then $I^{(n)}_{\varphi,g} f_\zeta \in \mathcal{B}_{\nu,0}$ for every function f_ζ, $\zeta \in \partial\mathbb{D}$, defined in (11), that is

$$\lim_{|z|\to 1} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}} = 0$$

for every $\zeta \in \partial\mathbb{D}$. Since $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is bounded, then $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu}$ is bounded too. Thus by Theorem 1, (5) follows, as claimed. □

Corollary 3. *Let ν be a weight, $n \in \mathbb{N}_0$ and $g \in H(\mathbb{D})$. Then $I^{(n)}_{g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is bounded if and only if condition* (14) *holds and*

$$\lim_{|z|\to 1} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}z|^{n+1}} = 0 \tag{31}$$

for every $\zeta \in \partial\mathbb{D}$.

Theorem 4. *Let ν be a typical weight, $n \in \mathbb{N}_0$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}} = 0. \tag{32}$$

Proof. By a known result (see, e.g. Lemma 1 in [25]), a closed set F in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1} \sup_{f\in F} \nu(z)|f'(z)| = 0.$$

Thus the set $\{I^{(n)}_{\varphi,g} f : f \in \mathcal{K}, \|f\|_{\mathcal{K}} \le 1\}$ has compact closure in $\mathcal{B}_{\nu,0}$ if and only if

$$\lim_{|z|\to 1} \sup\{\nu(z)|(I^{(n)}_{\varphi,g} f)'(z)| : f \in \mathcal{K}, \|f\|_{\mathcal{K}} \le 1\} = 0. \tag{33}$$

Let $f \in B_{\mathcal{K}}$, then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{K}}$ and

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}.$$

From (8), we easily get that for each $f \in B_{\mathcal{K}}$

$$\nu(z)|(I^{(n)}_{\varphi,g}f)'(z)| \leq n!\|\mu\| \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}} \leq n! \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+1}}. \tag{34}$$

Using (32) in (34), we get (33). Hence $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is compact.

Conversely, suppose that $I^{(n)}_{\varphi,g} : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is compact. Taking the test functions in (11) and using (12), we obtain that (32) follows from (33). □

Corollary 7. *Let ν be a typical weight, $n \in \mathbb{N}_0$ and $g \in H(\mathbb{D})$. Then $I^{(n)}_g : \mathcal{K} \to \mathcal{B}_{\nu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|g(z)|}{|1-\overline{\zeta}z|^{n+1}} = 0. \tag{35}$$

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Some Remarks on Extended Hypergeometric, Extended Confluent Hypergeometric and Extended Appell's Functions

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Abstract

Recently, Chaudhry et al. have introduced the extended hypergeometric functions (EHF) and extended confluent hypergeometric functions (ECHF) by using the generalized beta functions [1]. In a similar way, Özarslan et al. extended the first two Appell's hypergeometric functions. In this paper, we show that these extended functions can be represented in terms of a finite number of well known higher transcendental functions, especially as an infinite series containing hypergeometric, confluent hypergeometric, Whittaker's, Lagrange functions, Laguerre polynomials, and products of them.

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1 Introduction

Recently, in [1], M.A. Chaudhry, A. Qadir, M. Rafique, S.M. Zubair have introduced the following extension of Euler's beta function

$$B_p(x,y) = B(x,y;p) := \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[-\frac{p}{t(1-t)}\right]dt; \tag{1}$$
$$(\mathrm{Re}(p)>0, \mathrm{Re}(x)>0, \mathrm{Re}(y)>0),$$

where the case $p=0$ gives the original beta function. They have shown that this extension has connection with Macdonald, error and Whittakers functions. Allen R. Miller obtained further representations of the extended beta functions as an infinite series of Whittaker's functions, simple Laguerre polynomials and their products [2].

In 2004, M. Aslam Chaudhry, Asghar Qadir, H.M. Srivastava, R.B. Paris [3], extended the hypergeometric functions and confluent hypergeometric functions as follows:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty}\frac{B_p(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!}; \quad (p\geq 0; |z|<1; \mathrm{Re}(c)>\mathrm{Re}(b)>0,)$$

and

$$\Phi_p(b;c;z) = \sum_{n=0}^{\infty}\frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}; \quad (p\geq 0; \mathrm{Re}(c)>\mathrm{Re}(b)>0,)$$

where $(\lambda)_\nu$ denotes the Pochhammer's symbol defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)};\ (\lambda)_0 = 1.$$

They called these functions as extended hypergeometric functions (EHF) and extended confluent hypergeometric functions (ECHF), respectively. They obtained the integral representation of EHF and ECHF as

$$F_p\left(a,b;c;z\right)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a}\exp\left[-\frac{p}{t(1-t)}\right]dt \tag{2}$$
$$\left(p>0;p=0 \text{ and } \left|arg\left(1-z\right)\right|<\pi;\mathrm{Re}\left(c\right)>\mathrm{Re}\left(b\right)>0\right),$$

and

$$\Phi_p(b;c;z)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}.\exp(zt-\frac{p}{t(1-t)})dt$$
$$\left(p\geq 0,\ \ \mathrm{Re}\left(c\right)>\mathrm{Re}\left(b\right)>0\right).$$

They also obtained the differentiation properties and Mellin transforms of $F_p\left(a,b;c;z\right)$, gave their transformations and recurrence relations and proved summation and asymptotic formulas for this function. Noticed that $F_0\left(a,b;c;z\right)=F\left(a,b;c;z\right):={}_2F_1\left(a,b;c;z\right)$.

Finally, in [4], M. A. Özarslan and E. Özergin introduced the extensions of the Appell's functions $F_1\left(a,b,c;d;x,y;p\right)$ and $F_2(a,b,c;d,e;x,y;p)$ by

$$F_1(a,b,c;d;x,y;p):=\sum_{n,m=0}^{\infty}\frac{B_p(a+m+n,d-a)}{B(a,d-a)}\left(b\right)_n\left(c\right)_m\frac{x^n}{n!}\frac{y^m}{m!}$$
$$\left(\max\left\{\left|x\right|,\left|y\right|\right\}<1\right),$$

$$F_2(a,b,c;d,e;x,y;p):=\sum_{n,m=0}^{\infty}\frac{\left(a\right)_{m+n}B_p\left(b+n,d-b\right)B_p\left(c+m,e-c\right)}{B\left(b,d-b\right)B\left(c;e-c\right)}\frac{x^n}{n!}\frac{y^m}{m!}$$
$$\left(\left|x\right|+\left|y\right|<1\right),$$

of which the case $p=0$ gives the original functions. They obtained the integral representations of the functions $F_1(a,b,c;d;x,y;p)$ and $F_2(a,b,c;d,e;x,y;p)$ as

$$F_1\left(a,b,c;d;x,y;p\right)=\frac{\Gamma\left(d\right)}{\Gamma\left(a\right)\Gamma\left(d-a\right)}\int_0^1 t^{a-1}\left(1-t\right)^{d-a-1}\left(1-xt\right)^{-b}\left(1-yt\right)^{-c}\exp\left[-\frac{p}{t(1-t)}\right]dt, \tag{3}$$
$$\left(p>0;p=0 \text{ and } \left|arg\left(1-x\right)\right|<\pi,\left|arg\left(1-y\right)\right|<\pi;\mathrm{Re}\left(d\right)>\mathrm{Re}\left(a\right)>0,\mathrm{Re}\left(b\right)>0,\mathrm{Re}\left(c\right)>0\right)$$

and

$$F_2(a,b,c;d,e;x,y;p) \tag{4}$$
$$=\frac{1}{B\left(b,d-b\right)B\left(c,e-c\right)}\int_0^1\int_0^1\frac{t^{b-1}\left(1-t\right)^{d-b-1}s^{c-1}\left(1-s\right)^{e-c-1}}{\left(1-xt-ys\right)^a}\exp\left[-\frac{p}{t(1-t)}-\frac{p}{s(1-s)}\right]dtds.$$
$$\left(p\geq 0 \text{ and } \left|x\right|+\left|y\right|<1,\mathrm{Re}\left(d\right)>\mathrm{Re}\left(b\right)>0,\mathrm{Re}\left(e\right)>\mathrm{Re}\left(c\right)>0,\mathrm{Re}\left(a\right)>0\right).$$

By introducing extended Riemann-Liouville fractional derivative operator as

$$D_z^{\mu,p}\{f\left(z\right)\}=\frac{1}{\Gamma\left(-\mu\right)}\int_0^z f\left(t\right)\left(z-t\right)^{-\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt\qquad\left(\mathrm{Re}\left(\mu\right)<0,\mathrm{Re}\left(p\right)>0\right) \tag{5}$$

and for $m-1<\mathrm{Re}\left(\mu\right)<m\ \ \left(m=1,2,3...\right)$,

$$D_z^{\mu,p}\{f\left(z\right)\}=\frac{d^m}{dz^m}D_z^{\mu-m}\{f\left(z\right)\}=\frac{d^m}{dz^m}\left\{\frac{1}{\Gamma\left(-\mu+m\right)}\int_0^z f\left(t\right)\left(z-t\right)^{-\mu+m-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt\right\}$$

where the path of integration is a line from 0 to z in complex $t-$plane, they obtained linear and bilinear generating functions for the extended hypergeometric functions in terms of the Appell's functions $F_1(a,b,c;d;x,y;p)$ and $F_2(a,b,c;d,e;x,y;p)$ by following the same method explained in [6]. We organize the paper as follows:

In section 2, using the similar technics used in [2], we show that the EHFs can be expressed as a product of simple Laguerre polynomials and hypergeometric functions and as a product of simple Laguerre polynomials and Whittaker's functions. Similar results for the ECHFs are also exhibited. In section 3, we obtain a representation of the extended Appel's functions $F_1(a,b,c;d;x,y;p)$ by means of Lagrange polynomials. Furthermore, using the extended fractional derivative operator, we show that the extended Appell's functions $F_2(a,b,c;d,e;x,y;p)$ can be represented as a product of simple Laguerre polynomials, the EHFs and Whittaker's functions.

2 Representations of the EHF, ECHF

We start by obtaining the representation of EHF in terms of simple Laguerre and the hypergeometric functions.

Theorem 1 *For the extended hypergeometric functions we have the following representation:*

$$\exp(2p)F_p(a,b;c;z)=\sum_{m,n=0}^{\infty}\frac{(b)_{m+1}(c-b)_{n+1}}{(c)_{n+m+2}}L_m(p)L_n(p)\,{}_2F_1(a,b+m;c+n+m+1;z)$$
$$(p>0;p=0\quad and\ \ |arg(1-z)|<\pi<p;\mathrm{Re}(c)>\mathrm{Re}(b)>0).$$

Proof. It is well known that the simple Laguerre polynomials are generated by the relation

$$\exp(\frac{-pt}{1-t})=(1-t)\sum_{n=0}^{\infty}L_n(p)t^n \tag{6}$$

where $|t|<1$. Replacing t by $1-t$ in (6), we have

$$\exp(\frac{-p}{t})=t\exp(-p)\sum_{n=0}^{\infty}L_n(p)(1-t)^n, \tag{7}$$

where $0<t<2$. Again replacing t by $1-t$ in (7), we have

$$\exp(\frac{-p}{1-t})=(1-t)\exp(-p)\sum_{m=0}^{\infty}L_m(p)t^m;\qquad |t|<1. \tag{8}$$

Multiplying both sides of (7) and (8), we get, as in [2],

$$\exp\left[-\frac{p}{t(1-t)}\right]=\exp(-2p)\sum_{m,n=0}^{\infty}L_m(p)L_n(p)t^{m+1}(1-t)^{n+1};\quad 0<t<1. \tag{9}$$

Using (9) in (2) and then interchanging the order of integration and summation, which is valid since the series involved is uniformly convergent and the integral involved is absolutely convergent for $p>0;p=0$ and $|arg(1-z)|<\pi;\mathrm{Re}(c)>\mathrm{Re}(b)>0$, we get

$$\exp(2p)F_p(a,b;c;z)=\frac{\exp(2p)}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\exp\left[-\frac{p}{t(1-t)}\right]dt \tag{10}$$
$$=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\sum_{m,n=0}^{\infty}L_m(p)L_n(p)t^{m+1}(1-t)^{n+1}dt$$
$$=\frac{1}{B(b,c-b)}\sum_{m,n=0}^{\infty}L_m(p)L_n(p)\int_0^1 t^{m+b}(1-t)^{n+c-b}(1-zt)^{-a}\,dt.$$

Now considering the Euler's integral representation of the hypergeometric functions

$$ {}_2F_1\left(a,b;c;z\right)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a}dt$$
$$ \left|arg\left(1-z\right)\right|<\pi;\mathrm{Re}\left(c\right)>\mathrm{Re}\left(b\right)>0$$

it is obvious that

$$\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{m+b}\left(1-t\right)^{n+c-b}\left(1-zt\right)^{-a}dt=\frac{(b)_{m+1}(c-b)_{n+1}}{(c)_{n+m+2}}\ {}_2F_1\left(a,b+m;c+n+m+1;z\right). \tag{11}$$

Hence the result follows by considering (11) in (10). ■

In a similar way, for the ECHF, we are led fairly easily to Theorem 2 below.

Theorem 2 *For the ECHF's we have the following representation:*

$$\exp(2p)\Phi_p(b;c;z)=\sum_{m,n=0}^{\infty}\frac{(b)_{m+1}(c-b)_{n+1}}{(c)_{n+m+2}}L_m(p)L_n(p){}_1F_1\left(b+m;c+n+m+1;z\right)$$
$$\left(p\geq 0\ ;\mathrm{Re}\left(c\right)>\mathrm{Re}\left(b\right)>0\right).$$

Now we give another representation of the extended hypergeometric functions in terms of the simple Laguerre polynomials and Whittakers functions.

Theorem 3 *For the extended hypergeometric functions, we have*

$$(\sqrt{p})^{1-b}\exp(\frac{3p}{2})F_p\left(a,b;c;z\right)=\frac{(c-b)\Gamma(c)}{\Gamma(b)}\sum_{m,n=0}^{\infty}(a)_nL_m(p)W_{\frac{b-1-n-m-2c}{2},\frac{n+m+b}{2}}(p)\frac{\left(\sqrt{p}z\right)^n p^{m/2}}{n!} \tag{12}$$
$$\left(p\geq 0,\mathrm{Re}\left(c\right)>\mathrm{Re}\left(b\right)>0,|z|<1\right).$$

Proof. Since $\exp\left[-\frac{p}{t(1-t)}\right]=\exp(\frac{-p}{1-t})\exp(\frac{-p}{t})$, we have by (8) that

$$\exp\left[-\frac{p}{t(1-t)}\right]=\exp(-p)\exp(\frac{-p}{t})(1-t)\sum_{m=0}^{\infty}L_m(p)t^m.$$

Therefore using (2) we have for $|z|<1$,

$$\exp(p)F_p\left(a,b;c;z\right)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a}\exp(\frac{-p}{t})(1-t)\sum_{m=0}^{\infty}L_m(p)t^m dt$$
$$=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\left(tz\right)^n\exp(\frac{-p}{t})\sum_{m=0}^{\infty}L_m(p)t^m dt.$$

Interchanging the order of summations and integration, we have

$$\exp(p)F_p\left(a,b;c;z\right)=\frac{1}{B\left(b,c-b\right)}\sum_{m,n=0}^{\infty}\frac{(a)_n}{n!}L_m(p)\left[\int_0^1 t^{n+m+b-1}\left(1-t\right)^{c-b}\exp(\frac{-p}{t})dt\right]z^n.$$

Finally, using the integral representation [5, Section 3.471 (2)]

$$\int_0^1 t^{\mu-1}\left(1-t\right)^{\nu-1}\exp(\frac{-p}{t})dt=\Gamma(\nu)p^{\frac{\mu-1}{2}}\exp(\frac{-p}{2})W_{\frac{1-\mu-2\nu}{2},\frac{\mu}{2}}(p)\qquad\left(\mathrm{Re}(\nu)>0,\ \mathrm{Re}(p)>0\right),$$

we get the result. ■

In a similar manner, we get the following result:

Theorem 4 *For the ECHF, we have*

$$(\sqrt{p})^{1-b}\exp(\frac{3p}{2})\Phi_p\left(b;c;z\right)=\frac{(c-b)\Gamma(c)}{\Gamma(b)}\sum_{m,n=0}^{\infty}L_m(p)W_{\frac{b-1-n-m-2c}{2},\frac{n+m+b}{2}}(p)\frac{\left(\sqrt{p}z\right)^n p^{m/2}}{n!}$$
$$\left(P\geq 0,\mathrm{Re}\left(c\right)>\mathrm{Re}\left(b\right)>0\right).$$

3 Representations of the Extended Appel's Functions

In this section we obtain some representations of the extended Appel's functions $F_1\left(a,b,c;d;x,y;p\right)$ and $F_2(a,b,c;d,e;x,y;p)$. We start with the following theorem.

Theorem 5 *For the extended Appell's functions $F_1\left(a,b,c;d;x,y;p\right)$, we have*

$$F_1\left(a,b,c;d;x,y;p\right)=\frac{\Gamma\left(d\right)}{\Gamma\left(a\right)\Gamma\left(d-a\right)}\sum_{n=0}^{\infty}g_n^{(b,c)}(x,y)B_p\left(n+a,d-a\right),$$
$$\left(p\geq 0\ \textit{and}\ |x|<1,|y|<1;\mathrm{Re}\left(d\right)>\mathrm{Re}\left(a\right)>0,\mathrm{Re}\left(b\right)>0,\mathrm{Re}\left(c\right)>0\right).$$

Proof. It is known that, for $|xt|<1,|yt|<1$ and $\mathrm{Re}\left(b\right)>0,\mathrm{Re}\left(c\right)>0$, we have

$$\left(1-xt\right)^{-b}\left(1-yt\right)^{-c}=\sum_{n=0}^{\infty}g_n^{(b,c)}(x,y)t^n \tag{13}$$

where $g_n^{(b,c)}(x,y)$ is the bivariate Lagrange polynomials [6, p. 441, equation 8.5 (12)]. Using (13) in (3) and then interchanging the order of summation and integration, which is valid since the series

$$\sum_{n=0}^{\infty}g_n^{(b,c)}(x,y)t^n$$

is uniformly convergent since $|x|<1,|y|<1$, $\mathrm{Re}\left(b\right)>0,\mathrm{Re}\left(c\right)>0$ and the integral

$$\int_0^1 t^{n+a-1}\left(1-t\right)^{d-a-1}\exp\left[-\frac{p}{t(1-t)}\right]dt$$

is absolutely convergent for $\mathrm{Re}\left(d\right)>\mathrm{Re}\left(a\right)>0,p\geq 0$, we get, by using (1),

$$\begin{aligned}F_1\left(a,b,c;d;x,y;p\right)&=\frac{\Gamma\left(d\right)}{\Gamma\left(a\right)\Gamma\left(d-a\right)}\int_0^1 t^{a-1}\left(1-t\right)^{d-a-1}\left(1-xt\right)^{-b}\left(1-yt\right)^{-c}\exp\left[-\frac{p}{t(1-t)}\right]dt\\&=\frac{\Gamma\left(d\right)}{\Gamma\left(a\right)\Gamma\left(d-a\right)}\sum_{n=0}^{\infty}g_n^{(b,c)}(x,y)\int_0^1 t^{n+a-1}\left(1-t\right)^{d-a-1}\exp\left[-\frac{p}{t(1-t)}\right]dt\\&=\frac{\Gamma\left(d\right)}{\Gamma\left(a\right)\Gamma\left(d-a\right)}\sum_{n=0}^{\infty}g_n^{(b,c)}(x,y)B_p\left(n+a,d-a\right).\end{aligned}$$

Whence the result. ■

In the next theorem by using the equalities, which are obtained in [4] ,

$$D_z^{\lambda-\mu,p}\{z^{\lambda-1}\left(1-z\right)^{-\alpha}\}=\frac{\Gamma\left(\lambda\right)}{\Gamma\left(\mu\right)}z^{\mu-1}F_p\left(\alpha,\lambda;\mu;z\right) \tag{14}$$
$$\left(\mathrm{Re}\left(\lambda\right)>0,\mathrm{Re}\left(\alpha\right)>0,\mathrm{Re}\left(\mu\right)<0;|z|<1\right),$$

and

$$D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}\left(1-z\right)^{-\alpha}F_p\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right)\right\}=\frac{z^{\mu-1}}{B\left(\beta,\gamma-\beta\right)\Gamma\left(\mu-\lambda\right)}F_2\left(\alpha,\beta,\lambda;\gamma,\mu;x,z;p\right) \tag{15}$$
$$\left(\mathrm{Re}\left(\mu\right)>\mathrm{Re}\left(\lambda\right)>0,\mathrm{Re}\left(\alpha\right)>0,\mathrm{Re}\left(\beta\right)>0,\mathrm{Re}\left(\gamma\right)>0;\left|\frac{x}{1-z}\right|<1\ \text{and}\ |x|+|z|<1\right),$$

where $D_z^{\lambda-\mu,p}$ is the extended Riemann-Liouville fractional derivative operator defined by (5), we give the representation of the function $F_2(a,b,c;d,e;x,y;p)$ in terms of the simple Laguerre, the Whittaker and the EHF.

Theorem 6 *For the extended Appell's functions $F_2(a,b,c;d,e;x,y;p)$, we have*

$$\begin{aligned}&\frac{p^{\frac{1-b}{2}}}{\Gamma(c-b+1)B(\lambda,\mu-\lambda)}\exp(\frac{3p}{2})F_2(a,b,\lambda;c,\mu;x,z;p)\\&=\sum_{m,n=0}^{\infty}(a)_n L_m(p)W_{\frac{b-1-n-m-2c}{2},\frac{n+m+b}{2}}(p)F_p\left(a+n,\lambda;\mu;z\right)\frac{\left(x\sqrt{p}\right)^n p^{\frac{m}{2}}}{n!},\end{aligned}$$

where $\operatorname{Re}(p)\geq 0,\ \operatorname{Re}(\mu)>\operatorname{Re}(\lambda)>0,\operatorname{Re}(a)>0,\operatorname{Re}(c)>\operatorname{Re}(b)>0;\left|\frac{x}{1-z}\right|<1$ *and* $|x|+|z|<1$.

Proof. Replacing z by $\frac{x}{1-z}$ in (12) and then multiplying both sides by $z^{\lambda-1}(1-z)^{-\alpha}$, we get

$$\begin{aligned}&p^{\frac{1-b}{2}}\exp(\frac{3p}{2})z^{\lambda-1}(1-z)^{-a}F_p\left(a,b;c;\frac{x}{1-z}\right)\\&=\frac{(c-b)\Gamma(c)}{\Gamma(b)}\sum_{m,n=0}^{\infty}(a)_n L_m(p)W_{\frac{b-1-n-m-2c}{2},\frac{n+m+b}{2}}(p)\frac{\left(x\sqrt{p}\right)^n p^{\frac{m}{2}}}{n!}z^{\lambda-1}(1-z)^{-a-n}.\end{aligned}\tag{16}$$

Now applying the extended fractional derivative operator $D_z^{\lambda-\mu,p}$ on both sides of (16), we have

$$\begin{aligned}&p^{\frac{1-b}{2}}\exp(\frac{3p}{2})D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}\left(1-z\right)^{-a}F_p\left(a,b;c;\frac{x}{1-z}\right)\right\}\\&=\frac{(c-b)\Gamma(c)}{\Gamma(b)}\sum_{m,n=0}^{\infty}(a)_n L_m(p)W_{\frac{b-1-n-m-2c}{2},\frac{n+m+b}{2}}(p)\frac{\left(x\sqrt{p}\right)^n p^{\frac{m}{2}}}{n!}D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-a-n}\right\}.\end{aligned}$$

Using (14) and (15), we get the result. ■

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Recurrence relations for the Harmonic mean Newton's method in Banach spaces*

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Abstract

In this paper, an attempt is made to using recurrence relations to establish the convergence of the Harmonic mean Newton's method used for nonlinear equations in Banach spaces. The recurrence relations for the method are derived and then an existence-uniqueness theorem is given to establish the R-order of the method to be three and a priori error bounds. Finally, some numerical applications is presented to demonstrate our approach.

Keywords: Nonlinear equations in Banach spaces; Recurrence relations; Semilocal convergence; Newton's method; A priori error bounds

MSC2000: 65D10; 65D99; 47H17

1 Introduction

Many scientific problems can be expressed in the form of a nonlinear equation

$$F(x) = 0, \tag{1}$$

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where $F : \Omega \subset X \to Y$ is a nonlinear operator on an open convex subset Ω of a Banach space X with values in a Banach space Y. This equation can represent differential equations, integral equations or a system of equations in the simplest case. The Newton's method is the most used iteration to solve those equations as a consequence of computational efficiency even less speed of convergence can be got.

Our goals in this paper is to increase the speed of convergence of Newton's method and not to increase its operational cost very much. Taking into account these goals, we consider a multipoint Newton-type method of order three called the Harmonic mean Newton's method studied by Özban [1] and Homeier [2] respectively. This method is defined for all $n \geq 0$ by

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ x_{n+1} &= x_n - \frac{1}{2}[\Gamma_n + \overline{\Gamma}_n]F(x_n) \end{aligned} \tag{2}$$

where $\Gamma_n = F'(x_n)^{-1}$ and $\overline{\Gamma}_n = F'(y_n)^{-1}$.

Recently, the convergence of iterative methods for solving nonlinear operator equations in Banach spaces is established from the convergence of majorizing sequences. An alternative approach is developed to establish this convergence by using recurrence relations. For many applications, third order methods are used in spite of high computational cost in evaluating the involved second order derivatives. They can also be used in stiff systems [4], where a quick convergence is required. Moreover, they are important from the theoretical standpoint as they provide results on existence and uniqueness of solutions that improve the results obtained from Newton's method [3]. Some of the well-known third order methods are Chebyshev's method, Halley's method and super Halley's method which convergence had studied by Candela, Gutuerrez and Hernandez et. al proposed in [5, 6, 7, 8, 9, 10, 11] using recurrence relations.

In this paper, we shall use recurrence relations to establish the convergence of third order Harmonic mean Newton's method for solving nonlinear operator equations (1). The recurrence relations based on two constants which depend on the operator F are derived. Then, based on this recurrence relations a priori error bounds are obtained for the said iterative method. Finally, some numerical examples are worked out for demonstrating our work.

2 Preliminary results

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega_0 \subseteq \Omega$. The Harmonic mean Newton's method to

solve the equation (1) given by (2) can be written in the following form:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ H(x_n, y_n) &= \overline{\Gamma}_n[F'(y_n) - F'(x_n)], \\ x_{n+1} &= y_n - \frac{1}{2}H(x_n, y_n)(y_n - x_n). \end{aligned} \tag{3}$$

where $\Gamma_n = F'(x_n)^{-1}$ and $\overline{\Gamma}_n = F'(y_n)^{-1}$.

Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in L(Y, X)$ exists at some $x_0 \in \Omega_0$, where $L(Y, X)$ is the set of bounded linear operators from Y into X.

Throughout this paper we assume that:

(I) $\|\Gamma_0\| \leq \beta, \quad \|\Gamma_0 F(x_0)\| \leq \eta, \quad \|F''(x) \leq M,\ x \in \Omega,$

(II) there exists a positive real number N such that $\|F''(x) - F''(y)\| \leq N\|x - y\|, \quad \forall x, y \in \Omega$.

We firstly give an approximation of the operator F in the following lemma, which will be used in the next derivation.

Lemma 1. *Assume that the nonlinear operator $F : \Omega \subset X \to Y$ is continuously second-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. Then we have*

$$\begin{aligned} F(x_{n+1}) = &\int_0^1 F''(x_n + t(y_n - x_n)(1-t)dt(y_n - x_n)^2 - \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n)dt(y_n - x_n)^2 \\ &+ \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2. \end{aligned} \tag{4}$$

Proof. By the definition of method given by (3), we obtain

$$F'(y_n)(x_{n+1} - y_n) = -\frac{1}{2}[F'(y_n) - F'(x_n)](y_n - x_n).$$

Using Taylor's formula, we have

$$\begin{aligned} F(x_{n+1}) =& F(y_n) + F'(y_n)(x_{n+1} - y_n) + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2 \\ =& F(y_n) - \frac{1}{2}[F'(y_n) - F'(x_n)](y_n - x_n) + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2. \end{aligned} \tag{5}$$

Similarly, we obtain

$$F(y_n) = F(x_n) + F'(x_n)(y_n - x_n) + \int_0^1 F''(x_n + t(y_n - x_n)(1-t)dt(y_n - x_n)^2, \tag{6}$$

$$F'(y_n) = F'(x_n) + \int_0^1 F''(x_n + t(y_n - x_n)dt(y_n - x_n).$$

It follows that

$$F'(y_n) - F'(x_n) = \int_0^1 F''(x_n + t(y_n - x_n)dt(y_n - x_n). \tag{7}$$

Substituting (6) and (7) into (5), we can obtain (4). □

Now, we denote $\eta_0 = \eta$, $\beta_0 = \beta$, $a_0 = M\beta_0\eta_0$, $b_0 = N\beta_0\eta_0^2$ and $c_0 = h(a_0)\varphi(a_0, b_0)$. Let $a_0 < \sigma$ and $h(a_0)c_0 < 1$, we can define the following sequences for $n \geq 0$.

$$\eta_{n+1} = c_n\eta_n, \quad \beta_{n+1} = h(a_n)\beta_n, \quad a_{n+1} = M\beta_{n+1}\eta_{n+1}, \tag{8}$$

$$b_{n+1} = N\beta_{n+1}\eta_{n+1}^2, \quad c_{n+1} = h(a_{n+1})\varphi(a_{n+1}, b_{n+1}) \tag{9}$$

where

$$g(t) = \frac{2-t}{2-2t}, \quad h(t) = \frac{1}{1-tg(t)}, \quad \varphi(t,s) = \frac{1}{8}\frac{t^3}{(1-t)^2} + \frac{5}{12}s \tag{10}$$

and $\sigma = 2 - \sqrt{2}$ is the smallest positive zero of the scalar function $g(t)t - 1$.

From the definition of a_{n+1}, b_{n+1}, (8) and (9), we also have

$$a_{n+1} = h(a_n)c_n a_n, \quad b_{n+1} = h(a_n)c_n^2 b_n. \tag{11}$$

Nextly we shall study some properties of the functions defined in (10) and the previous scalar sequences defined in (8)-(9), (11), later developments will require the following lemma.

Lemma 2. *Let the real functions g, h and φ be given in (10). Then*

(a) $g(t)$ and $h(t)$ are increasing and $g(t) > 1$, $h(t) > 1$ for all $t \in (0, \sigma)$ and for all $s > 0$,

(b) $\varphi(t, s)$ is increasing for all $t \in (0, \sigma)$ and for all $s > 0$,

(c) $g(\theta t) < g(t)$, $h(\theta t) < h(t)$ and $\varphi(\theta t, \theta^2 s) < \theta^2\varphi(t, s)$ for $\theta \in (0, 1)$, $t \in (0, \sigma)$ and $s > 0$.

Lemma 3. *Let the real functions g, h and φ be given in (10). If $a_0 < \sigma$ and $h(a_0)c_0 < 1$, then we have*

(a) $h(a_0) > 1$ and $c_n < 1$ for $n \geq 0$,

(b) the sequence $\{\eta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are decreasing while $\{\beta_n\}$ is increasing,

(c) $g(a_n)a_n < 1$ and $h(a_n)c_n < 1$ for $n \geq 0$.

Proof. By Lemma 2 and assumption, $h(a_0) > 1$ and $c_n < 1$ hold. It follows from the definitions that $\eta_1 < \eta_0$, $a_1 < a_0$, $b_1 < b_0$. Moreover, by Lemma 2, we have $1 < h(a_1) < h(a_0)$ and $\varphi(a_1, b_1) < \varphi(a_0, b_0)$. This yields $c_1 < c_0$ and (b) holds. Based on these results we obtain $g(a_1)a_1 < g(a_0)a_0 < 1$ and $h(a_1)c_1 < h(a_0)c_0 < 1$ and (c) holds. By induction we can derive that the items (a), (b) and (c) hold. □

Lemma 4. *Under the assumptions of Lemma 3 and define $\gamma = h(a_0)c_0$, then*

$$c_n \le \lambda\gamma^{3^n}, \quad n \ge 0, \tag{12}$$

where $\lambda = 1/h(a_0)$. Also for $n \ge 0$, we have

$$\prod_{i=0}^{n} c_i \le \lambda^{n+1}\gamma^{\frac{3^{n+1}-1}{2}}. \tag{13}$$

Proof. By the definition of a_{n+1} and b_{n+1} given in (11) we obtain $a_1 = h(a_0)c_0a_0 = \gamma a_0$, $b_1 = h(a_0)c_0^2b_0 < \gamma^2 b_0$, by Lemma 2 we have

$$c_1 < h(\gamma a_0)\varphi(\gamma a_0, \gamma^2 b_0) < \gamma^2 h(a_0)\varphi(a_0, b_0) = \gamma^{3^1-1}c_0 = \lambda\gamma^{3^1}.$$

Suppose $c_k \le \lambda\gamma^{3^k}$, $k \le 1$. Then by Lemma 3, we have $a_{k+1} < a_k$, $b_{k+1} < b_k$ and $h(a_k, b_k)c_k < 1$. Thus

$$\begin{aligned} c_{k+1} &< h(a_k)\varphi(h(a_k)c_ka_k, h(a_k)c_k^2b_k) < h(a_k)\varphi(h(a_k)c_ka_k, h^2(a_k)c_k^2b_k) \\ &< h^3(a_k)c_k^2\varphi(a_k, b_k) = h^2(a_k)c_k^3 < \lambda\gamma^{3^{k+1}}. \end{aligned}$$

Therefore it holds that $c_n \le \lambda\gamma^{3^n}$, $n \ge 0$.

By (12), we get

$$\prod_{i=0}^{n} c_i \le \prod_{i=0}^{n} \lambda\gamma^{3^i} = \lambda^{n+1}\gamma^{\sum_{i=0}^{n} 3^i} = \lambda^{n+1}\gamma^{\frac{3^{n+1}-1}{2}}, \quad n \ge 0.$$

This shows (13) holds. The proof is completed. □

Lemma 5. *Under the assumptions of Lemma 3. Let $\gamma = h(a_0)c_0$ and $\lambda = 1/h(a_0)$. The sequence $\{\eta_n\}$ satisfies $\eta_n \le \eta\lambda^n\gamma^{\frac{3^n-1}{2}}$, $n \ge 0$. Hence the sequence $\{\eta_n\}$ converges to 0. Moreover, for any $n \ge 0$, $m \ge 1$, it holds*

$$\sum_{i=n}^{n+m} \eta_i \le \eta\lambda^n\gamma^{\frac{3^n-1}{2}} \frac{1 - \lambda^{m+1}\gamma^{\frac{3^n(3^m+1)}{2}}}{1 - \lambda\gamma^{3^n}}.$$

Proof. From the definition of sequence $\{\eta_n\}$ given in (8) and (13), we have

$$\eta_n = c_{n-1}\eta_{n-1} = c_{n-1}c_{n-2}\eta_{n-2} = \cdots = \eta\left(\prod_{i=0}^{n-1} c_i\right) \le \eta\lambda^n\gamma^{\frac{3^n-1}{2}}.$$

Because $\lambda < 1$ and $\gamma < 1$, it follows that $\eta_n \to 0$ as $n \to \infty$, hence the sequence $\{\eta_n\}$ converges to 0.

Since

$$\begin{aligned} \sum_{i=n}^{n+m} \lambda^i\gamma^{\frac{3^i}{2}} &\le \lambda^n\gamma^{\frac{3^n}{2}} + \gamma^{3^n}\left(\sum_{i=n+1}^{n+m} \lambda^i\gamma^{\frac{3^{i-1}}{2}}\right) = \lambda^n\gamma^{\frac{3^n}{2}} + \lambda\gamma^{3^n}\left(\sum_{i=n}^{n+m-1} \lambda^i\gamma^{\frac{3^i}{2}}\right) \\ &= \lambda^n\gamma^{\frac{3^n}{2}} + \lambda\gamma^{3^n}\left(\sum_{i=n}^{n+m} \lambda^i\gamma^{\frac{3^i}{2}} - \lambda^{n+m}\gamma^{\frac{5^{n+m}}{2}}\right), \end{aligned}$$

where $n \geq 0$, $m \geq 1$. we can obtain $\sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{3^i}{2}} \leq \lambda^n \gamma^{\frac{3^n}{2}} \frac{1-\lambda^{m+1}\gamma^{\frac{3^n(3^m+1)}{2}}}{1-\lambda\gamma^{3^n}}$. Furthermore,

$$\sum_{i=n}^{n+m} \eta_i \leq \eta \sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{3^i-1}{2}} = \eta\gamma^{-\frac{1}{2}} \sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{3^i}{2}} \leq \eta\lambda^n \gamma^{\frac{3^n-1}{2}} \frac{1-\lambda^{m+1}\gamma^{\frac{3^n(3^m+1)}{2}}}{1-\lambda\gamma^{3^n}}.$$

Therefore $\sum_{n=0}^{\infty} \eta_n$ exists. The proof is completed. □

3 Recurrence relations

We denote $B(x,r) = \{y \in X : \|y-x\| < r\}$ and $\overline{B(x,r)} = \{y \in X : \|y-x\| \leq r\}$ in this paper.

In the following, the recurrence relations are derived for the method given by (3) under the assumptions mentioned in the previous section.

For $n = 0$, the existence of Γ_0 implies the existence of y_0. This gives us $\|u_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta_0$, this means that $y_0 \in B(x_0, R\eta)$ where $R = \frac{g(a_0)}{1-c_0}$. By the initial hypotheses (I)-(II), we have

$$\|I - \Gamma_0 F'(y_0)\| \leq \|\Gamma_0\|\|F'(x_0) - F'(y_0)\| \leq M\|\Gamma_0\|\|y_0 - x_0\| \leq a_0 < \sigma \tag{14}$$

and, by the Banach lemma [3], $\overline{\Gamma}_0$ exists and $\|\overline{\Gamma}_0\| \leq \frac{1}{1-a_0}\|\Gamma_0\|$. Consequently we have

$$\|x_1 - y_0\| \leq \frac{1}{2}\|H(x_0,y_0)\|\|y_0 - x_0\| \leq \frac{1}{2}\|\overline{\Gamma}_0\|\|F'(x_0) - F'(y_0)\|\|y_0 - x_0\| \leq \frac{a_0}{2(1-a_0)}\eta_0 \tag{15}$$

and x_1 is well defined

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq \frac{a_0}{2(1-a_0)}\eta_0 + \eta_0 = g(a_0)\eta_0.$$

By $a_0 < \sigma$ and $g(t)$ is increasing in $t \in (0, \sigma)$, we have

$$\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leq M\beta_0\|x_1 - x_0\| \leq a_0 g(a_0) < 1,$$

it follows by the Banach lemma [3] that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\|\Gamma_1\| \leq \frac{\beta_0}{1 - a_0 g(a_0)} = h(a_0)\beta_0 = \beta_1. \tag{16}$$

Also

$$\begin{aligned}
&\left\| \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt - \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))dt \right\| \\
&\leq \left\| \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt - \frac{1}{2}F''(x_n) \right\| + \left\| \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))dt - \frac{1}{2}F''(x_n) \right\| \\
&\leq \left\| \int_0^1 [F''(x_n + t(y_n - x_n)) - F''(x_n)](1-t)dt \right\| + \frac{1}{2}\left\| \int_0^1 [F''(x_n + t(y_n - x_n)) - F''(x_n)]dt \right\| \\
&\leq N\int_0^1 (t - t^2)dt\|y_n - x_n\| + \frac{N}{2}\int_0^1 tdt\|y_n - x_n\| = \frac{5}{12}N\|y_n - x_n\|.
\end{aligned} \tag{17}$$

By Lemma 1, we can get

$$\|F(x_1)\| \leq \frac{1}{2}M\|x_1 - y_0\|^2 + \frac{5}{12}N\|y_0 - x_0\|^3. \tag{18}$$

Then from (16) and (18), we have

$$\|y_1 - x_1\| = \|\Gamma_1 F(x_1)\| \leq \|\Gamma_1\|\|F(x_1)\| \leq h(a_0)\varphi(a_0, b_0)\eta_0 = c_0\eta_0 = \eta_1.$$

Because of $g(a_0) > 1$, we obtain

$$\|y_1 - x_0\| \leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq (g(a_0) + c_0)\eta_0 < g(a_0)(1 + c_0)\eta < R\eta,$$

which shows $y_1 \in B(x_0, R\eta)$.

In addition, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leq h(a_0)c_0 a_0 = a_1, \quad N\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \leq h(a_0)c_0^2 b_0 = b_1.$$

Repeating the above derivation, we can obtain the system of recurrence relations given in the next lemma.

Lemma 6. *Let the assumptions and the conditions (I)-(II) hold. Then the following items are true for all $n \geq 0$:*

(I) *There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leq \beta_n$,*

(II) $\|\Gamma_n F(x_n)\| \leq \eta_n$,

(III) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leq a_n$,

(IV) $N\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 \leq b_n$,

(V) $\|x_{n+1} - x_n\| \leq g(a_n)\eta_n$,

(VI) $\|x_{n+1} - x_0\| \leq R\eta$, *where* $R = g(a_0)/(1 - c_0)$.

Proof. The proof of (I)-(V) follows by using the above mentioned way and invoking the induction hypothesis. We only consider (VI). By (V) and Lemma 5 we obtain

$$\|x_{n+1} - x_0\| \leq \sum_{i=0}^{n}\|x_{i+1} - x_i\| \leq \sum_{i=0}^{n} g(a_i)\eta_i \leq g(a_0)\sum_{i=0}^{n}\eta_i \leq g(a_0)\eta\frac{1 - \lambda^{n+1}\gamma^{\frac{3^n+1}{2}}}{1 - c_0} < R\eta.$$

So the lemma is proved. □

4 Semilocal Convergence

Lemma 7. *Let $R = \frac{g(a_0)}{1-c_0}$. If $h(a_0)c_0 < 1$, then $R < \frac{1}{a_0}$.*

Proof. Since $c_0 < \frac{1}{h(a_0)} = 1 - a_0 g(a_0)$, we obtain $R < \frac{1}{a_0}$. □

Now we give a theorem to establish the semilocal convergence of (2), the existence and uniqueness of the solution and the domain in which it is located, along with a priori error bounds, which lead to the R-order of convergence at least three of iteration (2).

Theorem 1. *Let $F : \Omega \subset X \to Y$ be a nonlinear two times Fréchet differentiable operator in an open convex subset Ω of a Banach space X with values in a Banach space Y. Assume that $x_0 \in \Omega$ and all conditions (I)-(II) hold. Let $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$ and $c_0 = h(a_0)\varphi(a_0, b_0)$ satisfy $a_0 < \sigma$ and $h(a_0)c_0 < 1$ where g, h, φ are defined by (10). Let $\overline{B(x_0, R\eta)} \subset \Omega$ where $R = g(a_0)/(1-c_0)$, then starting from x_0, the sequence $\{x_n\}$ generated by the method (2) converges to a solution x^* of $F(x)$ with x_n, x^* belong to $\overline{B(x_0, R\eta)}$ and x^* is the unique solution of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$.*

Moreover, a priori error estimate is given by

$$\|x_n - x^*\| \leq \frac{g(a_0)\eta\lambda^n\gamma^{\frac{3^n-1}{2}}}{1-\lambda\gamma^{3^n}}, \tag{19}$$

where $\gamma = h(a_0)c_0$ and $\lambda = 1/h(a_0)$.

Proof. By Lemma 6, the sequences $\{x_n\}$ is well-defined in $\overline{B(x_0, R\eta)}$. Now we prove that $\{x_n\}$ is a Cauchy sequence. Since

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \leq \sum_{i=n}^{n+m-1} g(a_i)\eta_i \leq g(a_0) \sum_{i=n}^{n+m-1} \eta_i \\ &\leq g(a_0)\eta\lambda^n\gamma^{\frac{3^n-1}{2}} \frac{1-\lambda^m\gamma^{\frac{3^n(3^{m-1}+1)}{2}}}{1-\lambda\gamma^{3^n}},\end{aligned} \tag{20}$$

it follow that $\{x_n\}$ is a Cauchy sequence, and hence the sequence $\{x_n\}$ is convergent. So there exists a x^* such that $\lim_{n\to\infty} x_n = x^*$.

By letting $n = 0$, $m \to \infty$ in (20), we obtain

$$\|x^* - x_0\| \leq R\eta.$$

This shows that $x^* \in \overline{B(x_0, R\eta)}$.

Now we prove that x^* is a solution of $F(x) = 0$. Since

$$\|\Gamma_0\|\|F(x_n)\| \leq \|\Gamma_n\|\|F(x_n)\| \leq \eta_n, \tag{21}$$

by letting $n \to \infty$ in (21), we obtain $\|F(x_n)\| \to 0$ since $g(a_n, b_n) < g(a_0)$ and $\eta_n \to 0$. Hence, by the continuity of F in Ω, we obtain $F(x^*) = 0$.

We prove the uniqueness of x^* in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. Firstly we can obtain $x^* \in B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$, since it follows by Lemma 6

$$\frac{2}{M\beta} - R\eta = \left(\frac{2}{a_0} - R\right)\eta > \frac{1}{a_0}\eta > R\eta,$$

and then $\overline{B(x_0, R\eta)} \subset B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. Let x^{**} be another zero of $F(x)$ in $B(x_0, \frac{2}{M\beta} -$

$R\eta) \bigcap \Omega$. By Taylor theorem, we have

$$0 = F(x^{**}) - F(x^*) = \int_0^1 F'((1-t)x^* + tx^{**})dt(x^{**} - x^*).$$

Since

$$\|\Gamma_0\| \left\| \int_0^1 [F'((1-t)x^* + tx^{**}) - F'(x_0)]dt \right\| \leq M\beta \int_0^1 [(1-t)\|x^* - x_0\| + t\|x^{**} - x_0\|]dt$$
$$< \frac{M\beta}{2}\left[R\eta + \frac{2}{M\beta} - R\eta\right] = 1,$$

it follows by the Banach lemma that $\int_0^1 F'((1-t)x^* + tx^{**})dt$ is invertible and hence $x^{**} = x^*$.

Finally, by letting $m \to \infty$ in (20), we obtain (19) and furthermore

$$\|x_n - x^*\| \leq \frac{g(a_0)\eta}{\gamma^{1/2}(1-c_0)}\left(\gamma^{1/2}\right)^{3^n}.$$

This means that the method given by (2) is of R-order of convergence at least three. This ends the proof. □

5 Numerical Example

Now, numerical example are given for demonstrating the semilocal convergence of our third order Harmonic mean Newton's method based on recurrence relations.

Example 1. *Let $X = C[0,1]$ be the space of continuous functions on $[0,1]$ and consider the integral equation $F(x) = 0$, where*

$$F(x)(s) = -1 + x(s) + \lambda x(s)\int_0^1 \frac{s}{s+t}x(t)dt, \tag{22}$$

with $s \in [0,1]$, $x \in C[0,1]$ and $0 < \lambda \leq 2$. Integral equations of this kind called Chandrasekhar equations arise in elasticity or neutron transport problems [4]. The norm is taken as max-norm.

Now it is easy to obtain the derivatives of F as

$$F'(x)u(s) = u(s) + \lambda u(s)\int_0^1 \frac{s}{s+t}x(t)dt + \lambda x(s)\int_0^1 \frac{s}{s+t}u(t)dt, \quad u \in \Omega$$

$$F''(x)uv(s) = \lambda u(s)\int_0^1 \frac{s}{s+t}v(t)dt + \lambda v(s)\int_0^1 \frac{s}{s+t}u(t)dt, \quad u, v \in \Omega$$

and the Lipschitz condition with $N = 0$, $\|[F''(x) - F''(y)](uv)\| = 0\|x - y\|$, $\forall x, y \in \Omega$.

Now one can easily compute

$$\|F(x_0)\| = \left\|-1 + x_0(s) + \lambda x_0(s)\int_0^1 \frac{s}{s+t}x_0(t)dt\right\|$$
$$\leq \|x_0 - 1\| + |\lambda| \max_{0\leq s\leq t}\left|\int_0^1 \frac{s}{s+t}x_0(t)dt\right| \|x_0\|^2 \leq \|x_0 - 1\| + |\lambda|\log 2\|x_0\|^2$$

and

$$\|F''(x)\| \leq 2|\lambda| \max_{0\leq s\leq t}\left|\int_0^1 \frac{s}{s+t}dt\right| \leq 2|\lambda|\log 2 = M.$$

Also notice that

$$\|I - F'(x_0)\| \leq \left\|\lambda\int_0^1 \frac{s}{s+t}x_0(t)dt + \lambda x_0(s)\int_0^1 \frac{s}{s+t}dt\right\|$$
$$\leq 2|\lambda| \max_{0\leq s\leq t}\left|\int_0^1 \frac{s}{s+t}x_0(t)dt\right| \|x_0\| \leq 2|\lambda|\log 2\|x_0\|,$$

if $2|\lambda|\log 2\|x_0\| < 1$*, by Banach Lemma [3] that* Γ_0 *exists and*

$$\|\Gamma_0\| \leq \frac{1}{1 - 2|\lambda|\log 2\|x_0\|} = \beta, \quad \|\Gamma_0 F(x_0)\| \leq \frac{\|x_0 - 1\| + |\lambda|\log 2\|x_0\|^2}{1 - 2|\lambda|\log 2\|x_0\|} = \eta.$$

Now for $\lambda = 1/4$*, and the initial point* $x_0 = x_0(x) = 1$*, we obtain*

$$M = 0.346573590279973, \quad N = 0, \quad \beta = 1.5303942190345,$$
$$\eta = 0.265197109517251, \quad a_0 = 0.14065901379261 < \sigma = 2 - \sqrt{2}$$

and

$$c_0 h(a_0) = 0.00065533723025431 < 1.$$

This means that the hypotheses of Theorem 1 is satisfied. Hence the recurrence relations for the method given by (2) is demonstrated in Table 1. Besides, the solution x^* *belongs to* $\overline{B(x_0, R\eta)} = \overline{B(1, 0.287060657036874)} \subset \Omega$ *and it is unique in* $B(1, 3.48371950651898)\bigcap\Omega$*. But in [10] and [11], the solution of (22) exists in* $\overline{B(1, 0.089298359)}$ *and* $\overline{B(1, 0.09355883)}$ *respectively, which are inferior to our result.*

Table 1: Results of recurrence relations of Example 1.

n	η	β	a_n	b_n	c_n
0	2.6520e-001	1.5304	1.4066e-001	0	5.5561e-004
1	1.4735e-004	1.8051	9.2179e-005	0	9.7933e-014
2	1.4430e-017	1.8052	9.0282e-018	0	9.1983e-053
3	1.3273e-069	1.8052	8.3044e-070	0	7.1588e-209
4	9.5020e-278	1.8052	5.9449e-278	0	0

6 Conclusions

A family of recurrence relations is developed for establishing the semilocal convergence of the Harmonic mean Newton's method (2) used for solving $F(x) = 0$ in Banach spaces. Based on these recurrence relations an existence-uniqueness theorem is established to show the R-order convergence of the method to be three. Also a priori error bounds are given. Some numerical example are worked out to demonstrate our approach and show our method can be of practical interest.

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A general iterative method with some control conditions for k-strictly pseudo-contractive mappings *

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Abstract

In this paper, we consider a general iterative scheme for finding fixed points of k-strictly pseudo-contractive mappings for some $0 \leq k < 1$ in Hilbert spaces. Then, under weaker control conditions than previous ones, we establish strong convergence of the sequence generated by proposed scheme to a fixed point of the mapping, which is a solution of a certain variational inequality. The main results substantially improve and develop the previous well-known results in this area.

MSC: 47H09, 47H10, 47J25, 49M05.

Key words: Iterative schemes; k-strictly pseudo-contractive mapping; Nonexpansive mapping; Fixed points; Contraction; Weakly asymptotically regular; Strongly positive linear operator; Hilbert space; Variational inequality

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to H$ is said to be *k-strictly pseudo-contractive* if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad x,\ y \in C), \tag{1.1}$$

and $F(T)$ denote the set of fixed points of the mapping T; that is, $F(T) = \{x \in C : Tx = x\}$. The mapping T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad x,\ y \in C.$$

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T is strongly pseudo-contractive if and only if there exists a constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq (1-\lambda)\|x-y\|^2, \quad x,\ y \in C.$$

Note that the class of k-strictly pseudo-contractive mappings includes the class of nonexpansive mappings T on C (that is, $\|Tx - Ty\| \leq \|x - y\|,\ x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k = 1$ and T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0,1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent of the class of k-strictly pseudo-contractive mappings (see [1–3]).

Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudo-contractive mappings. W can refer, for example, [4–7] and the references therein. In particular, in 2009, as the viscosity iteration method, Cho et al. [5] considered the following iterative scheme for k-strictly pseudo-contractive mapping T:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) P_C S x_n, \quad n \geq 1, \tag{1.2}$$

where $\alpha_n \in (0,1)$, $f : C \to C$ is a contraction with constant $\alpha \in (0,1)$ (i.e., $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in C$), A is a strongly positive linear operator with constant $\overline{\gamma} > 0$ (i.e., there exist a constant $\overline{\gamma} > 0$ with the property: $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H$), $0 < \gamma\alpha < \overline{\gamma}$, $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$ and P_C is the metric projection of H onto C. They proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a fixed point q of T which is the unique solution of the variational inequality related to the linear operator A

$$\langle (A - \gamma f)q, p - q\rangle \geq 0, \quad \forall p \in F(T), \tag{1.3}$$

and is also the optimality condition for the minimization problem:

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf. They utilized the following as control conditions on $\{\alpha_n\}$:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or, $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,

Their result extended the result of Marino and Xu [8] from the class of nonexpansive mappings to the class of k-strictly pseudo-contractive mappings.

In 2010, in order to improve the corresponding results of Cho et al. [5] as well as Marino and Xu [8] by removing the condition (C3), Jung [6] studied the following iterative scheme

for the k-strictly pseudo-contractive mapping T along with the same f, A, γ, P_C and S as in [5]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n) P_C S x_n), \quad n \geq 1, \tag{1.4}$$

where α_n, $\beta_n \in (0, 1)$. Under conditions (C1) and (C2) on $\{\alpha_n\}$ and the condition $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ on $\{\beta_n\}$, he proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a fixed point q of T, which is the unique solution of a variational inequality (1.3).

In this paper, motivated by the above-mentioned results, we consider the following iterative scheme (1.2) for the k-strictly pseudo-contractive mapping T. Under weaker control conditions than Cho et al. [5] (Marino and Xu [8]), we establish the strong convergence of the sequence $\{x_n\}$ generated by (1.2) to a fixed point of T, which is a solution of the variational inequality (1.3). The main results improve and develop the corresponding results of Cho et al. [5] and Jung [6] as well as Marino and Xu [8]. Our results also extend the corresponding results of Halpern [9], Moudafi [10], Wittmann [11] and Xu [12].

2. Preliminaries and Lemmas

Throughout this paper, when $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H to C. It is well known that P_C is nonexpansive.

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([13]). *Let H be a Hilbert space, C be a closed convex subset of H. If T is a k-strictly pseudo-contractive mapping on C, then the fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.2 ([13]). *Let H be a Hilbert space and C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.*

Lemma 2.3 ([13]). *Let H be a Hilbert space, C be a closed convex subset of H, and $T : C \to H$ be a k-strictly pseudo-contractive mapping. Define a mapping $S : C \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

The following Lemmas 2.4 and 2.5 can be obtained from the Proposition 2.6 of Acedo and Xu [4].

Lemma 2.4. *Let H be a Hilbert space and C be a closed convex subset of H. For any $N \geq 1$, assume that for each $1 \leq i \leq N$, $T_i : C \to H$ is a k_i-strictly pseudo-contractive mapping for some $0 \leq k_i < 1$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a nonself-k-strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 2.5. *Let $\{T_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ be given as in Lemma 2.4. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.6 ([8])**.** *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with constant $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.7 ([14,15])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + r_n, \quad n \geq 1,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=1}^\infty \lambda_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^\infty \lambda_n\delta_n < \infty$,

(iii) $r_n \geq 0$ $(n \geq 0)$, $\sum_{n=1}^\infty r_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8. *In a Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad x,\ y \in H.$$

Let μ be a mean on positive integers N, that is, a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in N\} \leq \mu(a) \leq \sup\{a_n : n \in N\}$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. If μ is a Banach limit, the following are well-known:

(i) for all $n \geq 1, a_n \leq c_n$ implies $\mu(a_n) \leq \mu(c_n)$,

(ii) $\mu(a_{n+N}) = \mu(a_n)$ for any fixed positive integer N,

(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_1, a_2, \cdots) \in l^\infty$.

The following lemma was given in [16, Proposition 2].

Lemma 2.9. *Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\mu_n(a_n) \leq a$ for all Banach limit μ. If $\limsup_{n\to\infty}(a_{n+1}-a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

Finally, we recall that the sequence $\{x_n\}$ in H is said to be *weakly asymptotically regular* if

$$w - \lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \text{that is,} \quad x_{n+1} - x_n \rightharpoonup 0$$

and *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

respectively.

3. Main results

We need the following result for the existence of solutions of a certain variational inequality, which is slightly an improvement of Theorem 3.2 of Marino and Xu [8].

Theorem MX. *Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$ and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let A be a strongly positive bounded linear operator on C with constant $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0,1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let x_t be a fixed point of a contraction $C_t \ni x \mapsto t\gamma f(x) + (I - tA)Tx$ for $t \in (0,1)$ and $t < \|A\|^{-1}$. Then $\{x_t\}$ converges strongly to a fixed point $\overline{x}$ of T as $t \to 0$, which solves the following variational inequality:*

$$\langle A\overline{x} - \gamma f(\overline{x}), \overline{x} - p\rangle \leq 0, \quad p \in F(T).$$

Now, we study the strong convergence results for a general iterative scheme for the k-strictly pseudo-contractive mapping.

Theorem 3.1. *Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ for some $0 \leq k < 1$. Let A be a strongly positive bounded linear operator on C with constant $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0,1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ which satisfies the condition:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

Let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)P_C S x_n, \quad n \geq 1, \end{cases}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$ and P_C is the metric projection of H onto C. Let μ be a Banach limit. Then

$$\mu_n(\langle Aq - \gamma f(q), q - x_n)\rangle) \le 0,$$

where $q = \lim_{t\to 0^+} x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)P_C Sx$, $0 < t < 1$ and $t < \|A\|^{-1}$.

proof. Let $\{x_t\}$ be the net generated by

$$x_t = t\gamma f(x_t) + (I - tA)P_C Sx_t, \quad 0 < t < 1 \text{ and } t < \|A\|^{-1}. \tag{3.1}$$

Since $P_C S$ is a nonexpansive mapping from C into itself, by Theorem MX and Lemmas 2.2 and 2.3, there exists $\lim_{t\to 0} x_t \in F(S) = F(T)$. Denote it by q. Moreover q is a solution of the variational inequality

$$\langle Aq - \gamma f(q), q - p\rangle \le 0, \quad p \in F(T).$$

By (3.1), we have

$$\|x_t - x_{n+1}\| = \|(I - tA)(P_C Sx_t - x_{n+1}) + t(\gamma f(x_t) - Ax_{n+1})\|.$$

Applying Lemma 2.6 and Lemma 2.8, we have

$$\|x_t - x_{n+1}\|^2 \le (1 - \overline{\gamma}t)^2\|P_C Sx_t - x_{n+1}\|^2 + 2t\langle \gamma f(x_t) - Ax_{n+1}, x_t - x_{n+1}\rangle. \tag{3.2}$$

Let $p \in F(T)$. Then we have

$$\begin{aligned}
\|x_t - p\| &= \|t(\gamma f(x_t) - Ap) + (I - tA)(P_C Sx_t - p)\| \\
&\le t\|\gamma f(x_t) - Ap\| + (1 - \overline{\gamma}t)\|x_t - p\| \\
&\le t(\|\gamma f(x_t) - \gamma f(p)\| + \|\gamma f(p) - Ap\|) + (1 - \overline{\gamma}t)\|x_t - p\| \\
&\le t\gamma\alpha\|x_t - p\| + t\|\gamma f(p) - Ap\| + (1 - \overline{\gamma}t)\|x_t - p\|
\end{aligned}$$

This gives that

$$\|x_t - p\| \le \frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|.$$

Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{P_C Sx_t\}$.

Now we show that $\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{\|Ap - \gamma f(p)\|}{\overline{\gamma} - \gamma\alpha}\}$ for all $n \ge 1$ and all $p \in F(T)$. Indeed, observing the condition (C1), we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ and $\alpha_n\overline{\gamma} < 1$ for all $n \ge 1$. From Lemma 2.6, we know that, if $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$. Therefore, taking a point $p \in F(T)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(x_n - p)\| \\
&\le (1 - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\|\gamma f(x_n) - Ap\| \\
&\le (1 - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\|) \\
&\le [1 - (\overline{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| + (\overline{\gamma} - \gamma\alpha)\alpha_n\frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}.
\end{aligned}$$

Using an induction, we have $\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{\|Ap - \gamma f(p)\|}{\overline{\gamma} - \gamma\alpha}\}$. Hence $\{x_n\}$ is bounded, and so are $\{\gamma f(x_n)\}$, $\{P_C S x_n\}$, $\{AP_C S x_n\}$, and $\{Ax_n\}$. As a consequence with the control condition (C1), we get

$$\|x_{n+1} - P_C S x_n\| = \alpha_n \|\gamma f(x_n) - AP_C S x_n\| \to 0 \quad (n \to \infty),$$

and

$$\|P_C S x_t - x_{n+1}\| \le \|x_t - x_n\| + e_n,$$

where $e_n = \|x_{n+1} - P_C S x_n\| \to 0$ as $n \to \infty$. Also observing that A is strongly positive linear, we have

$$\langle Ax_t - Ax_n, x_t - x_n\rangle = \langle A(x_t - x_n), x_t - x_n\rangle \ge \overline{\gamma}\|x_t - x_n\|^2. \tag{3.3}$$

So, combining (3.2) and (3.3), we obtain

$$\begin{aligned}
&\|x_t - x_{n+1}\|^2 \\
&\le (1 - \overline{\gamma}t)^2(\|x_t - x_n\| + e_n)^2 \\
&\quad + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_{n+1}\rangle + 2t\langle Ax_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&\le (\overline{\gamma}^2 t^2 - 2\overline{\gamma}t)\|x_t - x_n\|^2 + \|x_t - x_n\|^2 + (1 - \overline{\gamma}t)^2(2\|x_t - x_n\|e_n + e_n^2) \\
&\quad + 2t\langle f(x_t) - Ax_t, x_t - x_{n+1})\rangle + 2t\langle Ax_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&\le (\overline{\gamma}t^2 - 2t)\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\
&\quad + (1 - \overline{\gamma}t)^2(2\|x_t - x_n\|e_n + e_n^2) \\
&\quad + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_{n+1})\rangle + 2t\langle Ax_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&= \overline{\gamma}t^2\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\
&\quad + (1 - \overline{\gamma}t)^2(2\|x_t - x_n\|e_n + e_n^2) + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_{n+1}\rangle \\
&\quad + 2t(\langle Ax_t - Ax_{n+1}, x_t - x_{n+1}\rangle - \langle Ax_t - Ax_n, x_t - x_n\rangle)
\end{aligned} \tag{3.4}$$

Applying the Banach limit μ to (3.4) together with $\lim_{n\to\infty} e_n = 0$, we have

$$\begin{aligned}
&\mu_n(\|x_t - x_{n+1}\|^2) \\
&\le \mu_n(\overline{\gamma}t^2\langle Ax_t - Ax_n, x_t - x_n\rangle) + \mu_n(\|x_t - x_n\|^2) \\
&\quad + 2t(\mu_n(\langle \gamma f(x_t) - Ax_t, x_t - x_{n+1})\rangle) \\
&\quad + 2t(\mu_n(\langle Ax_t - Ax_{n+1}, x_t - x_{n+1}\rangle) - \mu_n(\langle Ax_t - Ax_n, x_t - x_n\rangle)).
\end{aligned} \tag{3.5}$$

Using the property $\mu_n(a_n) = \mu_n(a_{n+1})$ of Banach limit in (3.5), we obtain

$$\begin{aligned}
&\mu_n(\langle Ax_t - \gamma f(x_t), x_t - x_n)\rangle) = \mu_n(\langle Ax_t - \gamma f(x_t), x_t - x_{n+1})\rangle) \\
&\le \frac{\overline{\gamma}t}{2}\mu_n(\langle Ax_t - Ax_n, x_t - x_n\rangle) + \frac{1}{2t}[\mu_n(\|x_t - x_n\|^2) - \mu_n(\|x_t - x_n\|^2)] \\
&\quad + \mu_n(\langle Ax_t - Ax_n, x_t - x_n\rangle) - \mu_n(\langle Ax_t - Ax_n, x_t - x_n\rangle) \\
&= \frac{\overline{\gamma}t}{2}\mu_n(\langle Ax_t - Ax_n, x_t - x_n\rangle).
\end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}
&t\langle Ax_t - Ax_n, x_t - x_n\rangle \\
&\le t\|A\|\|x_t - x_n\|^2 \\
&\le t\|A\|(\|x_t - p\| + \|p - x_n\|)^2 \\
&\le t\|A\|\left(\frac{2}{\overline{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\| + \|x_0 - p\|\right)^2 \to 0 \quad (\text{as } t \to 0),
\end{aligned} \tag{3.7}$$

we conclude from (3.6) and (3.7) that

$$\begin{aligned}\mu_n(\langle Aq-\gamma f(q), q-x_n\rangle) &\le \limsup_{t\to 0}\mu_n(\langle Ax_t-\gamma f(x_t), x_t-x_n\rangle)\\ &\le \limsup_{t\to 0}\frac{\overline{\gamma}t}{2}\mu_n(\langle Ax_t-Ax_n, x_t-x_n\rangle)\le 0,\end{aligned}$$

where $q=\lim_{t\to 0}x_t$. This completes the proof. □

Using Theorem 3.1, we give the following main result.

Theorem 3.2. *Let H be a Hilbert space, C be a closed convex subset of H such that $C\pm C\subset C$, $T:C\to H$ be a k-strictly pseudo-contractive mapping with $F(T)\neq\emptyset$ for some $0\le k<1$. Let A be a strongly positive bounded linear operator on C with constant $\overline{\gamma}>0$ and $f:C\to C$ be a contraction with the contractive constant $\alpha\in(0,1)$ such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ which satisfies the conditions:*

(C1) $\lim_{n\to\infty}\alpha_n=0$;

(C2) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases}x_1\in C\\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_nA)P_CSx_n, \quad n\ge 1,\end{cases}\tag{3.8}$$

where $S:C\to H$ is a mapping defined by $Sx=kx+(1-k)Tx$ and P_C is the metric projection of H onto C. If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $q\in F(T)$, where solves the following variational inequality:

$$\langle Aq-\gamma f(q), q-p\rangle\le 0, \qquad p\in F(T).\tag{3.9}$$

Proof. First, note that from the condition (C1), without loss of generality, we assume that $\alpha_n\le\|A\|^{-1}$, $\alpha_n\overline{\gamma}<1$ and $\frac{2(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\alpha\gamma}\alpha_n<1$ for $n\ge 0$.

Let x_t be defined by (3.1), that is, $x_t=t\gamma f(x_t)+(I-tA)P_cSx_t$ for $0<t<1$ and $\lim_{t\to 0}x_t:=q\in F(S)=F(T)$ (by using Theorem MX and Lemmas 2.2 and 2.3). Then q is a solution of a variational inequality

$$\langle Aq-\gamma f(q), q-p\rangle\le 0 \quad p\in F(T).$$

We divides the proof several steps:

Step 1. We show that $\|x_n-p\|\le\max\left\{\|x_1-p\|, \frac{\|Ap-\gamma f(p)\|}{\overline{\gamma}-\gamma\alpha}\right\}$ for all $n\ge 1$ and all $p\in F(T)$ as in the proof of Theorem 3.1. Hence $\{x_n\}$ is bounded and so are $\{P_CSx_n\}$ and $\{f(x_n)\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle Aq-\gamma f(q), q-x_n\rangle \le 0$. To this end, put

$$a_n := \langle Aq-\gamma f(q), q-x_n)\rangle, \quad n\ge 1.$$

Then Theorem 3.1 implies that $\mu_n(a_n)\le 0$ for any Banach limit μ. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1}-a_n) = \lim_{j\to\infty}(a_{n_j+1}-a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w-\lim_{j\to\infty}(q-x_{n_j+1}) = w-\lim_{j\to\infty}(q-x_{n_j}) = (q-v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1}-a_n) = \lim_{j\to\infty}\langle Aq-\gamma f(q), (q-x_{n_j+1})-(q-x_{n_j})\rangle = 0.$$

Then Lemma 2.9 implies that $\limsup_{n\to\infty} a_n \le 0$, that is,

$$\limsup_{n\to\infty}\langle Aq-\gamma f(q), q-x_n)\rangle \le 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-q\|=0$. By using (3.8), we have

$$x_{n+1}-q = \alpha_n(\gamma f(x_n)-Aq)+(I-\alpha_n A)(x_n-q).$$

Applying Lemma 2.8, we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(I-\alpha_n A)(x_n-q)+\alpha_n(\gamma f(x_n)-Aq)\|^2\\
&\le \|(I-\alpha_n A)(x_n-q)\|^2 + 2\alpha_n\langle \gamma f(x_n)-Aq, x_{n+1}-q\rangle\\
&\le (1-\alpha_n\overline{\gamma})^2\|x_n-q\|^2 + 2\alpha_n\gamma\alpha\|x_n-q\|\|x_{n+1}-q\|\\
&\quad + 2\alpha_n\langle \gamma f(q)-Aq, x_{n+1}-q\rangle\\
&\le (1-\alpha_n\overline{\gamma})^2\|x_n-q\|^2 + \alpha_n\gamma\alpha(\|x_n-q\|^2+\|x_{n+1}-q\|^2)\\
&\quad + 2\alpha_n\langle \gamma f(q)-Aq, x_{n+1}-q\rangle.
\end{aligned} \tag{3.10}$$

It then follows from (3.10) that

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\le \frac{(1-\alpha_n\overline{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(q)-Aq, x_{n+1}-q\rangle\\
&\le \left(1-\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\right)\|x_n-Q(f)\|^2\\
&\quad + \frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\left(\frac{1}{\overline{\gamma}-\alpha\gamma}\langle \gamma f(q)-Aq, x_{n+1}-q\rangle + \frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\alpha\gamma)}M_2\right),
\end{aligned} \tag{3.11}$$

where $M_2 = \sup_{n\ge 1}\|x_n-q\|^2$. Put

$$\lambda_n = \frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\alpha\gamma} \quad \text{and}$$

$$\delta_n = \frac{1}{\overline{\gamma}-\alpha\gamma}\langle Aq-\gamma f(q), q-x_{n+1}\rangle + \frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\alpha\gamma)}M_2.$$

By (C1), (C2) and Step 2, we have $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Since (3.11) reduces to

$$\|x_{n+1} - q\|^2 \le (1 - \lambda_n)\|x_n - q\|^2 + \lambda_n \delta_n,$$

from Lemma 2.7 with $r_n = 0$, we conclude that $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

Theorem 3.3. *Let H be a Hilbert space and C be a closed convex subset of H such that $C \pm C \subset C$. Let $T_i : C \to H$ be a k_i-strictly pseudo-contractive mapping for some $0 \le k_i < 1$ and $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let A be a strongly positive bounded linear operator on C with constant $\overline{\gamma}$ and $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0, 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{\alpha_n\} \subset (0, 1)$ be sequence which satisfies the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) P_C S x_n, \end{cases}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)\sum_{i=1}^{N} \eta_i T_i x$ with $k = \max\{k_i : 1 \le i \le N\}$ and $\{\eta_i\}$ is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$. If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to a common fixed point q of $\{T_i\}_{i=1}^{N}$, which is a solution of the following variational inequality:

$$\langle Aq - \gamma f(q), q - p \rangle \le 0, \qquad p \in \cap_{i=1}^{N} F(T_i).$$

Proof. Define a mapping $T : C \to H$ by $Tx = \sum_{i=1}^{N} \eta_i T_i x$. By Lemmas 2.4 and 2.5, we conclude that $T : C \to H$ is a k-strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \le i \le N\}$ and $F(T) = F(\sum_{i=1}^{N} \eta_i T_i) = \cap_{i=1}^{N} F(T_i)$. Then the result follows from Theorem 3.2 immediately. □

Corollary 3.4. *Let H, C, T, A, f and γ be as in Theorem 3.2. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) P_C S x_n, \quad n \ge 0, \end{cases} \tag{3.12}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$ and P_C is the metric projection of H onto C. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ which satisfies the conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $q \in F(T)$, where is a solution of the variational inequality (3.9).

Remark 3.1. If $\{\alpha_n\}$ in Corollary 3.4 satisfies conditions (C1), (C2) and

(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or

(C4) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$; or

(C5) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ (the perturbed control condition),

then the sequence $\{x_n\}$ generated by (3.12) is asymptotically regular. Now we give only the proof in case when $\{\alpha_n\}$ satisfies the conditions (C1), (C2) and (C5). By Step 1 in the proof of Theorem 3.2, there exists a constant $L > 0$ such that for all $n \ge 0$,

$$\|AP_CSx_n\| + \gamma\|f(x_n)\| \le L.$$

So,we obtain, for all $n \ge 0$,

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
=&\|(I - \alpha_n A)(P_CSx_n - P_CSx_{n-1}) + (\alpha_n - \alpha_{n-1})AP_CSx_{n-1} \\
&\quad + \gamma[\alpha_n(f(x_n) - f(x_{n-1})) + f(x_{n-1})(\alpha_n - \alpha_{n-1})]\| \\
\le& (1 - \alpha_n\overline{\gamma})\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|AP_CSx_{n-1}\| \\
&\quad + \gamma[\alpha_n\alpha\|x_n - x_{n-1}\| + \|f(x_{n-1}\||\alpha_n - \alpha_{n-1}] \\
\le& (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - x_{n-1}\| + L|\alpha_n - \alpha_{n-1}| \\
\le& (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - x_{n-1}\| + (o(\alpha_n) + \sigma_{n-1})L.
\end{aligned} \tag{3.13}$$

By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\lambda_n = \alpha_n(\overline{\gamma} - \gamma\alpha)$, $\lambda_n\delta_n = o(\alpha_n)L$ and $r_n = \sigma_{n-1}L$, from (3.13) we have

$$s_{n+1} \le (1 - \lambda_n)s_n + \lambda_n\delta_n + r_n.$$

Hence, by (C1), (C2), (C5) and Lemma 2.7, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

In view of this observation, we have the following:

Corollary 3.5. *Let H, C, T, A, f and γ be as in Theorem 3.2. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ which satisfies the conditions* (C1), (C2) *and* (C5) *(or the conditions* (C1), (C2) *and* (C3)*, or the conditions* (C1), (C2) *and* (C4)*). Let $\{x_n\}$ be a sequence in C generated by (3.10).*

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)P_CSx_n, \quad n \ge 0, \end{cases}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k)Tx$ and P_C is the metric projection of H onto C. Then $\{x_n\}$ converges strongly to $q \in F(T)$, where is a solution of the variational inequality (3.9).

Remark 3.2. (1) Theorem 3.2 and Theorem 3.3 improve the corresponding results of Cho et al. [5] by using the weak asymptotic regularity on $\{x_n\}$ instead of the condition (C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

(2) Theorem 3.2 also includes the corresponding results of Halpern [9], Marino and Xu [8], and Wittmann [11] as some special cases.

(3) The condition (C5) on $\{\alpha_n\}$ in Corollary 3.5 is independent of the condition (C3) or (C4) in Remark 3.1, which was imposed in Theorem 2.1 of Cho et al. [5]. For this fact, see [17,18].

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Approximation of an additive-quadratic functional equation in RN-spaces

Hassan Azadi Kenary, Sun-Young Jang* and Choonkil Park

Abstract. In this paper, using the fixed point and direct methods, we prove the Hyers-Ulam stability of the following additive-quadratic functional equation:

$$af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)=cf(x)+cf(y)+cf(z)$$

where a, b and c are positive real numbers, in random normed spaces.

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, then we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [34] in 1940. In the next year, Hyers [14] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Th.M. Rassias [26] proved a generalization of the Hyers' theorem for additive mappings. The result of Th.M. Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept (see [1]-[5]). Furthermore, in 1994, a generalization of the Th.M. Rassias' theorem was obtained by Găvruta [13] by replacing the bound $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\phi(x,y)$.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, a Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [33] for mappings $f: X\to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation.

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The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([8]-[31]).

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [32].
Throughout this paper (in random stability section), let Γ^+ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set

$$D^+ = \{F \in \Gamma^+ : l^- F(-\infty) = 1\},$$

where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0, & \text{if } \ t \leq a, \\ 1, & \text{if } \ t > a. \end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

Definition 1.1. A function $T : [0, 1]^2 \to [0, 1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(x, 1) = x$ for all $x \in [0, 1]$;
(d) $T(x, y) \leq T(z, w)$ whenever $x \leq z$ and $y \leq w$ for all $x, y, z, w \in [0, 1]$.

Three typical examples of continuous t-norms are as follows: $T_P(x, y) = xy$, $T(x, y) = \max\{a + b - 1, 0\}$, $T_M(x, y) = \min(a, b)$. Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined by $T_{i=1}^{\infty} x_{n+i}$.

Definition 1.2. A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:
(a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;
(b) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(c) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_u(t) = \frac{t}{t + \|u\|}$$

for all $t > 0$ and T_M is the minimum t-norm. This space X is called the *induced random normed space*.
If the t-norm T is such that $\sup_{0<a<1} T(a, a) = 1$, then every RN-space (X, μ, T) is a metrizable linear topological space with the topology τ (called the *μ-topology* or the *(ϵ, δ)-topology*, where $\epsilon > 0$ and $\lambda \in (0, 1)$) induced by the base $\{U(\epsilon, \lambda)\}$ of neighborhoods of θ, where

$$U(\epsilon, \lambda) = \{x \in X : \Psi_x(\epsilon) > 1 - \lambda\}.$$

Definition 1.3. Let (X, μ, T) be an RN-space.
(a) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$.
(b) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.
(c) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 1.1. ([32]) *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Definition 1.4. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:
(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.2. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(a) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*
(b) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(c) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
(d) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In this paper, using the fixed point and direct methods, we prove the Hyers-Ulam stability of the following functional equation

$$\begin{aligned} af\left(\frac{x+y+z}{b}\right) &+ af\left(\frac{x-y+z}{b}\right) + af\left(\frac{x+y-z}{b}\right) + af\left(\frac{-x+y+z}{b}\right) \\ &= cf(x) + cf(y) + cf(z) \end{aligned} \tag{1.1}$$

in random normed spaces.

2. Hyers-Ulam stability of the functional equation (1.1): a fixed point method

In this section, using fixed point method, we prove the Hyers-Ulam stability of the functional equation (1.1) in random normed space for odd and even cases.

Theorem 2.1. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x, y, z)$ is denoted by $\Phi_{x,y,z}$) such that there exists $0 < \alpha < \frac{1}{2}$ such that*

$$\Phi_{2x,2y,2z}(t) \leq \Phi_{x,y,z}(\alpha t) \tag{2.1}$$

for all $x, y, z \in X$ and $t > 0$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t) \geq \Phi_{x,y,z}(t) \tag{2.2}$$

for all $x, y, z \in X$ and $t > 0$. Then the limit

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq T_M\left(\Phi_{x,x,0}\left(\frac{(1-2\alpha)ct}{2\alpha}\right), \Phi_{2x,0,0}\left(\frac{(1-2\alpha)ct}{2\alpha}\right)\right) \tag{2.3}$$

for all $x \in X$ and $t > 0$.

Proof. Replacing x, y, z by $\frac{x}{2}, \frac{x}{2}, 0$, respectively, in (2.2) and letting $y = z = 0$ in (2.2), one can easily obtain that

$$\begin{aligned} \mu_{f(x)-2f\left(\frac{x}{2}\right)}(t) &\geq T_M\left(\Phi_{\frac{x}{2},\frac{x}{2},0}\left(\frac{ct}{2}\right), \Phi_{x,0,0}\left(\frac{ct}{2}\right)\right) \\ &\geq T_M\left(\Phi_{x,x,0}\left(\frac{ct}{2\alpha}\right), \Phi_{2x,0,0}\left(\frac{ct}{2\alpha}\right)\right). \end{aligned} \tag{2.4}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S := \{g : X \to Y\}$$

and the generalized metric d in S defined by

$$d(f, g) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq T_M\left(\Phi_{x,x,0}(t), \Phi_{2x,0,0}(t)\right), \forall x \in X, t > 0\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [20, Lemma 2.1]).

Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right)$$

for all $x \in X$.

First, we prove that J is a strictly contractive mapping with the Lipschitz constant 2α. In fact, let $g, h \in S$ be such that $d(g,h) < \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq T_M\left(\Phi_{x,x,0}(t), \Phi_{2x,0,0}(t)\right)$$

for all $x \in X$ and $t > 0$ and so

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(2\alpha\epsilon t) &= \mu_{2g(\frac{x}{2})-2h(\frac{x}{2})}(2\alpha\epsilon t) \\
&= \mu_{g(\frac{x}{2})-h(\frac{x}{2})}(\alpha\epsilon t) \\
&\geq T_M\left(\Phi_{\frac{x}{2},\frac{x}{2},0}(\alpha t), \Phi_{x,0,0}(\alpha t)\right) \\
&\geq T_M\left(\Phi_{x,x,0}(t), \Phi_{2x,0,0}(t)\right)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) < \epsilon$ implies that $d(Jg, Jh) < 2\alpha\epsilon$. This means that

$$d(Jg, Jh) \leq 2\alpha d(g,h)$$

for all $g, h \in S$. It follows from (2.4) that

$$d(f, Jf) \leq \frac{2\alpha}{c}.$$

By Theorem 1.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g,h) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.5) such that there exists $u \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(ut) \geq T_M\left(\Phi_{x,x,0}(t), \Phi_{2x,0,0}(t)\right)$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-2\alpha}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{2\alpha}{c - 2c\alpha}$$

and so

$$\mu_{f(x)-A(x)}\left(\frac{2\alpha t}{c-2c\alpha}\right) \geq T_M\left(\Phi_{x,x,0}(t), \Phi_{2x,0,0}(t)\right)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (2.3) holds. Replacing x, y and z by $\frac{x}{2^n}$, $\frac{y}{2^n}$ and $\frac{z}{2^n}$, respectively, in (2.2), we obtain

$$\mu_{2^n\left[af\left(\frac{x+y+z}{2^n b}\right)+af\left(\frac{x-y+z}{2^n b}\right)+af\left(\frac{x+y-z}{2^n b}\right)+af\left(\frac{-x+y+z}{2^n b}\right)-cf\left(\frac{x}{2^n}\right)-cf\left(\frac{y}{2^n}\right)-cf\left(\frac{z}{2^n}\right)\right]}(t) \geq \Phi_{\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}}\left(\frac{t}{2^n}\right)$$

for all $x, y, z \in X$, $t > 0$ and $n \geq 1$ and so, from (2.1), it follows that

$$\Phi_{\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}}\left(\frac{t}{2^n}\right) \geq \Phi_{x,y,z}\left(\frac{t}{2^n\alpha^n}\right).$$

Since

$$\lim_{n\to\infty} \Phi_{x,y,z}\left(\frac{t}{2^n\alpha^n}\right) = 1$$

for all $x, y, z \in X$ and $t > 0$, we have

$$\mu_{aA\left(\frac{x+y+z}{b}\right)+aA\left(\frac{x-y+z}{b}\right)+aA\left(\frac{x+y-z}{b}\right)+aA\left(\frac{-x+y+z}{b}\right)-cA(x)-cA(y)-cA(z)}(t) = 1$$

for all $x, y, z \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ is additive. This completes the proof. □

Corollary 2.1. *Let X be a real normed space, $\theta \geq 0$ and r be a real numbers with $r > 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t) \geq \frac{t}{t+\theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)}$$

for all $x, y, z \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq T_M\left(\frac{(2^r-2)ct}{(2^r-2)ct+4\theta\|x\|^r}, \frac{(2^r-2)ct}{(2^r-2)ct+2^{r+1}\theta\|x\|^r}\right)$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 2.1 if we take

$$\Phi_{x,y,z}(t) = \frac{t}{t+\theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)}$$

for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^{-r}$, then we get the desired result. □

Theorem 2.2. *Let X be a linear space, (Y,μ,T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x,y,z)$ is denoted by $\Phi_{x,y,z}$) such that for some $0<\alpha<2$*

$$\Phi_{\frac{x}{2},\frac{y}{2},\frac{z}{2}}(t)\leq\Phi_{x,y,z}(\alpha t)$$

for all $x,y,z\in X$ and $t>0$. Let $f:X\to Y$ be an odd mapping satisfying (2.2). Then the limit $A(x):=\lim_{n\to\infty}\frac{f(2^nx)}{2^n}$ exists for all $x\in X$ and $A:X\to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t)\geq T_M\left(\Phi_{x,x,0}\left(\frac{(2-\alpha)ct}{2}\right),\Phi_{2x,0,0}\left(\frac{(2-\alpha)ct}{2}\right)\right)$$

for all $x\in X$ and $t>0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1. Consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{1}{2}g(2x)$$

for all $x\in X$.
By (2.4), we obtain that

$$\mu_{\frac{f(2x)}{2}-f(x)}\left(\frac{t}{c}\right)\geq T_M\left(\Phi_{x,x,0}(t),\Phi_{2x,0,0}(t)\right).$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.2. *Let X be a real normed space, $\theta\geq 0$ and r be a real numbers with $0<r<1$. Let $f:X\to Y$ be an odd mapping satisfying*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t)\geq\frac{t}{t+\theta\Big(\|x\|^r+\|y\|^r+\|z\|^r\Big)}$$

for all $x,y,z\in X$ and $t>0$. Then the limit $A(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n}$ exists for all $x\in X$ and $A:X\to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t)\geq T_M\left(\frac{(2-2^r)ct}{(2-2^r)ct+4\theta\|x\|^r},\frac{(2-2^r)ct}{(2-2^r)ct+2^{r+1}\theta\|x\|^r}\right)$$

for all $x\in X$ and $t>0$.

Proof. The proof follows from Theorem 2.2 if we take

$$\Phi_{x,y,z}(t)=\frac{t}{t+\theta(\|x\|^r+\|y\|^r+\|z\|^r)}$$

for all $x,y,z\in X$ and $t>0$. In fact, if we choose $\alpha=2^r$, then we get the desired result. □

Theorem 2.3. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x,y,z)$ is denoted by $\Phi_{x,y,z}$) such that there exists $0 < \alpha < \frac{1}{4}$ such that*

$$\Phi_{2x,2y,2z}(t) \leq \Phi_{x,y,z}(\alpha t)$$

for all $x, y, z \in X$ and $t > 0$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then the limit $Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic mapping such that

$$\mu_{f(x)-Q(x)}(t) \geq T_M\left(\Phi_{x,x,0}\left(\frac{(1-4\alpha)ct}{4\alpha}\right), \Phi_{2x,0,0}\left(\frac{(1-4\alpha)ct}{2\alpha}\right)\right)$$

for all $x \in X$ and $t > 0$.

Proof. Consider the set

$$S^* := \{g : X \to Y : \ g(0) = 0\}$$

and the generalized metric d^* in S^* defined by

$$d^*(f,g) = \inf\left\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq T_M\left(\Phi_{x,x,0}\left(\frac{t}{2}\right), \Phi_{2x,0,0}(t)\right), \forall x \in X,\, t > 0\right\}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S^*, d^*) is complete (see [20, Lemma 2.1]).

By the same method as in the proof of Theorem 2.1, we obtain that

$$\mu_{f(x)-4f\left(\frac{x}{2}\right)}\left(\frac{2\alpha t}{c}\right) \geq T_M\left(\Phi_{x,x,0}\left(\frac{t}{2}\right), \Phi_{2x,0,0}(t)\right).$$

for all $x \in X$ and $t > 0$.
Now, we consider a linear mapping $J^* : (S^*, d^*) \to (S^*, d^*)$ such that

$$J^*h(x) := 4h\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.3. *Let X be a real normed linear space, $\theta \geq 0$ and r be real numbers with $r \in (1, \infty)$. If $f : X \to Y$ be an even mapping with $f(0) = 0$ such that*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t) \geq \frac{t}{t+\theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)}$$

for all $x, y, z \in X$ and $t > 0$, then the limit $Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_M\left(\frac{(4^r-4)ct}{(4^r-4)ct+8\theta\|x\|^r}, \frac{(4^r-4)ct}{(4^r-4)ct+2^{r+2}\theta\|x\|^r}\right)$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 2.3 if we take

$$\Phi_{x,y,z}(t) = \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)}$$

for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 4^{-r}$, then we get the desired result. □

Theorem 2.4. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x, y, z)$ is denoted by $\Phi_{x,y,z}$) such that there exists $0 < \alpha < 4$ such that*

$$\Phi_{\frac{x}{2},\frac{y}{2},\frac{z}{2}}(t) \leq \Phi_{x,y,z}(\alpha t)$$

for all $x, y, z \in X$ and $t > 0$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then the limit $Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic mapping such that

$$\mu_{f(x)-Q(x)}(t) \geq T_M\left(\Phi_{x,x,0}\left(\frac{(4-\alpha)ct}{4}\right), \Phi_{2x,0,0}\left(\frac{(4-\alpha)ct}{2}\right)\right)$$

for all $x \in X$ and $t > 0$.

Proof. Let (S^*, d^*) be the generalized metric space defined in the proof of Theorem 2.3. Consider the linear mapping $J^* : (S^*, d^*) \to (S^*, d^*)$ such that

$$J^* g(x) := \frac{1}{4} g(2x).$$

By the same method as in the proof of Theorem 2.1, we obtain that

$$\mu_{\frac{f(2x)}{4}-f(x)}\left(\frac{t}{2c}\right) \geq T_M\left(\Phi_{x,x,0}\left(\frac{t}{2}\right), \Phi_{2x,0,0}(t)\right).$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.3. □

Corollary 2.4. *Let X be a normed vector space with norm $\|\cdot\|$, $\theta \geq 0$ and r be a real number with $r \in (0, 1)$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ such that*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t) \geq \frac{t}{t + \theta\Big(\|x\|^r + \|y\|^r + \|z\|^r\Big)}$$

for all $x, y, z \in X$ and $t > 0$. Then $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic mapping such that

$$\mu_{f(x)-Q(x)}(t) \geq T_M\left(\frac{(4-4^r)ct}{(4-4^r)ct + 8\theta\|x\|^r}, \frac{(4-4^r)ct}{(4-4^r)ct + 2^{r+2}\theta\|x\|^r}\right)$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 2.4 if we take

$$\Phi_{x,y,z}(t) = \frac{t}{t + \theta\Big(\|x\|^r + \|y\|^r + \|z\|^r\Big)}$$

for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 4^r$, then we get the desired result. □

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (1.1) . Let $f_e(x) := \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then f_e is an even mapping satisfying (1.1) and f_o is an odd mapping satisfying (1.1) such that $f(x) = f_e(x) + f_o(x)$. So we obtain the following.

Theorem 2.5. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x, y, z)$ is denoted by $\Phi_{x,y,z}$) such that there exists $0 < \alpha < \frac{1}{4}$ such that*

$$\Phi_{2x,2y,2z}(t) \leq \Phi_{x,y,z}(\alpha t)$$

for all $x, y, z \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
&\mu_{2f(x)-A(x)-Q(x)}(t) \\
&\geq T_M\left(\mu_{f(x)-A(x)}\left(\frac{t}{2}\right), \mu_{f(x)-Q(x)}\left(\frac{t}{2}\right)\right) \\
&\geq T_M\left(T_M\left(\Phi_{x,x,0}\left(\frac{(1-2\alpha)ct}{4\alpha}\right), \Phi_{2x,0,0}\left(\frac{(1-2\alpha)ct}{4\alpha}\right)\right),\right. \\
&\geq \left. T_M\left(\Phi_{x,x,0}\left(\frac{(1-4\alpha)ct}{8\alpha}\right), \Phi_{2x,0,0}\left(\frac{(1-4\alpha)ct}{4\alpha}\right)\right)\right)
\end{aligned}$$

for all $x \in X$ and $t > 0$.

3. **Random stability of the functional equation (1.1): a direct method**

In this section, using direct method, we prove the Hyers-Ulam stability of the functional equation (1.1) in random normed space for odd and even cases.

Theorem 3.1. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\phi : X^3 \to Z$ be a function such that there exists $0 < \alpha < 2$ such that*

$$\mu'_{\phi(2x,2y,2z)}(t) \geq \mu'_{\alpha\phi(x,y,z)}(t) \tag{3.1}$$

for all $x, y, z \in X$ and $t > 0$ and

$$\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y, 2^n z)}(2^n t) = 1$$

for all $x, y, z \in X$ *and* $t > 0$. *Let* $(Y, \mu, \min)$ *be a complete RN-space. If* $f : X \to Y$ *is an odd mapping such that*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t) \geq \mu'_{\phi(x,y,z)}(t) \tag{3.2}$$

for all $x, y, z \in X$ *and* $t > 0$, *then the limit*

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(2-\alpha)ct}{2}\right), \mu'_{\phi(2x,0,0)}\left(\frac{(2-\alpha)ct}{2}\right)\right) \tag{3.3}$$

for all $x \in X$ *and* $t > 0$.

Proof. Putting $y = z = 0$ in (3.2), we see that

$$\mu_{2af\left(\frac{x}{b}\right)-cf(x)}(t) \geq \mu'_{\phi(x,0,0)}(t) \tag{3.4}$$

for all $x \in X$. Replacing x by $2x$ in (3.4), we obtain

$$\mu_{2af\left(\frac{2x}{b}\right)-cf(2x)}(t) \geq \mu'_{\phi(2x,0,0)}(t) \tag{3.5}$$

for all $x \in X$. Putting $y = x$ and $z = 0$ in (3.2), we have

$$\mu_{2af\left(\frac{2x}{b}\right)-2cf(x)}(t) \geq \mu'_{\phi(x,x,0)}(t) \tag{3.6}$$

By (3.5) and (3.6), we obtain that

$$\begin{aligned}
\mu_{cf(2x)-2cf(x)}(t) &= \mu_{cf(2x)\pm 2af\left(\frac{2x}{b}\right)-2cf(x)}(t) \\
&= \mu_{cf(2x)-2af\left(\frac{2x}{b}\right)+2af\left(\frac{2x}{b}\right)-2cf(x)}(t) \\
&\geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{t}{2}\right), \mu'_{\phi(2x,0,0)}\left(\frac{t}{2}\right)\right).
\end{aligned}$$

So

$$\mu_{\frac{f(2x)}{2}-f(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}(ct), \mu'_{\phi(2x,0,0)}(ct)\right). \tag{3.7}$$

Replacing x by $2^n x$ in (3.7) and using (3.1), we obtain

$$\begin{aligned}
\mu_{\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}}(t) &\geq T_M\left(\mu'_{\phi(2^n x,2^n x,0)}(2^n ct), \mu'_{\phi(2^{n+1}x,0,0)}(2^n ct)\right) \\
&\geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{2^n ct}{\alpha^n}\right), \mu'_{\phi(2x,0,0)}\left(\frac{2^n ct}{\alpha^n}\right)\right).
\end{aligned}$$

Since $\frac{f(2^n x)}{2^n} - f(x) = \sum_{k=0}^{n-1} \frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}$,

$$\begin{aligned}
\mu_{\frac{f(2^n x)}{2^n}-f(x)}\left(\sum_{k=0}^{n-1}\frac{t\alpha^k}{2^k}\right) &= \mu_{\sum_{k=0}^{n-1}\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}}\left(\sum_{k=0}^{n-1}\frac{t\alpha^k}{2^k}\right)\\
&\geq T_{k=0}^{n-1}\left(\mu_{\frac{f(2^{k+1}x)}{2^{k+1}}-2^k f(\frac{x}{2^k})}\left(\frac{t\alpha^k}{2^k}\right)\right)\\
&\geq T_{k=0}^{n-1}\left(T_M\left(\mu'_{\phi(x,x,0)}(ct),\mu'_{\phi(2x,0,0)}(ct)\right)\right)\\
&= T_M\left(\mu'_{\phi(x,x,0)}(ct),\mu'_{\phi(2x,0,0)}(ct)\right).
\end{aligned}$$

This implies that

$$\mu_{\frac{f(2^n x)}{2^n}-f(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{ct}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2^k}}\right),\mu'_{\phi(2x,0,0)}\left(\frac{ct}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2^k}}\right)\right). \tag{3.8}$$

Replacing x by $2^p x$ in (3.8), we obtain

$$\mu_{\frac{f(2^{n+p}x)}{2^{n+p}}-\frac{f(x^p)}{2^p}}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{ct}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{2^k}}\right),\mu'_{\phi(2x,0,0)}\left(\frac{ct}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{2^k}}\right)\right). \tag{3.9}$$

Since

$$\lim_{p,n\to\infty}\mu'_{\phi(x,x,0)}\left(\frac{ct}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{2^k}}\right) = \lim_{p,n\to\infty}\mu'_{\phi(2x,0,0)}\left(\frac{ct}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{2^k}}\right) = 1$$

it follows that $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence in complete RN-space $(Y,\mu,\min)$ and so there exists a point $A(x)\in Y$ such that

$$\lim_{n\to\infty}\frac{f(2^n x)}{2^n} = A(x).$$

Fix $x\in X$ and put $p=0$ in (3.9). Then we obtain

$$\mu_{\frac{f(2^n x)}{2^n}-f(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{ct}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2^k}}\right),\mu'_{\phi(2x,0,0)}\left(\frac{ct}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2^k}}\right)\right).$$

and so, for any $\epsilon>0$,

$$\begin{aligned}
&\mu_{A(x)-f(x)}(t+\epsilon) \qquad (3.10)\\
&\geq T\left(\mu_{A(x)-\frac{f(2^n x)}{2^n}}(\epsilon),\mu_{\frac{f(2^n x)}{2^n}-f(x)}(t)\right)\\
&\geq T\left(\mu_{A(x)-\frac{f(2^n x)}{2^n}}(\epsilon),T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{ct}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2^k}}\right),\mu'_{\phi(2x,0,0)}\left(\frac{ct}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2^k}}\right)\right)\right).
\end{aligned}$$

Taking $n\to\infty$ in (3.10), we get

$$\mu_{C(x)-f(x)}(t+\epsilon) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(2-\alpha)ct}{2}\right),\mu'_{\phi(2x,0,0)}\left(\frac{(2-\alpha)ct}{2}\right)\right). \tag{3.11}$$

Since ϵ is arbitrary, by taking $\epsilon \to 0$ in (3.11), we get

$$\mu_{C(x)-f(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(2-\alpha)ct}{2}\right), \mu'_{\phi(2x,0,0)}\left(\frac{(2-\alpha)ct}{2}\right)\right).$$

Replacing x, y and z by $2^n x, 2^n y$ and $2^n z$, respectively, in (3.2), we get

$$\begin{aligned}&\mu_{\frac{1}{2^n}\left[af\left(\frac{2^n(x+y+z)}{b}\right)+af\left(\frac{2^n(x-y+z)}{b}\right)+af\left(\frac{2^n(x+y-z)}{b}\right)+af\left(\frac{2^n(-x+y+z)}{b}\right)-cf(2^n x)-cf(2^n y)-cf(2^n z)\right]}(t)\\ &\geq \mu'_{\phi(2^n x,2^n y,2^n z)}(2^n t)\end{aligned}$$

for all $x, y, z \in X$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\phi(2^n x,2^n y,2^n z)}(2^n t) = 1$, we conclude that A satisfies (1.1).
Furthermore

$$\begin{aligned}A(2x) - 2A(x) &= \lim_{n\to\infty} \frac{f(2^{n+1}x)}{2^n} - 2\lim_{n\to\infty} \frac{f(2^n x)}{2^n}\\ &= 2\left[\lim_{n\to\infty} \frac{f(2^{n+1}x)}{2^{n+1}} - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}\right]\\ &= 0.\end{aligned}$$

So, $A : X \to Y$ is an additive mapping.
To prove the uniqueness of the additive mapping A, assume that there exists another additive mapping $L : X \to Y$ which satisfies (3.3). Then we have

$$\begin{aligned}\mu_{A(x)-L(x)}(t) &= \lim_{n\to\infty} \mu_{\frac{A(2^n x)}{2^n}-\frac{L(2^n x)}{2^n}}(t)\\ &\geq \lim_{n\to\infty} \min\left\{\mu_{\frac{A(2^n x)}{2^n}-\frac{f(2^n x)}{2^n}}\left(\frac{t}{2}\right), \mu_{\frac{f(2^n x)}{2^n}-\frac{L(2^n x)}{2^n}}\left(\frac{t}{2}\right)\right\}\\ &\geq \lim_{n\to\infty} T_M\left(\mu'_{\phi(2^n x,2^n x,0)}\left(\frac{2^n(2-\alpha)ct}{4}\right), \mu'_{\phi(2^{n+1}x,0,0)}\left(\frac{2^n(2-\alpha)ct}{4}\right)\right)\\ &\geq \lim_{n\to\infty} T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{2^n(2-\alpha)ct}{4\alpha^n}\right), \mu'_{\phi(2x,0,0)}\left(\frac{2^n(2-\alpha)ct}{4\alpha^n}\right)\right).\end{aligned}$$

Since

$$\lim_{n\to\infty}\left(\frac{2^n(2-\alpha)ct}{4\alpha^n}\right) = \lim_{n\to\infty}\left(\frac{2^n(2-\alpha)ct}{4\alpha^n}\right) = \infty$$

we get

$$\lim_{n\to\infty} T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{2^n(2-\alpha)ct}{4\alpha^n}\right), \mu'_{\phi(2x,0,0)}\left(\frac{2^n(2-\alpha)ct}{4\alpha^n}\right)\right) = 1.$$

Therefore, it follows that $\mu_{A(x)-L(x)}(t) = 1$ for all $t > 0$ and so $A(x) = L(x)$. This completes the proof. □

Corollary 3.1. *Let X be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $0 < r < 1$, $z_0 \in Z$ and $f : X \to Y$ be an odd*

mapping satisfying

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t) \geq \mu'_{(\|x\|^r+\|y\|^r+\|z\|^r)z_0}(t)$$

for all $x, y, z \in X$ *and* $t > 0$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq T_M\left(\mu'_{\|x\|^r z_0}\left(\frac{(2-2^r)ct}{4}\right), \mu'_{\|x\|^r z_0}\left(\frac{(2-2^r)ct}{2^{r+1}}\right)\right)$$

for all $x \in X$ *and* $t > 0$.

Proof. Let $\alpha = 2^r$ and $\phi : X^3 \to Z$ be a mapping defined by $\phi(x, y, z) = (\|x\|^r + \|y\|^r + \|z\|^r)z_0$. Then, from Theorem 3.1, the conclusion follows. □

Corollary 3.2. *Let* X *be a real normed linear space,* $(Z, \mu', \min)$ *be an RN-space and* $(Y, \mu, \min)$ *be a complete RN-space. Let* $z_0 \in Z$ *and* $f : X \to Y$ *be an odd mapping satisfying*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t) \geq \mu'_{\delta z_0}(t)$$

for all $x, y, z \in X$ *and* $t > 0$ *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\delta z_0}\left(\frac{ct}{2}\right)$$

for all $x \in X$ *and* $t > 0$.

Proof. Let $\alpha = 1$ and $\phi : X^3 \to Z$ be a mapping defined by $\phi(x, y, z) = \delta z_0$. Then, from Theorem 3.1, the conclusion follows. □

Theorem 3.2. *Let* X *be a real linear space,* $(Z, \mu', \min)$ *be an RN-space and* $\phi : X^3 \to Z$ *be a function such that there exists* $0 < \alpha < \frac{1}{2}$ *such that* $\mu'_{\phi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)}(t) \geq \mu'_{\alpha\phi(x,y,z)}(t)$ *for all* $x, y, z \in X$ *and* $t > 0$ *and*

$$\lim_{n\to\infty} \mu'_{\phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)}\left(\frac{t}{2^n}\right) = 1$$

for all $x, y, z \in X$ *and* $t > 0$. *Let* $(Y, \mu, \min)$ *be a complete RN-space and* $f : X \to Y$ *be an odd mapping satisfying (3.2). Then the limit* $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ *exists for all* $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(1-2\alpha)ct}{2\alpha}\right), \mu'_{\phi(2x,0,0)}\left(\frac{(1-2\alpha)ct}{2\alpha}\right)\right)$$

for all $x \in X$ *and* $t > 0$.

Proof. By (3.7), we obtain

$$\begin{aligned}\mu_{f(x)-2f\left(\frac{x}{2}\right)}(t) &\geq T_M\left(\mu'_{\phi\left(\frac{x}{2},\frac{x}{2},0\right)}\left(\frac{ct}{2}\right),\mu'_{\phi(x,0,0)}\left(\frac{ct}{2}\right)\right)\\ &\geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{ct}{2\alpha}\right),\mu'_{\phi(2x,0,0)}\left(\frac{ct}{2\alpha}\right)\right).\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.3. *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let $r\in(1,+\infty)$, $z_0\in Z$ and $f:X\to Y$ be an odd mapping satisfying*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t)\geq\mu'_{(\|x\|^r+\|y\|^r+\|z\|^r)z_0}(t)$$

for all $x,y,z\in X$ and $t>0$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\mu_{f(x)-A(x)}(t)\geq T_M\left(\mu'_{\|x\|^r z_0}\left(\frac{(2^r-2)ct}{4}\right),\mu'_{\|x\|^r z_0}\left(\frac{(2^r-2)ct}{2^{r+1}}\right)\right)$$

for all $x\in X$ and $t>0$.

Proof. Let $\alpha=2^{-r}$ and $\phi:X^3\to Z$ be a mapping defined by $\phi(x,y,z)=(\|x\|^r+\|y\|^r+\|z\|^r)z_0$. Then, from Theorem 3.2, the conclusion follows. □

Theorem 3.3. *Let X be a real linear space, $(Z,\mu',\min)$ be an RN-space and $\phi:X^3\to Z$ be a function such that there exists $0<\alpha<4$ such that $\mu'_{\phi(2x,2y,2z)}(t)\geq\mu'_{\alpha\phi(x,y,z)}(t)$ for all $x,y,z\in X$ and $t>0$ and*

$$\lim_{n\to\infty}\mu'_{\phi(2^n x,2^n y,2^n z)}(4^n t)=1$$

for all $x,y,z\in X$ and $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f:X\to Y$ is an even mapping satisfyingf$(0)=0$ and (3.2), then the limit $A(x)=\lim_{n\to\infty}\frac{f(2^n x)}{4^n}$ exists for all $x\in X$ and defines a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{f(x)-Q(x)}(t)\geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(4-\alpha)ct}{4}\right),\mu'_{\phi(2x,0,0)}\left(\frac{(4-\alpha)ct}{2}\right)\right)$$

for all $x\in X$ and $t>0$.

Proof. Putting $y=x$ and $z=0$ in (3.2), we have

$$\mu_{2af\left(\frac{2x}{b}\right)-2cf(x)}(t)\geq\mu'_{\phi(x,x,0)}(t)\tag{3.12}$$

for all $x\in X$. Putting $y=z=0$ and then replacing x by $2x$, we have

$$\mu_{4af\left(\frac{2x}{b}\right)-cf(2x)}(t)\geq\mu'_{\phi(2x,0,0)}(t)\tag{3.13}$$

By (3.12) and (3.13), we see that

$$\begin{aligned}\mu_{cf(2x)-4cf(x)}(t) &= \mu_{2(2af\left(\frac{2x}{b}\right)-2cf(x))-4af\left(\frac{2x}{b}\right)+cf(2x)}(t)\\ &\geq T_M\left(\mu_{2af\left(\frac{2x}{b}\right)-2cf(x)}\left(\frac{t}{4}\right),\mu_{4af\left(\frac{2x}{b}\right)-cf(2x)}\left(\frac{t}{2}\right)\right)\\ &\geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{t}{4}\right),\mu'_{\phi(2x,0,0)}\left(\frac{t}{2}\right)\right).\end{aligned}$$

So

$$\mu_{\frac{f(2x)}{4}-f(x)}(t)\geq T_M\left(\mu'_{\phi(x,x,0)}(ct),\mu'_{\phi(2x,0,0)}(2ct)\right). \tag{3.14}$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let $0<r<1$, $z_0\in Z$ and $f:X\to Y$ be an even mapping satisfying $f(0)=0$ and*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t)\geq\mu'_{(\|x\|^r+\|y\|^r+\|z\|^r)z_0}(t)$$

for all $x,y,z\in X$ and $t>0$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{f(x)-Q(x)}(t)\geq T_M\left(\mu'_{\|x\|^r z_0}\left(\frac{(4-4^r)ct}{8}\right),\mu'_{\|x\|^r z_0}\left(\frac{(4-4^r)ct}{2^{r+1}}\right)\right)$$

for all $x\in X$ and $t>0$.

Proof. Let $\alpha=4^r$ and $\phi:X^3\to Z$ be a mapping defined by $\phi(x,y,z)=(\|x\|^r+\|y\|^r+\|z\|^r)z_0$. Then, from Theorem 3.3, the conclusion follows. □

Corollary 3.5. *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let $z_0\in Z$ and $f:X\to Y$ be an even mapping with $f(0)=0$ such that*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t)\geq\mu'_{\delta z_0}(t)$$

for all $x,y,z\in X$ and $t>0$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{f(x)-Q(x)}(t)\geq T_M\left(\mu'_{\delta z_0}\left(\frac{3ct}{4}\right),\mu'_{\delta z_0}\left(\frac{3ct}{2}\right)\right)$$

for all $x\in X$ and $t>0$.

Proof. Let $\alpha=1$ and $\phi:X^3\to Z$ be a mapping defined by $\phi(x,y,z)=\delta z_0$. Then, from Theorem 3.3, the conclusion follows. □

Theorem 3.4. *Let X be a real linear space, $(Z,\mu',\min)$ be an RN-space and $\phi: X^3\to Z$ be a function such that there exists $0<\alpha<\frac{1}{4}$ such that $\mu'_{\phi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)}(t)\geq\mu'_{\alpha\phi(x,y,z)}(t)$ for all $x,y,z\in X$ and $t>0$ and*

$$\lim_{n\to\infty}\mu'_{\phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)}\left(\frac{t}{4^n}\right)=1$$

for all $x,y,z\in X$ and $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space and $f:X\to Y$ be an even mapping satisfying $f(0)=0$ and (3.2). Then the limit $Q(x)=\lim_{n\to\infty}4^nf\left(\frac{x}{2^n}\right)$ exists for all $x\in X$ and defines a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{f(x)-Q(x)}(t)\geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(1-4\alpha)ct}{4\alpha}\right),\mu'_{\phi(2x,0,0)}\left(\frac{(1-4\alpha)ct}{2\alpha}\right)\right)$$

for all $x\in X$ and $t>0$.

Proof. By (3.14), we obtain

$$\begin{aligned}\mu_{f(x)-4f\left(\frac{x}{2}\right)}(t) &\geq T_M\left(\mu'_{\phi\left(\frac{x}{2},\frac{x}{2},0\right)}\left(\frac{ct}{4}\right),\mu'_{\phi(x,0,0)}\left(\frac{ct}{2}\right)\right)\\ &\geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{ct}{4\alpha}\right),\mu'_{\phi(2x,0,0)}\left(\frac{ct}{2\alpha}\right)\right).\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.6. *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let $r\in(1,+\infty)$, $z_0\in Z$ and $f:X\to Y$ be an even mapping satisfying $f(0)=0$ and*

$$\mu_{af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)+af\left(\frac{-x+y+z}{b}\right)-cf(x)-cf(y)-cf(z)}(t)\geq\mu'_{(\|x\|^r+\|y\|^r+\|z\|^r)z_0}(t)$$

for all $x,y,z\in X$ and $t>0$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{f(x)-Q(x)}(t)\geq T_M\left(\mu'_{\|x\|^rz_0}\left(\frac{(4^r-4)ct}{8}\right),\mu'_{\|x\|^rz_0}\left(\frac{(4^r-4)ct}{2^{r+4}}\right)\right)$$

for all $x\in X$ and $t>0$.

Proof. Let $\alpha=4^{-r}$ and $\phi:X^3\to Z$ be a mapping defined by $\phi(x,y,z)=(\|x\|^r+\|y\|^r+\|z\|^r)z_0$. Then, from Theorem 3.4, the conclusion follows. □

Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (1.1) . Let $f_e(x):=\frac{f(x)+f(-x)}{2}$ and $f_o(x)=\frac{f(x)-f(-x)}{2}$. Then f_e is an even mapping satisfying (1.1) and f_o is an odd mapping satisfying (1.1) such that $f(x)=f_e(x)+f_o(x)$. So we obtain the following.

Theorem 3.5. *Let X be a real linear space, $(Z,\mu',\min)$ be an RN-space and $\phi:X^3\to Z$ be a function such that there exists $0<\alpha<2$ such that $\mu'_{\phi(2x,2y,2z)}(t)\geq\mu'_{\alpha\phi(x,y,z)}(t)$ for all $x,y,z\in X$ and $t>0$ and $\lim_{n\to\infty}\mu'_{\phi(2^nx,2^ny,2^nz)}(2^nt)=1$ for all $x,y,z\in X$ and $t>0$.*

Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\mu_{\frac{f(x)-f(-x)}{2}-A(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(2-\alpha)ct}{2}\right), \mu'_{\phi(2x,0,0)}\left(\frac{(2-\alpha)ct}{2}\right)\right)$$

$$\mu_{\frac{f(x)+f(-x)}{2}-Q(x)}(t) \geq T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(4-\alpha)ct}{4}\right), \mu'_{\phi(2x,0,0)}\left(\frac{(4-\alpha)ct}{2}\right)\right)$$

and

$$\begin{aligned}
&\mu_{2f(x)-A(x)-Q(x)}(t)\\
&= \mu_{\frac{f(x)-f(-x)}{2}+\frac{f(x)+f(-x)}{2}-A(x)-Q(x)}(t)\\
&\geq T_M\left(\mu_{\frac{f(x)-f(-x)}{2}-A(x)}\left(\frac{t}{2}\right), \mu_{\frac{f(x)+f(-x)}{2}-Q(x)}\left(\frac{t}{2}\right)\right)\\
&\geq T_M\Bigg(T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(2-\alpha)ct}{4}\right), \mu'_{\phi(2x,0,0)}\left(\frac{(2-\alpha)ct}{4}\right)\right),\\
&\quad T_M\left(\mu'_{\phi(x,x,0)}\left(\frac{(4-\alpha)ct}{8}\right), \mu'_{\phi(2x,0,0)}\left(\frac{(4-\alpha)ct}{4}\right)\right)\Bigg)
\end{aligned}$$

for all $x \in X$ and $t > 0$.

4. **Conclusion**

We linked here three different disciplines, namely, random normed spaces, functional equations and fixed point theory. We established the Hyers-Ulam stability of the functional equation (1.1) in random normed spaces.

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Mixed-Stable Models for Analyzing High-Frequency Financial Data

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Abstract In this paper, we propose the mixed-stable model for analyzing high-frequency stock return data that usually contain a large number of zeros. Based on the data of German Dax component stocks, we apply the stable and mix-stable laws (both with dependent and independent states) to model the data. We also investigate the self-similarity and multifractality of the data. We show the empirical results of the model performance.

Keywords: high frequency data; stable models, mixed-stable models; financial modeling; self-similarity; Hurst index.

1 Introduction

With the introduction of electronic trading, an enormous quantity of trading data became available. Intra-daily price oscillation, trade duration distribution analysis and overall market microstructure investigations attract more and more attention from the researchers. Containing all transactions of the financial market high frequency data can reveal events and laws that are impossible to identify with monthly, weekly or daily data. A summary of the literature covering intra-daily data research is presented in Sun et al. (2008) and the reference therein.

In our work we propose a new modeling methodology, that is, mixed-stable model to model the intra-daily data from German DAX component stocks returns. In practice, we often observe a large number of zero returns in the high-frequency return data due to the underlying asset price does not change at

a given very short time intervals. Our model is designed to capture this unique features observed in the high-frequency return data.

This paper is organized as follows. The first part is the introduction. Section 2 describe the data we investigated. In Section 3, we introduce our modeling methodology applied for modeling the data (i.e., stable and mixed-stable models, self-similarity and multifractality analysis, Hurst exponent calculation, stagnation intervals modeling etc.). We present our empirical results in Section 4 and conclude in Section 5.

2 The data

In this paper we analyze high-frequency data series of 29 stocks from DAX, representing August 17, 2007 (one of typical business-active days of the year). We aggregate raw inhomogeneous intra-daily data into the equally-spaced homogeneous 10 seconds intra-daily time series. To mitigate opening auction effect, transactions of the first 10 minutes of the day were omitted. The aggregation was done with previous-tick interpolation (see Wasserfallen and Zimmermann (1985)). Dacorogna et al. (2001) point out that linear interpolation relies on the future information whereas previous-tick interpolation is based on the information already known.

Previous-tick interpolation. We denote times of raw intra-daily series as $\{t_i\}$ and the corresponding price as $\{P_i\}$. The aggregated homogeneous high frequency series is obtained at times $t_0 + j\Delta t$ with $\Delta t = 10$ sec. The index j identifies the regularly spaced sequence. The time $t_0 + j\Delta t$ is bounded by two times t_i of the irregularly spaced series, $t_I \leq t_0 + j\Delta t < t_{I+1}$, here

$$I = \max\{i | t_i \leq t_0 + j\Delta\}$$

.

With previous-tick interpolation we obtain that $P(t_0 + j) = P_I$. Having obtained homogeneous price series, we can calculate corresponding logarithmic returns series $\{X_i\}$: $X_i = \ln \frac{P_{i+1}}{P_i}$.

The length of all series is 3060. The number of zero 10-sec stock returns differs from 19% to 69%, with the average 47%.

Almost all data series are asymmetric, and the empirical kurtosis (see Table 1) shows that density functions of the series are more peaked than that of Gaussian. That is why we make an assumption that Gaussian models are not applicable to these financial series. Moreover, it has been observed by researchers, that intra-daily data are characterized by fat tails and long-range dependence (see Dacorogna et al. (2001)).

3 Methodology

Methodology of this research follows the main aspects of stable and mixed-stable modeling, developed by Kabasinskas et al. (2009, 2010). We start the analysis

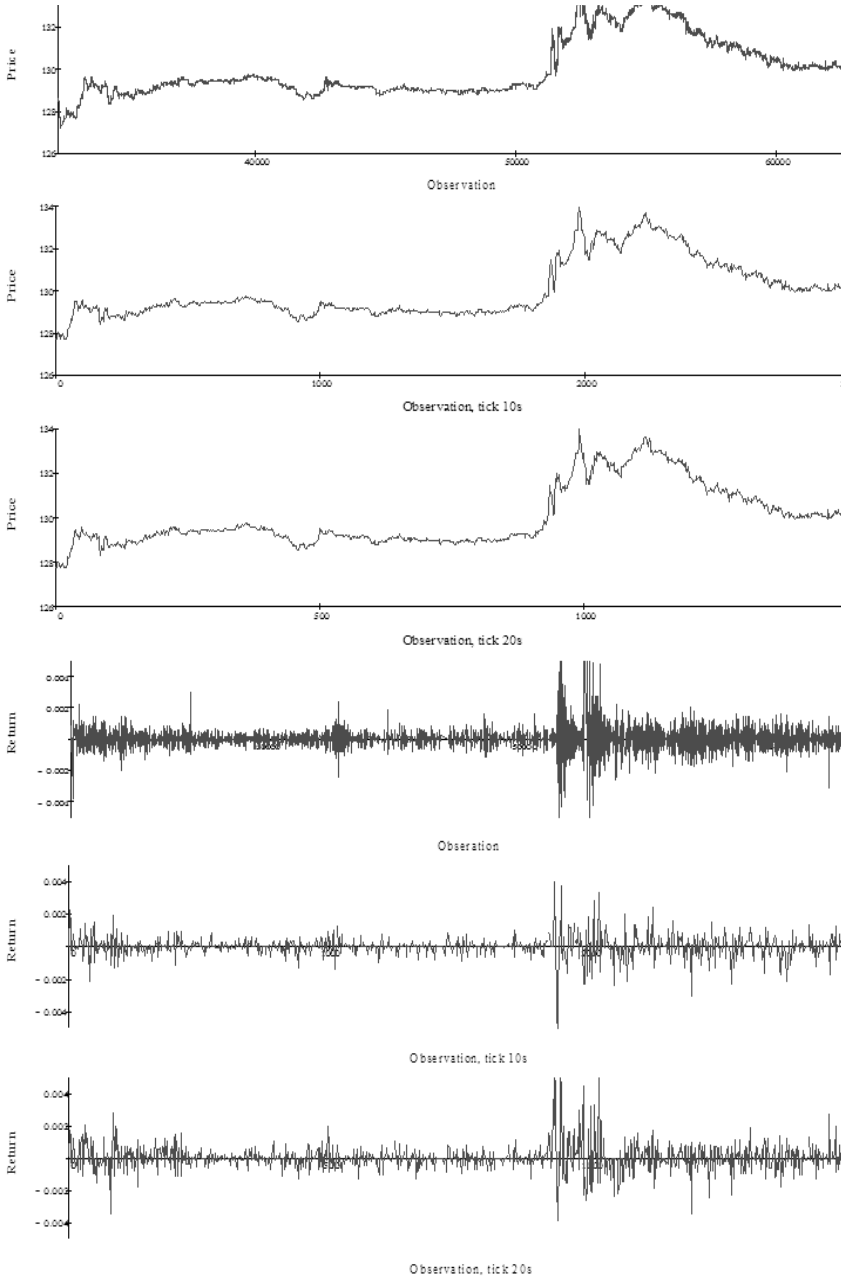


Figure 1: Example of intraday and corresponding high frequency data.

Table 1: Empirical moments

Company name	Mean	St.dev.	Skewness	Kurtosis	Zeros, %
Adidas AG	9.027E-06	0.0008	0.1757	31.12	36.18
Deutsche Bank AG	9.780E-06	0.0007	-0.1265	6.65	78.63
BASF SE	5.711E-06	0.0008	-0.6836	31.52	68.56
BMW AG St	4.417E-06	0.0006	0.2197	6.83	48.10
Continental AG	5.677E-06	0.0009	-0.8912	122.42	48.40
Deutsche Post AG	3.682E-06	0.0006	-0.0232	8.73	37.97
Deutsche Telekom AG	4.738E-06	0.0007	0.2499	3.83	41.73
Bayer AG O.N.	6.468E-06	0.0007	0.3973	7.82	65.72
Fresenius Medical Care AG and Co. KGaA St	5.774E-06	0.0008	-0.2354	10.49	34.58
Deutsche Borse AG	1.548E-05	0.0008	0.0732	14.57	54.58
MAN SE St	5.520E-06	0.0010	0.5253	11.44	53.14
Henkel AG and Co. KGaA Vz	3.280E-06	0.0008	0.0879	13.97	36.11
Infineon Technologies AG	4.891E-06	0.0008	0.3245	8.53	31.47
Linde AG	8.652E-06	0.0008	-0.4563	16.87	39.93
Merck KGaA	1.473E-05	0.0008	0.1754	17.56	38.86
RWE AG St	1.166E-05	0.0008	0.0881	6.99	69.15
Daimler AG	8.151E-06	0.0008	-0.3854	14.23	77.65
SAP AG	8.250E-06	0.0007	0.2107	7.14	62.94
Siemens AG	9.405E-06	0.0007	-0.1094	7.23	77.39
METRO AG St	1.221E-05	0.0009	-0.1969	20.98	55.39
ThyssenKrupp AG	5.698E-06	0.0004	-1.6274	50.38	56.41
Volkswagen AG St	1.776E-05	0.0010	-0.1207	16.45	32.12
Deutsche Postbank AG	1.544E-05	0.0009	0.6914	15.37	44.77
HYPO REAL ESTATE	1.575E-05	0.0010	-0.1406	21.22	57.12
Commerzbank AG	9.627E-06	0.0007	-1.0132	27.87	37.84
Deutsche Lufthansa AG	1.510E-05	0.0006	0.1229	5.79	81.21
Allianz SE	1.510E-05	0.0006	0.1229	5.79	81.21
Munchener Ruck AG	5.706E-06	0.0006	0.3447	16.50	61.37
TUI AG	5.038E-06	0.0009	-0.1190	11.48	41.60

of our log-return series with stable and mixed-stable parameter estimation and goodness-of-fit hypothesis testing. Next we analyze the self-similarity and multifractality of our data and calculate Hurst exponents. Recall that for Gaussian processes, Hurst exponent $H = 0.5$ indicates Brownian motion; $0.5 < H < 1$ indicates long time memory processes and persistent behavior; $0 < H < 0.5$ shows anti-persistent behavior, see Samorodnitsky and Taqqu (2000). Finally we are applying mixed-stable models and analyze the behavior of zeros in the series (how they are distributed and occur in the series, are they random, how they can be simulated, etc.).

3.1 Stable and mixed stable models

The stable distribution is usually described by its characteristic function

$$\log \varphi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - i\beta sign(t) \tan \frac{\pi\alpha}{2}\} + i\mu t, & \alpha \neq 1, \\ -\sigma|t|\{1 + i\beta sign(t) \frac{2}{\pi} \log|t|\} + i\mu t, & \alpha = 1, \end{cases}$$

Each stable distribution is described by four parameters: the first one and the most important is the stability index $\alpha \in (0, 2]$, which is essential when characterizing financial data. The others are: skewness $-1 \leq \beta \leq 1$, location $\mu \in \mathbb{R}$, and scale $\sigma > 0$. Let us denote the parameter vector $\Theta = (\alpha, \beta, \mu, \sigma)$.

In financial modeling it is generally assumed that $1 < \alpha \leq 2$. An overview of stable distributions properties can be found in Rachev and Mittnik (2002), Samorodnitsky and Taqqu (2000), and Zolotarev (1986).

The probability density function of the stable laws cannot be expressed in elementary functions (except few cases: Levy, Cauchy and Gaussian distributions). Exact and fast calculation of stable densities is a nontrivial task [1, 2, 3, 11, 21, 22, 23]. To calculate the density we use Zolotarev integral representation [2, 3]:

$$p(x, \Theta) = \begin{cases} \frac{\alpha|\frac{x-\mu}{\sigma}|^{\frac{1}{\alpha-1}}}{2\sigma|\alpha-1|} \int_{-\theta}^{1} U_\alpha(\varphi, \theta) \exp\{-|\frac{x-\mu}{\sigma}|^{\frac{\alpha}{\alpha-1}} U_\alpha(\varphi, \theta)\} d\varphi, & x \neq \mu, \\ \frac{1}{\pi\sigma} \Gamma\left(1 + \frac{1}{\alpha}\right) \cos\left(\frac{1}{\alpha} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right)\right), & x = \mu, \end{cases}$$

where

$$U_\alpha(\varphi, \theta) = \left(\frac{\sin\left(\frac{\pi\alpha}{2}(\varphi + \theta)\right)}{\cos\frac{\pi\varphi}{2}} \right)^{\frac{\alpha}{1-\alpha}} \frac{\cos\left(\frac{\pi}{2}((\alpha-1)\varphi + \alpha\theta)\right)}{\cos\frac{\pi\varphi}{2}}$$

and

$$\theta = \arctan\left(\beta \tan\frac{\pi\alpha}{2}\right) \frac{2}{\pi\alpha} sign(x - \mu).$$

The problem of parameter estimation in stable modeling is hampered by the lack of known closed form of density functions for almost all stable distributions, hence many statistical methods depending on the explicit form of the probability density function cannot be applied. However, there are several numerical methods that have been found useful in practice: McCulloch method [20], method

of moments [26], regression methods [15, 16, 18]. Comparative studies [2, 24] confirm that the most accurate method of estimation is the Maximum Likelihood method. However, it is the most time consuming, but the implementation of parallel algorithms can allow us to get result in an adequate time even for a large amount of long data series [4].

Stable parameters can be estimated from the returns by maximizing the log-likelihood function

$$L(\Theta) = \sum_{i=1}^{n} \log p(X_i, \Theta).$$

To optimize the function we use Davidon-Fletcher-Powell quasi-Newton method [28]. We also verify two hypotheses: the first one is that our sample follows Gaussian distribution. The is that our sample follows the stable non-Gaussian distribution. Both hypotheses are examined by Anderson-Darling and Kolmogorov-Smirnov tests. The first test is more sensitive to the difference between empirical and theoretical distribution functions in far quantiles (tails), in contrast to the Kolmogorov-Smirnov test which is more sensitive to the difference in the central part of distribution.

The mixed-stable model was introduced by Belovas et al. [3] to cope with the problem of daily zero returns. Cumulative distribution function of mixed-stable random variable is

$$F(x) = \frac{n-k}{n} P(x, \Theta_{max}) + \frac{k}{n} \varepsilon(x),$$

where $P(x, \Theta)$ the the cumulative distribution function of the stable distribution, $\varepsilon(x)$ is the cumulative distribution function of the degenerate distribution

$$\varepsilon(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0, \end{cases}$$

the vector Θ_{max} is estimated with nonzero returns, and k is the number of zero returns in the given data set $X = \{X_1, X_2, \ldots, X_n\}$. The probability density function of mixed-stable random variable is

$$f(x) = \frac{n-k}{n} p(x, \Theta_{max}) + \frac{k}{n} \delta(x),$$

where $\delta(x)$ is Dirac delta function.

There arises a problem when we are trying to test the adequacy hypothesis for these models. Since we have a discontinuous distribution function, classic methods for continuous distributions (Kolmogorov-Smirnov, Anderson–Darling) do not work. So we have to choose a special goodness-of-fit test suitable for discontinuous distributions, ex. Koutrouvelis empirical characteristic function test. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having a common characteristic function $\varphi(t) = C(t) + iS(t)$ and let

$$\varphi_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itX_j}$$

be the empirical characteristic function. Denote by $\varphi_0(t) = C_0(t) + iS_0(t)$ a completely specified characteristic function. I. Koutrouvelis has proposed a goodness-of-fit test for the simple hypothesis $H_0 : \varphi(t) = \varphi_0(t)$ based on the quadratic form

$$Q_n^0 = 2n(\xi_n - \xi_0)^T \Omega_0^{-1}(\xi_n - \xi_0),$$

where ξ_l is the $2m$-dimensional vector

$$\xi_l^T = (C_l(t_1), \ldots, C_l(t_m), S_l(t_1), \ldots, S_l(t_m)),$$

for $l = 0, n$, and $\Omega_0/2n$ is the covariance matrix of ξ_n under H_0. The values $t_1, t_2, \ldots, t_n$ a suitably chosen in a region of t near zero, and are such that Ω_0 is non-singular, see [17, 19]. Another alternative is modified χ^2) [14].

3.2 Self-similarity, multifractality and Hurst exponent

Financial time series do often exhibit fractionality (characterized by Hurst exponent) or self-similarity [12, 30]. Continuous time process $Y = \{Y(t), t \in T\}$ is self-similar, with the self-similarity parameter H (Hurst index), if it satisfies the condition:

$$Y(t) = a^{-H} Y(at), \quad \forall t \in T,\ \forall a > 0,\ 0 \leq H < 1,$$

where the equality is in the sense of finite-dimensional distributions. This definition of a self-similar process given above can be generalized to that of multifractal processes. Methodology of self-similarity and multifractality analysis is fully described in Section 2.2 of Kabasinskas et al. [13].

Hurst exponent was estimated by R/S [9] and the ratio of variance of residuals method. Given a time-series $X_1, X_2, \ldots, X_n$, the rescaled range (R/S) statistic is defined as the ratio of the maximal range normalized to the standard deviation:

$$R/S(n) = \frac{\max_{1 \leq k \leq n} \sum_{i=1}^{k} (X_i - \overline{X}) - \min_{1 \leq k \leq n} \sum_{i=1}^{k} (X_i - \overline{X})}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2}}.$$

The expected value of R/S scales like cn^H as $n \to \infty$, where H is the Hurst exponent. To estimate the Hurst exponent, we plot $R/S(n)$ versus n in log-log axes. The slope of the regression line approximates the Hurst exponent.

The ratio of variance of residuals method is based on the variance of residuals method for estimating long-ranged dependence. This method, proposed by Peng et al. [25] and further studied by Taqqu et al. [31], includes following procedures.

First, we calculate the integrated series

$$Y_k = \sum_{i=1}^{k} (X_i - \overline{X})$$

The integrated series is divided into d sub-series of equal length m. In each sub-series, we fit a least-squares line to the partial sums within each block and

compute the sample variance of the residuals. We repeat this procedure for each of the blocks, and average the resulting sample variances. We should get a straight line with a slope of $2H$ if the result is plotted on a log-log plot versus m.

Estimates were calculated using SELFIS software, which is freeware and can be found on the web page http://www.cs.ucr.edu/~tkarag.

3.3 Modeling of stagnation intervals

Let us define the set of discrete states $\{X_i\}$, corresponding given time series $\{P_i\}$ following way: $X_i = 0$, if $P_{i+1} = P_i$ and $X_i = 1$, if $P_{i+1} \neq P_i$. A set of zeros between two units in $\{X_i\}$ we call a run. The first run is a set of zeros before the first unit and the last one after the last unit. The length of the run is equal to the number of zeros between two units. If there are no zeros between two units, then an empty set has zero length.

Theoretically, if states are independent (Bernoulli scheme), then the series of lengths of zero state runs should be distributed by the geometric law. Obviously, if the probability of zero is p, then the probability of k-length run (before the first unit) is $p_k = p^k(1-p),\ k \in \mathbb{N}_0$. However, the results of empirical tests do not corroborate this theoretic assumption. To determine the law, we will fit with χ^2 goodness-of-fit test the series distribution of lengths of zero state runs by geometric, generalized logarithmic, Poisson and generalized Poisson, Hurwitz zeta and generalized Hurwitz zeta, discrete stable laws. The statistic of the test is

$$\chi^2 = \sum_{i=1}^{m} \frac{(O_i - E_i)^2}{E_i},$$

here m is the number of cells for n states to divide, O_i - observed frequencies, E_i - expected theoretical frequency, asserted by the null hypothesis [5].

The probability mass function of the generalized logarithmic distribution is defined by

$$P(\xi = k) = \left(\sum_{i=0}^{\infty} \frac{p^i}{i+m} \right)^{-1} \frac{p^k}{k+m}, \qquad k \in \mathbb{N}_0,\ 0 < p < 1,\ m > 0.$$

The probability mass function of the generalized Poisson law is defined by

$$P(\xi = k) = \left(\sum_{i=0}^{\infty} \frac{\lambda^i}{\Gamma(i+\mu+1)} \right)^{-1} \frac{\lambda^k}{\Gamma(k+\mu+1)}, \qquad k \in \mathbb{N}_0,\ \lambda > 0,\ m > -1.$$

The probability mass function of Hurwitz distribution is defined by

$$P(\xi = k) = \left(\sum_{i=0}^{\infty} \frac{1}{(i+q)^s} \right)^{-1} \frac{1}{(k+q)^s}, \qquad k \in \mathbb{N}_0,\ q > 0,\ s > 1.$$

The probability mass function of the generalized Hurwitz distribution is defined by

$$P(\xi = k) = \left(\sum_{i=0}^{\infty} \frac{\lambda^i}{(i+q)^s} \right)^{-1} \frac{\lambda^k}{(k+q)^s}, \qquad k \in \mathbb{N}_0,\ q > 0,\ s > 1,\ 0 < \lambda < 1.$$

Discrete stable distribution with the probability mass function

$$P(\xi = k) = \frac{(-1)^k}{e^\lambda} \sum_{m=0}^{k} \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) \left(\begin{array}{c} \gamma j \\ k \end{array} \right) \frac{(-1)^j \lambda^m}{m!},\ k \in \mathbb{N}_0,\ \lambda > 0,\ \gamma \in (0, 1]$$

was introduced by Christoph and Schreiber [6]. Parameters of generalized logarithmic, generalized Poisson, Hurwitz zeta, generalized Hurwitz zeta and discrete stable laws are estimated by the maximal likelihood method numerically.

3.4 Model of stagnation intervals with dependent states

In order to examine the interdependencies between states further, we tried the Markov chain model. With the Hoel [10] criterion we test the order m, $m \in \mathbb{N}_0$, of the chain: the hypothesis $H_0^{m:m+1}$, that the series represents the m^{th}-order Markov chain, with the alternative $H_1^{m:m+1}$, that the series represents $(m{+}1)^{\text{th}}$-order Markov chain. The statistic

$$L = 2 \sum_{i \ldots l} n_{ij \ldots kl} \left(\log \frac{n_{ij \ldots kl}}{n_{ij \ldots k\cdot}} - \log \frac{n_{\cdot j \ldots kl}}{n_{\cdot j \ldots k\cdot}} \right)$$

is distributed by the χ^2 law with $s^{m-1}(s-1)^2$ degrees of freedom. Here s is the number of the Markov chain series states,$n_{ij \ldots kl}$ is the number of visits in state $ij \ldots kl$, symbol $\cdot$ indicates the summation index. Number of indexes in this equation depends on the order of the chain: if the order is m, then number of indexes is equal to $m{+}1$. To test if series is generated by the Bernullian scheme we have to test the hypothesis $H_0^{0:1}$. In this research five hypothesis were tested: $H_0^{0:1}$, $H_0^{1:2}$, $H_0^{2:3}$, $H_0^{3:4}$, $H_0^{4:5}$. Since the runs test rejects the randomness hypothesis (see e.g. [12, 13]) of the sequence of states, the probability of state depends on the position n in the sequence:

$$P(X_n = 1 | \ldots, X_{n-k-1} = 0, X_{n-k} = 1, \ldots, X_{n-1} = 1) = \frac{p_k}{1 - \sum_{j=0}^{k-1} p_j}, \quad (1)$$

where p_k are probabilities of a model law, $P(X_0 = 1) = p_0$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$. It should be noted that $P(X_n = 0 | \ldots) = 1 - P(X_n = 1 | \ldots)$, $n, k \in \mathbb{N}_0$.

With the probabilities of states and the distribution of nonzero returns we can generate sequences of stock returns, interchanging in the state sequence ones with stable r.v.

4 Empirical results

4.1 Stable and mixed-stable models

First of all we have estimated parameters Θ of the α-stable distribution for full series. The results are presented in Table 2.

Table 2: Estimates of parameters of α-stable distribution for full series

Data set	α	β	μ	σ
Adidas AG	1.409	0.1327	0.0001	0.0002
Deutsche Bank AG	1.479	-0.7460	-0.0002	0.0003
BASF SE	1.118	0.3844	0.0003	0.0001
BMW AG St	1.281	0.0949	0.0001	0.0002
Continental AG	1.787	0.7234	0.0001	0.0003
Deutsche Post AG	1.640	-0.4147	-0.0002	0.0002
Deutsche Telekom AG	1.587	-0.2994	-0.0001	0.0003
Bayer AG O.N.	1.153	0.7240	0.0004	0.0001
Fresenius Medical Care AG	1.654	0.3097	0.0001	0.0002
Deutsche Borse AG	1.238	-0.5806	-0.0002	0.0002
MAN SE St	1.267	0.3916	0.0003	0.0002
Henkel AG	1.441	-0.6664	-0.0002	0.0002
Infineon Technologies AG	1.171	0.0335	0.0000	0.0001
Linde AG	1.809	-0.2578	0.0000	0.0002
Merck KGaA	1.352	0.1420	0.0001	0.0001
RWE AG St	1.139	0.0923	0.0001	0.0003
Daimler AG	1.506	0.0607	0.0000	0.0004
SAP AG	1.177	-0.4520	-0.0002	0.0002
Siemens AG	1.169	0.7111	0.0008	0.0003
METRO AG St	1.207	-0.0115	0.0000	0.0001
ThyssenKrupp AG	1.364	-0.0874	-0.0001	0.0004
Volkswagen AG St	1.281	-0.0574	-0.0001	0.0001
Deutsche Postbank AG	1.345	0.8168	0.0002	0.0002
HYPO REAL ESTATE	1.323	0.2379	0.0001	0.0002
Commerzbank AG	1.436	0.7541	0.0002	0.0004
Deutsche Lufthansa AG	1.376	-0.2873	-0.0001	0.0001
Allianz SE	1.439	0.0474	0.0000	0.0003
Munchener Ruck AG	1.126	-0.3917	-0.0005	0.0001
TUI AG	1.620	0.4667	0.0001	0.0003

Anderson-Darling test rejected hypotheses of full series stability in all cases.

Since we have found a lot of zeros in our series (see Table 2, and Fig. 2) we proceed with the mixed-stable model parameters estimation (see Section 3.1). Distribution of zero returns in a series is given in Fig. 2. The results of parameter estimation for 29 DAX series are given in Table 3.

The results of goodness-of-fit test based on the empirical characteristic function are presented in Table 4. The results of Table 4 show that in 75.86% cases the goodness-of-fit hypothesis were not rejected for mixed-stable law.

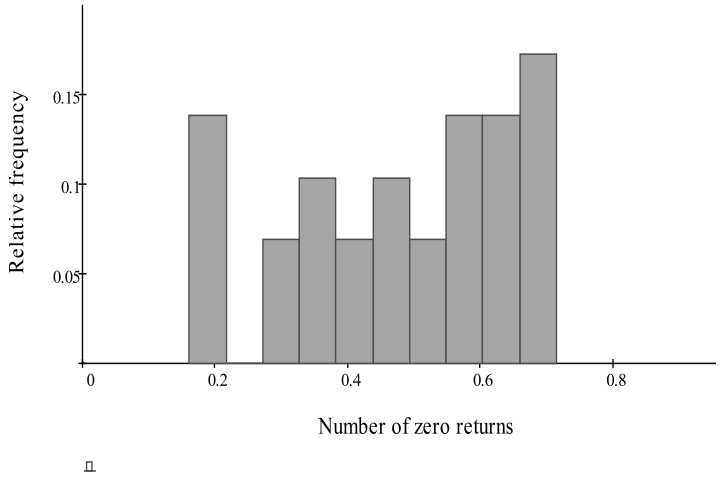


Figure 2: Distribution of zero returns.

Table 3: Estimates of parameters of mixed-stable distribution and and value of Anderson-Darling test statistics.

Data set	α	β	μ	σ	p	A-D
Adidas AG	1.8059	-0.2105	0.0000	0.0008	0.6382	3.54
Deutsche Bank AG	1.7252	0.5040	0.0001	0.0005	0.2134	15.97
BASF SE	1.7317	0.0802	0.0002	0.0005	0.3144	75.55
BMW AG St	1.8498	0.0114	0.0000	0.0006	0.5190	8.60
Continental AG	1.2659	0.7174	0.0004	0.0003	0.5160	375.26
Deutsche Post AG	1.8895	0.1019	0.0000	0.0007	0.6203	31.66
Deutsche Telekom AG	1.9716	-0.1529	0.0000	0.0007	0.5827	94.50
Bayer AG O.N.	1.7746	0.1342	0.0000	0.0005	0.3428	7.71
Fresenius Medical Care AG	1.8121	-0.1385	0.0000	0.0009	0.6542	3.62
Deutsche Borse AG	1.6818	-0.0214	0.0000	0.0006	0.4542	2.24
MAN SE St	1.4455	-0.7883	-0.0002	0.0008	0.4686	42.58
Henkel AG	1.1397	-0.5828	-0.0016	0.0006	0.6389	1290.29
Infineon Technologies AG	1.9904	0.8684	0.0000	0.0009	0.6853	61.19
Linde AG	1.7621	-0.1606	0.0000	0.0007	0.6007	2.01
Merck KGaA	1.7315	-0.1406	0.0000	0.0006	0.6114	2.56
RWE AG St	1.7206	0.0329	0.0000	0.0005	0.3085	3.33
Daimler AG	1.6322	0.6379	-0.0001	0.0005	0.2235	123.59
SAP AG	1.2294	-0.1224	0.0000	0.0006	0.3706	81.73
Siemens AG	1.8571	0.4400	0.0000	0.0004	0.2261	10.26
METRO AG St	1.3243	-0.7434	-0.0004	0.0004	0.5552	255.92
ThyssenKrupp AG	1.7265	-0.0046	0.0000	0.0006	0.4461	8.65
Volkswagen AG St	1.6523	0.0912	0.0000	0.0003	0.4359	2.84
Deutsche Postbank AG	1.7997	-0.2273	0.0000	0.0011	0.6791	1.82
HYPO REAL ESTATE	1.4384	0.6894	0.0002	0.0007	0.5523	15.87
Commerzbank AG	1.7772	-0.1563	0.0000	0.0007	0.4288	14.46
Deutsche Lufthansa AG	1.9294	0.1672	0.0000	0.0007	0.6216	29.65
Allianz SE	1.4425	0.6277	0.0002	0.0004	0.1879	39.64
Munchener Ruck AG	1.3572	-0.1744	-0.0001	0.0005	0.3863	25.53
TUI AG	1.7978	0.1681	0.0000	0.0008	0.5840	31.93

Table 4: Part of accepted hypothesis with given significance level, for mixed-stable and mixed-Gaussian distributions, by Koutrouvelis test.

Significance level	Mixed-stable	Mixed-Gaussian
0.01	62.07%	3.45%
0.05	75.86%	6.90%
0.1	75.86%	6.90%

4.2 Modeling of stagnation intervals: mixed stable model with dependent states

As mentioned above, theoretically (if states are independent) the series of zeros should be distributed by the binomial law and the lengths of these series should be distributed by the geometrical law, however, from Table 5 we can see that other laws fit our data (82.76% series) much better. It means that zero state series from our data are better described by the Hurwitz zeta distribution.

Table 5: Modelling of the distribution of runs.

Significance level	0.01	0.05	0.10
Hurwitz distribution	93.10%	82.76%	68.97%
Generalized Hurwitz distribution	82.76%	72.41%	65.52%
Generalized logarithmic distribution	68.97%	44.83%	34.48%
Discrete stable distribution	0	0	0
Poisson distribution	0	0	0
Generalized Poisson distribution	0	0	0
Geometrical distribution	0	0	0

This result allows us to assume that zero-unit states are not purely independent. The WaldWolfowitz runs test corroborates this assumption for almost all series from the given shares. The inner series dependence was tested by the Hoel [10] criterion on the order of the Markov chain. It has been concluded that there are one zero order series or Bernoulli scheme series. 82% of given series are higher than 4^{th}-order Markov chains with 0.05 significance level (see Table 6).

Table 6: Orders of Markov chains.

Order of Markov chain	Percent of the series
0	3.45
1	13.79
2	58.62
3	75.86
≥ 4	82.76

4.3 Self-similarity, multifractality and Hurst exponent

Having these results we may then proceed to analysis of self-similarity and multifractality. Financial series studies have shown that asset prices exhibit strong variability and burstiness on many time scales. This was the reason to introduce fractal models, which were able to capture the discovered scaling properties [2]. Self-similarity expresses the monofractal property, that is, fluctuations look statistically similar on many time scales. Long range dependence is revealed by the power. The degree of this slow decay is determined by the Hurst exponent. Multufractality expresses a more complex scaling behavior, which cannot be explained in a self-similar framework. It would be next logical step to try and analyze applicability of fractal models for high frequency financial data. As we can see (results of absolute moments method for full series and series without zeros are given in Table 7), this approach is validated by the empirical studies.

Table 7: Multifractality and self-similarity analysis: 0 indicates that series do not have the property, 1 indicated when it does

	Full series		Without zero returns	
Data set	Multifractal	Self-similar	Multifractal	Self-similar
Adidas AG	1	0	1	0
Deutsche Bank AG	1	1	1	1
BASF SE	1	0	1	0
BMW AG St	1	1	1	1
Continental AG	0	0	0	0
Deutsche Post AG	1	0	1	0
Deutsche Telekom AG	1	0	1	1
Bayer AG O.N.	1	0	1	0
Fresenius Medical Care AG	1	1	1	1
Deutsche Borse AG	1	1	1	0
MAN SE St	1	1	1	1
Henkel AG	1	0	1	1
Infineon Technologies AG	1	1	1	0
Linde AG	1	1	1	1
Merck KGaA	1	0	1	0
RWE AG St	1	0	1	0
Daimler AG	1	0	0	0
SAP AG	1	1	1	0
Siemens AG	1	1	1	1
METRO AG St	0	0	0	0
ThyssenKrupp AG	1	0	1	0
Volkswagen AG St	1	0	1	0
Deutsche Postbank AG	1	0	1	0
HYPO REAL ESTATE	1	0	1	0
Commerzbank AG	1	0	1	0
Deutsche Lufthansa AG	1	1	1	0
Allianz SE	1	1	1	0
Munchener Ruck AG	1	0	0	0
TUI AG	1	0	1	0

As we can see, 93% of full financial series and 90% series without zeros exibit multifractality. However, only 42% of full series are self-similar (see Table 7).

Table 8 presents results about the Hurst exponent of full series. The Hurst exponent was estimated by two methods: R/S and ratio of variance residuals

(RVR). The results are given with the corresponding correlation cofficent ρ.

Table 8: Hurst exponent estimates

	Full series				Without zero returns			
Data set	R/S	ρ	RVR	ρ	R/S	ρ	RVR	ρ
Adidas AG	0.457	0.9957	0.518	0.9763	0.479	0.9952	0.491	0.9826
Deutsche Bank AG	0.629	0.9942	0.632	0.9777	0.622	0.9975	0.617	0.9826
BASF SE	0.551	0.9988	0.575	0.9719	0.550	0.9996.	0.599	0.9827
BMW AG St	0.486	0.9934	0.437	0.9455	0.472	0.9988	0.445	0.9537
Continental AG	0.565	0.9981	0.636	0.9859	0.551	0.9996	0.600	0.9790
Deutsche Post AG	0.419	0.9394	0.477	0.9953	0.151	0.2870	0.432	0.9872
Deutsche Telekom AG	0.514	0.9953	0.475	0.9611	0.307	0.6897	0.467	0.9853
Bayer AG O.N.	0.552	0.9993	0.600	0.9729	0.529	0.9987	0.508	0.9969
Fresenius Medical Care AG	0.498	0.9976	0.470	0.9847	0.232	0.5719	0.447	0.9822
Deutsche Borse AG	0.596	0.9973	0.629	0.9901	0.623	0.9964	0.652	0.9895
MAN SE St	0.562	0.9995	0.605	0.9874	0.563	0.9996	0.610	0.9875
Henkel AG	0.496	0.9938	0.498	0.9841	0.175	0.3555	0.464	0.9776
Infineon Technologies AG	0.484	0.9879	0.504	0.9807	0.238	0.5889	0.441	0.9860
Linde AG	0.380	0.9125	0.464	0.9858	0.158	0.3197	0.443	0.9821
Merck KGaA	0.531	0.9991	0.516	0.9914	0.525	0.9984	0.518	0.9896
RWE AG St	0.492	0.9981	0.413	0.9993	0.476	0.9974	0.408	0.9988
Daimler AG	0.555	0.9951	0.553	0.9494	0.377	0.7981	0.552	0.9633
SAP AG	0.333	0.8382	0.431	0.9636	0.431	0.9976	0.298	0.9750
Siemens AG	0.575	0.9990	0.569	0.9958	0.355	0.7389	0.529	0.9970
METRO AG St	0.504	0.9960	0.469	0.9882	0.477	0.9903	0.412	0.9908
ThyssenKrupp AG	0.395	0.8612	0.544	0.9853	0.302	0.6278	0.539	0.9921
Volkswagen AG St	0.496	0.9953	0.488	0.9854	0.480	0.9976	0.462	0.9733
Deutsche Postbank AG	0.547	0.9948	0.574	0.9952	0.552	0.9976	0.461	0.9901
HYPO REAL ESTATE	0.527	0.9993	0.529	0.9873	0.280	0.6117	0.460	0.9906
Commerzbank AG	0.548	0.9950	0.555	0.9756	0.258	0.5031	0.518	0.9639
Deutsche Lufthansa AG	0.537	0.9987	0.529	0.9917	0.276	0.6032	0.524	0.9934
Allianz SE	0.608	0.9965	0.545	0.9837	0.593	0.9992	0.539	0.9947
Munchener Ruck AG	0.566	0.9984	0.557	0.9761	0.568	0.9991	0.488	0.9923
TUI AG	0.434	0.9456	0.524	0.9924	0.232	0.4889	0.517	0.9825

5 Conclusions

- As we can see from Table 1 empirical data contain many repeating values (in average 47% of data) and exhibit significant kurtosis. Classical α-stable models in this case are not applicable, the fact is reflected by Table 2.
- Mixed-stable models can be used to describe intraday high frequency data, see Table 3.
- The returns series exhibit the dependences between states (the price change indicators), which is corroborated by Wald-Wolfowitz runs test and Hoel test. Therefore the intraday (high frequency) series should be modeled by mixed-stable law with dependent states. The runs (sets of zero returns in a series) are best described by the Hurwitz zeta distribution (see Table 5). The conditional probability of the price change in the n^{th} time tick can be calculated by formula (1).

- Analysis of series behavior showed that less than a half of them are self-similar, however 93% are multifractal. The range of the Hurst index, calculated by RVR method, is [0.413; 0.636]. Since the Hurst index in some cases is less that 0.5 the series implies antipersistency.

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NEW REPRESENTATIONS FOR THE EULER-MASCHERONI CONSTANT AND INEQUALITIES FOR THE GENERALIZED-EULER-CONSTANT FUNCTION

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Abstract. (i) We establish new representations for the Euler-Mascheroni constant γ in terms of the gamma and psi functions. (ii) A class of two-sided inequalities for the generalized-Euler-constant function are presented. These inequalities provide the lower and upper bounds on the error of approximating the generalized-Euler-constant function by means of a power series.

1. Introduction

The Euler constant (or, more popularly, the Euler-Mascheroni constant) γ is defined by the limit

$$\gamma = \lim_{n\to\infty} D_n = 0.57721566\ldots, \tag{1}$$

where

$$D_n = \sum_{j=1}^{n} \frac{1}{j} - \ln n, \qquad n \in \mathbb{N} := \{1, 2, 3, \ldots\}. \tag{2}$$

The constant γ is closely related to the celebrated gamma function $\Gamma(x)$ by means of the familiar Weierstrass formula [1, p. 255, Equation (6.1.3)] (see also [32, Chapter 1, Section 1.1]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right)e^{-z/n}\right] \quad (|z| < \infty).$$

The logarithmic derivative of the gamma function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t$$

is known as the psi (or digamma) function.

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Sondow and Hadjicostas [31] introduced and studied the generalized-Euler-constant function $\gamma(z)$, defined by

$$\gamma(z) = \sum_{n=1}^{\infty} z^{n-1} \left(\frac{1}{n} - \ln \frac{n+1}{n} \right), \tag{3}$$

where the series converges when $|z| \leq 1$. Pilehrood and Pilehrood [23] considered the function $z\gamma(z)$ ($|z| \leq 1$). The function $\gamma(z)$ generalizes both Euler's constant $\gamma(1)$ and the alternating Euler constant $\ln \frac{4}{\pi} = \gamma(-1)$ [28, 29], where γ is defined by (1). The limit definition (1) of Euler's constant is equivalent to the series formula

$$\gamma = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln \frac{n+1}{n} \right) \tag{4}$$

(see [28]). The corresponding alternating series gives the alternating Euler constant [28, 29] (see also [15])

$$\gamma(-1) = \ln \frac{4}{\pi} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \ln \frac{n+1}{n} \right) = 0.24156447\ldots. \tag{5}$$

It is known (see [31]) that

$$\gamma(0) = 1 - \ln 2 \quad \text{and} \quad \gamma\left(\frac{1}{2}\right) = 2 \ln \frac{2}{\sigma}, \tag{6}$$

where

$$\sigma = \sqrt{1\sqrt{2\sqrt{3\cdots}}} = 1^{1/2} 2^{1/4} 3^{1/8} \cdots = \prod_{n=1}^{\infty} n^{1/2^n} = 1.66168794\ldots \tag{7}$$

is the Somos's quadratic recurrence constants [27] (see also [38] and [14, p. 446]).

An alternative method in estimating the generalized-Euler-constant function $\gamma(z)$ was proposed by Lampret [18] who exploited the Euler-Maclaurin (Boole/Hermite) summation formula. Mortici [21] provided some estimates about the Somos's quadratic recurrence constant.

This paper is organized as follows. In Section 2 we establish new representations for the Euler-Mascheroni constant γ in terms of the gamma and psi functions. In Section 3, A class of two-sided inequalities for the generalized-Euler-constant function are presented.

2. New representations for the Euler-Mascheroni constant

Several bounds for $D_n - \gamma$ have been given in the literature [2, 3, 24, 33, 34, 35, 39]. For example, the following bounds for $D_n - \gamma$ were established in [24, 39]:

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}, \qquad n \in \mathbb{N}. \tag{8}$$

Alzer [2] obtained sharp form of the inequality (8). The convergence of the sequence D_n to γ is very slow. Some quicker approximations to the Euler-Mascheroni constant were established in [4, 7, 6, 8, 5, 12, 16, 19, 20, 22, 25, 26, 36, 37]. For example, DeTemple [12] studied in 1993 a modified sequence which converges faster and established the following inequality:

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \tag{9}$$

where

$$R_n = \sum_{j=1}^{n} \frac{1}{j} - \ln\left(n+\frac{1}{2}\right). \tag{10}$$

Recently, Chen [7] obtained sharp form of the inequalities (9).

Let D_n and R_n be defined by (2) and (10), respectively. Considering the sums $\sum_{k=1}^{n} D_k$ and $\sum_{k=1}^{n} R_k$, we find new analytical representations for the Euler-Mascheroni constant γ in terms of the gamma and psi functions.

Theorem 1. *Let n be a positive integer. Then we have*

$$\gamma = \frac{1}{n+1}\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{1}{j} - \ln k\right) - \psi(n+2) + 1 + \frac{1}{n+1}\ln\Gamma(n+1) \tag{11}$$

and

$$\begin{aligned}\gamma = \frac{1}{n+1}\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{1}{j} - \ln\left(k+\frac{1}{2}\right)\right) - \psi(n+2) + 1 \\ + \frac{1}{n+1}\ln\Gamma\left(n+\frac{3}{2}\right) - \frac{\ln(\sqrt{\pi}/2)}{n+1}.\end{aligned} \tag{12}$$

Proof. It is easily observed that the formula (11) is equivalent to the following result:

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{1}{j} - \ln k\right) = (n+1)\big(\psi(n+2)+\gamma-1\big) - \ln\Gamma(n+1), \qquad n\in\mathbb{N}. \tag{13}$$

We now prove the representation formula (13) by using the principle of mathematical induction. In our proof of the representation formula (13), we require the integer values [1, p. 258, Equation (6.3.2)]:

$$\psi(1) = -\gamma \qquad \text{and} \qquad \psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}, \tag{14}$$

the recurrence formulas [1, p. 258, Equation (6.3.5)]:

$$\Gamma(z+1) = z\Gamma(z) \qquad \text{and} \qquad \psi(z+1) = \psi(z) + \frac{1}{z}. \tag{15}$$

For $n=1$, we find from (13) that

$$\sum_{k=1}^{1}\left(\sum_{j=1}^{k}\frac{1}{j}-\ln k\right)=2\big(\psi(3)+\gamma-1\big)-\ln\Gamma(2)=1,$$

which shows that the formula (13) holds true for $n=1$. We assume now that the formula (13) holds true for a fixed positive integer n. Then, for $n\mapsto n+1$ in (13), we have

$$\begin{aligned}\sum_{k=1}^{n+1}\left(\sum_{j=1}^{k}\frac{1}{j}-\ln k\right)&=\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{1}{j}-\ln k\right)+\sum_{j=1}^{n+1}\frac{1}{j}-\ln(n+1)\\&=(n+1)\big(\psi(n+2)+\gamma-1\big)-\ln\Gamma(n+1)\\&\quad+\psi(n+2)+\gamma-\ln(n+1)\\&=(n+2)\big(\psi(n+3)+\gamma-1\big)-\ln\Gamma(n+2).\end{aligned}$$

The proof of the formula (13) is thus completed by means of the principle of mathematical induction on n.

It is easily observed that the formula (12) is equivalent to the following result:

$$\begin{aligned}\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{1}{j}-\ln\left(k+\frac{1}{2}\right)\right)&=(n+1)\big(\psi(n+2)+\gamma-1\big)\\&\quad-\ln\Gamma\left(n+\frac{3}{2}\right)+\ln(\sqrt{\pi}/2),\qquad n\in\mathbb{N}.\end{aligned}\tag{16}$$

We now prove the representation formula (16) by using the principle of mathematical induction. For $n=1$, we find from (16) that

$$\begin{aligned}\sum_{k=1}^{1}\left(\sum_{j=1}^{k}\frac{1}{j}-\ln\left(k+\frac{1}{2}\right)\right)&=2\big(\psi(3)+\gamma-1\big)-\ln\Gamma\left(\frac{5}{2}\right)+\ln(\sqrt{\pi}/2)\\&=1+\ln 2-\ln 3.\end{aligned}$$

which shows that the formula (16) holds true for $n=1$. We assume now that the formula (16) holds true for a fixed positive integer n. Then, for $n\mapsto n+1$ in (16), we have

$$\begin{aligned}\sum_{k=1}^{n+1}\left(\sum_{j=1}^{k}\frac{1}{j}-\ln\left(k+\frac{1}{2}\right)\right)&=\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{1}{j}-\ln\left(k+\frac{1}{2}\right)\right)+\sum_{j=1}^{n+1}\frac{1}{j}-\ln\left(n+\frac{3}{2}\right)\\&=(n+1)\big(\psi(n+2)+\gamma-1\big)-\ln\Gamma\left(n+\frac{3}{2}\right)+\ln(\sqrt{\pi}/2)\\&\quad+\psi(n+2)+\gamma-\ln\left(n+\frac{3}{2}\right)\\&=(n+2)\big(\psi(n+3)+\gamma-1\big)-\ln\Gamma\left(n+\frac{5}{2}\right)+\ln(\sqrt{\pi}/2).\end{aligned}$$

The proof of the formula (16) is thus completed by means of the principle of mathematical induction on n. □

Remark 1. Choi [10] summarized some known representations for the Euler-Mascheroni constant γ. For a rather impressive collection of various classes of integral representations for the Euler-Mascheroni constant γ, the interested reader may be referred to a recent paper by Choi and Srivastava [11].

Very recent, Chen and Srivastava [9] presented the following representations for the Euler-Mascheroni constant γ:

$$\gamma = -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j} + \ln 2 - 1 + \left(n+\frac{1}{2}\right)\psi\left(n+\frac{1}{2}\right) - \left(n+\frac{3}{2}\right)\psi(n) + (2\ln 2)n - \frac{3}{2n}. \tag{17}$$

3. A class of two-sided inequalities for the generalized-Euler-constant function

Theorem 2 presents a class of two-sided inequalities for the generalized-Euler-constant function. These inequalities provide the lower and upper bounds on the error of approximating the generalized-Euler-constant function by means of a power series.

Theorem 2. *Let $|z| \leq 1$ and $n = 1, 2, \ldots$. Then*

$$\frac{1}{n+1}\sum_{k=1}^{\infty}\frac{1}{(k+\theta(\infty))^{n+1}}z^{k-1} < (-1)^{n+1}\left[\gamma(z) - \sum_{i=1}^{n-1}\frac{(-1)^{i+1}}{i+1}\sum_{k=1}^{\infty}\frac{1}{k^{i+1}}z^{k-1}\right] < \frac{1}{n+1}\sum_{k=1}^{\infty}\frac{1}{(k+\theta(1))^{n+1}}z^{k-1}, \tag{18}$$

where

$$\theta(\infty) = \frac{1}{n+2}$$

and

$$\theta(1) = \left\{(-1)^n(n+1)\left[\ln 2 - \sum_{i=0}^{n-1}\frac{(-1)^i}{i+1}\right]\right\}^{-1/(n+1)} - 1.$$

Proof. By Taylor's theorem, there exists a $\theta = \theta(k)$ such that $0 < \theta(k) < 1$ and

$$\ln\left(1+\frac{1}{k}\right) = \sum_{i=0}^{n-1}\frac{(-1)^i}{(i+1)k^{i+1}} + \frac{(-1)^n}{(n+1)(k+\theta(k))^{n+1}}\ . \tag{19}$$

Upon substituting from (19) into (3), we obtain

$$\gamma(z) = \sum_{k=1}^{\infty} z^{k-1} \left[\sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{(i+1)k^{i+1}} + \frac{(-1)^{n+1}}{(n+1)(k+\theta(k))^{n+1}} \right]$$
$$= \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i+1} \sum_{k=1}^{\infty} \frac{1}{k^{i+1}} z^{k-1} + \frac{(-1)^{n+1}}{n+1} \sum_{k=1}^{\infty} \frac{1}{(k+\theta(k))^{n+1}} z^{k-1} .$$

which leads us to

$$(-1)^{n+1} \left[\gamma(z) - \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i+1} \sum_{k=1}^{\infty} \frac{1}{k^{i+1}} z^{k-1} \right]$$
$$= \frac{1}{n+1} \sum_{k=1}^{\infty} \frac{1}{(k+\theta(k))^{n+1}} z^{k-1} \qquad (|z| \le 1) . \tag{20}$$

It can be shown that $\theta(k)$ is strictly increasing in k. By letting $t = k$ in (19), we get

$$\theta(t) = [(n+1)g(t)]^{-1/(n+1)} - t , \tag{21}$$

where

$$g(t) = (-1)^n \left[\ln\left(1 + \frac{1}{t}\right) - \sum_{i=0}^{n-1} \frac{(-1)^i}{(i+1)t^{i+1}} \right] .$$

Equivalently, we need to show that $\theta(t)$ increases in t, that is, that

$$\frac{d\theta(t)}{dt} = \frac{1}{(1+t)t^{n+1}} [(n+1)g(t)]^{-(n+2)/(n+1)} - 1 > 0 . \tag{22}$$

However, since $g(t) > 0$, (22) is equivalent to the following inequality:

$$h(t) = g(t) - \frac{1}{(n+1)(1+t)^{(n+1)/(n+2)} t^{(n+1)^2/(n+2)}} < 0 ,$$

which follows from the observations that $h(\infty) = 0$ and

$$h'(t) = (1+t)^{-\frac{2n+3}{n+2}} t^{-\frac{n^2+3n+3}{n+2}}$$
$$\left[\left(\frac{1}{n+2}\right) t + \left(\frac{n+1}{n+2}\right)(1+t) - t^{\frac{1}{n+2}} (1+t)^{\frac{n+1}{n+2}} \right] > 0 \qquad (t > 0) ,$$

by using the weighted arithmetic-geometric mean inequality. We thus find from (20) that

$$\frac{1}{n+1} \sum_{k=1}^{\infty} \frac{1}{(k+\theta(\infty))^{n+1}} z^{k-1} < (-1)^{n+1} \left[\gamma(z) - \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i+1} \sum_{k=1}^{\infty} \frac{1}{k^{i+1}} z^{k-1} \right]$$
$$< \frac{1}{n+1} \sum_{k=1}^{\infty} \frac{1}{(k+\theta(1))^{n+1}} z^{k-1} . \tag{23}$$

Next, by applying (21), we get

$$\theta(1) = \left\{ (-1)^n (n+1) \left[\ln 2 - \sum_{i=0}^{n-1} \frac{(-1)^i}{i+1} \right] \right\}^{-1/(n+1)} - 1$$

and

$$\theta(t) = \left[\frac{1}{t^{n+1}} - \frac{n+1}{n+2}\frac{1}{t^{n+2}} + O(t^{-n-3})\right]^{-1/(n+1)} - t$$
$$= \frac{1}{n+2} + O(t^{-1}) \quad \text{as} \quad t \longrightarrow \infty \ ,$$

so that

$$\theta(\infty) = \lim_{k\to\infty} \theta(k) = \frac{1}{n+2} \ .$$

By combining each of these observations with (23), we complete the proof of theorem. □

Remark 2. Clearly, we have for $|z| \leq 1$,

$$\frac{1}{n+1}\sum_{k=1}^{\infty} \frac{1}{(k+\theta(k))^{n+1}} z^{k-1} \longrightarrow 0, \qquad n \longrightarrow \infty. \tag{24}$$

Proceeding to the limit when $n \to \infty$ in (20), we obtain the following series expansion of the generalized-Euler-constant function $\gamma(z)$:

$$\gamma(z) = \sum_{n=2}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^n}{n}\frac{1}{k^n} z^{k-1} \ . \tag{25}$$

In particular, taking $z = 1$ in (25), we obtain the following well-known (rather classical) series expansion of Euler's constant (see, for example, [13, p.45, Equation 1.17 (3)], [17, p.355, Entry (54.2.3)] and [32, p.161, Equation 3.4 (23)]):

$$\gamma = \sum_{n=2}^{\infty} \frac{(-1)^n \zeta(n)}{n} \ , \tag{26}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1$$

is the Riemann zeta function.

In fact, (25) can be written as [31, Theorem 1]:

$$z\gamma(z) = \sum_{k=2}^{\infty} (-1)^k \frac{\mathrm{Li}_k(z)}{k} \ , \tag{27}$$

where $\mathrm{Li}_k(z)$ ($|z| \leq 1$) denotes the polylogarithm (see [15, 30]), defined for $k = 2, 3, \ldots$ by the convergent series

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \ . \tag{28}$$

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CAUCHY-JENSEN FUNCTIONAL INEQUALITY IN BANACH SPACES AND NON-ARCHIMEDEAN BANACH SPACES *

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the following Cauchy-Jensen functional inequality

$$\Big\| \sum_{i=1}^{l} f(x_i) + m \sum_{j=1}^{n} f(y_j) \Big\| \quad \leq \quad \Big\| mf\Big(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{i=1}^{n} y_j \Big) \Big\|,$$

in the class of mappings from normed spaces to Banach spaces and non-Archimedean Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [21] concerning the stability of group homomorphisms. We are given a group G_1 and a metric group G_2 with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$? In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it.

First of all, D. H. Hyers [8] considered the case of approximately additive mappings between Banach spaces and proved the following result. The method which was provided by Hyers, and which produces the additive mapping h, was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' Theorem was generalized by T. Aoki [1] and D.G. Bourgin [3] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [17] also provided a generalization of Hyers Theorem for linear mappings which allows the Cauchy difference to be unbounded like this $\|x\|^p + \|y\|^p$. A generalized result of Th.M. Rassias' theorem was obtained by P. Găvruta in [6] and S. Jung in [9]. In 1990, Th.M. Rassias [18] during the 27th International Symposium on Functional Equations

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asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [5] following the same approach as in [17], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [5], as well as by Th.M. Rassias and P. Šemrl [19], that one cannot prove a Th.M. Rassias type theorem when $p = 1$. The counterexamples of Z. Gajda [5], as well as of Th.M. Rassias and P. Šemrl [19], have stimulated several mathematicians to invent new approximately additive or approximately linear mappings. In particular, J.M. Rassias [15, 16] proved a similar stability theorem in which he replaced the unbounded Cauchy difference by this factor $\|x\|^p\|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. In addition, in the paper [4] the authors proves that if a mapping f satisfies the following functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \left\|kf\left(\frac{x+y+z}{k}\right)\right\|, \qquad k \geqq 3,\ |3| > |k|,$$

in non-Archimedean Banach spaces then f is additive, and they prove the generalized Hyers-Ulam stability of the functional inequality in non-Archimedean Banach spaces. The stability problems of several functional inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem [10, 11, 13, 14].

In this paper, we generalize the functional inequality to the following generalized Cauchy-Jensen functional inequality

$$\left\|\sum_{i=1}^{l} f(x_i) + m\sum_{j=1}^{n} f(y_j)\right\| \ \leq \ \left\|mf\left(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{i=1}^{n} y_j\right)\right\|, \tag{1.1}$$

where $l \geq 2, m \geq 1, n \geq 0$ are integers and $\sum_{i=1}^{0} E(i) := 0$ by notational convenience. Now, it is easy to see that if a mapping f satisfies the generalized Cauchy-Jensen inequality (1.1), then f is additive. In this work, we are going to improve the theorems given in the paper [4] without using the oddness of approximate additive functions concerning the functional inequality (1.1) for a more general case. Moreover, we obtain stability results of the functional inequality (1.1) in non-Archimedean Banach spaces in the last section.

2. Stability of (1.1) in Banach spaces

In this section, let X be a normed space, Y a Banach space and let $l \geq 2, m \geq 1, n \geq 1$ be integers.

Theorem 2.1. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\left\|\sum_{i=1}^{l} f(x_i) + m\sum_{j=1}^{n} f(y_j)\right\| \ \leq \ \left\|mf\left(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j\right)\right\| + \varphi(x_1, \cdots, x_l, y_1, \cdots, y_n) \tag{2.1}$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$ *and there exists a constant* L *with* $0 < L < 1$ *for which the perturbing function* $\varphi : X^{l+n} \to \mathbb{R}^+$ *satisfies*

$$\varphi\Big(\frac{l}{mn}(x_1, \cdots, x_l, y_1, \cdots, y_n)\Big) \leq L \cdot \frac{l}{mn}\varphi(x_1, \cdots, x_l, y_1, \cdots, y_n) \tag{2.2}$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$. *Then there exists a unique additive mapping* $h_1 : X \to Y$, *defined as* $h_1(x) = \lim_{k\to\infty}(\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in X)$ *such that*

$$\|f(x) - h_1(x)\| \leq \frac{1}{1-L}\Psi(x) \tag{2.3}$$

for all $x \in X$, *where*

$$\begin{aligned}\Psi(x) := \frac{1}{K}\varphi(\underbrace{x, -x, \cdots, x, -x}_{2\lceil\frac{l}{2}\rceil}, 0, \cdots, 0, \underbrace{x, -x, \cdots, x, -x}_{2\lceil\frac{n}{2}\rceil}) \\ + \frac{1}{l}\varphi(\underbrace{-x, \cdots, -x}_{l}, \underbrace{\frac{l}{mn}x, \cdots, \frac{l}{mn}x}_{n}), (x \in X),\end{aligned}$$

and $K := \lceil\frac{l}{2}\rceil + m\lceil\frac{n}{2}\rceil$ *and* $\lceil\cdot\rceil$ *denotes Gaussian notation.*

Proof. Putting

$$(x_1, \cdots, x_l, y_1, \cdots, y_n) := (\underbrace{x, -x, \cdots, x, -x}_{2\lceil\frac{l}{2}\rceil}, 0, \cdots, 0, \underbrace{x, -x, \cdots, x, -x}_{2\lceil\frac{n}{2}\rceil})$$

in (2.1), we have an approximate odd condition

$$\begin{aligned}&\|f(x) + f(-x)\| \\ &\leq \frac{1}{K}\varphi(\underbrace{x, -x, \cdots, x, -x}_{2\lceil\frac{l}{2}\rceil}, 0, \cdots, 0, \underbrace{x, -x, \cdots, x, -x}_{2\lceil\frac{n}{2}\rceil})\end{aligned} \tag{2.4}$$

for all $x \in X$, where $K := \lceil\frac{l}{2}\rceil + m\lceil\frac{n}{2}\rceil$ and $\lceil\cdot\rceil$ denotes Gaussian notation. Replacing

$$(x_1, \cdots, x_l, y_1, \cdots, y_n) := (\underbrace{-x, \cdots, -x}_{l}, \underbrace{\frac{l}{mn}x, \cdots, \frac{l}{mn}x}_{n})$$

in (2.1), we lead to

$$\Big\|lf(-x) + mnf(\frac{l}{mn}x)\Big\| \leq \varphi(\underbrace{-x, \cdots, -x}_{l}, \underbrace{\frac{l}{mn}x, \cdots, \frac{l}{mn}x}_{n}) \tag{2.5}$$

for all $x \in X$. Associating (2.4) with (2.5) yields

$$\|f(x) - \frac{mn}{l}f(\frac{l}{mn}x)\| \leq \Psi(x) \tag{2.6}$$

for all $x \in X$. Thus, it follows from (2.6) that for all nonnegative integers k and j with $j > k \geq 0$ and $x \in X$

$$\begin{aligned}
&\left\|(\frac{mn}{l})^k f\Big((\frac{l}{mn})^k x\Big) - (\frac{mn}{l})^{k+j} f\Big((\frac{l}{mn})^{k+j} x\Big)\right\| \\
&\leq \sum_{i=k}^{k+j-1} \left\|(\frac{mn}{l})^i f\Big((\frac{l}{mn})^i x\Big) - (\frac{mn}{l})^{i+1} f\Big((\frac{l}{mn})^{i+1} x\Big)\right\| \\
&\leq \sum_{i=k}^{k+j-1} (\frac{mn}{l})^i \Psi\Big((\frac{l}{mn})^i x\Big) \leq \sum_{i=k}^{k+j-1} L^i \Psi(x),
\end{aligned}$$

which tends to zero as $k \to \infty$. Hence the sequence $\left\{(\frac{mn}{l})^k f\Big((\frac{l}{mn})^k x\Big)\right\}$ is Cauchy for all $x \in X$, and so we can define a function $h_1 : X \to Y$ by

$$h_1(x) = \lim_{k\to\infty} (\frac{mn}{l})^k f\Big((\frac{l}{mn})^k x\Big), \quad x \in X.$$

Moreover, letting $k = 0$ and $j \to \infty$ in the last inequality yields

$$\|f(x) - h_1(x)\| \leq \frac{1}{1-L}\Psi(x) \tag{2.7}$$

for all $x \in X$, which yields the estimation (2.3).

Next, let $h_1{}' : G \to Y$ be another additive mapping satisfying the inequality (2.7). Then it is obvious that $h_1'((\frac{l}{mn})^k x) = (\frac{mn}{l})^k h_1'(x)$ and $h_1((\frac{l}{mn})^k x) = (\frac{mn}{l})^k h_1(x)$ for all $k \in \mathbb{N}$ and all $x \in X$. Thus, we have

$$\begin{aligned}
\| \quad & h_1(x) - h_1{}'(x)\| = (\frac{mn}{l})^k \left\|h_1\Big((\frac{l}{mn})^k x\Big) - h_1{}'\Big((\frac{l}{mn})^k x\Big)\right\| \\
\leq \quad & (\frac{mn}{l})^k \left\{\left\|h_1\Big((\frac{l}{mn})^k x\Big) - f\Big((\frac{l}{mn})^k x\Big)\right\| + \left\|f\Big((\frac{l}{mn})^k x\Big) - h_1{}'\Big((\frac{l}{mn})^k x\Big)\right\|\right\} \\
\leq \quad & \frac{2}{1-L}(\frac{mn}{l})^k \Psi((\frac{l}{mn})^k x) \leq \frac{2}{1-L} L^k \Psi(x)
\end{aligned}$$

for all $k \in \mathbb{N}$ and all $x \in X$. Taking the limit as $k \to \infty$, we lead to the uniqueness of the mapping h_1 near f satisfying the inequality (2.7).

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&(\frac{mn}{l})^k \left\|\sum_{i=1}^{l} f\Big((\frac{l}{mn})^k x_i\Big) + m\sum_{j=1}^{n} f\Big((\frac{l}{mn})^k y_j\Big)\right\| \\
&\quad \leq (\frac{mn}{l})^k \left\|mf\Big(\frac{1}{m}\sum_{i=1}^{l}(\frac{l}{mn})^k x_i + \sum_{j=1}^{n}(\frac{l}{mn})^k y_j\Big)\right\| \\
&\quad\quad + L^k \varphi(x_1, \cdots, x_l, y_1, \cdots, y_n)
\end{aligned}$$

for all $k \in \mathbb{N}$ and all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$. Taking $k \to \infty$ in the last relation, we see that

$$\left\|\sum_{i=1}^{l} h_1(x_i) + m\sum_{j=1}^{n} h_1(y_j)\right\| \quad \leq \quad \left\|mh_1\Big(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j\Big)\right\|$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$. This implies that the mapping h_1 is additive. This completes the proof. □

Remark 2.2. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.1) for which the perturbing function $\varphi : X^{l+n} \to \mathbb{R}^+$ satisfies

$$\sum_{i=0}^{\infty} (\frac{mn}{l})^i \varphi\Big((\frac{l}{mn})^i (x_1, \cdots, x_l, y_1, \cdots, y_n)\Big) < \infty$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$ instead of the condition (2.2). Then it follows from the similar argument to Theorem 2.1 that there exists a unique additive mapping $h_1 : X \to Y$, defined as $h_1(x) = \lim_{k\to\infty} (\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in X)$ such that

$$\|f(x) - h_1(x)\| \leq \sum_{i=0}^{\infty} (\frac{mn}{l})^i \Psi\Big((\frac{l}{mn})^i x\Big)$$

for all $x \in X$, where Ψ is defined as in Theorem 2.1.

Theorem 2.3. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.1) and there exists a constant L with $0 < L < 1$ for which the perturbing function $\varphi : X^{l+n} \to \mathbb{R}^+$ satisfies*

$$\varphi\Big(\frac{mn}{l}(x_1, \cdots, x_l, y_1, \cdots, y_n)\Big) \leq L \cdot \frac{mn}{l} \varphi(x_1, \cdots, x_l, y_1, \cdots, y_n) \tag{2.8}$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$. Then there exists a unique additive mapping $h_2 : X \to Y$, defined as $h_2(x) = \lim_{k\to\infty} (\frac{l}{mn})^k f((\frac{mn}{l})^k x), (x \in X)$ such that

$$\|f(x) - h_2(x)\| \leq \frac{L}{1-L} \Psi(x) \tag{2.9}$$

for all $x \in X$, where Ψ is given as in Theorem 2.1.

Proof. It follows from the inequality (2.6) that

$$\begin{aligned} &\Big\|(\frac{l}{mn})^k f\Big((\frac{mn}{l})^k x\Big) - (\frac{l}{mn})^{k+j} f\Big((\frac{mn}{l})^{k+j} x\Big)\Big\| \\ &\leq \sum_{i=k}^{k+j-1} (\frac{l}{mn})^{i+1} \Psi\Big((\frac{mn}{l})^{i+1} x\Big) \leq \sum_{i=k}^{k+j-1} L^{i+1} \Psi(x), \end{aligned}$$

which tends to zero as $k \to \infty$.

The remaining proof is similar to the corresponding proof of Theorem 2.1. This completes the proof. □

Remark 2.4. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.1) for which the perturbing function $\varphi : X^{l+n} \to \mathbb{R}^+$ satisfies

$$\sum_{i=0}^{\infty} (\frac{l}{mn})^i \varphi\Big((\frac{mn}{l})^i (x_1, \cdots, x_l, y_1, \cdots, y_n)\Big) < \infty$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$ instead of the condition (2.8). Then it follows from the similar argument to Theorem 2.3 that there exists a unique additive mapping $h_2 : X \to Y$, defined as $h_2(x) = \lim_{k\to\infty}(\frac{l}{mn})^k f((\frac{mn}{l})^k x), (x \in X)$ such that

$$\|f(x) - h_2(x)\| \leq \sum_{i=0}^{\infty}(\frac{l}{mn})^{i+1}\Psi\Big((\frac{mn}{l})^{i+1}x\Big)$$

for all $x \in X$, where Ψ is defined as in Theorem 2.1.

Corollary 2.5. *Let $0 < p \neq 1, l \neq mn$ and $\theta > 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$\Big\|\sum_{i=1}^{l} f(x_i) + m\sum_{j=1}^{n} f(y_j)\Big\| \leq \Big\|mf\Big(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j\Big)\Big\| + \theta\Big(\sum_{i=1}^{l}\|x_i\|^p + \sum_{j=1}^{n}\|y_j\|^p\Big)$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$, then there exists a unique additive mapping $h : X \to Y$, defined as

$$h(x) = \begin{cases} \lim_{k\to\infty}(\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in X), & \text{if } l < mn, p > 1, (\text{or } l > mn, 0 < p < 1); \\ \lim_{k\to\infty}(\frac{l}{mn})^k f((\frac{mn}{l})^k x), (x \in X), & \text{if } l < mn, 0 < p < 1, (\text{or } l > mn, p > 1), \end{cases}$$

such that

$$\|f(x)-h(x)\| \leq \begin{cases} \frac{(mn)^{p-1}}{(mn)^{p-1}-l^{p-1}}\Big(\frac{2\lceil\frac{l}{2}\rceil+2\lceil\frac{n}{2}\rceil}{\lceil\frac{l}{2}\rceil+m\lceil\frac{n}{2}\rceil} + 1 + \frac{1}{m}(\frac{l}{mn})^{p-1}\Big)\theta\|x\|^p, & \text{if } l < mn, p > 1, \\ & (\text{or } l > mn, 0 < p < 1); \\ \frac{(mn)^{p-1}}{l^{p-1}-(mn)^{p-1}}\Big(\frac{2\lceil\frac{l}{2}\rceil+2\lceil\frac{n}{2}\rceil}{\lceil\frac{l}{2}\rceil+m\lceil\frac{n}{2}\rceil} + 1 + \frac{1}{m}(\frac{l}{mn})^{p-1}\Big)\theta\|x\|^p, & \text{if } l < mn, 0 < p < 1, \\ & (\text{or } l > mn, p > 1), \end{cases}$$

for all $x \in X$.

Corollary 2.6. *Let $l \neq mn$ and $\theta > 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$\Big\|\sum_{i=1}^{l} f(x_i) + m\sum_{j=1}^{n} f(y_j)\Big\| \leq \Big\|mf\Big(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j\Big)\Big\| + \theta$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$, then there exists a unique additive mapping $h : X \to Y$, defined as

$$h(x) = \begin{cases} \lim_{k\to\infty}(\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in X), & \text{if } l > mn; \\ \lim_{k\to\infty}(\frac{l}{mn})^k f((\frac{mn}{l})^k x), (x \in X), & \text{if } l < mn, \end{cases}$$

such that

$$\|f(x) - h(x)\| \leq \begin{cases} \frac{l}{l-mn}(\frac{1}{K} + \frac{1}{l})\theta, & \text{if } l > mn; \\ \frac{l}{mn-l}(\frac{1}{K} + \frac{1}{l})\theta, & \text{if } l < mn, \end{cases}$$

for all $x \in X$, where $K := \lceil\frac{l}{2}\rceil + m\lceil\frac{n}{2}\rceil$.

3. Stability of (1.1) in non-Archimedean Banach spaces

We recall that a field $\mathbb{K}$, equipped with a function (non-Archimedean absolute value, valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$, is called a non-Archimedean field if the function $|\cdot|:\mathbb{K}\to[0,\infty)$ satisfies the following conditions:

(1) $|r|=0$ if and only if $r=0$;
(2) $|rs|=|r||s|$;
(3) the strong triangle inequality, namely, $|r+s|\le\max\{|r|,|s|\}$ for all $r,s\in\mathbb{K}$.

Clearly, $|1|=1=|-1|$ and $|n|\le 1$ for all nonzero integer n.

Let Y be a vector space over the non-Archimedean field $\mathbb{K}$ with a non-trivial non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|:Y\to[0,\infty)$ is called a non-Archimedean norm (valuation) if it satisfies the following conditions:

(1) $\|x\|=0$ if and only if $x=0$;
(2) $\|rx\|=|r|\|x\|$ for all $x\in Y$ and all $r\in\mathbb{K}$;
(3) the strong triangle inequality, namely,

$$\|x+y\|\le\max\{\|x\|,\|y\|\}$$

for all $x,y\in Y$.

In this case, the pair $(Y,\|\cdot\|)$ is called a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. It follows from the strong triangle inequality that

$$\|x_n-x_m\|\le\max\{\|x_{j+1}-x_j\|:m\le j<n-1\}$$

for all $x_n,x_m\in Y$ and all $m,n\in\mathbb{N}$ with $n>m$. Therefore, a sequence $\{x_n\}$ is a Cauchy sequence in non-Archimedean space $(Y,\|\cdot\|)$ if and only if the sequence $\{x_{n+1}-x_n\}$ converges to zero in the space $(Y,\|\cdot\|)$. In this section, let X be a non-Archimedean normed space and Y a complete non-Archimedean space. Now, we will investigate the generalized Hyers–Ulam stability problem for the functional inequality (1.1) with non-Archimedean valuations $|l+mn|>|m|$ in a complete non-Archimedean space Y.

Theorem 3.1. *If a mapping $f:X\to Y$ with $f(0)=0$ satisfies the functional inequality (2.1) for which there exists a constant L with $0<L<1$ for which the perturbing function $\varphi:X^{l+n}\to\mathbb{R}^+$ satisfies*

$$\varphi\Big(\frac{l}{mn}(x_1,\cdots,x_l,y_1,\cdots,y_n)\Big)\le L\cdot\Big|\frac{l}{mn}\Big|\varphi(x_1,\cdots,x_l,y_1,\cdots,y_n)\tag{3.1}$$

for all $x_1,\cdots,x_l,y_1,\cdots,y_n\in X$, then there exists a unique additive mapping $h_1:X\to Y$ which satisfies the inequality (1.1) and the inequality

$$\|f(x)-h_1(x)\|\le\Psi(x)\tag{3.2}$$

for all $x \in X$, where $|\cdot|$ is a non-Archimedean valuation and

$$\Psi(x) := \max\Big\{\frac{1}{|K|}\varphi(\underbrace{x,-x,\cdots,x,-x}_{2\lceil\frac{l}{2}\rceil},0,\cdots,0,\underbrace{x,-x,\cdots,x,-x}_{2\lceil\frac{n}{2}\rceil}),$$
$$\frac{1}{|l|}\varphi(\underbrace{-x,\cdots,-x}_{l},\underbrace{\frac{l}{mn}x,\cdots,\frac{l}{mn}x}_{n})\Big\},\quad K:=\lceil\frac{l}{2}\rceil+m\lceil\frac{n}{2}\rceil$$

Proof. Associating (2.4) with (2.5) in the complete non-Archimedean space Y yields

$$\|f(x)-\frac{mn}{l}f(\frac{l}{mn}x)\|\le\Psi(x) \tag{3.3}$$

for all $x \in X$. It follows from the inequality (3.3) that

$$\begin{aligned}&\Big\|(\frac{mn}{l})^k f\Big((\frac{l}{mn})^k x\Big)-(\frac{mn}{l})^{k+j}f\Big((\frac{l}{mn})^{k+j}x\Big)\Big\|\\ &\le\max\Big\{\Big|\frac{mn}{l}\Big|^i\Psi\Big((\frac{l}{mn})^i x\Big):k\le i\le k+j-1\Big\}\\ &\le\max\Big\{L^i\Psi(x):k\le i\le k+j-1\Big\}=L^k\Psi(x),\end{aligned}$$

which tends to zero as $k\to\infty$. Hence the sequence $\Big\{(\frac{mn}{l})^k f\Big((\frac{l}{mn})^k x\Big)\Big\}$ is Cauchy for all $x \in X$, and so one can define a mapping $h_1 : X \to Y$ by

$$h_1(x)=\lim_{k\to\infty}(\frac{mn}{l})^k f\Big((\frac{l}{mn})^k x\Big),\quad x\in X.$$

The remaining proof is similar to the corresponding proof of Theorem 2.1. This completes the proof. □

Theorem 3.2. *If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.1) and there exists a constant L with $0 < L < 1$ for which the perturbing function $\varphi : X^{l+n} \to \mathbb{R}^+$ satisfies*

$$\varphi\Big(\frac{mn}{l}(x_1,\cdots,x_l,y_1,\cdots,y_n)\Big)\le L\cdot\Big|\frac{mn}{l}\Big|\varphi(x_1,\cdots,x_l,y_1,\cdots,y_n) \tag{3.4}$$

for all $x_1,\cdots,x_l,y_1,\cdots,y_n \in X$, then there exists a unique additive mapping $h_2 : X \to Y$ which satisfies the inequality (1.1) and the inequality

$$\|f(x)-h_2(x)\|\le L\Psi(x) \tag{3.5}$$

for all $x \in X$, where Ψ is given as in Theorem 3.1.

Proof. It follows from the inequality (2.6) that

$$\begin{aligned}&\Big\|(\frac{l}{mn})^k f\Big((\frac{mn}{l})^k x\Big)-(\frac{l}{mn})^{k+j}f\Big((\frac{mn}{l})^{k+j}x\Big)\Big\|\\ &\le\max\Big\{\Big|\frac{l}{mn}\Big|^{i+1}\Psi\Big((\frac{mn}{l})^{i+1}x\Big):k\le i\le k+j-1\Big\}\\ &\le\max\Big\{L^{i+1}\Psi(x):k\le i\le k+j-1\Big\}=L^{k+1}\Psi(x),\end{aligned}$$

which tends to zero as $k \to \infty$.

The remaining assertion is similar to that of Theorem 3.1. This completes the proof. □

Corollary 3.3. *Let $p \neq 1, \left|\frac{l}{mn}\right| \neq 1$ and $\theta > 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$\Big\| \sum_{i=1}^{l} f(x_i) + m \sum_{j=1}^{n} f(y_j) \Big\| \leq \Big\| m f\Big(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j \Big) \Big\| + \theta \Big(\sum_{i=1}^{l} \|x_i\|^p + \sum_{j=1}^{n} \|y_j\|^p \Big)$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$, then there exists a unique additive mapping $h : X \to Y$, defined as

$$h(x) = \begin{cases} \lim_{k\to\infty} (\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in X), & \text{if } \left|\frac{l}{mn}\right| < 1 \ \& \ p > 1, \\ & (\text{or } \left|\frac{l}{mn}\right| > 1 \ \& \ 0 < p < 1); \\ \lim_{k\to\infty} (\frac{l}{mn})^k f((\frac{mn}{l})^k x), (x \in X), & \text{if } \left|\frac{mn}{l}\right| < 1 \ \& \ p > 1, \\ & (\text{or } \left|\frac{mn}{l}\right| > 1 \ \& \ 0 < p < 1), \end{cases}$$

such that

$$\|f(x) - h(x)\| \leq \begin{cases} \max\left\{ \frac{2\lceil \frac{l}{2} \rceil + 2\lceil \frac{n}{2} \rceil}{|\lceil \frac{l}{2} \rceil + m \lceil \frac{n}{2} \rceil|}, \ \frac{l + n|\frac{l}{mn}|^p}{|l|} \right\} \theta \|x\|^p, & \text{if } \left|\frac{l}{mn}\right| < 1 \ \& \ p > 1, \\ & (\text{or } \left|\frac{l}{mn}\right| > 1 \ \& \ 0 < p < 1); \\ \max\left\{ \frac{2\lceil \frac{l}{2} \rceil + 2\lceil \frac{n}{2} \rceil}{|\lceil \frac{l}{2} \rceil + m \lceil \frac{n}{2} \rceil|}, \ \frac{l + n|\frac{l}{mn}|^p}{|l|} \right\} \left|\frac{mn}{l}\right|^{p-1} \theta \|x\|^p, & \text{if } \left|\frac{mn}{l}\right| < 1 \ \& \ p > 1, \\ & (\text{or } \left|\frac{mn}{l}\right| > 1 \ \& \ 0 < p < 1), \end{cases}$$

for all $x \in X$.

Corollary 3.4. *Let r_i, s_i be positive reals with $\sum_{i=1}^{l} r_i := r, \sum_{j=1}^{n} s_i := s$ and $r + s \neq 1$, and let $\left|\frac{l}{mn}\right| \neq 1$ and $\theta > 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$\Big\| \sum_{i=1}^{l} f(x_i) + m \sum_{j=1}^{n} f(y_j) \Big\| \leq \Big\| m f\Big(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j \Big) \Big\| + \theta \Big(\prod_{i=1}^{l} \|x_i\|^{r_i} \prod_{j=1}^{n} \|y_j\|^{s_j} \Big)$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$, then there exists a unique additive mapping $h : X \to Y$, defined as

$$h(x) = \begin{cases} \lim_{k\to\infty} (\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in X), & \text{if } \left|\frac{l}{mn}\right| < 1 \ \& \ r + s > 1, \\ & (\text{or } \left|\frac{l}{mn}\right| > 1 \ \& \ 0 < r + s < 1); \\ \lim_{k\to\infty} (\frac{l}{mn})^k f((\frac{mn}{l})^k x), (x \in X), & \text{if } \left|\frac{mn}{l}\right| < 1 \ \& \ r + s > 1, \\ & (\text{or } \left|\frac{mn}{l}\right| > 1 \ \& \ 0 < r + s < 1), \end{cases}$$

such that

$$\|f(x)-h(x)\| \leq \begin{cases} \max\left\{\frac{I_{l+n}(2\lceil\frac{l}{2}\rceil+2\lceil\frac{n}{2}\rceil)}{|\lceil\frac{l}{2}\rceil+m\lceil\frac{n}{2}\rceil|}, \frac{|\frac{l}{mn}|^s}{|l|}\right\}\theta\|x\|^{r+s}, & \text{if } \left|\frac{l}{mn}\right| < 1 \,\&\, r+s>1, \\ & (\text{or } \left|\frac{l}{mn}\right| > 1 \,\&\, 0<r+s<1); \\ \max\left\{\frac{I_{l+n}(2\lceil\frac{l}{2}\rceil+2\lceil\frac{n}{2}\rceil)}{|\lceil\frac{l}{2}\rceil+m\lceil\frac{n}{2}\rceil|}, \frac{|\frac{l}{mn}|^s}{|l|}\right\}\left|\frac{mn}{l}\right|^{r+s-1}\theta\|x\|^p, & \text{if } \left|\frac{mn}{l}\right| < 1 \,\&\, r+s>1, \\ & (\text{or } \left|\frac{mn}{l}\right| > 1 \,\&\, 0<r+s<1), \end{cases}$$

for all $x \in X$, where

$$I_c(x) := \begin{cases} 0, & \text{if } x < c; \\ 1, & \text{if } x \geq c. \end{cases}$$

We remark that $I_{l+n}(2\lceil\frac{l}{2}\rceil + 2\lceil\frac{n}{2}\rceil) = 0$ if l or n is odd, and $I_{l+n}(2\lceil\frac{l}{2}\rceil + 2\lceil\frac{n}{2}\rceil) = 1$ if both l and n are even.

Corollary 3.5. *Let $\left|\frac{l}{mn}\right| \neq 1$ and $\theta > 0$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following functional inequality*

$$\left\|\sum_{i=1}^{l} f(x_i) + m\sum_{j=1}^{n} f(y_j)\right\| \leq \left\|mf\left(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j\right)\right\| + \theta$$

for all $x_1, \cdots, x_l, y_1, \cdots, y_n \in X$, then there exists a unique additive mapping $h : X \to Y$, defined as

$$h(x) = \begin{cases} \lim_{k\to\infty}(\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in X), & \text{if } |\frac{mn}{l}| < 1; \\ \lim_{k\to\infty}(\frac{l}{mn})^k f((\frac{mn}{l})^k x), (x \in X), & \text{if } |\frac{l}{mn}| < 1, \end{cases}$$

such that

$$\|f(x) - h(x)\| \leq \begin{cases} \max\left\{\frac{1}{|\lceil\frac{l}{2}\rceil+m\lceil\frac{n}{2}\rceil|}, \frac{1}{|l|}\right\}\theta, & \text{if } |\frac{mn}{l}| < 1; \\ \max\left\{\frac{1}{|\lceil\frac{l}{2}\rceil+m\lceil\frac{n}{2}\rceil|}, \frac{1}{|l|}\right\}|\frac{l}{mn}|\theta, & \text{if } |\frac{l}{mn}| < 1, \end{cases}$$

for all $x \in X$.

We remark that stability results of the functional inequality (1.1) in non-Archimedean Banach spaces are very different from those of the inequality (1.1) in Banach spaces.

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SOME PROPERTIES OF CERTAIN CLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

ADRIANA CĂTAŞ AND CĂLIN DUBĂU

Abstract. The aim of this paper is to derive several interesting properties of a new class denoted by $W^n_{p,\lambda}(\alpha,\gamma)$ consisting of analytic functions with negative coefficients for which coefficient inequalities, distortion theorems, closure theorems are determined. Furthermore, integral operators and modified Hadamard products of several functions belonging to the class $W^n_{p,\lambda}(\alpha,\gamma)$ are studied. The results are obtained using a generalized Sălăgean operator and they are improvement of a known results.

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1. Introduction and preliminaries

Let V_p denote the class of functions of the form

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j}, \quad (p \in \mathbb{N} = \{1, 2, \dots\}) \tag{1}$$

which are analytic and p-valent on the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

We also denote by W_p the subclass of V_p containing functions f which can be expressed in the form

$$f(z) = z^p - \sum_{j=1}^{\infty} a_{p+j} z^{p+j} \tag{2}$$

with $a_{p+j} \geq 0$ for all $p \in \mathbb{N}$.

The differential operator D^n_λ is defined by

$$D^0_\lambda f(z) = f(z) \tag{3}$$

$$D^1_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda, \quad \lambda > 0 \tag{4}$$

$$D^n_\lambda f(z) = D_\lambda (D^{n-1}_\lambda f(z)). \tag{5}$$

This operator was introduced by Al-Oboudi [1] and when $\lambda = 1$ we get the Sălăgean differential operator [3].

It can be easily seen that for a function $f \in W_p$ one obtains

$$D_\lambda^n f(z) = (1-\lambda+\lambda p)^n z^p - \sum_{j=1}^{\infty}[1-\lambda+\lambda(p+j)]^n a_{p+j} z^{p+j}. \tag{6}$$

2. Coefficient estimates

Now we propose

Definition 1. We say that a function $f(z) \in V_p$ is in the class $V_{p,\lambda}^n(\alpha,\gamma)$, $p \in \mathbb{N}^*$, $n \in \mathbb{N}$, $\alpha \in [0,p)$, $\gamma \in [-1,1)$, $\lambda > 0$ if

$$\left| \frac{\dfrac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - p}{\dfrac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - \gamma p - (1-\gamma)\alpha} \right| < 1. \tag{7}$$

Let

$$W_{p,\lambda}^n(\alpha,\gamma) = V_{p,\lambda}^n(\alpha,\gamma) \cap W_p. \tag{8}$$

Remark 1. The class $W_{p,1}^n(\alpha,\gamma)$ has been studied by G.S. Sălăgean and F.I. Stan [4] and the class $W_p^*(\alpha,\gamma) := W_{p,1}^0(\alpha,\gamma)$ has been studied by D.Z. Pashkouleva and K.V. Vasilev [2].

Theorem 1. *Let $p \in \mathbb{N}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\gamma \in [-1,1)$, $\lambda \geq 1$ and $\alpha \in [0,p)$. Then a function f is in the class $W_{p,\lambda}^n(\alpha,\gamma)$ if and only if*

$$\sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]a_{p+j} \leq \tag{9}$$
$$\leq (1-\lambda+\lambda p)^n (1-\gamma)(p-\lambda)$$

and

$$c_{p,j}(n,\lambda) = [1+(p+j-1)\lambda]^n. \tag{10}$$

Proof. Assume that the inequality (9) holds and let $|z| = 1$. Then we have

$$|D_\lambda^{n+1} f(z) - p D_\lambda^n f(z)| - |D_\lambda^{n+1} f(z) - [\gamma p + (1-\gamma)\alpha] D_\lambda^n f(z)| =$$

$$= |(1-\lambda+\lambda p)^n (\lambda-1)(p-1) z^p - \sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[1-p+\lambda(p+j-1)]a_{p+j} z^{p+j}| -$$

$$-|(1-\lambda+\lambda p)^n [(p-1)(\lambda-1)+(1-\gamma)(p-\alpha)] z^p -$$

$$- \sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[(p-1)(\lambda-1)+\lambda j+(1-\gamma)(p-\alpha)]a_{p+j} z^{p+j}| \leq$$

$$\leq \sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]a_{p+j}-$$

$$-(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)\leq 0.$$

Consequently, by the maximum modulus theorem, the function $f(z)$ is in the class $W^n_{p,\lambda}(\alpha,\gamma)$.

Conversely, if $f\in W^n_{p,\lambda}(\alpha,\gamma)$ then from (7) we have

$$\left|\frac{D^{n+1}_\lambda f(z)-pD^n_\lambda f(z)}{D^{n+1}_\lambda f(z)-[\gamma p+(1-\gamma)\alpha]D^n_\lambda f(z)}\right|=$$

$$=\left|\frac{(1-\lambda+\lambda p)^n(\lambda-1)(p-1)z^p-\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[1-p+\lambda(p+j-1)]a_{p+j}z^{p+j}}{(1-\lambda+\lambda p)^n b_p(\alpha,\gamma,\lambda)z^p-\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[b_p(\alpha,\gamma,\lambda)+\lambda j]a_{p+j}z^{p+j}}\right|<1.$$

where

$$b_p(\alpha,\gamma,\lambda)=(p-1)(\lambda-1)+(1-\gamma)(p-\alpha)$$

Since $|\mathrm{Re}(z)|\leq|z|$, $z\in\mathbb{C}$, we obtain

$$\mathrm{Re}\left\{\frac{(1-\lambda+\lambda p)^n(\lambda-1)(p-1)z^p-\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[1-p+\lambda(p+j-1)]a_{p+j}z^{p+j}}{(1-\lambda+\lambda p)^n b_p(\alpha,\gamma,\lambda)z^p-\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[b_p(\alpha,\gamma,\lambda)+\lambda j]a_{p+j}z^{p+j}}\right\}<1.$$

Choosing z on the real axis such that $\dfrac{D^{n+1}_\lambda f(z)}{D^n_\lambda f(z)}$ is real and letting $z\to 1^-$ through real values we get

$$(1-\lambda+\lambda p)^n(\lambda-1)(p-1)-\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[1-p+\lambda(p+j-1)]a_{p+j}\leq$$

$$\leq(1-\lambda+\lambda p)^n[(p-1)(\lambda-1)+(1-\gamma)(p-\alpha)]-$$

$$-\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[(p-1)(\lambda-1)+\lambda j+(1-\gamma)(p-\alpha)]a_{p+j}$$

and this inequality gives the required condition.

This completes the proof of Theorem 1.

Remark 2. The result obtained in Theorem 1 is sharp as can be seen by (11)

$$f_j(z)=z^p-\frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]c_{p,j}(n,\lambda)}z^{p+j},\ j\in\mathbb{N}.$$

Each function on the form (11) is an extremal function for the theorem.

Remark 3. Theorem 1 improves the result obtained in [[4], Theorem 1].

COROLLARY 1. *Let $f(z)\in W^n_{p,\lambda}(\alpha,\gamma)$. Then*

$$a_{p+j}\le\frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]},\quad j\in\mathbb{N}^*. \tag{12}$$

The equality in (12) is attained for the function $f_j(z)$ given in (11).

3. Growth and distortion theorems

Next we prove the following growth and distortion properties for the class $W^n_{p,\lambda}(\alpha,\gamma)$.

THEOREM 2. *Let the function $f(z)$ be in the class $W^n_{p,\lambda}(\alpha,\gamma)$. Then we have*

$$|D^k_\lambda f(z)|\le(1-\lambda+\lambda p)^k|z|^p+ \tag{13}$$

$$+\frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)}|z|^{p+1}$$

and

$$|D^k_\lambda f(z)|\ge(1-\lambda+\lambda p)^k|z|^p- \tag{14}$$

$$-\frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)}|z|^{p+1}$$

for $z\in U$, where $0\le k\le n$.

The equalities (13) and (14) are attained for the function $f(z)$ defined by

$$D^kf(z)=(1-\lambda+\lambda p)^kz^p-\frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)}z^{p+1}. \tag{15}$$

Proof. From Theorem 1, for $0\le k\le n$, one obtains

$$\sum_{j=1}^{\infty}c_{p,j}(k,\lambda)a_{p+j}\le\frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)}. \tag{16}$$

Furthermore, we note from (6) that

$$|D^k_\lambda f(z)|\le(1-\lambda+\lambda p)^k|z|^p+\sum_{j=1}^{\infty}[1+\lambda(p+j-1)]^ka_{p+j}|z|^{p+j}$$

and

$$|D_\lambda^k f(z)| \geq (1-\lambda+\lambda p)^k |z|^p - \sum_{j=1}^{\infty} [1+\lambda(p+j-1)]^k a_{p+j} |z|^{p+j}.$$

Using the fact that $|z| < 1$ the assertions of (13) and (14) of Theorem 2 follow immediately.

COROLLARY 2. *If $f(z)$ belongs to the class $W_{p,\lambda}^n(\alpha,\gamma)$ then*

$$|f(z)| \leq |z|^p + \frac{(1-\lambda+\lambda p)^n (1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)} |z|^{p+1} \tag{17}$$

and

$$|f(z)| \geq |z|^p - \frac{(1-\lambda+\lambda p)^n (1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)} |z|^{p+1} \tag{18}$$

for $|z| < 1$.

The equalities in (17) and (18) hold for the function

$$f(z) = z^p - \frac{(1-\lambda+\lambda p)^n (1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)} z^{p+1}, \quad z \in U. \tag{19}$$

COROLLARY 3. *If f belongs to the class $W_{p,\lambda}^n(\alpha,\gamma)$ then*

$$|f'(z)| \leq p|z|^{p-1} + \frac{(p+1)(1-\lambda+\lambda p)^n (1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)} |z|^p \tag{20}$$

and

$$|f'(z)| \geq p|z|^{p-1} - \frac{(p+1)(1-\lambda+\lambda p)^n (1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)} |z|^p. \tag{21}$$

The equalities in (20) and (21) hold for the function $f(z)$ given by (19).

COROLLARY 4. *The unit disc is mapped into a domain which contains*

$$U\left(0, 1 + \frac{(1-\lambda+\lambda p)^n (1-\gamma)(p-\alpha)}{2(p-1)(\lambda-1)+2\lambda+(1-\gamma)(p-\alpha)}\right)$$

by any $f \in W_{p,\lambda}^n(\alpha,\gamma)$. The sharpness is assured by the function (19).

4. Closure theorems

Let the functions $f_k(z)$ be defined, for $k = 1, 2, \ldots, m$ by

$$f_k(z) = z^p - \sum_{j=1}^{\infty} a_{p+j,k} z^{p+j}, \quad a_{p+j,k} \geq 0. \tag{22}$$

We shall prove the following results for the closure of functions in the class $W_{p,\lambda}^n(\alpha,\gamma)$.

THEOREM 3. *Let the functions $f_k(z)$ defined by (22) be in the class $W^n_{p,\lambda}(\alpha,\gamma)$ for every $k=1,2,\ldots,m$. Then the function $h(z)$ defined by*

$$h(z)=\sum_{k=1}^{m} c_k f_k(z), \quad c_k \geq 0 \tag{23}$$

is also in the class $W^n_{p,\lambda}(\alpha,\gamma)$ where

$$\sum_{k=1}^{m} c_k = 1. \tag{24}$$

Proof. According to the definition of the function $h(z)$, we can write

$$h(z)=z^p-\sum_{j=1}^{\infty}\left(\sum_{k=1}^{m} c_k a_{p+j,k}\right) z^{p+j}. \tag{25}$$

Further, since f_k are in $W^n_{p,\lambda}$ for every $k=1,2,\ldots,m$, we get from Theorem 1

$$\sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]a_{p+j,k} \leq \tag{26}$$

$$\leq (1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha).$$

Hence we can see that

$$\sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]\left(\sum_{k=1}^{m} c_k a_{p+j,k}\right)=$$

$$=\sum_{k=1}^{\infty} c_k\left(\sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]a_{p+j,k}\right)\leq$$

$$\leq\left(\sum_{k=1}^{m} c_k\right)(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)=(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)$$

which implies that $h(z)$ belongs to the class $W^n_{p,\lambda}(\alpha,\gamma)$.

THEOREM 4. *The class $W^n_{p,\lambda}(\alpha,\gamma)$ is closed under a convex linear combinations.*

Proof. Let the functions $f_k(z)$ $(k=1,2)$ defined by (22) be in the class $W^n_{p,\lambda}(\alpha,\gamma)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=\eta f_1(z)+(1-\eta)f_2(z), \quad 0\leq\eta\leq 1 \tag{27}$$

is in the class $W^n_{p,\lambda}(\alpha,\gamma)$. But, taking $m=2$, $c_1=\eta$ and $c_2=1-\eta$ in Theorem 3 we get the required result.

Remark 4. As a consequence of Theorem 4, there exists the extreme points of the class $W^n_{p,\lambda}(\alpha,\gamma)$.

THEOREM 5. *Let f_j be defined as in (11), $j \in \mathbb{N}$ and*

$$f_0(z) = z^p, \quad z \in U. \tag{28}$$

Then $f(z)$ is in the class $W^n_{p,\lambda}(\alpha,\gamma)$ if and only if it can be expressed as

$$f(z) = \sum_{j=0}^{\infty} \mu_j f_j(z) \tag{29}$$

where $\mu_j \geq 0$ for $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\sum_{j=0}^{\infty} \mu_j = 1$.

Proof. Suppose that f can be expressed as in (29), namely

$$f(z) = z^p - \sum_{j=1}^{\infty} \mu_j \frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]c_{p,j}(n,\lambda)} z^{p+j}$$

then it follows that

$$\sum_{j=1}^{\infty} c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]\cdot$$

$$\cdot \frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]c_{p,j}(n,\lambda)}\mu_j \leq$$

$$\leq (1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)$$

is equivalent to $\sum_{j=1}^{\infty} \mu_j \leq 1$ and from Theorem 1 $f(z)$ belongs to the class $W^n_{p,\lambda}(\alpha,\gamma)$.

Conversely, assume that the function $f(z)$ belongs to the class $W^n_{p,\lambda}(\alpha,\gamma)$. Then

$$\sum_{j=1}^{\infty} \frac{c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]}{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)} a_{p+j} \leq 1.$$

We denote

$$\mu_j = \frac{c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]}{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)} a_{p+j}$$

for $j \in \mathbb{N}$ and $\mu_0 = 1 - \sum_{j=1}^{\infty} \mu_j$.

We notice that $f(z)$ can be expressed in the form (29). This completes the proof of Theorem 5.

COROLLARY 5. *The extreme points of the class $W^n_{p,\lambda}(\alpha,\gamma)$ are the functions $f_j(z)$, $j \in \mathbb{N}$, given by Theorem 5.*

5. Modified Hadamard product

Let the functions f_k $(k = 1, 2)$ defined by (22). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z^p - \sum_{j=1}^{\infty} a_{p+j,1} a_{p+j,2} z^{p+j}. \tag{30}$$

Theorem 6. *Let the functions $f_k(z)$ $(k = 1, 2)$ defined by (22) be in the class $W^n_{p,\lambda}(\alpha, \gamma)$. Then the modified Hadamard product belongs to the class $W^n_{p,\lambda}(\alpha, \gamma)$ where*

$$\beta = \beta(\alpha, \gamma, \lambda, n, p) = \tag{31}$$

$$= p - \frac{2(1 - \lambda + \lambda p)^n (1 - \gamma)(p - \alpha)^2 [(p - 1)(\gamma - 1) + \lambda]}{(1 + p\lambda)^n [d_p(\alpha, \gamma, \lambda) + 2\lambda]^2 - (1 - \lambda + \lambda p)^n (1 - \gamma)^2 (p - \alpha)^2}$$

where

$$d_p(\alpha, \gamma, \lambda) = 2(p - 1)(\lambda - 1) + (1 - \gamma)(p - \alpha).$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [5], we need to find the largest $\beta = \beta(\alpha, \gamma, \lambda, n, p)$ such that

$$\sum_{j=1}^{\infty} \frac{c_{p,j}(n, \lambda)[2(p - 1)(\lambda - 1) + 2\lambda j + (1 - \gamma)(p - \beta)]}{(1 - \lambda + \lambda p)^n (1 - \gamma)(p - \beta)} a_{p+j,1} a_{p+j,2} \leq 1. \tag{32}$$

Since

$$\sum_{j=1}^{\infty} \frac{c_{p,j}(n, \lambda)[2(p - 1)(\lambda - 1) + 2\lambda j + (1 - \gamma)(p - \alpha)]}{(1 - \lambda + \lambda p)^n (1 - \gamma)(p - \alpha)} a_{p+j,i} \leq 1, \ i = 1, 2 \tag{33}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{j=1}^{\infty} \frac{c_{p,j}(n, \lambda)[2(p - 1)(\lambda - 1) + 2\lambda j + (1 - \gamma)(p - \alpha)]}{(1 - \lambda + \lambda p)^n (1 - \gamma)(p - \alpha)} \sqrt{a_{p+j,1} a_{p+j,2}} \leq 1. \tag{34}$$

It is sufficient to show that

$$\frac{c_{p,j}(n, \lambda)[2(p - 1)(\lambda - 1) + 2\lambda j + (1 - \gamma)(p - \beta)]}{(1 - \lambda + \lambda p)^n (1 - \gamma)(p - \beta)} a_{p+j,1} a_{p+j,2} \leq$$

$$\leq \frac{c_{p,j}(n, \lambda)[2(p - 1)(\lambda - 1) + 2\lambda j + (1 - \gamma)(p - \alpha)]}{(1 - \lambda + \lambda p)^n (1 - \gamma)(p - \alpha)} \sqrt{a_{p+j,1} a_{p+j,2}}$$

thus

$$\sqrt{a_{p+j,1} a_{p+j,2}} \leq \frac{[2(p - 1)(\lambda - 1) + 2\lambda j + (1 - \gamma)(p - \alpha)](p - \beta)}{[2(p - 1)(\lambda - 1) + 2\lambda j + (1 - \gamma)(p - \beta)](p - \alpha)}, \quad j \in \mathbb{N}.$$

We note that

$$\sqrt{a_{p+j,1}a_{p+j,2}} \leq \frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]}.$$

Consequently, we only need to prove that

$$\frac{(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)}{c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]} \leq$$
$$\leq \frac{[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)](p-\beta)}{[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\beta)](p-\alpha)}$$

which is equivalent to

$$\beta \leq p - \frac{2(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)^2[(p-1)(\lambda-1)+\lambda j]}{c_{p,j}(n,\lambda)[d_p(\alpha,\gamma,\lambda)+2\lambda j]^2-(1-\lambda+\lambda p)^n(1-\gamma)^2(p-\alpha)^2}$$

We denote

$$E(\alpha,\gamma,\lambda,n,p;j) =$$
$$p - \frac{2(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)^2[(p-1)(\lambda-1)+\lambda j]}{c_{p,j}(n,\lambda)[d_p(\alpha,\gamma,\lambda)+2\lambda j]^2-(1-\lambda+\lambda p)^n(1-\gamma)^2(p-\alpha)^2}.$$

But

$$E(\alpha,\gamma,\lambda,n,p;j+1)-E(\alpha,\gamma,\lambda,n,p;j) =$$
$$= \frac{2(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)^2}{A(\alpha,\gamma,\lambda,n,p;j)}[B(\alpha,\gamma,\lambda,n,p;j)+\lambda(1-\lambda+\lambda p)^n(1-\gamma)^2(p-\alpha)^2]$$

where

$$A(\alpha,\gamma,\lambda,n,p;j) =$$
$$= \{c_{p,j}(n,\lambda)[d_p(\alpha,\gamma,\lambda)+2\lambda j]^2-(1-\lambda+\lambda p)^n(1-\gamma)^2(p-\alpha)^2\}\cdot$$
$$\cdot\{c_{p,j+1}(n,\lambda)[d_p(\alpha,\gamma,\lambda)+2\lambda(j+1)]^2-$$
$$-(1-\lambda+\lambda p)^n(1-\gamma)^2(p-\alpha)^2\} > 0$$

and

$$B(\alpha,\gamma,\lambda,n,p;j) =$$
$$= c_{p,j+1}(n,\lambda)[(p-1)(\lambda-1)+\lambda j]\cdot[d_p(\alpha,\gamma,\lambda)+2\lambda(j+1)]^2-$$
$$-c_{p,j}(n,\lambda)[(p-1)(\lambda-1)+\lambda(j+1)]\cdot[d_p(\alpha,\gamma,\lambda)+2\lambda j]^2 > 0.$$

Thus, we deduce that $E(\alpha,\gamma,\lambda,n,p;j)$ is an increasing function of $j \in \mathbb{N}$.

Finally, we obtain $\beta \leq E(\alpha,\gamma,\lambda,n,p;1)$ which is

$$\beta \leq p - \frac{2(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha)^2[(p-1)(\gamma-1)+\lambda]}{c_{p,1}(n,\lambda)[d_p(\alpha,\gamma,\lambda)+2\lambda]^2-(1-\lambda+\lambda p)^n(1-\gamma)^2(p-\alpha)^2}.$$

For the proof of the sharpness we take the functions f_1 and f_2 of the form (19).

6. Integral property of the class $W^n_{p,\lambda}(\alpha,\gamma)$

Theorem 7. *Let f be in the class $W^n_{p,\lambda}(\alpha,\gamma)$ and c a real number such that $c>-1$. Then the integral operator of Libera-Bernardi type*

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{35}$$

also belongs to the class $W^n_{p,\lambda}(\alpha,\gamma)$.

Proof. We note that

$$F(z)=(c+p)\int_0^1 \tau^{c-1}f(t\tau)d\tau=z^p-\sum_{j=1}^{\infty}\frac{c+p}{c+p+j}a_{p+j}z^{p+j}$$

and it follows

$$\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]\frac{c+p}{c+p+j}a_{p+j}\leq$$

$$\leq\sum_{j=1}^{\infty}c_{p,j}(n,\lambda)[2(p-1)(\lambda-1)+2\lambda j+(1-\gamma)(p-\alpha)]a_{p+j}\leq$$

$$\leq(1-\lambda+\lambda p)^n(1-\gamma)(p-\alpha).$$

Using Theorem 1 the proof is complete.

Remark 5. The result obtained in Theorem 7 improves the result obtained in [[4], Theorem 6].

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Smooth and optimal shape of aeolian blade profile by splines

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Abstract

A minimal property of cubic splines generated by initial conditions is obtained in order to generate smooth and optimal aerodynamic shape for the aeolian blade profile.

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1 Introduction

Within aero-electrical aggregates the wine turbine is the component that ensures the conversion of kinetic energy of wind into mechanical energy useable to turbine shaft, through the interaction between air current and moving blade.

Wind turbine is composed mainly of an rotator fixed on a support shaft, comprising a hub and a moving blade consisting of one or more blades. Active body of aeolian turbines which made the quantity of converted energy is the blade. The achieving of aerodynamic performances, kinematics and energy curves of the aeolian turbines depend on the choice of a certain geometry. Wind energy conversion is achieved by the interaction between air currents and solid surface of the blade. For design of the blade profile are used optimized shape (aerodynamic profile) selected and positioned such that the obtained performances for certain conditions, proper to the location, to be optimal. Interaction moment

between moving blade and fluid current is deduced from lifted aerodynamic forces and resistance. Calculation of the blade profile was realized by determining the geometric contour of the aerodynamic profile deduced by an analytical method, combining two mathematical functions, a framework function and a gauge function, respectively. Presented model was synthesized by a section calculation which set the final shape of the blade, using a computer algorithm, taking into account the behavior of aerodynamic profile which was placed in a stream of air.

The aerodynamic profile behavior depends mainly on profile position compared to the air current speed, through the angle of incidence value.

In order to generate an aerodynamic shape for the aeolian blade profile with initial minimal ecart from the segment line we can consider cubic splines with partial minimal quadratic oscillation in average on the first subinterval.

The notion of quadratic oscillation in average is defined in [3] and [5] and the partial minimal quadratic oscillation in average on the first and on the last two subintervals was obtained in [3] in order to improve the Akima's method (see [2]) for the Hermite type interpolation.

In the definition of the notion of quadratic oscillation in average is considered in [5] a partition Δ of an interval $[a,b]$,

$$\Delta : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

a set of values $y_0, y_1, \ldots, y_n \in \mathbb{R}$ and an arbitrary function $f : [a,b] \to \mathbb{R}$, $f \in C[a,b]$ satisfying the interpolation conditions:

$$f(x_i) = y_i,\ \forall i = \overline{0,n}.$$

Let $L : [a,b] \to \mathbb{R}$ such that $L_i = L\big|_{[x_{i-1},x_i]}$, $i = \overline{1,n}$ and

$$L_i(x) = y_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot (y_i - y_{i-1}), \quad x \in [x_{i-1}, x_i]$$

the polygonal line joining the points (x_i, y_i),$i = \overline{0,n}$.

It is denoted by f_i the restriction of f to each interval $[x_{i-1}, x_i]$, $i = \overline{1,n}$ and $h_i = x_i - x_{i-1}$, $i = \overline{1,n}$.

Definition 1.1 *[5] The quadratic oscillation of the function f corresponding to the partition Δ and to the values y_i, $i = \overline{0,n}$, is the following functional $\rho : C[a,b] \times \mathbb{R}^{n+1} \to \mathbb{R}$*

$$\rho(f; \Delta, y) = \sqrt{\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [f_i(x) - L_i(x)]^2 dx} \tag{1.1}$$

where $y = (y_0, y_1, \ldots, y_n)$.

Remark 1.1 If there exists a set $K \subset \{1, 2, \ldots, n\}$ such that in (1.1) the sum is

$$\sum_{i \in K} \int_{x_{i-1}}^{x_i} [f_i(x) - L_i(x)]^2 dx$$

then, the functional is called partial quadratic oscillation in average (see [3], where the set is $K = \{1, n-1, n\}$).

In this paper, we consider the cubic splines generated by initial conditions (see [10]) with minimal partial quadratic oscillation in average corresponding to the set $K = \{1\}$.

The paper is organized as follows. In Section 2 we construct the algorithm of cubic splines generated by initial conditions with minimal partial quadratic oscillation in average on the first subinterval. Section 3 is devoted to the error estimation of the interpolation with this spline, in the class of continuous functions. A numerical experiment to illustrate an application in the construction of aerodynamic profile of aeolian blade is presented in Section 4.

2 The cubic spline procedure

The cubic spline generated by initial conditions was introduced by C. Iancu in [10] and it is constructed integrating on each subinterval $[x_{i-1}, x_i]$, $i = \overline{1,n}$ the initial value problem.

$$\begin{cases} s_i''(x) &= M_{i-1} + \frac{x-x_{i-1}}{x_i-x_{i-1}} \cdot (M_i - M_{i-1}) \\ s_i(x_{i-1}) &= y_{i-1} \\ s_i'(x_{i-1}) &= m_{i-1}, \quad x \in [x_{i-1}, x_i],\ i = \overline{1,n} \end{cases}.$$

The expression is

(2.1)
$$s_i(x) = \frac{M_i - M_{i-1}}{6h_i} \cdot (x - x_{i-1})^3 + \frac{M_{i-1}}{2} \cdot (x - x_{i-1})^2 + m_{i-1} \cdot (x - x_{i-1}) + y_{i-1},$$

and imposing the conditions $s_i(x_i) = y_i, i = \overline{1,n}$, we obtain a continuous $S \in C[a,b]$, $S : [a,b] \to \mathbb{R}$ with the restrictions S_i to the intervals $[x_{i-1}, x_i]$, $i = \overline{1,n}$.

Since a cubic spline without deficiency is $S \in C^2[a,b]$, imposing the condition $S \in C^1[a,b]$ that is $S_i'(x_i) = m_i$, $i = \overline{1,n}$ we obtain together with $S_i(x_i) = y_i$, $i = \overline{1,n}$ the relations:

(2.2)
$$M_i = \frac{6}{h_i^2}(y_i - y_{i-1}) - \frac{6m_{i-1}}{h_i} - 2M_{i-1}$$

$$m_i = \frac{3}{h_i}(y_i - y_{i-1}) - 2m_{i-1} - \frac{h_i}{2} \cdot M_{i-1}, \quad i = \overline{1,n}.$$

which uniquely determine the values m_i, M_i, $i = \overline{1,n}$ in recurrent way starting from $y_0, \ldots, y_n, m_0$ and M_0.

So, the values m_0 and M_0 remains free and can be specified in order to obtain a supplementary property of the cubic spline. In [5], these values are obtained such that the spline to be natural cubic spline. In [4], these values are obtained such that the cubic spline generated by initial conditions to be with minimal quadratic oscillation in average.

For the cubic spline generated by boundary conditions, the first and the last moment are computed such that the corresponding spline to have minimal quadratic oscillation in average, in [7].

Here, we determine m_0 and M_0 such that the cubic spline generated by initial conditions such that S will be with minimal partial quadratic oscillation in average on the first interval $[x_0, x_1]$. So, the set in Remark 1 is $K = 1$ and the functional is

$$\rho_p(S; \Delta, y) = \sqrt{\int_{x_0}^{x_1} [S_1(x) - L_1(x)]^2 dx}.$$

Then, we can consider the residual

$$R(m_0, M_0) = \int_{x_0}^{x_1} [S_1(x) - L_1(x)]^2 dx,$$

where S_1 is obtained by the interpolation conditions:

$$S_1(x_0) = y_0, \quad S_1(x_1) = y_1, \quad S_1'(x_0) = m_0, \quad S_1''(x_0) = M_0,$$

having the expression

$$S_1(x) = \left[1 - \left(\frac{x - x_0}{h_1}\right)^3\right] \cdot y_0 + \left(\frac{x - x_0}{h_1}\right)^3 \cdot y_1 + \tag{2.3}$$

$$+(x - x_0) \cdot \left[1 - \left(\frac{x - x_0}{h_1}\right)^2\right] \cdot m_0 + \frac{(x - x_0)^2}{2} \cdot \left[1 - \left(\frac{x - x_0}{h_1}\right)\right] \cdot M_0,$$

$x \in [x_0, x_1]$ and $h_1 = x_1 - x_0$.

So, the residual is, $R : \mathbb{R}^2 \to \mathbb{R}$

$$R(m_0, M_0) = \int_{x_0}^{x_1} [A(x)y_0 + B(x)y_1 + C(x)m_0 + D(x)M_0 - L(x)]^2 dx,$$

where

$$A(x) = 1 - \left(\frac{x - x_0}{h_1}\right)^3, \; B(x) = \left(\frac{x - x_0}{h_1}\right)^3,$$

$$C(x) = (x - x_0) \cdot \left[1 - \left(\frac{x - x_0}{h_1}\right)^2\right], \; D(x) = \frac{(x - x_0)^2}{2} \cdot \left[1 - \left(\frac{x - x_0}{h_1}\right)\right],$$

$$L_1(x) = \frac{x - x_0}{h_1} \cdot y_1 + \frac{x_1 - x}{h_1} \cdot y_0.$$

The values m_0 and M_0 will be determined in order to minimize the residual R and consequently, the functional ρ_p will be minimized.

Proposition 2.1 *There exist unique values* $\overline{m_0}, \overline{M_0} \in \mathbb{R}$ *such that*

$$R(\overline{m_0}, \overline{M_0}) = min\left\{R(m_0, M_0) : (m_0, M_0) \in \mathbb{R}^2\right\}.$$

The values are

$$m_0 = \frac{y_1 - y_0}{9h_1},\ M_0 = \frac{44(y_1 - y_0)}{9h_1^2} \tag{2.4}$$

Proof. Applying the least squares method, the system of normal equations is:

$$\left\{\begin{array}{l} \frac{\partial R}{\partial m_0} = 0 \\ \frac{\partial R}{\partial M_0} = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \left(\int_{x_0}^{x_1} [C(x)]^2 dx\right) \cdot m_0 + \left(\int_{x_0}^{x_1} C(x) \cdot D(x) dx\right) \cdot M_0 = \\ \left(\int_{x_0}^{x_1} C(x) \cdot D(x) dx\right) \cdot m_0 + \left(\int_{x_0}^{x_1} [D(x)]^2 dx\right) \cdot M_0 = \end{array}\right.$$

$$\left\{\begin{array}{l} = -\int_{x_0}^{x_1} [A(x)y_0 + B(x)y_1 - L(x)]C(x)dx \\ \\ = -\int_{x_0}^{x_1} [A(x)y_0 + B(x)y_1 - L(x)]D(x)dx \end{array}\right.$$

and after elementary calculus,

$$\left\{\begin{array}{l} \frac{23}{210}h_1^3 \cdot m_0 + \frac{11}{840}h_1^4 \cdot M_0 = \frac{8}{105}h_1^2 \cdot (y_1 - y_0) \\ \frac{11}{840}h_1^4 \cdot m_0 + \frac{1}{420}h_1^5 \cdot M_0 = \frac{11}{840}h_1^3 \cdot (y_1 - y_0) \end{array}\right. .$$

Since the determinant of the system is $\Delta = \frac{63}{705600}h_1^8 > 0$, there exist an unique solution $(\overline{m_0}, \overline{M_0})$ of this system

$$\overline{m_0} = \frac{y_1 - y_0}{9h_1},\ \overline{M_0} = \frac{44(y_1 - y_0)}{9h_1^2}.$$

The hessian of R has the diagonal minors:

$$\delta = det\left(\frac{\partial^2 R}{\partial m_0^2}(\overline{m_0}, \overline{M_0})\right) = 2\int_{x_0}^{x_1} [C(x)]^2 dx > 0$$

and

$$\overline{D} = det\begin{pmatrix} \frac{\partial^2 R}{\partial m_0^2}(\overline{m_0}, \overline{M_0}) & \frac{\partial^2 R}{\partial m_0 \partial M_0}(\overline{m_0}, \overline{M_0}) \\ \frac{\partial^2 R}{\partial m_0 \partial M_0}(\overline{m_0}, \overline{M_0}) & \frac{\partial^2 R}{\partial M_0^2}(\overline{m_0}, \overline{M_0}) \end{pmatrix} = 4\Delta > 0.$$

□

So, $(\overline{m_0}, \overline{M_0})$ is the unique point which minimizes R.

Remark 2.1 Replacing in (2.3) the values $\overline{m_0}, \overline{M_0}$ given in (2.4) we get

$$S_1(x) = [1 - Q(x)] \cdot y_0 + Q(x) \cdot y_1, \quad x \in [x_0, x_1], \tag{2.5}$$

where

$$Q(x) = \frac{1}{9} \cdot \left(\frac{x - x_0}{h_1}\right) + \frac{22}{9} \cdot \left(\frac{x - x_0}{h_1}\right)^2 - \frac{14}{9} \cdot \left(\frac{x - x_0}{h_1}\right)^3.$$

If we introduce the values $\overline{m_0}, \overline{M_0}$ in the relation

$$m_1 = \frac{3}{h_1} \cdot (y_1 - y_0) - 2m_0 - \frac{h_1}{2} \cdot M_0$$

we get $m_1 = \frac{y_1 - y_0}{3h_1}$, the same value as $S_1'(x) = \frac{y_1 - y_0}{3h_1}$.

We construct on the interval $[x_1, x_n]$, the complete cubic spline generated by initial conditions $S : [x_1, x_n] \to \mathbb{R}$ with the conditions

$$S'(x_1) = m_1 = \frac{y_1 - y_0}{3h_1} \quad \text{and} \quad S'(x_n) = \frac{y_n - y_{n-1}}{h_n},\ S \in C^2[a, b].$$

Defining $\overline{S} : [a, b] \to \mathbb{R}$,

$$\overline{S}(x) = \begin{cases} S_1(x), & x \in [x_0, x_1] \\ \\ S(x), & x \in [x_1, x_n] \end{cases} \tag{2.6}$$

we see that $\overline{S} \in C^1[a, b]$ and $\overline{S}$ has minimal partial quadratic oscillation in average on the interval $[x_0, x_1]$.

Using the algorithm of complete cubic spline presented in [5], given in

$$m_1 = \frac{y_1 - y_0}{3h_1},\ M_1 = \frac{S'(x_n) - b_n}{a_n},\ S'(x_n) = \frac{y_n - y_{n-1}}{h_n} \tag{2.7}$$

where a_n, b_n, c_n, d_n are obtained recurrently in

$$\begin{cases} a_i = -2a_{i-1} - \frac{h_i}{2} \cdot c_{i-1}, \\ c_i = -2c_{i-1} - \frac{6}{h_i} \cdot a_{i-1}, \\ b_i = \frac{3}{h_i} \cdot (y_i - y_{i-1}) - 2 \cdot b_{i-1} - \frac{h_i}{2} \cdot d_{i-1}, \\ d_i = \frac{6}{h_i^2} \cdot (y_i - y_{i-1}) - 2 \cdot d_{i-1} - \frac{6}{h_i} \cdot b_{i-1},\ n \geq 2,\ i = \overline{3, n} \end{cases} \tag{2.8}$$

starting from

$$a_2 = -\frac{h_2}{2}, c_2 = -2, b_2 = \frac{3}{h_2} \cdot (y_2 - y_1) - 2m_1, d_2 = \frac{6}{h_2^2} \cdot (y_2 - y_1) - \frac{6}{h_2} \cdot m_1$$

and

$$\begin{cases} m_i = \frac{3}{h_i} \cdot (y_i - y_{i-1}) - 2m_{i-1} - \frac{h_i}{2} \cdot M_{i-1},\ \ i = \overline{2, n-1} \\ M_i = \frac{6}{h_i^2} \cdot (y_i - y_{i-1}) - \frac{6}{h_i} \cdot m_{i-1} - 2 \cdot M_{i-1},\ \ i = \overline{2, n} \end{cases}, \tag{2.9}$$

this complete cubic spline is uniquely obtained according to the following result:

Proposition 2.2 *There exist unique values for m_1 and M_1, given in (2.7), such that the obtained cubic spline generated by initial conditions is complete cubic spline $[x_1, x_n]$.*

Proof. For the shape of aeolian blade profile we use the cubic spline $\overline{S} \in C^1[a,b]$ given in (2.6), with S_1 as in (2.5) and S being complete cubic spline generated by initial conditions with the restrictions (2.1) to the intervals $[x_{i-1}, x_i]$, $i = \overline{2,n}$ and the parameters m_i, M_i, $i = \overline{2,n}$ given in (2.9) and

$$m_1 = \frac{y_1 - y_0}{3h_1}, \quad M_1 = \frac{\frac{y_n - y_{n-1}}{h_n} - b_n}{a_n},$$

a_n, b_n as in (2.8). We see that from (2.8) we can write

$$\begin{pmatrix} a_i \\ c_i \end{pmatrix} = \begin{pmatrix} -2 & \frac{h_i}{2} \\ -\frac{6}{h_i} & -2 \end{pmatrix} \cdot \begin{pmatrix} a_{i-1} \\ c_{i-1} \end{pmatrix}, \quad \forall\, i = \overline{3,n}$$

and since $a_1 = -\frac{h_2}{2} < 0$, $c_1 = -2 < 0$ we obtain $a_2 > 0$, $c_2 > 0$ and generally $a_i > 0$, $c_i > 0$ for even i, $a_i < 0$, $c_i < 0$ for odd i. Then, $a_n \neq 0$ and M_1 is uniquely given in (2.7). The parameter m_1 has unique value, $m_1 = \frac{y_1 - y_0}{3h_1}$. With the recurrent relations (2.9), these uniquely determine the complete cubic spline S. $\square$

3 The error estimation

In order to obtain the error estimation in the interpolation of a continuous function $f : [a,b] \to \mathbb{R}$, with $f(x_i) = y_i$, $\forall i = \overline{0,n}$, by $\overline{S}$ we observe that

$$|S_1(x) - f(x)| = |[1 - Q(x)] \cdot y_0 + Q(x) \cdot y_1 - f(x)| \leq$$

$$\leq |1 - Q(x)| \cdot |f(x) - y_0| + |Q(x)| \cdot |f(x) - y_1| \leq \tag{3.1}$$

$$[|1 - Q(x)| + |Q(x)|] \cdot \omega(f, h_1) = \omega(f, h_1),$$

$\forall x \in [x_0, x_1]$, since $0 \leq Q(x) \leq 1$, $\forall x \in [x_0, x_1]$ and it remains to estimate $|S(x) - f(x)|$ for $x \in [x_1, x_n]$.

The condition $S_i(x_i) = y_i$, $\forall i = \overline{2,n}$ lead us to

$$m_{i-1} = \frac{y_i - y_{i-1}}{h_i} - \frac{(M_i + 2M_{i-1})h_i}{6}, \quad i = \overline{2,n} \tag{3.2}$$

and replacing (3.2) in (2.1) we obtain

$$S_i(x) = \left[\frac{(x - x_{i-1})^2}{2} - \frac{(x - x_{i-1})^3}{6h_i} - \frac{h_i(x - x_{i-1})}{3}\right] \cdot M_{i-1} + \tag{3.3}$$

$$+ \left[\frac{(x - x_{i-1})^3}{6h_i} - \frac{h_i(x - x_{i-1})}{6}\right] \cdot M_i + \frac{x_i - x}{h_i} \cdot y_{i-1} + \frac{x - x_{i-1}}{h_i} \cdot y_i,$$

$\forall x \in [x_{i-1}, x_i], i = \overline{2,n}$.

The smoothness conditions on the interior knots

$$S_i'(x_i) = S_{i+1}'(x_i), \quad \forall i = \overline{2,n-1},$$

lead us to

$$\frac{h_i}{6} \cdot M_{i-1} + \frac{h_i + h_{i+1}}{3} \cdot M_i + \frac{h_{i+1}}{6} \cdot M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}. \tag{3.4}$$

The conditions of complete cubic spline

$$S'(x_1) = m_1 = \frac{y_1 - y_0}{3h_1}, \quad S'(x_n) = m_n = \frac{y_n - y_{n-1}}{h_n}$$

provide the equations:

$$\begin{cases} \frac{h_2}{3} \cdot M_1 + \frac{h_2}{6} \cdot M_2 = \frac{2(y_1 - y_0)}{3h_1} \\ \frac{h_n}{6} \cdot M_{n-1} + \frac{h_n}{3} \cdot M_n = 0 \end{cases} \tag{3.5}$$

which together with (3.4) form a linear system in n equations and n unknowns $M_1, \ldots, M_n$. The system (3.4)+(3.5) is tridiagonal, diagonally dominant and therefore has unique solution which can be found using the iterative algorithm presented in [8] and [1] pages 14-15. Here, we use this system in order to derive the error estimation.

Theorem 3.1 *If $f \in C[a,b]$ with $f(x_i) = y_i$, $\forall i = \overline{0,n}$ and there exists $\beta \geq 1$ such that*

$$\frac{max\{h_i : i = \overline{2,n}\}}{min\{h_i : i = \overline{2,n}\}} \leq \beta,$$

then for $S \in C^2[x_1, x_n]$ complete cubic spline interpolating the values $y_i, i = \overline{1,n}$, where $S'(x_1)$ and $S'(x_n)$ are given in (2.7), the following error estimate holds:

$$|S(x) - f(x)| \leq \left(1 + \frac{3}{4}\beta^2\right) \cdot \omega(f,h), \quad \forall x \in [x_1, x_n], \tag{3.6}$$

where $h = max\{h_i : i = \overline{2,n}\}$ and $\omega(f,h)$ is the modulus of continuity.

Proof. Multiplying (3.4) by $\frac{3}{h_i + h_{i+1}}$, $i = \overline{2,n-1}$ and the equations (3.5) by $\frac{3}{h_2}$ and $\frac{3}{h_n}$ respectively, the system (3.4)+(3.5) takes the form $G \cdot M = d$, where $M = (M_1, \ldots M_n)$

$$G = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \frac{h_2}{2(h_2+h_3)} & 1 & \frac{h_3}{2(h_2+h_3)} & 0 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \frac{h_i}{2(h_i+h_{i+1})} & 1 & \frac{h_{i+1}}{2(h_i+h_{i+1})} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

and $d=(d_1,\dots,d_n)$, $d_1=\frac{2(y_1-y_0)}{h_1h_2}$, $d_n=0$, $d_i=\frac{3(y_{i+1}-y_i)}{h_{i+1}(h_i+h_{i+1})}-\frac{3(y_i-y_{i-1})}{h_i(h_i+h_{i+1})}$, $i=\overline{2,n-1}$.

We see that $G=I+A$ with $\|A\|_\infty=\frac{1}{2}$ and therefore G is invertible and $M=G^{-1}\cdot d$ with $\|M\|_\infty\le\|G^{-1}\|_\infty\cdot\|d\|_\infty$, $\|G^{-1}\|_\infty\le\frac{1}{1-\|A\|_\infty}=2$ and

$$|d_1|\le\frac{2}{h_1h_2}\cdot\omega(f,h_1)\le\frac{2}{\underline{h}^2}\cdot\omega(f,h),\quad |d_n|=0,$$

$$|d_i|\le\frac{3\omega(f,h_{i+1})}{2\underline{h}^2}+\frac{3\omega(f,h_i)}{2\underline{h}^2}\le\frac{3\omega(f,h)}{\underline{h}^2},\ \forall i=\overline{2,n-1}.$$

So, $\|d\|_\infty\le\frac{3}{\underline{h}^2}\cdot\omega(f,h)$, where $\underline{h}=min\{h_i:i=\overline{2,n}\}$ and $\|M\|_\infty\le\frac{6}{\underline{h}^2}\cdot\omega(f,h)$.

Let $x\in[x_1,x_n]$ arbitrary. Then, there exists $i\in\{2,\dots,n\}$ such that $x\in[x_{i-1},x_i]$. Using (3.3), we obtain:

$$|S(x)-f(x)|\le\left|\frac{(x-x_{i-1})^2}{2}-\frac{(x-x_{i-1})^3}{6h_i}-\frac{h_i(x-x_{i-1})}{3}\right|\cdot|M_{i-1}|+$$

$$\left|\frac{(x-x_{i-1})^3}{6h_i}-\frac{h_i(x-x_{i-1})}{6}\right|\cdot M_i+\left|\frac{x_i-x}{h_i}\right|\cdot|y_{i-1}-f(x)|+\left|\frac{x-x_{i-1}}{h_i}\right|\cdot|y_i-f(x)|$$

and because

$$\frac{(x-x_{i-1})^2}{2}-\frac{(x-x_{i-1})^3}{6h_i}-\frac{h_i(x-x_{i-1})}{3}\le0,\ \frac{(x-x_{i-1})^3}{6h_i}-\frac{h_i(x-x_{i-1})}{6}\le0$$

$\forall x\in[x_{i-1},x_i]$ and $|M_{i-1}|\le\frac{6}{\underline{h}^2}\cdot\omega(f,h)$, $|M_i|\le\frac{6}{\underline{h}^2}\cdot\omega(f,h)$ we infer that,

$$|S(x)-f(x)|\le\omega(f,h_i)+\frac{6}{\underline{h}^2}\cdot\omega(f,h)\cdot\frac{|(x-x_{i-1})(x_i-x)|}{2}\le$$

$$\omega(f,h)+\frac{6}{\underline{h}^2}\cdot\omega(f,h)\cdot\frac{h_i^2}{8}\le\omega(f,h)+\frac{3}{4}\cdot\left(\frac{h}{\underline{h}}\right)^2\cdot\omega(f,h)=\left(1+\frac{3}{4}\beta^2\right)\cdot\omega(f,h),$$

$\forall x\in[x_{i-1},x_i]$, $\forall i=\overline{2,n}$. □

Remark 3.1 For equidistant partitions $h=\underline{h}$ and $\beta=1$ the inequality (3.6) becomes $|f(x)-S(x)|\le\frac{7}{h}\cdot\omega(f,h)$, $\forall x\in[x_1,x_n]$. The error estimate in the interpolation of continuous functions by cubic splines generated by boundary conditions, the error estimates were obtained in [12]. For the case of smooth functions $f\in C^1[a,b]$ and $f\in C^2[a,b]$ optimal error estimates were obtained in [11].

Corollary 3.1 *The error estimate is*

$$\left|f(x)-\overline{S}(x)\right|\le\begin{cases}\omega(f,h), & x\in[x_0,x_1]\\ \left[1+\frac{3}{4}\cdot\left(\frac{h}{\underline{h}}\right)^2\right]\cdot\omega(f,h), & x\in[x_1,x_n].\end{cases}$$

Remark 3.2 To obtain shape with high order of smoothness we can construct an interpolation procedure based on quartic splines (see [6]).

4 Numerical experiments

In order to illustrate the interpolation procedure we consider a transversal section of an aeolian blade (see [9]). We consider at an cross section the points $x_i,\ i = \overline{0,10}$ situated on the axis of the section and the corresponding values on the surface of the blade $y_i,\ i = \overline{0,10}$. There are: $x_0 = 0$, $x_1 = 35.5$, $x_2 = 71.00$, $x_3 = 106.50$, $x_4 = 142.00$, $x_5 = 177.50$, $x_6 = 213.00$, $x_7 = 248.50$, $x_8 = 284.00$, $x_9 = 319.50$, $x_{10} = 355.00$ (for top shape). As we can see in Figure 1, the inflection point near the first node increases aerodynamic profile properties.

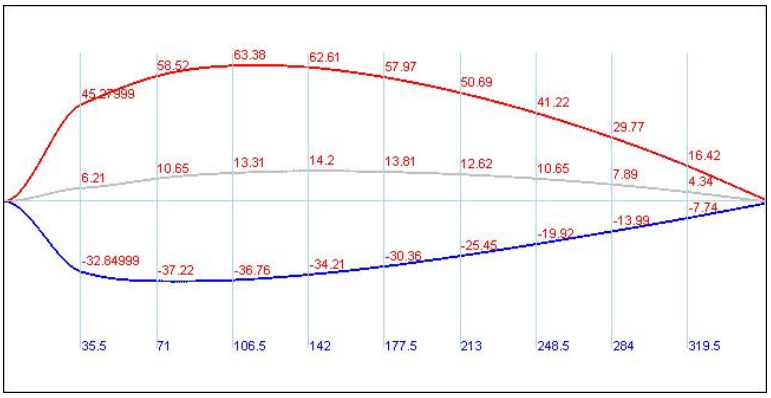


Figure 1: Aeolian blade profile

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Viscosity approximations with weak contraction for finding a common solution of fixed points and a general system of variational inequalities for two accretive operators*

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Abstract

In this paper, we prove a strong convergence theorem for finding a common solutions of a general system of variational inequalities involving two different inverse-strongly accretive operators and solutions of fixed point problems involving the nonexpansive mapping in a Banach space by using a modified viscosity extragradient method with weak contraction. Moreover, using the above results, we can apply to finding solutions of zeros of accretive operators and the class of k-strictly pseudocontractive mappings. The results presented in this paper extend and improve the corresponding results of Qin et al. [Convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mappings with applications, J. Comput. Appl. Math., 233 (2009), 231-240.], Aoyama et al. [Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed Point Theory and Applications, vol. 2006, Article ID 35390, 1-13.], Yao et al. [Modified extragradient methods for a system of variational inequalities in Banach spaces, Acta. Appl. Math., 110(3) (2010), 1211-1224.], Katchang et al. [A viscosity iterative scheme for inverse-strongly accretive operators in Banach spaces, Journal of Computational Analysis and Applications, 12(3) (2010), 678–686.] and many others.

Keywords: Inverse-strongly accretive operator; Fixed point; General system of variational inequalities; Modified viscosity extragradient approximation method; Sunny nonexpansive retraction; Weak contraction

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1 Introduction

Variational inequality has become a rich of inspiration in pure and applied mathematics. In recent years, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences and have withnessed an explosive growth in theoretical advances, algorithmic development, etc; see for e.g. [5, 7, 24] and the references therein. It has been shown a wide class of nonsymmetric and odd-order moving, obstacle, free, equilibrium problems arising in various branches of pure and applied sciences can be studied via the general variational inequalities. In brief, this field is dynamic and is experiencing an explosive growth in both theory and applications. Consequently, several numerical techniques including projection, the Wiener-Hopf equations, auxiliary principle, decomposition and descent are being developed for solving various classes of variational inequalities and related optimization problems. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [4, 8, 21, 23, 39] and the references therein.

To convey an idea of the *variational inequality*, let C be a closed and convex set in a real Hilbert space H. For a given operator A, we consider the problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0$$

for all $x \in C$, which is known as the variational inequality, introduced and studied by Stampacchia [30] in 1964 in the field of potential theory.

Let E be a real Banach space with norm $\|\cdot\|$ and C be a nonempty closed convex subset of E. Let E^* be the dual space of E and $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. We recall the following concepts (see also [32] for). Let $S : C \to C$ a nonlinear mapping. We use $F(S)$ to denote the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. A mapping S is called *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. A mapping $f : C \to C$ is said to be *contraction* if there exists a constant $\alpha \in [0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$. In the cases f is any contraction have recently been studied by Moudafi [20] for a nonexpansive selfmapping S defined on a Hilbert space. Very recently, Suzuki [29] replaced a contraction f by a Meir-Keeler contractions Φ to find a fixed point of a nonexpansive mapping S. A mapping f is called *weakly contractive* on a closed convex set C in the Banach space E if there exists $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function such that φ is positive on $(0, \infty)$, $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ and $x, y \in C$

$$\|f(x) - f(y)\| \leq \|x - y\| - \varphi(\|x - y\|). \tag{1.1}$$

If $\varphi(t) = (1 - k)t$, then f is called to be *contractive* with the contractive coefficient k. If $\varphi(t) \equiv 0$, then f is said to be *nonexpansive*.

For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in E$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and, usually, write $J_2 = J$. Further, we have the following properties of the generalized duality mapping J_q: (i) $J_q(x) =$

$\|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$; (ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$; and (iii) $J_q(-x) = -J_q(x)$ for all $x \in E$. It is known that if X is smooth, then J is single-valued, which is denoted by j . Recall that the duality mapping j is said to be weakly sequentially continuous if for each $x_n \longrightarrow x$ weakly, we have $j(x_n) \longrightarrow j(x)$ weakly-*. We know that if X admits a weakly sequentially continuous duality mapping, then X is smooth (for the details, see [32, 33, 39]).

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if, for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$: see, for instance, [1, 32].

We note that E is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of E. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$. Note also that no Banach space is q-uniformly smooth for $q > 2$; see [32, 36] for more details.

Recall that an operator $A : C \to E$ is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0$$

for all $x, y \in C$. A mapping $A : C \to E$ is said to be *β-strongly accretive* if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \beta\|x - y\|^2 \quad \forall x, y \in C.$$

An operator $A : C \to E$ is said to be *β-inverse strongly accretive* if, for any $\beta > 0$,

$$\langle Ax - Ay, j(x - y)\rangle \geq \beta\|Ax - Ay\|^2$$

for all $x, y \in C$. Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator.

Recently, Aoyama et al. [1] first considered the following generalized variational inequality problem in a smooth Banach space. Let A be an accretive operator of C into E. Find a point $x \in C$ such that

$$\langle Ax, j(y - x)\rangle \geq 0, \tag{1.2}$$

for all $y \in C$. This problem is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [10, 11] . In order to find a solution

of the variational inequality (1.2), Aoyama et al. [1] proved the strong convergence theorem in the framework of Banach spaces which is generalized Iiduka et al. [8] from Hilbert spaces.

In 2006, Aoyama, Iiduka and Takahashi [1] proved the following weak convergence theorem.

Theorem AIT.([1, Aoyama et al. Theorem 3.1]) *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$, and A be an α-inverse strongly accretive operator of C into E with $S(C,A) \neq \emptyset$, where*

$$S(C,A) = \{x^* \in C : \langle Ax^*, j(x - x^*)\rangle \geq 0, \quad x \in C\}.$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$, for some $a > 0$ and $\alpha_n \in [b,c]$, for some b,c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners: $x_1 - x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n),$$

converges weakly to some element z of $S(C,A)$, where K is the 2-uniformly smoothness constant of E and Q_C is a sunny nonexpansive retraction.

Motivated by Aoyama et al. [1] and also Ceng et al. [4], Qin et al. [23] and Yao et al. [39] first considered the following system of general variational inequalities in Banach spaces:

Let $A : C \to E$ be an β-inverse strongly accretive mapping. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, j(x - y^*)\rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{1.3}$$

Let C be nonempty closed convex subset of a real Banach space E. For given two operators $A, B : C \to E$, consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0 \quad \forall x \in C, \end{cases} \tag{1.4}$$

where λ and μ are two positive real numbers. This system is called the *system of general variational inequalities* in a real Banach spaces. If we add up the requirement that $A = B$, then the problem (1.4) is reduced to the system (1.3).

Recently, the so-called viscosity approximation methods have been studied by many author. They are very important because they are applied to convex optimization, linear programming, monotone inclusions and elliptic differential equations. In [20], Moudafi proposed the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. Other investigations of a viscosity approximation method can be found in Reference [3, 6, 34] and many results not cited here.

An interesting problem to extend the above results to find a solution of a general system of variational inequalities. In 2008, Ceng et al. [4] introduced a relaxed extragradient method for finding solutions of a general system of variational inequalities with inverse-strongly monotone mappings in a real Hilbert space.

Suppose $x_1 = u \in C$ and x_n is generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \end{cases} \tag{1.5}$$

for all $n \geq 1$ where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, S is a nonexpansive mapping and A and B are α and β-inverse-strongly monotone, respectively. They proved the strong convergence theorem under quite mild conditions. In 2010, Katchang et al. [13] introduced a viscosity iterative schemes for an inverse-strongly accretive operator in Banach spaces as follows:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Q_C(x_n - \lambda_n A x_n), \tag{1.6}$$

we proved the strong convergent theorems under some parameters controlling conditions. In the same time, Yao et al. [39] introduce the following iteration scheme for solving a general system of variational inequality problem (1.4) and some fixed point problem involving the nonexpansive mapping in Banach spaces. For arbitrarily given $x_0 = u \in C$ and $\{x_n\}$ is given by

$$\begin{cases} y_n = Q_C(x_n - B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - A y_n), \end{cases} \tag{1.7}$$

for all $n \geq 0$ where $C \subset E$, Q_C is a sunny nonexpansive retraction from E onto C and A and B are inverse-strongly accretive mappings. They obtained a strong convergence theorem in Banach spaces. Recently, Katchang and Kumam [12] introduced a viscosity iterative scheme for finding solutions of a general system of variational inequalities (1.4) for two inverse-strongly accretive operators with a viscosity of modified extragradient methods and solutions of fixed point problems involving the nonexpansive mapping in Banach spaces. We prove that the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ v_n = Q_C(y_n - \lambda A y_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\delta S x_n + (1 - \delta) v_n], \end{cases} \tag{1.8}$$

converge strongly to $\bar{x} = Q_{F(G) \cap F(S)} f(\bar{x})$ which $(\bar{x}, \bar{y})$ is a solution of the system of general variational inequalities (1.4), where $\bar{y} = Q_C(\bar{x} - \mu B \bar{x})$.

This naturally brings us to the following questions:

Question 1. Can the results of Ceng et al. [4] be extend from a Hilbert space to a general Banach space?

Question 2. Can we extend and modify the iterative scheme (1.7) to a general algorithm?

Question 3. Can we extend and modify the iterative algorithm (1.7) for finding a common element of the set of solutions of (1.4) and the set of fixed points for nonexpansive mappings?

In this paper, motivated and inspired by the idea of Ceng et al. [4], Yao et al. [39], Iiduka, Takahashi and Toyoda [8], and Iiduka and Takahashi [9] we introduce a new viscosity iterative scheme for finding solutions of a general system of variational inequalities (1.4) involving two different inverse-strongly accretive operators

and solutions of fixed point problems involving the nonexpansive mapping in a Banach space by using a modified viscosity extragradient method. Consequently, we obtain new strong convergence theorems for fixed point problems which solves the system of general variational inequalities (1.3) and (1.4). Moreover, using the above theorem, we can apply to finding solutions of zeros of accretive operators and the class of k-strictly pseudocontractive mappings. The results presented in this paper extend and improve the corresponding results of Yao et al. [39], Ceng et al. [4], Qin et al. [23], Katchang and Kumam [12], Katchang et al. [13] and many others.

2 Preliminaries

We always assume that E is a real Banach space and C be a nonempty closed convex subset of E. Let D be a subset of C and $Q : C \to D$. Then Q is said to *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q. For example, see [1, 31] for more details. The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.1. ([25]) *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;
(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.

Proposition 2.2. ([14]) *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

We need the following lemmas for proving our main results.

Lemma 2.3. ([36]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \qquad \forall x, y \in E.$$

Lemma 2.4. ([28]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5. (Lemma 2.2 in [16]) *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real number sequences and $\{\alpha_n\}$ a positive real number sequence satisfying the conditions: $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\lim_{n\longrightarrow\infty}\frac{b_n}{\alpha_n}=0$. Let the recursive inequality*

$$a_{n+1}\leq a_n-\alpha_n\varphi(a_n)+b_n,\ n\geq 0$$

where $\varphi(a)$ is a continuous and strict increasing function for all $a\geq 0$ with $\varphi(0)=0$. Then $\lim_{n\longrightarrow\infty}a_n=0$.

Lemma 2.6. ([22]) *Let E be a Banach space. Then for all $x,y,z\in E$ and $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma=1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2=\alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.7. ([2]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n\to x$ weakly and $x_n-Tx_n\to 0$ strongly, then x is s fixed point of T.*

Lemma 2.8. (Yao et al. [39, Lemma 3.1]; see also [1, Lemma 2.8]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A:C\to E$ be β-inverse-strongly accretive. Then, we have*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2\leq\|x-y\|^2+2\lambda(\lambda K^2-\beta)\|Ax-Ay\|^2.$$

If $\beta\geq\lambda K^2$, then $I-\lambda A$ is nonexpansive.

Proof. For any $x,y\in C$, from Lemma 2.3, we have

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|(x-y)-\lambda(Ax-Ay)\|^2\\ &\leq \|x-y\|^2-2\lambda\langle Ax-Ay,j(x-y)\rangle+2\lambda^2K^2\|Ax-Ay\|^2\\ &\leq \|x-y\|^2-2\lambda\beta\|Ax-Ay\|^2+2\lambda^2K^2\|Ax-Ay\|^2\\ &= \|x-y\|^2+2\lambda(\lambda K^2-\beta)\|Ax-Ay\|^2.\end{aligned}$$

If $\beta\geq\lambda K^2$, then $I-\lambda A$ is nonexpansive. □

In order to prove our main results, we need the following two lemmas which is proved along the proof of Yao et al.'s lemmas as it appears in [39].

Lemma 2.9. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let the mapping $A,B:C\to E$ be β-inverse-strongly accretive and γ-inverse-strongly accretive, respectively. Let $\Omega:C\to C$ be a mapping defined by*

$$\Omega(x)=Q_C(Q_C(x-\mu Bx)-\lambda AQ_C(x-\mu Bx))\quad\forall x\in C.$$

If $\beta\geq\lambda K^2$ and $\gamma\geq\mu K^2$, then Ω is nonexpansive.

Proof. For any $x, y \in C$, from Lemma 2.8 and Q_C is nonexpansive, we obtain

$$\begin{aligned}
\|\Omega(x) - \Omega(y)\| &= \|Q_C[Q_C(I-\mu B)x - \lambda A Q_C(I-\mu B)x] - Q_C[Q_C(I-\mu B)y - \lambda A Q_C(I-\mu B)y]\| \\
&\leq \|[Q_C(I-\mu B)x - \lambda A Q_C(I-\mu B)x] - [Q_C(I-\mu B)y - \lambda A Q_C(I-\mu B)y]\| \\
&= \|(I-\lambda A)Q_C(I-\mu B)x - (I-\lambda A)Q_C(I-\mu B)y\| \\
&\leq \|Q_C(I-\mu B)x - Q_C(I-\mu B)y\| \\
&\leq \|(I-\mu B)x - (I-\mu B)y\| \\
&\leq \|x-y\|.
\end{aligned}$$

Therefore Ω is nonexpansive. □

Lemma 2.10. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be two possibly nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.4) if and only if $x^* = Q_C(y^* - \lambda A y^*)$ where $y^* = Q_C(x^* - \mu B x^*)$.*

Proof. From (1.4), we rewrite as

$$\begin{cases} \langle x^* - (y^* - \lambda A y^*), j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle y^* - (x^* - \mu B x^*), j(x - y^*)\rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{2.1}$$

Using Proposition 2.1 (iii), the system (2.1) equivalent to

$$\begin{cases} x^* = Q_C(y^* - \lambda A y^*), \\ y^* = Q_C(x^* - \mu B x^*). \end{cases} \tag{2.2}$$

□

Remark 2.11. From Lemma 2.10, we observe that

$$x^* = Q_C(Q_C(x^* - \mu B x^*) - \lambda A Q_C(x^* - \mu B x^*)),$$

which implies that x^* is a fixed point of the mapping Ω.

Throughout this paper, the set of fixed points of the mapping Ω is denoted by $F(\Omega)$.

3 Main results

In this section, we prove a strong convergence theorem. The next result states the main result of this work.

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonexpansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with $\gamma \geq \mu K^2$, respectively and K be the best smooth constant. Let f be a weakly contractive of C into itself with function φ. Suppose*

$\mathcal{F} := F(\Omega) \cap F(S) \neq \emptyset$ where Ω defined by Lemma 2.9. For arbitrary given $x_0 = x \in C$, the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S Q_C(y_n - \lambda A y_n). \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and λ, μ are positive real numbers. The following conditions are satisfied:
(C1). $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
Then $\{x_n\}$ converges strongly to $\bar{x} = Q_{\mathcal{F}} f(\bar{x})$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.4), where $\bar{y} = Q_C(\bar{x} - \mu B\bar{x})$ and $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. First, we prove that $\{x_n\}$ is bounded. Let $x^* \in \mathcal{F}$, from Lemma 2.10, we see that

$$x^* = Q_C(Q_C(x^* - \mu B x^*) - \lambda A Q_C(x^* - \mu B x^*)).$$

Put $y^* = Q_C(x^* - \mu B x^*)$ and $v_n = Q_C(y_n - \lambda A y_n)$. Then $x^* = Q_C(y^* - \lambda A y^*)$. By nonexpansiveness of $I - \lambda A$, $I - \mu B$ and Q_C, we have

$$\begin{aligned} \|v_n - x^*\| &= \|Q_C(y_n - \lambda A y_n) - Q_C(y^* - \lambda A y^*)\| \\ &\leq \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\| \\ &= \|(I - \lambda A)y_n - (I - \lambda A)y^*\| \\ &\leq \|y_n - y^*\| \\ &= \|Q_C(x_n - \mu B x_n) - Q_C(x^* - \mu B x^*)\| \\ &\leq \|(x_n - \mu B x_n) - (x^* - \mu B x^*)\| \\ &= \|(I - \mu B)x_n - (I - \mu B)x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.2}$$

It follows from (3.1) and (3.2), we also have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S v_n - x^*\| \\ &\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|S v_n - x^*\| \\ &\leq \alpha_n \Big[\|x_n - x^*\| - \varphi(\|x_n - x^*\|)\Big] + \beta_n \|x_n - x^*\| + \gamma_n \|v_n - x^*\| \\ &\leq \alpha_n \|x_n - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &= \|x_n - x^*\| \\ &\leq \max\{\|x_1 - x^*\|\}. \end{aligned} \tag{3.3}$$

This implies that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{y_n\}$, $\{v_n\}$, $\{Sv_n\}$, $\{Ay_n\}$ and $\{Bx_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Notice that

$$\begin{aligned}
\|v_{n+1}-v_n\| &= \|Q_C(y_{n+1}-\lambda Ay_{n+1})-Q_C(y_n-\lambda Ay_n)\| \\
&\leq \|(y_{n+1}-\lambda Ay_{n+1})-(y_n-\lambda Ay_n)\| \\
&= \|(I-\lambda A)y_{n+1}-(I-\lambda A)y_n\| \\
&\leq \|y_{n+1}-y_n\| \\
&= \|Q_C(x_{n+1}-\mu Bx_{n+1})-Q_C(x_n-\mu Bx_n)\| \\
&\leq \|(x_{n+1}-\mu Bx_{n+1})-(x_n-\mu Bx_n)\| \\
&= \|(I-\mu B)x_{n+1}-(I-\mu B)x_n\| \\
&\leq \|x_{n+1}-x_n\|.
\end{aligned}$$

Setting $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n$ for all $n\geq 0$, we see that $z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$, then we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n Sv_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}+\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}-\frac{\gamma_{n+1}Sv_n}{1-\beta_{n+1}}+\frac{\gamma_{n+1}Sv_n}{1-\beta_{n+1}} \\
&\quad -\frac{\alpha_n f(x_n)+\gamma_n Sv_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(Sv_{n+1}-Sv_n)+(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})f(x_n) \\
&\quad +(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})Sv_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\
&\quad +|\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}-\frac{1-\beta_n-\alpha_n}{1-\beta_n}|\|Sv_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big[\|x_{n+1}-x_n\|-\varphi(\|x_{n+1}-x_n\|)\Big]+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\| \\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|Sv_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Sv_n\|) \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\|v_{n+1}-v_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Sv_n\|) \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Sv_n\|).
\end{aligned}$$

Therefore

$$\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Sv_n\|).$$

It follow from the condition (C1) and (C2), which implies that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Applying Lemma 2.4, we obtain $\lim_{n\to\infty}\|z_n-x_n\|=0$ and also

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|z_n-x_n\|\to 0$$

as $n\to\infty$. Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

Next, we show that $\lim_{n\to\infty}\|Sv_n-v_n\|=0$. Since $x^*\in\mathcal{F}$, from Lemma 2.6, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n Sv_n-x^*\|^2\\ &\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n-\gamma_n)\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2\\ &= \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\gamma_n(\|x_n-x^*\|^2-\|v_n-x^*\|^2)\\ &\leq \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\|x_n-v_n\|(\|x_n-x^*\|+\|v_n-x^*\|).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\gamma_n\|x_n-v_n\|(\|x_n-x^*\|+\|v_n-x^*\|) &\leq \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\leq \alpha_n\|f(x_n)-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|.\end{aligned}$$

From the condition (C1) and (3.4), this implies that $\|x_n-v_n\|\longrightarrow 0$ as $n\longrightarrow\infty$. Now, we note that

$$\begin{aligned}\|x_n-Sv_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Sv_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n Sv_n-Sv_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_n(f(x_n)-Sv_n)+\beta_n(x_n-Sv_n)\|\\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-Sv_n\|+\beta_n\|x_n-Sv_n\|.\end{aligned}$$

Therefore, we get

$$\|x_n-Sv_n\| \leq \frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-Sv_n\|.$$

From the condition (C1), (C2) and (3.4), this implies that $\|x_n-Sv_n\|\longrightarrow 0$ as $n\longrightarrow\infty$. Since

$$\|Sv_n-v_n\| \leq \|Sv_n-x_n\|+\|x_n-v_n\|,$$

and hence it follows that $\lim_{n\to\infty}\|Sv_n-v_n\|=0$.

Next, we show that $\limsup_{n\to\infty}\langle(f-I)\bar{x},J(x_n-\bar{x})\rangle\leq 0$, where $\bar{x}=Q_{\mathcal{F}}f(\bar{x})$. Since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i}\rightharpoonup x^*$ such that

$$\limsup_{n\to\infty}\langle(f-I)\bar{x},J(x_n-\bar{x})\rangle=\lim_{i\to\infty}\langle(f-I)\bar{x},J(x_{n_i}-\bar{x})\rangle. \tag{3.5}$$

Next, we prove that $x^* \in \mathcal{F} := F(\Omega) \cap F(S)$.
(a) First, we show that $x^* \in F(S)$. To show this, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$. Since $\{v_{n_i}\}$ is bounded, we have that a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ converges weakly to x^*. We may assume without loss of generality that $v_{n_i} \rightharpoonup x^*$. Since $\|Sv_n - v_n\| \to 0$, we obtain $Sv_{n_i} \rightharpoonup x^*$. Then we can obtain $x^* \in \mathcal{F}$. Assume that $x^* \notin F(S)$. Since $v_{n_i} \rightharpoonup x^*$ and $Sx^* \neq x^*$, from Opial's condition, we get

$$\begin{aligned} \liminf_{i\longrightarrow\infty}\|v_{n_i} - x^*\| &< \liminf_{i\longrightarrow\infty}\|v_{n_i} - Sx^*\| \\ &\leq \liminf_{i\longrightarrow\infty}(\|v_{n_i} - Sv_{n_i}\| + \|Sv_{n_i} - Sx^*\|) \\ &\leq \liminf_{i\longrightarrow\infty}\|v_{n_i} - x^*\|. \end{aligned}$$

This is a contradiction. Thus, we have $x^* \in F(S)$.
(b) Next, we show that $x^* \in F(\Omega)$. From Lemma 2.9, we know that Ω is nonexpansive, it follows that

$$\begin{aligned} \|v_n - \Omega(v_n)\| &= \|Q_C(Q_C(x_n - \mu Bx_n) - \lambda AQ_C(x_n - \mu Bx_n)) - G(v_n)\| \\ &= \|\Omega(x_n) - \Omega(v_n)\| \\ &\leq \|x_n - v_n\| \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \end{aligned}$$

Thus $\lim_{n\to\infty} \|v_n - \Omega(v_n)\| = 0$. Since Ω is nonexpansive, we get

$$\begin{aligned} \|x_n - \Omega(x_n)\| &\leq \|x_n - v_n\| + \|v_n - \Omega(v_n)\| + \|\Omega(v_n) - \Omega(x_n)\| \\ &\leq 2\|x_n - v_n\| + \|v_n - \Omega(v_n)\|, \end{aligned}$$

and so

$$\lim_{n\to\infty} \|x_n - \Omega(x_n)\| = 0. \tag{3.6}$$

By Lemma 2.7 and (3.6), we have $x^* \in F(\Omega)$. Therefore $x^* \in \mathcal{F}$.

Now, from (3.5), Proposition 2.1 (iii) and the weakly sequential continuity of the duality mapping J, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle (f - I)\bar{x}, J(x_n - \bar{x})\rangle &= \lim_{i\to\infty}\langle (f - I)\bar{x}, J(x_{n_i} - \bar{x})\rangle \\ &= \langle (f - I)\bar{x}, J(x^* - \bar{x})\rangle \leq 0. \end{aligned} \tag{3.7}$$

From (3.4), it follows that

$$\limsup_{n\to\infty}\langle (f - I)\bar{x}, J(x_{n+1} - \bar{x})\rangle \leq 0. \tag{3.8}$$

Finally, we show that $\{x_n\}$ converges strongly to $\bar{x} = Q_{\mathcal{F}} f(\bar{x})$. We compute that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \langle x_{n+1} - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n S v_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&= \langle \alpha_n (f(x_n) - \bar{x}) + \beta_n (x_n - \bar{x}) + \gamma_n (S v_n - \bar{x}), J(x_{n+1} - \bar{x})\rangle \\
&= \alpha_n \langle f(x_n) - f(\bar{x}), J(x_{n+1} - \bar{x})\rangle + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \beta_n \langle x_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\quad + \gamma_n \langle S v_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\leq \alpha_n \Big[\|x_n - \bar{x}\| - \varphi(\|x_n - \bar{x}\|)\Big] \|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\quad + \beta_n \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| + \gamma_n \|v_n - \bar{x}\| \|x_{n+1} - \bar{x}\| \\
&\leq \alpha_n \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| - \alpha_n \varphi(\|x_n - \bar{x}\|) \|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\quad + \beta_n \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| + \gamma_n \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| \\
&= \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| - \alpha_n \varphi(\|x_n - \bar{x}\|) \|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&= \frac{1}{2}\Big(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\Big) - \alpha_n \varphi(\|x_n - \bar{x}\|) \|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle.
\end{aligned}$$

By (3.3) and since $\{x_{n+1} - \bar{x}\}$ bounded i.e., there exist $M > 0$ such that $\|x_{n+1} - \bar{x}\| \leq M$, which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 - 2\alpha_n M \varphi(\|x_n - \bar{x}\|) + 2\alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle. \tag{3.9}$$

Now, from (C1) and applying Lemma 2.5 to (3.9), we get $\|x_n - \bar{x}\| \to 0$ as $n \to \infty$. This completes the proof. □

Corollary 3.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonepansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with $\gamma \geq \mu K^2$, respectively and K be the best smooth constant. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Let the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and satisfy the condition (C1) and (C2) in Theorem 3.1. Suppose $\mathcal{F} := F(\Omega) \cap F(S) \neq \emptyset$ where Ω defined by Lemma 2.9 and let λ, μ are positive real numbers. For arbitrary given $x_0 = x \in C$, the sequences $\{x_n\}$ generated by*

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S Q_C(y_n - \lambda A y_n). \end{cases} \tag{3.10}$$

Then $\{x_n\}$ converges strongly to $Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. Replacing a weakly contractive f in Theorem 3.1 by an α-contraction f, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.3. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonepansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with $\gamma \geq \mu K^2$, respectively and K*

be the best smooth constant. Let the sequences $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ *and satisfy the condition (C1) and (C2) in Theorem 3.1. Suppose* $\mathcal{F} := F(\Omega) \cap F(S) \neq \emptyset$ *where* Ω *defined by Lemma 2.9 and let* λ, μ *are positive real numbers. For arbitrary given* $x_0 = x \in C$, *the sequences* $\{x_n\}$ *generated by*

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S Q_C(y_n - \lambda A y_n). \end{cases} \tag{3.11}$$

Then $\{x_n\}$ *converges strongly to* $Q_{\mathcal{F}} u$, *where* $Q_{\mathcal{F}}$ *is the sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof. Put $f(x) = u$ for all $x \in C$. Then, by Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.4. *[39, Theorem 3.1,] Let* E *be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and* C *be a nonempty closed convex subset of* E. *Let* Q_C *be a sunny nonexpansive retraction from* E *onto* C. *Let* $A, B : C \to E$ *be* β*-inverse-strongly accretive with* $\beta \geq K^2$ *and* γ*-inverse-strongly accretive with* $\gamma \geq K^2$, *respectively and* K *be the best smooth constant. Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ *and satisfy the condition (C1) and (C2) in Theorem 3.1. Assume* $F(\Omega) \neq \emptyset$ *where* Ω *defined by Lemma 2.9. For arbitrary given* $x_1 = u \in C$, *the sequences* $\{x_n\}$ *generated by (1.7). Then* $\{x_n\}$ *converges strongly to* $Q_{F(\Omega)} u$, *where* $Q_{F(\Omega)}$ *is the sunny nonexpansive retraction of* C *onto* $F(\Omega)$.

Proof. Taking $f(x) = u$ for all $x \in C$, $S = I$ and $\lambda = \mu = 1$ in (3.1). Then, from Theorem 3.1, we can conclude the desired conclusion easily. □

Remark 3.5. Theorem 3.1, Corollary 3.2, Corollary 3.3 and Corollary 3.4 extend, improve and unify the results of Aoyama et al. [1] and also Ceng et al. [4], Qin et al. [23] and Yao et al. [39].

4 Applications

(I) Application to finding zeros of accretive operators.

In Banach space E, we always assume that E is a uniformly convex and 2-uniformly smooth. Recall that an accretive operator T is m-accretive if $R(I + rT) = E$ for each $r > 0$. We assume that T is m-accretive and has a zero (i.e., the inclusion $0 \in T(z)$ is solvable) [15, 35, 37]. The set of zeros of T is denoted by $T^{-1}(0)$, that

$$T^{-1}(0) = \{z \in D(T) : 0 \in T(z)\}.$$

The resolvent of T, i.e., $J_r^T = (I + rT)^{-1}$, for each $r > 0$. If T is m-accretive, then $J_r^T : E \longrightarrow E$ is nonexpansive and $F(J_r^T) = T^{-1}(0), \forall r > 0$. For example, see Rockafellar [27] and [17, 18, 19, 25, 26] for more details.

From the main result Theorem 3.1, we can conclude the following result immediately.

Theorem 4.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with $\gamma \geq \mu K^2$, respectively, K is the 2-uniformly smoothness constant of E and let T be an m-accretive mapping. Let f be a weakly contractive of C into itself with function φ and suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Suppose $\Theta := T^{-1}(0) \cap A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$ and let λ, μ are positive real numbers. The following conditions are satisfied:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
The sequences $\{x_n\}$ generated by $x_0 = x \in C$ and

$$\begin{cases} y_n = x_n - \mu B x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_r^T(y_n - \lambda A y_n). \end{cases} \tag{4.1}$$

Then $\{x_n\}$ converges strongly to $\bar{x} = Q_\Theta f(\bar{x})$, where Q_Θ is the sunny nonexpansive retraction of E onto Θ.

(II) Application to strictly pseudocontractive mappings

Let E be a Banach space and let C be a subset of E. Recall that a mapping $T : C \longrightarrow C$ is said to be *k-stricly pseudocontractive* if there exist $k \in [0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \frac{1-k}{2}\|(I-T)x - (I-T)y\|^2 \tag{4.2}$$

for all $x, y \in C$. Then (4.2) can be written in the following form

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \frac{1-k}{2}\|(I-T)x - (I-T)y\|^2. \tag{4.3}$$

We know that, A is $\frac{1-k}{2}-$ inverse strongly monotone and $A^{-1}0 = F(T)$ (see [9]).

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $S : C \to C$ be a nonepansive mapping and a sunny nonexpansive retraction of E. Let $T, U : C \to C$ be k-stricly pseudocontractive and l-stricly pseudocontractive with $\lambda \leq \frac{(1-k)}{2K^2}$ and $\mu \leq \frac{(1-l)}{2K^2}$, respectively. Let f be a weakly contractive of C into itself with function φ and suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Suppose $\Theta := F(S) \cap F(T) \cap F(U) \neq \emptyset$ and let λ, μ are positive real numbers. The following conditions are satisfied:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
The sequences $\{x_n\}$ generated by $x_0 = x \in C$ and

$$\begin{cases} y_n = (1-\mu)x_n + \mu U x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S((1-\lambda)y_n + \lambda T y_n). \end{cases} \tag{4.4}$$

Then $\{x_n\}$ converges strongly to Q_Θ, where Q_Θ is the sunny nonexpansive retraction of E onto Θ.

Proof. Put $A = I - T$ and $B = I - U$. Form (4.3), we get A, B are $\frac{1-k}{2}$ and $\frac{1-l}{2}-$ inverse strongly accretive operators, respectively. It follows that $VI(C, A) = VI(C, I-T) = F(T) \neq \emptyset$, $CI(C, B) = VI(C, I-U) =$

$F(U) \neq \emptyset$ and $CI(C, I-T) \cap VI(C, I-U) = F(U) = F(\Omega) \Leftrightarrow$ is the solution of problems (1.3) $\Leftrightarrow$ problems (1.4) (see also Ceng et al. [4, Theorem 4.1 pp. 388–389]) and also have (see Aoyama et al.[1, Theorem 4.1 pp. 10.])

$$(1-\lambda)y_n + \lambda T y_n = Q_C((1-\lambda)y_n - \lambda T y_n) \text{ and } (1-\lambda)x_n + \lambda U x_n = Q_C((1-\lambda)x_n - \lambda U x_n).$$

Therefore, by Theorem 3.1, $\{x_n\}$ converges strongly to some element x^* of Θ. □

(III) Application to Hilbert spaces.

In real Hilbert spaces, by Lemma 2.10 and Remark 2.11 it follow from Lemma 4.1 of [23], we obtain the following Lemma:

Lemma 4.3. *For given $(x^*, y^*) \in C$, where $y^* = P_C(x^* - \mu B x^*)$, (x^*, y^*) is a solution of problem (1.4) if and only if x^* is a fixed point of the mapping $\Omega' : C \longrightarrow C$ defined by*

$$\Omega'(x) = P_C[P_C(x - \mu B x) - \lambda A P_C(x - \mu B x)],$$

where P_C is a metric projection H onto C.

It is well known that the smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces. From Theorem 3.1, we can obtain the following result immediately.

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B : C \longrightarrow H$ are an β-inverse-strongly monotone mapping with $\lambda \in (0, 2\beta)$ and γ-inverse-strongly monotone mapping with $\mu \in (0, 2\gamma)$, respectively. Let f be a weakly contractive of C into itself with function φ. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Assume that $F(\Omega') \cap F(S) \neq \emptyset$ where G' defined by Lemma 4.3 and let λ, μ are positive real numbers. The following conditions are satisfied:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
For arbitrary given $x_0 = x \in C$, the sequences $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n). \end{cases} \tag{4.5}$$

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(\Omega')\cap F(S)} f(\bar{x})$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.4), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

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Characterization on Simultaneous Approximation for Left Gamma Quasi-Interpolants in L_p Spaces *

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Abstract:Recently M.W.Müller [9] gave the Gamma quasi-interpolants and obtained approximation equivalence theorems with $\omega_{\varphi}^{2r}(f,t)_p$. In this paper we obtain characterizations on simultaneous approximation for left Gamma quasi-interpolants with $\omega_{\varphi}^{2r}(f,t)_p$.

Key words:Gamma operator; quasi-interpolants; simultaneous approximation; characterization; modulus of smoothness

AMS (2000): 41A25, 41A35, 41A36

1. Introduction

The Gammaoperators

$$G_n(f,x)=\int_0^{\infty} g_n(x,t)f(\frac{n}{t})dt, g_n(x,t)=\frac{x^{n+1}}{n!}e^{-xt}t^n, x\in I:=(0,\infty), \tag{1.1}$$

were introduced by M. Müller and A. Lupas [8]. The other representation of this operator is

$$G_n(f,x)=\frac{1}{n!}\int_0^{\infty} e^{-t}t^n f(\frac{nx}{t})dt, x\in I. \tag{1.2}$$

These operators have been investigated in subsequent papers (e.g.[3],[6],[7],[13],[14]) and their references. In order to obtain much faster convergence rate, quasi-interpolants $G_n^{(k)}$ of G_n in the sense of Sablonnière[11] are considered (cf.[9]), and some approximation equivalence theorem for the quasi-interpolants of Bernstein-type operators and Gamma-type operators were given(cf.[4],[5],[10]). First, we recall their construction. Π_n denotes the space of algebraic polynomials of degree at most n, on Π_n the Gamma operator G_n

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and its inverse G_n^{-1} can be expressed as linear differential operators with polynomials coefficients in the form $G_n = \sum_{j=0}^{n} \beta_j^n D^j$ and $G_n^{-1} = \sum_{j=0}^{n} \alpha_j^n D^j$ with $D = d/dx$ and $D^0 = id$. The left Gamma quasi-interpolants of k degree are defined by (cf.[9])

$$G_n^{(k)}(f,x) = \sum_{j=0}^{k} \alpha_j^n(x) D^j G_n(f,x), 0 \le k \le n. \tag{1.3}$$

In 2001, Müller[9] obtained an approximation equivalence theorem:for $f \in L_p(I), 1 \le p \le \infty, \varphi(x) = x, n \ge 4r, r \in N$, and $0 < \alpha < r$, the following statements are equivalent, that is

$$\|G_n^{(2r-1)} f - f\|_p = O(n^{-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f,t)_p = O(t^{2\alpha}). \tag{1.4}$$

In 2005, Guo[5] gave a weighted approximation equivalence theorem: for $f \in L_\infty(I), 0 \le \lambda \le 1, \varphi(x) = x, w(x) = x^a(1+x)^b (a \ge 0, b$ is arbitrary), $n \ge 4r$, and $0 < \alpha < 2r$, then

$$|w(x)(G_n^{(2r-1)}(f,x) - f(x))| = O((n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^\alpha) \Leftrightarrow \omega_{\varphi^\lambda}^{2r}(f,t)_w = O(t^\alpha). \tag{1.5}$$

The aim of the present article is to prove the analogous result in the L_p-metri. It is easy to see that for every function $f \in L_p(I)$, $G_n^{(2r-1)}(f,x)$ is defined for $x \in I$ and $n \ge 4r$. The analogous result in L_p space is as follow

Theorem 1.1. Let $f^{(s)} \in L_p(I), 0 \le s \le 2r-1, 1 \le p \le \infty, \varphi(x) = x, n \ge 4r$, and $0 < \alpha < 2r - s$. Then the following statements are equivalent

$$\|D^s G_n^{(2r-1)}(f,x) - f^{(s)}(x)\|_p = O(n^{-\alpha/2}) \Leftrightarrow \omega_\varphi^{2r-s}(f^{(s)},t)_p = O(t^\alpha). \tag{1.6}$$

Remark 1.2. when $s = 0$, Theorem 1.1 is the Müller[9] results. Throughout this paper, C denotes a positive constant independent of $n.x$ and not necessarily the same at each occurrence.

2. Preliminaries

In this section, we first give the definition of the modulus and k-functional(cf.[3]):

$$\omega_\varphi^r(f,t)_p = \sup_{0<h<t} \|\Delta_{h\varphi}^r f(x)\|_p, \varphi(x) = x. \tag{2.1}$$

$$K_\varphi^r(f,t^r)_p = \inf_{g \in W_p^r(\varphi,I)} \{\|f-g\|_p + t^r \|\varphi^r g^{(r)}\|_p\}. \tag{2.2}$$

Where $W_p^r(\varphi,I) =: \{g | g \in L_p(I), g^{(r-1)} \in A.C._{loc}(I), \varphi^r g^{(r)} \in L_p(I)\}$. It was proved in [3] that

$$\omega_\varphi^r(f,t)_p \sim K_\varphi^r(f,t^r)_p. \tag{2.3}$$

Some basic properties can be found in [9].

(1)For $j \in N_0, n \geq j$ and $x \in I$

$$\alpha_j^n(x) = (\frac{x}{n})^j L_j^{(n-j)}(n), \alpha_0^n(x) = 1, \alpha_1^n(x) = 0, \tag{2.4}$$

where for $\alpha \in R$

$$L_j^{(\alpha)}(x) = \sum_{r=0}^{j}(-1)^r \binom{j+\alpha}{j-r}\frac{x^r}{r!} \tag{2.5}$$

is the Laguerre polynomial of degree j.

(2)For $j \in N_0$ and $n \geq j$

$$|\frac{1}{n^j}L_j^{(n-j)}(n)| \leq Cn^{-\frac{j}{2}}. \tag{2.6}$$

(3)If $p \in \Pi_k$, then

$$G_n^{(k)}(p, x) = p(x). \tag{2.7}$$

(4)

$$\frac{\partial^m}{\partial x^m}g_n(x,t) = \frac{m!}{x^m}g_n(x,t)L_m^{(n+1-m)}(xt). \tag{2.8}$$

(5)

$$\begin{aligned}(G_n f)^{(2r)}(x) &= \frac{n^{2r}}{n!}\int_0^\infty e^{-t}t^{n-2r}f^{(2r)}(\frac{nx}{t})dt \\ &= \frac{n^{2r}(n-2r)!}{n!}\int_0^\infty g_{n-2r}(x,u)f^{(2r)}(\frac{n}{u})du.\end{aligned} \tag{2.9}$$

(6)For $\alpha > -1$,

$$\int_0^\infty e^{-t}t^\alpha |L_j^{(\alpha)}(t)|^2 dt = \frac{\Gamma(j+\alpha+1)}{j!}. \tag{2.10}$$

(7)For $m, n, l \in N_0$

$$\frac{1}{(n+l)!}\int_0^\infty e^{-t}t^{n+l}(\frac{nx}{t}-x)^m dt \leq C\frac{x^m}{n^{[\frac{(m+1)}{2}]}}. \tag{2.11}$$

For the sake of convenience, utilizing (1.1) and (2.9) we introduce the auxiliary operator for $x \in I$,

$$G_{n,s}^{(k)}f^{(s)} =: \sum_{j=0}^{k}\alpha_j^n(x)\sum_{i=0}^{\hat{s}}\binom{\hat{s}}{i}(j)_i x^{-i}(G_{n,s}f^{(s)})^{(j-i)} \tag{2.12}$$

Where $G_{n,s}f^{(s)} =: \int_0^\infty g_{n-s}(x,t)f^{(s)}(\frac{n}{t})dt = \frac{n!}{n^s(n-s)!}D^s G_n f, \hat{s} = min\{s, j\}, (j)_0 = 1, (j)_i = j(j-1)\cdots(j-i+1)$ for $i \geq 1$. Hence, $G_{n,s}^{(k)}(1,x) = 1 + O(\frac{1}{n})$. Moreover, for $f^{(s)} \in L_p(I)$, there holds $G_{n,s}^{(k)}(f^{(s)}, x) = \frac{n!}{n^s(n-s)!}D^s G_n^{(k)}(f,x)$.

Next we give two lemmas.

Lemma 2.1. Let $r \in N, \varphi(x) = x, 0 \le s \le 2r-1, 1 \le p \le \infty$. Then for $n \ge max\{2, r\}$

$$\|\varphi^r(G_{n,s}f^{(s)})^{(r)}\|_p \le Cn^{r/2}\|f^{(s)}\|_p, f^{(s)} \in L_p(I). \tag{2.13}$$

Proof. Let us use the fact that the norm of an integral is not greater the integral of the corresponding norms(cf.[12,p.271]) and (2.8), we have

$$\begin{aligned} I: &= \|\varphi^r(G_{n,s}f^{(s)})^{(r)}\|_p \\ &= \{\int_0^\infty |\varphi^r(x)\int_0^\infty \frac{r!}{x^r}g_{n-s}(x,t)L_r^{(n-s+1-r)}(xt)f^{(s)}(\frac{n}{t})dt|^p dx\}^{1/p} \\ &= \{\int_0^\infty |\frac{r!}{(n-s)!}\int_0^\infty e^{-u}u^{n-s}L_r^{(n-s+1-r)}(u)f^{(s)}(\frac{nx}{u})du|^p dx\}^{1/p} \\ &\le \frac{C}{(n-s)!n^{1/p}}\int_0^\infty e^{-u}u^{n-s+1/p}|L_r^{(n-s+1-r)}(u)|du\|f^{(s)}\|_p, \end{aligned}$$

by the Cauchy-Schwarz inequality and the orthogonality of Laguerre polynomials (2.10)

$$\begin{aligned} I &\le \frac{C}{(n-s)!n^{1/p}}(\int_0^\infty e^{-u}u^{n-s+1-r}|L_r^{(n-s+1-r)}(u)|du)^{1/2}(\int_0^\infty e^{-u}u^{n-s-1+r+2/p}du)^{1/2}\|f^{(s)}\|_p \\ &\le \frac{C}{(n-s)!n^{1/p}}[\frac{(n-s+1)!}{r!}]^{1/2}[\Gamma(n-s+r+2/p)]^{1/2}\|f^{(s)}\|_p \\ &\le \frac{C}{(n-s)!n^{1/p}}\{\Gamma(n-s+r)^{1-1/p}\Gamma(n-s+r+2)^{1/p}\frac{(n-s+1)!}{r!}\}^{1/2}\|f^{(s)}\|_p \\ &\le \frac{C}{n^{1/p}}\{\frac{(n-s+r-1)!}{(n-s)!}[\frac{(n-s+r+1)!}{(n-s+r-1)!}]^{1/p}\frac{(n-s+1)!}{(n-s)!}\}^{1/2}\|f^{(s)}\|_p \\ &\le \frac{C}{n^{1/p}}(n-s)^{(r-1)/2}(n-s)^{1/p}(n-s)^{1/2}\|f^{(s)}\|_p \\ &\le Cn^{r/2}\|f^{(s)}\|_p. \end{aligned}$$

Where, at the last but one step we also applied the convexity of $\log\Gamma(x)$ (cf.[1]). The proof of lemma 2.1 is complete.

Lemma 2.2. The operator $G_{n,s}^{(2r-1)}(f^{(s)}, x)$ is bounded, that is

$$\|G_{n,s}^{(k)}f^{(s)}\|_p \le C\|f^{(s)}\|_p. \tag{2.14}$$

Proof. Utilizing (2.12), (2.4), (2.6) and (2.13)

$$\begin{aligned} \|G_{n,s}^{(k)}f^{(s)}\|_p &= \|G_{n,s}f^{(s)}\|_p + \sum_{j=2}^{k}|\frac{1}{n^j}L_j^{(n-j)}(n)\sum_{i=0}^{\hat{s}}\binom{\hat{s}}{i}(j)_i\|\varphi^{j-i}(G_{n,s}f^{(s)})^{j-i}\|_p \\ &\le C(\|f^{(s)}\|_p + \sum_{j=2}^{k}n^{-j/2}\sum_{i=0}^{\hat{s}}n^{(j-i)/2}\|f^{(s)}\|_p) \\ &\le C\|f^{(s)}\|_p. \end{aligned}$$

Thus, (2.14) is valid. The proof of lemma 2.2 is complete.

3. The Direct Theorem

In this section we will give an direct approximation theorem.

Theorem 3.1. Let $n \geq 4r, 0 \leq s \leq 2r-1, \varphi(x) = x$, then for all function $f^{(s)} \in L_p(I), 1 \leq p \leq \infty$, we have

$$\|D^s G_n^{(2r-1)}(f,x) - f^{(s)}(x)\|_p \leq C\omega_\varphi^{2r-s}(f^{(s)}, n^{-1/2})_p. \tag{3.1}$$

Proof. By the definition of $K_\varphi^{2r-s}(f, t^{2r-s})$, for fixed n, x we can choose $g(t) = g_{n,x}(t)$ such that

$$\|f^{(s)} - g\|_p + (\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s} g^{(2r-s)}\|_p \leq 2K_\varphi^{2r-s}(f^{(s)}, (\frac{1}{\sqrt{n}})^{2r-s})_p. \tag{3.2}$$

By (2.12) and (2.14), we have

$$\begin{aligned}
&\|D^s G_n^{(2r-1)}(f,x) - f^{(s)}(x)\|_p = \|G_{n,s}^{(2r-1)}(f^{(s)},x) - f^{(s)}(x)\|_p \\
&\leq \|G_{n,s}^{(2r-1)}(f^{(s)} - g, x)\|_p + \|f^{(s)} - g\|_p + \|G_{n,s}^{(2r-1)}(g,x) - g(x)\|_p \\
&\leq C(\|f^{(s)} - g\|_p + \|G_{n,s}^{(2r-1)}(g,x) - g(x)\|_p) \\
&\leq C(K_\varphi^{2r-s}(f^{(s)}, (\frac{1}{\sqrt{n}})^{2r-s})_p + \|G_{n,s}^{(2r-1)}(g,x) - g(x)\|_p).
\end{aligned} \tag{3.3}$$

Here we only need to estimate $\|G_{n,s}^{(2r-1)}(g,x) - g(x)\|_p$.

Since $G_{n,s}^{(2r-1)}(f,x)$ is exact on Π_{2r-1}(cf.[9]), we have for all $1 \leq j \leq 2r-1$,

$$G_n^{(2r-1)}((t-x)^j, x) = 0. \tag{3.4}$$

Note that $D^s G_n^{(2r-1)}(f,x) = \dfrac{n^s(n-s)!}{n!} G_{n,s}^{(2r-1)}(f^{(s)}, x)$, so

$$D^s G_n^{(2r-1)}((t-x)^j, x) = \frac{n^s(n-s)!}{n!}(j)_s G_{n,s}^{(2r-1)}((t-x)^{j-s}, x).$$

We expand an arbitrary function $g \in W_p^{2r-s}(\varphi, I)$ by the Taylor formula

$$g(t) = \sum_{j=0}^{2r-s} \frac{1}{j!}(t-x)^j g^{(j)}(x) + R_{2r-s}(g,t,x). \tag{3.5}$$

with the integral remainder $R_{2r-s}(g,t,x) = \dfrac{1}{(2r-s-1)!}\displaystyle\int_x^t (t-u)^{2r-s-1} g^{(2r-s)}(u)du, x.t \in I$. By (3.5) and (3.4), we obtain $G_{n,s}^{(2r-1)}(g,x) - g(x) = G_{n,s}^{(2r-1)}(R_{2r-s}(g,\cdot,x),x)$, and, con-

sequently, with (2.12), (2.4), and (2.6)

$$\begin{aligned}
&\|G_{n,s}^{(2r-1)}g-g\|_p=\|\sum_{j=0}^{2r-1}\alpha_j^n(x)\sum_{i=0}^{\hat{s}}\binom{\hat{s}}{i}(j)_i x^{-i}D^{j-i}G_{n,s}(R_{2r-s}(g,\cdot,x),x)\|_p\\
&\le\|G_{n,s}(R_{2r-s}(g,\cdot,x),x)\|_p+C\sum_{j=2}^{2r-1}n^{-j/2}\sum_{i=0}^{\hat{s}}\|\varphi^{j-i}(x)D^{j-i}G_{n,s}(R_{2r-s}(g,\cdot,x),x)\|_p\\
&=:I+C\sum_{j=2}^{2r-1}n^{-j/2}\sum_{i=0}^{\hat{s}}\|\varphi^{j-i}(x)D^{j-i}G_{n,s}(R_{2r-s}(g,\cdot,x),x)\|_p.
\end{aligned}\tag{3.6}$$

we first estimate the I. Of this the L_p-norm is estimated for $p=1$ and $p=\infty$ separately. The case $1<p<\infty$ follow then by Stein generalization of the Riesz-Thorin interpolation theorem (cf.[2, Theorem 3.6]). We start with $p=\infty$ and make use of the estimate (cf. [3, Lemma 9.6.1]) and (2.11)

$$\begin{aligned}
&|G_{n,s}(R_{2r-s}(g,\cdot,x),x)|\le\frac{1}{(2r-s-1)!}\|\varphi^{2r-s}g^{(2r-s)}\|_\infty\\
&\times\{\frac{1}{(n-s)!}\int_0^\infty e^{-t}t^{n-s}\frac{(nx/t-x)^{2r-s}}{x^{2r-s}}dt+\frac{1}{(n-s)!}\int_0^\infty e^{-t}t^{n-s+1}\frac{(nx/t-x)^{2r-s}}{x^{2r-s}n}dt\}\\
&\le\frac{1}{(2r-s-1)!}\|\varphi^{2r-s}g^{(2r-s)}\|_\infty((\frac{1}{\sqrt{n}})^{2r-s}+\frac{n-s+1}{n}(\frac{1}{\sqrt{n}})^{2r-s})\\
&\le C(\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s}g^{(2r-s)}\|_\infty.
\end{aligned}\tag{3.7}$$

Taking the L_1-norm of I, by (2.11), we have

$$\begin{aligned}
&\|G_{n,s}(R_{2r-s}(g,\cdot,x),x)\|_1\\
&=\frac{1}{(n-s)!(2r-s-1)!}\int_0^\infty|\int_0^\infty(\int_x^{nx/t}(nx/t-u)^{2r-s-1}g^{(2r-s)}(u)du)e^{-t}t^{n-s}dt|dx\\
&\le\frac{n-s+1}{n(2r-s-1)!}\int_0^\infty|g^{(2r-s)}(u)|\frac{1}{(n-s+1)!}\int_0^\infty(nu/t-u)^{2r-s}e^{-t}t^{n-s+1}dtdu\\
&\le\frac{n-s+1}{n(2r-s-1)!}\int_0^\infty|g^{(2r-s)}(u)|u^{2r-s}(\frac{1}{\sqrt{n}})^{2r-s}du\\
&\le C(\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s}g^{(2r-s)}\|_1.
\end{aligned}\tag{3.8}$$

By (3.7), (3.8), and Stein interpolation theorem imply the general estimate

$$\|G_{n,s}(R_{2r-s}(g,\cdot,x),x)\|_p\le(\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s}g^{(2r-s)}\|_p.\tag{3.9}$$

Next we estimate the second term of (3.6). We start with $p=\infty$ and utilize [3, Lemma

9.6.1] and (2.8),

$$
\begin{aligned}
&|\varphi^{j-i}(x)D^{j-i}G_{n,s}(R_{2r-s}(g,\cdot,x),x)|\\
&=\frac{(j-i)!}{(2r-s-1)!}\|\varphi^{2r-s}g^{(2r-s)}\|_\infty\{\frac{1}{(n-s)!}\int_0^\infty e^{-t}t^{n-s}\frac{(nx/t-x)^{2r-s}}{x^{2r-s}}|L_{j-i}^{(n-s+1-j+i)}(t)|dt\\
&+\frac{1}{(n-s)!}\int_0^\infty e^{-t}t^{n-s+1}\frac{(nx/t-x)^{2r-s}}{x^{2r-s}n}|L_{j-i}^{(n-s+1-j+i)}(t)|dt\}\\
&=:\frac{(j-i)!}{(2r-s-1)!}\|\varphi^{2r-s}g^{(2r-s)}\|_\infty\{J_1(x)+J_2(x)\}.
\end{aligned}
\tag{3.10}
$$

By the Cauchy-Schwarz inequality, and (2.10), (2.11), we have

$$
\begin{aligned}
&J_1(x)=\frac{1}{x^{2r-s}}\frac{1}{(n-s)!}\int_0^\infty(e^{-t}t^{n-s+1-j+i})^{1/2}|L_{j-i}^{(n-s+1-j+i)}(t)|\\
&\times(e^{-t}t^{n-s-1+j-i})^{1/2}(nx/t-x)^{2r-s}dt\\
&\le\frac{1}{x^{2r-s}}\{\frac{1}{(n-s)!}\int_0^\infty e^{-t}t^{n-s-1+j-i}(nx/t-x)^{4r-2s}dt\}^{1/2}\\
&\times\{\frac{1}{(n-s)!}\int_0^\infty e^{-t}t^{n-s+1-j+i}|L_{j-i}^{(n-s+1-j+i)}(t)|^2dt\}^{1/2}\\
&\le C\{(n-s)^{j-i-1}n^{-(2r-s)}(n-s)\}^{1/2}\\
&\le C(\frac{1}{\sqrt{n}})^{2r-s-j+i}.
\end{aligned}
$$

In the similarly way from above procedure we can obtain $J_2(x)\le C(\frac{1}{\sqrt{n}})^{2r-s-j+i}$. Thus we get by (3.6), (3.10) and (3.9), the required estimate for $p=\infty$.

$$
\|G_{n,s}^{(2r-1)}g-g\|_\infty\le C(\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s}g^{(2r-s)}\|_\infty. \tag{3.11}
$$

Taking the L_1-norm, and proceeding in the same way as in [6,p.164]

$$
\begin{aligned}
&\|\varphi^{j-i}D^{j-i}G_{n,s}(R_{2r-s}(g,\cdot,x),x)\|_1=\frac{(j-i)!}{(n-s)!(2r-s-1)!}\\
&\times\int_0^\infty|\int_0^\infty e^{-t}t^{n-s}\int_x^{nx/t}(nx/t-u)^{2r-s-1}g^{(2r-s)}(u)L_{j-i}^{(n-s+1-j+i)}(t)dudt|dx\\
&\le\frac{(j-i)!}{(2r-s-1)!}\frac{1}{n}\int_0^\infty|g^{(2r-s)}(u)|\frac{1}{(n-s)!}\int_0^\infty(nu/t-u)^{2r-s}e^{-t}t^{n-s+1}L_{j-i}^{(n-s+1-j+i)}(t)dtdu\\
&=:\frac{(j-i)!}{(2r-s-1)!}\int_0^\infty|g^{(2r-s)}(u)|I(u)du.
\end{aligned}
\tag{3.12}
$$

By the Cauchy-Schwarz inequality, and (2.10), (2.11),

$$
\begin{aligned}
&I(u) \le \frac{1}{n(n-s)!}\int_0^\infty (e^{-t}t^{n-s+1+j-i})^{1/2}(nu/t-u)^{2r-s}(e^{-t}t^{n-s+1-j+i})^{1/2}|L_{j-i}^{(n-s+1-j+i)}(t)|dt\\
&\le \frac{1}{n(n-s)!}\{\int_0^\infty e^{-t}t^{n-s+1+j-i}(nu/t-u)^{4r-2s}\}^{1/2}\{\int_0^\infty e^{-t}t^{n-s+1-j+i}|L_{j-i}^{(n-s+1-j+i)}(t)|^2\}^{1/2}\\
&\le \frac{1}{n}\varphi^{2r-s}(u)(n-s)^{(j-i+1)/2}(n-s)^{1/2}n^{-(2r-s)/2}\\
&\le C\varphi^{2r-s}(u)(\frac{1}{\sqrt{n}})^{2r-s-j+i}.
\end{aligned}
$$

Combining (3.6), (3.10) and (3.12), we obtain

$$\|(G_{n,s}^{(2r-1)}(g,x)-g(x)\|_1 \le C(\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s}g^{(2r-s)}\|_1. \tag{3.13}$$

Together with (3.11), (3.13), and the Riesz-Thorin interpolation theorem imply the general estimate

$$\|G_{n,s}^{(2r-1)}g-g\|_p \le C(\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s}g^{(2r-s)}\|_p, 1\le p\le\infty. \tag{3.14}$$

Therefore, from (2.2), (3.3) ,(2.3), and (3.14), we obtain

$$\|G_{n,s}^{(2r-1)}f^{(s)}-f^{(s)}\|_p \le CK_\varphi^{2r-s}(f^{(s)},(\frac{1}{\sqrt{n}})^{2r-s})_p \le C\omega_\varphi^{2r-s}(f^{(s)},\frac{1}{\sqrt{n}})_p.$$

The proof of the theorem 3.1 is complete.

4. The Inverse Theorem

In this section we will give an inverse result as follow.

Theorem 4.1. Let $f^{(s)} \in L_p(I), 0\le s\le 2r-1, n\ge 4r$, and $0<\alpha<2r-s$, then

$$\|D^sG_{n,s}^{(2r-1)}f-f^{(s)}\|_p = O(n^{-\alpha/2}), \tag{4.1}$$

implies

$$\omega_\varphi^{2r-s}(f^{(s)},t)_p = O(t^\alpha).$$

To prove theorem 4.1, we need the following two lemmas.

Lemma 4.2. Let $r,m\in N, \varphi(x)=x$, and for $1\le p\le\infty$

$$U =: U_p^{2r}(\varphi,I) =: \{g|g^{(2r-1)}\in A.C.loc(I), g^{(2r)}and\varphi^{2r}g^{(2r)}\in L_p(I)\}$$

a linear manifold in $L_p(I)$. Then for $f^{(s)}\in U, n\ge 2r+m$

$$\|\varphi^{2r-s+m}(G_{n,s}f^{(s)})^{(2r-s+m)}\|_p \le Cn^{m/2}\|\varphi^{2r-s}f^{(2r)}\|_p. \tag{4.2}$$

Proof. By (2.8) and (2.9)

$$
\begin{aligned}
&(G_{n,s}f^{(s)})^{(2r-s+m)} = [(G_{n,s}f^{(s)})^{(2r-s)}]^{(m)}(x)\\
&= \frac{(n-s)^{2r-s}(n-2r)!}{(n-s)!}\int_0^\infty \frac{\partial^m}{\partial x^m} g_{n-2r}(x,u) f^{(2r)}(\frac{n}{u})du\\
&= \frac{(n-s)^{2r-s}m!}{(n-s)!n^{2r-s}x^{2r-s+m}}\int_0^\infty e^{-t}t^{n-s}L_m^{(n-2r+1-m)}(t)(\frac{nx}{t})^{2r-s}f^{(2r)}(\frac{n}{u})du,
\end{aligned}
$$

then by the properties of integral (cf.[12,p.271]), and the Cauchy-Schwarz inequlity, and (2.10), (2.11)

$$
\begin{aligned}
&\|\varphi^{2r-s+m}(G_{n,s}f^{(s)})^{(2r-s+m)}\|_p\\
&= \{\int_0^\infty |\frac{m!(n-s)^{2r-s}}{(n-s)!n^{2r-s}}\int_0^\infty e^{-t}t^{n-s}L_m^{(n-2r+1-m)}(t)(\frac{nx}{t})^{2r-s}f^{(2r)}(\frac{nx}{t})dt|^p\}^{1/p}\\
&\le \frac{m!(n-s)^{2r-s}}{(n-s)!n^{2r-s}}\int_0^\infty e^{-t}t^{n-s}|L_m^{(n-2r+1-m)}(t)|\{\int_0^\infty |(\frac{nx}{t})^{2r-s}f^{(2r)}(\frac{nx}{t})|^p dx\}^{1/p}dt\\
&\le \frac{m!(n-s)^{2r-s}}{(n-s)!n^{2r-s}n^{1/p}}\|\varphi^{2r-s}f^{(2r)}\|_p\int_0^\infty e^{-t}t^{n-s+1/p}|L_m^{(n-2r+1-m)}(t)|dt\\
&\le \frac{m!(n-s)^{2r-s}}{(n-s)!n^{2r-s}n^{1/p}}\|\varphi^{2r-s}f^{(2r)}\|_p\{\int_0^\infty e^{-t}t^{n+2r-2s+m-1+2/p}\}^{1/2}\\
&\times\{\int_0^\infty e^{-t}t^{n-2r+1-m}|L_m^{(n-2r+1-m)}(t)|^2\}^{1/2}\\
&\le \frac{m!(n-s)^{2r-s}}{(n-s)!n^{2r-s}n^{1/p}}\|\varphi^{2r-s}f^{(2r)}\|_p\Gamma(n+2r-2s+m+2/p)^{1/2}(\frac{(n-2r+1)!}{m!})^{1/2}\\
&\le \frac{m!(n-s)^{2r-s}}{(n-s)!n^{2r-s}n^{1/p}}\|\varphi^{2r-s}f^{(2r)}\|_p\\
&\times\{\Gamma(n+2r-2s+m)^{1-1/p}\Gamma(n+2r-2s+m+2)^{1/p}\frac{(n-2r+1)!}{m!}\}^{1/2}\\
&\le \frac{C}{n^{1/p}}\|\varphi^{2r-s}f^{(2r)}\|_p\{\frac{(n+2r-2s+m+1)!}{(n-s)!}[\frac{(n+2r-2s+m+1)!}{(n+2r-2s+m-1)!}]^{1/p}\frac{(n-2r+1)!}{(n-s)!}\}^{1/2}\\
&\le \frac{C}{n^{1/p}}\|\varphi^{2r-s}f^{(2r)}\|_p(n-s)^{1/p+m/2}\\
&\le Cn^{m/2}\|\varphi^{2r-s}f^{(2r)}\|_p.
\end{aligned}
$$

Where we have applied the convexity of $\log\Gamma(x)$. Now, on the linear manifold $U = U_p^{2r}(\varphi, I)$ we consider the semi-norm $|g|_U =: \|\varphi^{2r-s}g^{(2r-s)}\|_p$.

Lemma 4.3. Let $1 \le p \le \infty, \varphi(x) = x, 0 \le s \le 2r-1$. Then for $n \ge 4r$

$$
|G_{n,s}^{(2r-1)}f^{(s)}|_U \le C(\frac{1}{\sqrt{n}})^{2r-s}\|f^{(s)}\|_p, f^{(s)} \in L_p(I). \tag{4.3}
$$

$$
|G_{n,s}^{(2r-1)}f^{(s)}|_U \le C\|f^{(s)}\|_U, f^{(s)} \in L_p(I). \tag{4.4}
$$

Proof. We first consider (4.3), by (2.12), (2.4), (2.6) and (2.13),

$$
\begin{aligned}
&|G_{n,s}^{(2r-1)}f^{(s)}|_U = \|\varphi^{2r-s}(\sum_{j=0}^{2r-1}\alpha_j^n(x)\sum_{i=0}^{\hat{s}}\binom{\hat{s}}{i}(j)_i x^{-i}D^{j-i}G_{n,s}f^{(s)})^{(2r-s)}\|_p \\
&\le \|\varphi^{2r-s}(G_{n,s}f^{(s)})^{(2r-s)}\|_p + C\sum_{j=2}^{2r-1}n^{-j/2}\sum_{i=0}^{\hat{s}}\sum_{k=0}^{j-i}\|\varphi^{2r-s+j-i-k}(G_{n,s}f^{(s)})^{(2r-s+j-i-k)}\|_p \\
&\le C(n^{(2r-s)/2}\|f^{(s)}\|_p + \sum_{j=2}^{2r-1}n^{-j/2}\sum_{i=0}^{\hat{s}}\sum_{k=0}^{j-i}n^{(2r-s)/2}n^{(j-i-k)/2}\|f^{(s)}\|_p) \\
&\le Cn^{(2r-s)/2}\|f^{(s)}\|_p(1+\sum_{j=2}^{2r-1}\sum_{i=0}^{\hat{s}}\sum_{k=0}^{j-i}n^{-(i+k)/2}) \\
&\le Cn^{(2r-s)/2}\|f^{(s)}\|_p.
\end{aligned}
$$

Which proves (4.3).

In the same way, by (4.2) we have

$$
\begin{aligned}
&|G_{n,s}^{(2r-1)}f^{(s)}|_U \\
&\le \|\varphi^{2r-s}(G_{n,s}f^{(s)})^{(2r-s)}\|_p + C\sum_{j=2}^{2r-1}n^{-j/2}\sum_{i=0}^{\hat{s}}\sum_{k=0}^{j-i}\|\varphi^{2r-s+j-i-k}(G_{n,s}f^{(s)})^{(2r-s+j-i-k)}\|_p \\
&\le C\|\varphi^{2r-s}f^{(2r)}\|_p(1+\sum_{j=2}^{2r-1}\sum_{i=0}^{\hat{s}}\sum_{k=0}^{j-i}n^{-(i+k)/2}) \\
&\le C\|\varphi^{2r-s}f^{(2r)}\|_p.
\end{aligned}
$$

Which proves (4.4). The proof of lemma 4.3 is complete.

Proof of Theorem 4.1 Using (4.3) and (4.4), and $\|D^sG_{k,s}^{(2r-1)}f - f^{(s)}\|_p = O(k^{-\alpha/2}), k \ge 4r$, by (2.2) we obtain

$$
\begin{aligned}
&K_\varphi^{2r-s}(f^{(s)},(\frac{1}{\sqrt{n}})^{2r-s})_p \\
&\le \|f^{(s)} - G_{k,s}^{(2r-1)}f^{(s)}\|_p + (\frac{1}{\sqrt{n}})^{2r-s}\|\varphi^{2r-s}(G_{k,s}^{(2r-1)}f^{(s)})^{(2r-s)}\|_p \\
&\le C_1k^{-\alpha/2} + (\frac{1}{\sqrt{n}})^{2r-s}(\|\varphi^{2r-s}(G_{k,s}^{(2r-1)}(f^{(s)}-g))^{(2r-s)}\|_p + \|\varphi^{2r-s}(G_{k,s}^{(2r-1)}g)^{(2r-s)}\|_p) \\
&\le C_1k^{-\alpha/2} + C_2(\frac{1}{\sqrt{n}})^{2r-s}(k^{(2r-s)/2}\|f^{(s)}-g)\|_p + \|\varphi^{2r-s}g^{(2r-s)}\|_p) \\
&\le C_1k^{-\alpha/2} + C_2(\frac{k}{n})^{(2r-s)/2}K_\varphi^{2r-s}(f^{(s)},k^{-(2r-s)/2})_p.
\end{aligned}
$$

By Berens-Lorentz lemma implies

$$K_\varphi^{2r-s}(f^{(s)},t^{-(2r-s)/2})_p \sim C\omega_\varphi^{2r-s}(f^{(s)},n^{-1/4})_p = O(n^{-\alpha/2}).$$

Hence we have $\omega_\varphi^{2r-s}(f^{(s)},t)_p = O(t^\alpha)$. The proof of Theorem 4.1 is complete.

Remark 4.4. Mainly by Theorem 3.1 and 4.1 the equivalence theorem 1.1 is valid.

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Comment on "On the stability of quadratic double centralizers on Banach algebras" [M. Eshaghi Gordji, A. Bodaghi, J. Comput. Anal. Appl. 13 (2011), 724-729]

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Abstract. Eshaghi Gordji and Bodaghi [2] proved the Hyers-Ulam stability of quadratic double centralizers on Banach algebras for the system of the functional equations

$$f(kx+y)+f(kx-y)=2k^2f(x)+2f(y) \qquad \& \qquad f(xy)=f(x)y$$

for a fixed integer k greater than 1. One can easily show that all the quadratic double centralizers (L,R) in the results are $(0,0)$. The results are trivial. In this paper, we correct the results.

Using the direct method, we prove the Hyers-Ulam stability of quadratic double centralizers on Banach algebras for the system of the functional equations

$$f(kx+y)+f(kx-y)=2k^2f(x)+2f(y) \qquad \& \qquad f(xy)=f(x)y^2$$

for a fixed integer k greater than 1.

1. Introduction

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: When is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

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The first stability problem concerning group homomorphisms was raised by Ulam [7] in 1940 and affirmatively solved by Hyers [3]. The result of Hyers was generalized by Aoki [1] for approximate additive mappings and by Th.M. Rassias [5] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\varepsilon(\| x \|^p + \| y \|^p)$.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to symmetric bi-additive function and is called a *quadratic functional equation* and every solution of the quadratic equation (1.1) is said to be a *quadratic mapping*. Skof [6] proved the Hyers-Ulam stability of the quadratic functional equation (1.1).

Definition 1.1. *Let A be a Banach algebra.*

(1) *A quadratic mapping $L : A \to A$ is said to be a quadratic left centralizer if $L(ab) = L(a)b^2$ for all $a, b \in A$.*

(2) *A quadratic mapping $R : A \to A$ is said to be a quadratic right centralizer if $R(ab) = a^2R(b)$ for all $a, b \in A$.*

(3) *A quadratic double centralizer on A is a pair (L, R), where L is a quadratic left centralizer, R is a quadratic right centralizer and $a^2L(b) = R(a)b^2$ for all $a, b \in A$.*

Assume that k is a fixed integer greater than 1.

2. Main results

Moslehian, Rahbarnia and Sahoo [4] proved the Hyers-Ulam stability of double centralizers in the framework of Banach algebras.

In this paper, we prove the same results for quadratic double centralizers.

Theorem 2.1. *Let A be a Banach algebra. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$. If there exist a mapping $g : A \to A$ with $g(0) = 0$ and functions $\chi : A^2 \to [0,\infty)$ and $\sigma : A^2 \to [0,\infty)$ such that*

$$\widetilde{\chi}(a,b) := \sum_{n=0}^{\infty} \frac{1}{k^{2n}} \chi(k^n a, k^n b) < \infty, \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{k^{4n}} \sigma(k^n a, k^n b) = 0,$$

$$\| f(ka+b)+f(ka-b)-2k^2 f(a)-2f(b) \| \le \chi(a,b), \tag{2.2}$$

$$\| f(ab)-f(a)b^2 \| \le \sigma(a,b), \tag{2.3}$$

$$\| g(ka+b)+g(ka-b)-2k^2 g(a)-2g(b) \| \le \chi(a,b), \tag{2.4}$$

$$\| g(ab)-a^2 g(b) \| \le \sigma(a,b), \tag{2.5}$$

$$\| a^2 f(b)-g(a)b^2 \| \le \sigma(a,b) \tag{2.6}$$

for all $a, b \in A$, then there exists a unique quadratic double centralizer (L, R) such that

$$\| f(a)-L(a) \| \le \frac{1}{2k^2} \widetilde{\chi}(a,0),$$

$$\| g(a)-R(a) \| \le \frac{1}{2k^2} \widetilde{\chi}(a,0)$$

for all $a \in A$

Proof. Setting $b = 0$ in (2.2), we have

$$\| 2f(ka) - 2k^2 f(a) \| \leq \chi(a, 0)$$

for all $a \in A$. Thus

$$\left\| \frac{1}{k^2} f(ka) - f(a) \right\| \leq \frac{1}{2k^2} \chi(a, 0)$$

for all $a \in A$. Hence for all nonnegative integers p, q with $q > p$ and all $a \in A$, we have

$$\left\| \frac{1}{k^{2p}} f(k^p a) - \frac{1}{k^{2q}} f(k^q a) \right\| \leq \sum_{n=p}^{q-1} \frac{1}{2k^{2n+2}} \chi(k^n a, 0). \tag{2.7}$$

It follows from (2.1) and (2.7) that sequence $\{\frac{1}{k^{2n}} f(k^n a)\}$ is Cauchy for all $a \in A$. By the completeness of A, the sequence $\{\frac{1}{k^{2n}} f(k^n a)\}$ converges. So we define the mapping $L : A \to A$ by

$$L(a) := \lim_{n \to \infty} \frac{1}{k^{2n}} f(k^n a)$$

for all $a \in A$. It follows from (2.1) and (2.2) that

$$\begin{aligned} &\| L(ka + b) - L(ka - b) - 2k^2 L(a) - 2L(b) \| \\ &= \lim_{n \to \infty} \frac{1}{k^{2n}} \| f(k^{n+1} a + k^n b) - L(k^{n+1} a - k^n b) - 2k^2 f(k^n a) - 2L(k^n b) \| \\ &\leq \lim_{n \to \infty} \frac{1}{k^{2n}} \chi(k^n a, k^n b) = 0. \end{aligned}$$

So the mapping L is quadratic.

Replacing a and b by $k^n a$ and $k^n b$ in (2.3), respectively, we have

$$\| 2f(k^{2n} ab) - 2f(k^n a) k^{2n} b^2 \| \leq \sigma(k^n a, k^n b).$$

Dividing the both sides of the above inequality by k^{4n}, we get

$$\left\| \frac{f(k^{2n} ab)}{k^{4n}} - \frac{f(k^n a)}{k^{2n}} b^2 \right\| \leq \frac{1}{2k^{4n}} \sigma(k^n a, k^n b).$$

Taking limit as $n \to \infty$, we conclude $L(ab) = L(a)b^2$. So the quadratic mapping L is a quadratic left centralizer.

Putting $p = 0$ and passing the limit $q \to \infty$ in (2.7), we get

$$\| f(a) - L(a) \| \leq \frac{1}{2k^2} \widetilde{\chi}(a, 0).$$

Let $L' : A \to A$ be another quadratic mapping satisfying $\| f(a) - L'(a) \| \leq \frac{1}{2k^2} \widetilde{\chi}(a, 0)$. Since L and L' are quadratic, for each $a \in A$, we have

$$\begin{aligned} \| L(a) - L'(a) \| &= \frac{1}{k^{2n}} \| L(k^n a) - L'(k^n a) \| \\ &\leq \frac{1}{k^{2n}} (\| L(k^n a) - f(k^n a) \| + \| L'(k^n a) - f(k^n a) \|) \\ &\leq \frac{1}{k^{2n+2}} \widetilde{\chi}(k^n a, 0), \end{aligned}$$

which tend to zero as $n \to \infty$ for all $a \in A$. So we have $L(a) = L'(a)$ for all $a \in A$.

A similar argument for g shows that there exists a unique quadratic right centralizer $R : A \to A$ satisfying

$$\| g(a) - R(a) \| \leq \frac{1}{2k^2} \widetilde{\chi}(a, 0) \qquad \& \qquad R(ab) = a^2 R(b)$$

for all $a, b \in A$. The quadratic right centralizer $R : A \to A$ is defined by $R(A) := \lim_{n\to\infty} \frac{g(k^n a)}{k^{2n}}$.

Replacing a and b by $k^n a$ and $k^n b$ in (2.6) and dividing by k^{4n}, we get

$$\| a^2 \frac{f(k^n b)}{k^{2n}} - \frac{g(k^n a)}{k^{2n}} b^2 \| \leq \frac{1}{k^{4n}} \sigma(k^n a, k^n b).$$

Passing to the limit as $n \to \infty$, we obtain $a^2 L(b) = R(a)b^2$ for all $a, b \in A$, as desired. □

Theorem 2.2. *Let A be a Banach algebra. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$. If there exist a mapping $g : A \to A$ with $g(0) = 0$ and functions $\chi : A^2 \to [0, \infty)$ and $\sigma : A^2 \to [0, \infty)$ satisfying (2.2)–(2.6) and*

$$\widetilde{\chi}(a, b) := \sum_{n=1}^{\infty} k^{2n} \chi(k^{-n} a, k^{-n} b) < \infty,$$

$$\lim_{n\to\infty} k^{4n} \sigma(k^{-n} a, k^{-n} b) = 0$$

for all $a, b \in A$, then there exists a unique quadratic double centralizer (L, R) such that

$$\| f(a) - L(a) \| \leq \frac{1}{2k^2} \widetilde{\chi}(a, 0),$$

$$\| g(a) - R(a) \| \leq \frac{1}{2k^2} \widetilde{\chi}(a, 0)$$

for all $a \in A$

Proof. Using the same method as in proof of Theorem 2.1, one can obtain the result. □

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A weighted bivariate blending rational interpolation function and visualization control

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Abstract

A new weighted bivariate blending rational spline with parameters is constructed based on function values of a function only. The interpolation is C^1 in the whole interpolating region under the condition which free parameters is not limited. This paper deals with the properties of the interpolation surface, including the properties of basis function, the properties of integral weighted coefficients and bounded property of the interpolation. In order to meet the needs of practical design, an interpolation technique is employed to control the shape of surfaces. The method of value control of the interpolation at any point in the interpolating region is developed. This control method can be applied to modify the local shape of an interpolating surface by selecting suitable parameters simply.

Keywords: Rational spline, bivariate blending interpolation, weighted interpolation, computer-aided geometric design

1. Introduction

The construction method of curve and surface and the mathematical description of them is a key issue in computer-aided geometric design. There are many ways to tackle this problem [2-4,9,10,12-14,17,18], for example, the polynomial spline method, the NURBS method and the Bèzier method. These methods are effective and applied widely in shape design of industrial

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products. Generally speaking, most of the polynomial spline methods are the interpolating methods. However, to construct the polynomial spline, usually derivative values are needed as the interpolating data besides the function values. Unfortunately, in many practical problems, such as the description of rainfall in some rainy region and some industrial geometric shapes, derivative values are difficult to obtain. On the other hand, one of the disadvantages of the polynomial spline method is its global property; it is not possible for the local modification for unchanged given data. The NURBS and Bèzier methods are the so-called "no-interpolating type" methods; this means that the constructed curve and surface do not pass through the given data, and the given points play the role of the control points. In order to meet the need of ever-increasing model complexity and to incorporate manufacturing requirement, the main difficulties to be solved are: (i) how to construct the spline interpolation with simple and explicit mathematical expressions so that they may be convenient to use both in practical application and in theoretical study; (ii) how to modify the shape of a curve or surface for unchanged given data. It seems that these two difficulties are contradictory in view of the uniqueness of the interpolation for the given interpolating data.

In recent years, a univariate rational spline interpolation with the parameters has received attention in the literature [1,5,11,15,16]. Since the parameters in the interpolation function are selective according to the control need, the constrained control of the shape becomes possible. Motivated by the univariate rational spline interpolation, the bivariate rational interpolation with parameters which has simple and explicit mathematical representation has been studied [6-8]. The construction of these kinds of interpolation splines can solve aforementioned difficulties. However, for the bivariate rational interpolation, in order to maintain smoothness of the interpolating surface, the parameters of y-direction for bivariate interpolating surface must be limited, they can not be selected freely for different interpolating subregion. In this paper, A new weighted bivariate blending rational interpolation will be constructed based only on function values of interpolated function, and the general properties and local shape control technique of this interpolation will be also discussed.

This paper is arranged as follows. In Section 2, a new weighted bivariate blending rational interpolation based only on the function values is constructed. Section 3 discusses the bases function of this bivariate interpolation. Section 4 is about some properties of the interpolation, especially, the properties of integral weights coefficients of the interpolation is derived. In Section

5, the bounded property of this interpolation is given. Section 6 deals with the modification of the interpolating surface, in the special case, the central point value control technique is developed. And numerical examples are presented to show the performance of the method in Section 7.

2. Interpolation

Let $\Omega : [a, b; c, d]$ be the plane region, and $\{(x_i, y_i, f_{i,j}, i = 1, 2, \cdots, n+1; j = 1, 2, \cdots, m+1\}$ be a given set of data points, where $a = x_1 < x_2 < \cdots < x_{n+1} = b$ and $c = y_1 < y_2 < \cdots < y_{m+1} = d$ are the knot spacings, $f_{i,j} = f(x_i, y_j)$. Let $h_i = x_{i+1} - x_i$, and $l_j = y_{j+1} - y_j$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the xy-plane, let $\theta = (x - x_i)/h_i$ and $\eta = (y - y_j)/l_j$. First, for each $y = y_j, j = 1, 2, \cdots, m+1$, construct the x-direction interpolant curve; this is given by

$$\overline{P}^*_{i,j}(x) = \frac{\overline{p}^*_{i,j}(x)}{\overline{q}^*_{i,j}(x)}, \quad i = 1, 2, \cdots, n-1, \tag{1}$$

where

$$\begin{aligned} \overline{p}^*_{i,j}(x) &= (1-\theta)^3 \alpha_{i,j} f_{i,j} + \theta(1-\theta)^2 \overline{V}^*_{i,j} \\ &+ \theta^2(1-\theta)\overline{W}^*_{i,j} + \theta^3 f_{i+1,j}, \\ \overline{q}^*_{i,j}(x) &= (1-\theta)\alpha_{i,j} + \theta, \end{aligned}$$

and

$$\begin{aligned} \overline{V}^*_{i,j} &= (\alpha_{i,j} + 1) f_{i,j} + \alpha_{i,j} f_{i+1,j}, \\ \overline{W}^*_{i,j} &= (\alpha_{i,j} + 2) f_{i+1,j} - h_i \overline{\Delta}^*_{i+1,j}, \end{aligned}$$

with $\alpha_{i,j} > 0$, and $\overline{\Delta}^*_{i,j} = (f_{i+1,j} - f_{i,j})/h_i$. This interpolation is called the rational cubic interpolation based on function values which satisfies

$$\overline{P}^*_{i,j}(x_i) = f_{i,j}, \quad \overline{P}^*_{i,j}(x_{i+1}) = f_{i+1,j},$$

$$\overline{P}^{*\prime}_{i,j}(x_i) = \overline{\Delta}^*_{i,j}, \quad \overline{P}^{*\prime}_{i,j}(x_{i+1}) = \overline{\Delta}^*_{i+1,j}.$$

Obviously, the interpolating function $\overline{P}^*_{i,j}(x)$ on $[x_i, x_{i+1}]$ is unique for the given data $\{x_r, f_{r,j},\ r = i, i+1, i+2\}$ and positive parameter $\alpha_{i,s}$.

For each pair of $(i,j), i = 1,2,\cdots,n-1$ and $j = 1,2,\cdots,m-1$, using the x-direction interpolation function $\overline{P}^*_{i,j}(x)$, define the interpolation function $\overline{P}(x,y)$ on each rectangular region $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$\begin{aligned}\overline{P}_{i,j}(x,y) &= (1-\eta)^3\overline{P}^*_{i,j}(x) + \eta(1-\eta)^2(2\overline{P}^*_{i,j}(x) \\ &+ \overline{P}^*_{i,j+1}(x)) + \eta^2(1-\eta)(3\overline{P}^*_{i,j+1}(x) \\ &- l_j\overline{\Delta}_{i,j+1}(x)) + \eta^3\overline{P}^*_{i,j+1}(x), \end{aligned} \tag{2}$$

where $\overline{\Delta}_{i,j} = (\overline{P}^*_{i,j+1}(x) - \overline{P}^*_{i,j}(x))/l_j$. $\overline{P}_{i,j}(x,y)$ is called the bivariate blending interpolating function based on function values which satisfies

$$\overline{P}_{i,j}(x_r, y_s) = f(x_r, y_s),\ r = i, i+1,\ s = j, j+1$$
$$\frac{\partial \overline{P}_{i,j}(x_r,y_s)}{\partial x} = \overline{\Delta}^*_{r,s},\ \frac{\partial \overline{P}_{i,j}(x_r,y_s)}{\partial y} = \overline{\Delta}_{r,s}.$$

This form of interpolation function $\overline{P}_{i,j}(x,y)$ on $[x_i, x_{i+1};\ y_j, y_{j+1}]$ is unique for the given data $\{(x_r, y_s, f_{r,s}), r = i, i+1, i+2; s = j, j+1, j+2\}$ and parameters $\alpha_{i,s}$, and it is easy to test that the interpolation is C^1 in the whole interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\alpha_{i,s}$ might be.

The interpolating scheme above begins in x-direction first. Now, let the interpolation begins with y-direction first. For each $x = x_i, i = 1,2,\cdots,n+1$, denote the y-direction interpolation in $[y_j, y_{j+1}]$ by

$$\overline{\overline{P}}^*_{i,j}(y) = \frac{\overline{\overline{p}}^*_{i,j}(y)}{\overline{\overline{q}}^*_{i,j}(y)}, \quad i = 1,2,\cdots,m-1, \tag{3}$$

where

$$\begin{aligned}\overline{\overline{p}}^*_{i,j}(y) &= (1-\eta)^3\beta_{i,j}f_{i,j} + \eta(1-\eta)^2\overline{\overline{V}}^*_{i,j} \\ &+ \eta^2(1-\eta)\overline{\overline{W}}^*_{i,j} + \eta^3 f_{i,j+1}, \\ \overline{\overline{q}}^*_{i,j}(y) &= (1-\eta)\beta_{i,j} + \eta, \end{aligned}$$

and

$$\overline{\overline{V}}^*_{i,j} = (\beta_{i,j}+1)f_{i,j} + \beta_{i,j}f_{i,j+1},$$
$$\overline{\overline{W}}^*_{i,j} = (\beta_{i,j}+2)f_{i,j+1} - l_j\overline{\overline{\Delta}}^*_{i,j+1},$$

with $\beta_{i,j} > 0$ and $\overline{\overline{\Delta}}^*_{i,j} = (f_{i,j+1} - f_{i,j})/l_j$.

For each pair $(i,j), i = 1,2,\cdots,n-1$ and $j = 1,2,\cdots,m-1$, using the y-direction interpolation function $\overline{\overline{P}}^*_{i,j}(y)$, define the bivariate blending interpolation function $\overline{\overline{P}}_{i,j}(x,y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$\begin{aligned}\overline{\overline{P}}_{i,j}(x,y) &= (1-\theta)^3\overline{\overline{P}}^*_{i,j}(y) + \theta(1-\theta)^2(2\overline{\overline{P}}^*_{i,j}(y) \\ &+ \overline{\overline{P}}^*_{i+1,j}(y)) + \theta^2(1-\theta)(3\overline{\overline{P}}^*_{i+1,j}(y) \\ &- h_i\overline{\overline{\Delta}}_{i+1,j}(y)) + \theta^3\overline{\overline{P}}^*_{i+1,j}(y), \end{aligned} \tag{4}$$

where $\overline{\overline{\Delta}}_{i,j} = (\overline{\overline{P}}^*_{i+1,j}(y) - \overline{\overline{P}}^*_{i,j}(y))/h_i$. The interpolation function $\overline{\overline{P}}_{i,j}(x,y)$ satisfies

$$\overline{\overline{P}}_{i,j}(x_r, y_s) = f(x_r, y_s),\ r = i, i+1,\ s = j, j+1$$
$$\frac{\partial\overline{\overline{P}}_{i,j}(x_r,y_s)}{\partial x} = \overline{\overline{\Delta}}_{r,s},\ \frac{\partial\overline{\overline{P}}_{i,j}(x_r,y_s)}{\partial y} = \overline{\overline{\Delta}}^*_{r,s}.$$

$\overline{\overline{P}}_{i,j}(x,y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ is unique for the given data $\{(x_r, y_s, f_{r,s}), r = i, i+1, i+2; s = j, j+1, j+2\}$ and parameters $\beta_{r,j}$, and it is easy to derive that the interpolation is C^1 in the whole interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\beta_{r,j}$ might be.

Where, we let

$$P_{i,j}(x,y) = \lambda\overline{P}_{i,j}(x,y) + (1-\lambda)\overline{\overline{P}}_{i,j}(x,y), \tag{5}$$

with the weighted coefficient $\lambda \in [0,1]$, then the weighted bivariate blending rational interpolation function $P_{i,j}(x,y)$ satisfies

$$P_{i,j}(x_r, y_s) = f(x_r, y_s), r = i, i+1,\ s = j, j+1,$$

$$\frac{\partial P_{i,j}(x_r,y_s)}{\partial x} = \lambda\overline{\Delta}^*_{r,s} + (1-\lambda)\overline{\overline{\Delta}}_{r,s},$$
$$\frac{\partial P_{i,j}(x_r,y_s)}{\partial y} = \lambda\overline{\Delta}_{r,s} + (1-\lambda)\overline{\overline{\Delta}}^*_{r,s},$$

and it is C^1 in the whole interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\alpha_{i,s}$ and $\beta_{r,j}$ might be. In the following of this paper, consider the interpolation defined in Eq. (5).

3. The Bases of the Interpolation

Consider the equally spaced knots case, namely, for all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, $h_i = h_j$ and $l_i = l_j$. From the Eq.(1)-(5), the interpolation function $P_{i,j}(x, y)$ defined in (5) can be written as follows:

$$P_{i,j}(x,y) = \sum_{r=0}^{2}\sum_{s=0}^{2} \omega_{r,s}(\theta,\eta) f_{i+r,j+s}, \tag{6}$$

where

$$\begin{aligned}
\omega_{0,0}(\theta,\eta) &= \frac{(1-\theta)^2(\theta+\alpha_{i,j})(1-\eta)^2(1+\eta)\lambda}{(1-\theta)\alpha_{i,j}+\theta} \\
&+ \frac{(1+\theta)(1-\theta)^2(1-\eta)^2(\eta+\beta_{i,j})(1-\lambda)}{(1-\eta)\beta_{i,j}+\eta}, \\
\omega_{0,1}(\theta,\eta) &= \frac{(1-\theta)^2(\theta+\alpha_{i,j+1})\eta(1+2\eta-2\eta^2)\lambda}{(1-\theta)\alpha_{i,j+1}+\theta} \\
&+ \frac{(1-\theta)^2(1+\theta)\eta(3\eta-2\eta^2+(1-\eta)\beta_{i,j})(1-\lambda)}{(1-\eta)\beta_{i,j}+\eta}, \\
\omega_{0,2}(\theta,\eta) &= -\frac{(1-\theta)^2(\theta+\alpha_{i,j+2})\eta^2(1-\eta)\lambda}{(1-\theta)\alpha_{i,j+2}+\theta} \\
&- \frac{(1-\theta)^2(1+\theta)\eta^2(1-\eta)(1-\lambda)}{(1-\eta)\beta_{i,j}+\eta}, \\
\omega_{1,0}(\theta,\eta) &= \frac{\theta(3\theta-2\theta^2+(1-\theta)\alpha_{i,j})(1-\eta)^2(1+\eta)\lambda}{(1-\theta)\alpha_{i,j}+\theta} \\
&+ \frac{\theta(1+2\theta-2\theta^2)(1-\eta)^2(\eta+\beta_{i+1,j})(1-\lambda)}{(1-\eta)\beta_{i+1,j}+\eta}, \\
\omega_{1,1}(\theta,\eta) &= \frac{\theta(3\theta-2\theta^2+(1-\theta)\alpha_{i,j+1})\eta(1+2\eta-2\eta^2)\lambda}{(1-\theta)\alpha_{i,j+1}+\theta} \\
&+ \frac{\theta(1+2\theta-2\theta^2)\eta(3\eta-2\eta^2+(1-\eta)\beta_{i+1,j})(1-\lambda)}{(1-\eta)\beta_{i+1,j}+\eta}, \\
\omega_{1,2}(\theta,\eta) &= -\frac{\theta(3\theta-2\theta^2+(1-\theta)\alpha_{i,j+2})\eta^2(1-\eta)\lambda}{(1-\theta)\alpha_{i,j+2}+\theta} \\
&- \frac{\theta(1+2\theta-2\theta^2)\eta^2(1-\eta)(1-\lambda)}{(1-\eta)\beta_{i+1,j}+\eta},
\end{aligned}$$

$$\begin{aligned}
\omega_{2,0}(\theta,\eta) &= -\frac{\theta^2(1-\theta)(1-\eta)^2(1+\eta)\lambda}{(1-\theta)\alpha_{i,j}+\theta} \\
&- \frac{\theta^2(1-\theta)(1-\eta)^2(\eta+\beta_{i+2,j})(1-\lambda)}{(1-\eta)\beta_{i+2,j}+\eta}, \\
\omega_{2,1}(\theta,\eta) &= -\frac{\theta^2(1-\theta)\eta(1+2\eta-2\eta^2)\lambda}{(1-\theta)\alpha_{i,j+1}+\theta} \\
&- \frac{\theta^2(1-\theta)\eta(3\eta-2\eta^2+(1-\eta)\beta_{i+2,j})(1-\lambda)}{(1-\eta)\beta_{i+2,j}+\eta}, \\
\omega_{2,2}(\theta,\eta) &= \frac{\theta^2(1-\theta)\eta^2(1-\eta)\lambda}{(1-\theta)\alpha_{i,j+2}+\theta} \\
&+ \frac{\theta^2(1-\theta)\eta^2(1-\eta)(1-\lambda)}{(1-\eta)\beta_{i+2,j}+\eta},
\end{aligned}$$

and $\omega_{r,s}(\theta,\eta), (r=,i,i+1; s=j,j+1)$ are called the basis of the bivariate interpolation defined by (6). These base functions satisfy

$$\sum_{r=0}^{2}\sum_{s=0}^{2}\omega_{r,s}(\theta,\eta)=1.$$

Furthermore, when $\alpha_{i,s}\to 0$ and $\beta_{r,j}\to 0$, the basis of the weighted bivariate interpolation $P_{i,j}(x,y)$ defined in (6) become

$$\begin{aligned}
\omega_{0,0}(\theta,\eta) &= (1-\theta)^2(1-\eta)^2(1+\theta(1-\lambda)+\eta\lambda), \\
\omega_{0,1}(\theta,\eta) &= (1-\theta)^2\eta((1+2\eta-2\eta^2)\lambda+(1+\theta)(3-2\eta)(1-\lambda)), \\
\omega_{0,2}(\theta,\eta) &= -(1-\theta)^2\eta(1-\eta)(1+\theta(1-\lambda)-(1-\eta)\lambda), \\
\omega_{1,0}(\theta,\eta) &= \theta(1-\eta)^2((3\theta-2\theta^2)(1+\eta)\lambda+(1+2\theta-2\theta^2)(1-\lambda)), \\
\omega_{1,1}(\theta,\eta) &= \theta\eta((3-2\theta)(1+2\eta-2\eta^2)\lambda+(1+2\theta-2\theta^2)(3-2\eta)(1-\lambda)), \\
\omega_{1,2}(\theta,\eta) &= -\theta\eta(1-\eta)((3-2\theta)\eta\lambda+(1+2\theta-2\theta^2)(1-\lambda)), \\
\omega_{2,0}(\theta,\eta) &= -\theta(1-\theta)(1-\eta)^2(\theta(1-\lambda)+(1+\eta)\lambda), \\
\omega_{2,1}(\theta,\eta) &= -\theta(1-\theta)\eta(\theta(3-2\eta)(1-\lambda)+(1+2\eta-2\eta^2)\lambda), \\
\omega_{2,2}(\theta,\eta) &= \theta(1-\theta)\eta(1-\eta)(\theta(1-\lambda)+\eta\lambda).
\end{aligned}$$

Also, $\alpha_{i,s}\to\infty$ and $\beta_{r,j}\to\infty$, the basis of the weighted bivariate interpolation $P_{i,j}(x,y)$ defined in (5) become

$$\omega_{0,0}(\theta,\eta) = (1-\theta)(1-\eta)(1-\theta^2(1-\lambda)-\eta^2\lambda),$$

$$\begin{aligned}
\omega_{0,1}(\theta,\eta) &= (1-\theta)\eta(1-\theta^2(1-\lambda)+2\eta(1-\eta)\lambda),\\
\omega_{0,2}(\theta,\eta) &= -(1-\theta)(1-\eta)\eta^2\lambda,\\
\omega_{1,0}(\theta,\eta) &= \theta(1-\eta)(1+2\theta(1-\theta)(1-\lambda)-\eta^2\lambda),\\
\omega_{1,1}(\theta,\eta) &= \theta\eta(1+2\theta(1-\theta)(1-\lambda)+2\eta(1-\eta)\lambda),\\
\omega_{1,2}(\theta,\eta) &= -\theta(1-\eta)\eta^2\lambda,\\
\omega_{2,0}(\theta,\eta) &= -\theta^2(1-\theta)(1-\eta)(1-\lambda),\\
\omega_{2,1}(\theta,\eta) &= -\theta^2(1-\theta)\eta(1-\lambda),\\
\omega_{2,2}(\theta,\eta) &= 0.
\end{aligned}$$

4. Some integral properties

For the weighted bivariate blending interpolation defined by (5), we use Eq.(7) to arrive at the following unity property.

Property 1. *Let $f(x,y)\equiv 1$, $\forall(x,y)\in\Omega$, and $P_{i,j}(x,y)$ is its interpolation function over $[x_i,x_{i+1};y_j,y_{j+1}]$ defined in (5). When $0\le\lambda\le 1$, no matter what positive number the parameters $\alpha_{i,s},\beta_{r,j}$ take, the unity property holds, namely*

$$\int\int_D P_{i,j}(x,y)dxdy = h_i l_j,$$

where D denotes the subregion $[x_i,x_{i+1};y_j,y_{j+1}]$.

Use Eq.(6) to define

$$\int\int_D P_{i,j}(x,y)dxdy = h_i l_j \sum_{r=0}^{2}\sum_{s=0}^{2}\omega^*_{r,s} f_{i+r,j+s},$$

where

$$\omega^*_{r,s} = \int\int_{[0,1;0,1]} \omega_{r,s}(\theta,\eta)d\theta d\eta,$$

$$r = i, i+1, i+2;\ \ s = j, j+1, j+2,$$

$\omega^*_{r,s}$ are called the integral weights coefficients of the interpolation defined in (5).

Furthermore, when $\lambda = 1$, the interpolation function $P_{i,j}(x,y)$, that is, $\overline{P}_{i,j}(x,y)$ defined in (2). Denote

$$u_{r,s} = \int_0^1 \omega_{r,s}(\theta,\eta)d\eta,$$

$$r = i, i+1, i+2;\ \ s = j, j+1, j+2.$$

It is easy to show that the following conclusions hold for any parameters $\alpha_{i,s} > 0$,

$$\begin{aligned}
&0 \le u_{0,0} \le \frac{2}{15}, \quad 0 \le u_{0,1} \le \frac{2}{3}, \quad -\frac{1}{12} \le u_{0,2} \le 0,\\
&0 \le u_{1,0} \le \frac{15}{32}, \quad 0 \le u_{1,1} \le \frac{3}{4}, \quad -\frac{3}{32} \le u_{1,2} \le 0,\\
&-\frac{5}{48} \le u_{2,0} \le 0, \quad -\frac{1}{6} \le u_{2,1} \le 0, \quad 0 \le u_{2,2} \le \frac{1}{48}.
\end{aligned}$$

Thus, it is easy to conclude the following property.

Property 2. *For the bivariate blending interpolation $\overline{P}_{i,j}(x,y)$ defined in (2), the integral weights coefficients of interpolation satisfy:*

$$\begin{aligned}
&0 < \omega^*_{0,0} < \frac{2}{15}, \quad 0 < \omega^*_{0,1} < \frac{2}{3}, \quad -\frac{1}{12} < \omega^*_{0,2} < 0,\\
&0 < \omega^*_{1,0} < \frac{15}{32}, \quad 0 < \omega^*_{1,1} < \frac{3}{4}, \quad -\frac{3}{32} < \omega^*_{1,2} < 0,\\
&-\frac{5}{48} < \omega^*_{2,0} < 0, \quad -\frac{1}{6} < \omega^*_{2,1} < 0, \quad 0 < \omega^*_{2,2} < \frac{1}{48}.
\end{aligned}$$

Also, when $\lambda = 0$, the interpolation function $P_{i,j}(x,y)$, that is, $\overline{\overline{P}}_{i,j}(x,y)$ defined in (4). Denote

$$u^*_{r,s} = \int_0^1 \omega_{r,s}(\theta,\eta)d\theta,$$
$$r = i, i+1, i+2;\ \ s = j, j+1, j+2.$$

Then for any parameters $\beta_{r,j} > 0$,

$$\begin{aligned}
&0 \le u^*_{0,0} \le \frac{2}{15}, \quad 0 \le u^*_{0,1} \le \frac{15}{32}, \quad -\frac{5}{48} \le u^*_{0,2} \le 0,\\
&0 \le u^*_{1,0} \le \frac{2}{3}, \quad 0 \le u^*_{1,1} \le \frac{3}{4}, \quad -\frac{1}{6} \le u^*_{1,2} \le 0,\\
&-\frac{1}{12} \le u^*_{2,0} \le 0, \quad -\frac{3}{32} \le u^*_{2,1} \le 0, \quad 0 \le u^*_{2,2} \le \frac{1}{48}.
\end{aligned}$$

This leads to the following property.

Property 3. *For the bivariate blending interpolation* $\overline{\overline{P}}_{i,j}(x,y)$ *defined in (4), the integral weights coefficients of interpolation satisfy:*

$$0<\omega^*_{0,0}<\frac{2}{15},\quad 0<\omega^*_{0,1}<\frac{15}{32},\quad -\frac{5}{48}<\omega^*_{0,2}<0,$$
$$0<\omega^*_{1,0}<\frac{2}{3},\quad 0<\omega^*_{1,1}<\frac{3}{4},\quad -\frac{1}{6}<\omega^*_{1,2}<0,$$
$$-\frac{1}{12}<\omega^*_{2,0}<0,\quad -\frac{3}{32}<\omega^*_{2,1}<0,\quad 0<\omega^*_{2,2}<\frac{1}{48}.$$

When $0<\lambda<1$, from Eq.(5) and Property 2 and Property 3, it is easy to derive the following property.

Property 4. *For the bivariate blending interpolation* $P_{i,j}(x,y)$ *defined in (5), when* $0<\lambda<1$*, no matter what positive number the parameters* $\alpha_{i,s},\beta_{r,j}$ *take, the integral weights coefficients of interpolation satisfy:*

$$0<\omega^*_{0,0}<\frac{2}{15},\quad 0<\omega^*_{0,1}<\frac{2}{3},\quad -\frac{5}{48}<\omega^*_{0,2}<0,$$
$$0<\omega^*_{1,0}<\frac{2}{3},\quad 0<\omega^*_{1,1}<\frac{3}{4},\quad -\frac{3}{32}<\omega^*_{1,2}<0,$$
$$-\frac{5}{48}<\omega^*_{2,0}<0,\quad -\frac{3}{32}<\omega^*_{2,1}<0,\quad 0<\omega^*_{2,2}<\frac{1}{48}.$$

5. Bounded property of the interpolation

The weighted bivariate blending interpolation defined in (5) is a local interpolation in each sub-rectangle, which only depends on the function values. There are some parameters in each interpolating region, and when the parameters vary, the interpolating surface changes accordingly. Now, we consider the bounded property of the interpolation.

Note that the interpolation defined in (5) is local, assume $\alpha_{i,j}=\alpha_{i,j+1}=\alpha_{i,j+2}$ and $\beta_{i,j}=\beta_{i+1,j}=\beta_{i+2,j}$.

Denote

$$N=\max\{|f_{i+r,j+s}|,r=0,1,2;j=0,1,2\}. \tag{7}$$

First, discuss the bounded property of the interpolation function $P_{i,j}(x,y)$ defined in (2).

From Eq.(6) and Prop.(2), it is easy to obtain

$$\begin{aligned}
|P_{i,j}(x,y)| &= |\sum_{r=0}^{2}\sum_{s=0}^{2}\omega_{r,s}(\theta,\eta)f_{i+r,j+s} \leq \sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|a_{r,s}(\theta,\eta)||f_{i+r,j+s}| \\
&\leq N\frac{(1+2\eta^2-2\eta^3)[(1-\theta)\alpha_{i,j}+\theta(1+2\theta-2\theta^2)]}{(1-\theta)\alpha_{i,j}+\theta} \\
&= N(1+2\eta^2-2\eta^3+\frac{2\theta^2(1-\theta)}{(1-\theta)\alpha_{i,j}+\theta}) \\
&\leq \frac{97}{54}N.
\end{aligned}$$

This leads to the following theorem.

Theorem 1. *For the given interpolation data* $\{f_{r,s}, r=i,i+1,i+2; s=j,j+1,j+2\}$, *let* $P_{i,j}(x,y)$ *be the bivariate blending interpolation over the rectangle region* $[x_i,x_{i+1};y_j,y_{j+1}]$ *defined in (2), then*

$$|P_{i,j}(x,y)| \leq \frac{97}{54}N,$$

where N *defined by (8).*

For the interpolation function defined in (4), similarly, we obtain the following theorem.

Theorem 2. *For the given interpolation data* $\{f_{r,s}, r=i,i+1,i+2; s=j,j+1,j+2\}$, *let* $P_{i,j}(x,y)$ *be the bivariate blending interpolation over the rectangle region* $[x_i,x_{i+1};y_j,y_{j+1}]$ *defined in (4), then*

$$|P_{i,j}(x,y)| \leq \frac{97}{54}N.$$

From Eq.(5) and theorem 1 and theorem 2, it is easy to derive the following theorem.

Theorem 3. *For the given interpolation data* $\{f_{r,s}, r=i,i+1,i+2; s=j,j+1,j+2\}$, *let* $P_{i,j}(x,y)$ *be the bivariate blending interpolation over the rectangle region* $[x_i,x_{i+1};y_j,y_{j+1}]$ *defined in (5), then*

$$|P_{i,j}(x,y)| \leq \frac{97}{54}N.$$

6. Local shape control of the interpolation

The shape of the interpolating surface on the interpolating region depends on the interpolating data. Generally speaking, when the interpolating data are given, the shape of interpolating surface is fixed, this is because of the uniqueness of the interpolation function. But, for the bivariate blending interpolation defined by (5), since there are parameters and weighted coefficient, when they vary, the interpolation function can be changed under the condition that the interpolating data are not changed. Thus, the interpolating surface varies as the parameters and the weighted coefficient vary. Based on this, the shape of the interpolating surface can be modified by selecting suitable parameters and weighted coefficient.

For any point (x, y) in the interpolating region $[x_i, x_{i+1}; y_j, y_{j+1}]$, let (θ, η) be its local coordinate. Now, we consider the following local point control issue: if the practical design requires the value of the interpolation function at the point (x, y) be equal to a real number M, how it can be achieved?

Let

$$M = \sum_{r=0}^{1}\sum_{s=0}^{1} \mu_{r,s} f_{i+r,j+s},$$

where $\mu_{r,s}$ $(r = i, i+1; s = j, j+1)$ are real number, and

$$\sum_{r=0}^{1}\sum_{s=0}^{1} \mu_{r,s} = 1.$$

Then

$$\sum_{r=0}^{2}\sum_{s=0}^{2} \omega_{r,s}(\theta, \eta) f_{i+r,j+s} = \sum_{r=0}^{1}\sum_{s=0}^{1} \mu_{r,s} f_{i+r,j+s}. \tag{8}$$

Eq.(9) is called the value control equation, where the local coordinate (θ, η) is given, only the parameters $\alpha_{i,s}, \beta_{r,j}$ and the weighted coefficient λ are unknown. If there exist parameters $\alpha_{i,s}, \beta_{r,j}$ and the weighted coefficient $\lambda \in [0, 1]$ which satisfy (9), then the problem is solved.

For example, the central point in the interpolating surface is more important than other points for shape control, consider the central point value

control problem. Note that the interpolation is local, assume $\alpha_{i,j} = \alpha_{i,j+1} = \alpha_{i,j+2}$, and denote by α_i; assume $\beta_{i,j} = \beta_{i+1,j} = \beta_{i+2,j}$, and denote by β_j. Let the weighted coefficient $\lambda = \frac{1}{2}$. Since (x, y) is the central point in $[x_i, x_{i+1}; y_j, y_{j+1}]$, $\theta = \frac{1}{2}, \eta = \frac{1}{2}$. In this case, Eq.(9) becomes

$$k_1\alpha_i + k_2\beta_j + k_3\alpha_i\beta_j + k_0 = 0, \tag{9}$$

where

$$\begin{aligned}
k_0 &= 2(3f_{i,j} + 9f_{i,j+1} - 2f_{i,j+2} + 9f_{i+1,j} + 24f_{i+1,j+1} - 5f_{i+1,j+2} - 2f_{i+2,j} \\
&\quad -5f_{i+2,j+1} + f_{i+2,j+2} - 32M), \\
k_1 &= 9f_{i,j} + 24f_{i,j+1} - 5f_{i,j+2} + 12f_{i+1,j} + 36f_{i+1,j+1} - 8f_{i+1,j+2} - f_{i+2,j} \\
&\quad -4f_{i+2,j+1} + f_{i+2,j+2} - 64M, \\
k_2 &= 9f_{i,j} + 12f_{i,j+1} - f_{i,j+2} + 24f_{i+1,j} + 36f_{i+1,j+1} - 4f_{i+1,j+2} - 5f_{i+2,j} \\
&\quad -8f_{i+2,j+1} + f_{i+2,j+2} - 64M, \\
k_3 &= 2(6f_{i,j} + 9f_{i,j+1} - f_{i,j+2} + 9f_{i+1,j} + 12f_{i+1,j+1} - f_{i+1,j+2} - f_{i+2,j} \\
&\quad -f_{i+2,j+1} - 32M).
\end{aligned}$$

From Eq.(10), it is interesting that the selection of parameters does not depend on the value of $\mu_{r,s}$ in point control problem. If there exist positive solutions α_i and β_j which satisfy Eq.(10), then the interpolation function $P_{i,j}(x, y)$ defined in (5) with these parameters is the solution of central point value control problem. This can be stated in the following theorem.

Theorem 2. *The sufficient condition for central point value control problem having solution is that the number of the variation sign of the sequence* $\{k_1, k_2, k_3, k_0\}$ *are not equal to zero.*

7. Numerical examples

Let $\Omega : [0, 2; 0, 2]$ be the plane region, and the interpolation data are given in Table 1.

Table 1. The interpolating data

(x, y)	$(0, 0)$	$(0, 1)$	$(0, 2)$
$f(x, y)$	4	2	3
(x, y)	$(1, 0)$	$(1, 1)$	$(1, 2)$
$f(x, y)$	2	3	4
(x, y)	$(2, 0)$	$(2, 1)$	$(2, 2)$
$f(x, y)$	3	3	5

Then the interpolation function $P_{i,j}(x,y)$ defined by (5) can be constructed in $[0,1;0,1]$ for the given positive parameters $\alpha_{i,s}, \beta_{r,j}$ and the weighted coefficient λ. In general case, one can select any positive real numbers as the values of $\alpha_{i,s}$ and $\beta_{r,j}$.

When $\alpha_{i,s} \to 0, \beta_{r,j} \to 0$ and $\lambda = \frac{1}{2}$, the graph of the weighted bivariate blending interpolation $P_{i,j}(x,y)$ defined in (5) is shown in the Figure 1.

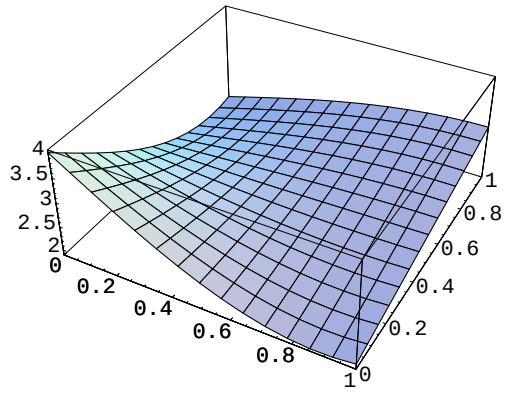


Figure 1: Graph of $P_{i,j}(x,y)$, where $\alpha_{i,s} \to 0, \beta_{r,j} \to 0, \lambda = \frac{1}{2}$.

Similarly, when $\alpha_{i,s} \to \infty, \beta_{r,j} \to \infty$ and $\lambda = \frac{1}{2}$, the graph of the weighted bivariate blending interpolation $P_{i,j}(x,y)$ defined in (5) is shown in the Figure 2.

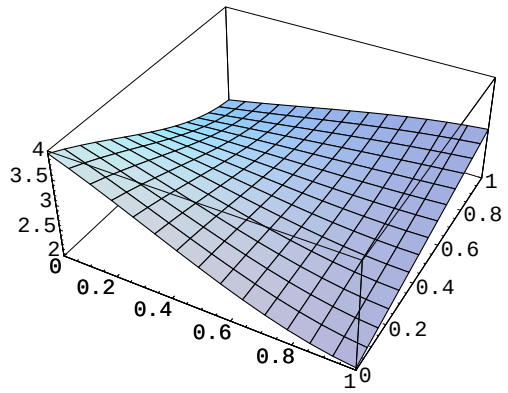


Figure 2: Graph of $P_{i,j}(x,y)$, where $\alpha_{i,s} \to \infty, \beta_{r,j} \to \infty, \lambda = \frac{1}{2}$.

Now, let $\alpha_i = 0.5, \beta_j = 2$ and $\lambda = \frac{1}{2}$, for the given interpolation data, denote the interpolation function by $P_1(x, y)$ which is defined over $[0, 1; 0, 1]$, the Figure 3 shows the graph of the weighted bivariate blending rational interpolation surface $P_1(x, y)$. In this case, the function value of central point is that $P_1(\frac{1}{2}, \frac{1}{2}) = \frac{485}{192} = 2.52604\cdots$.

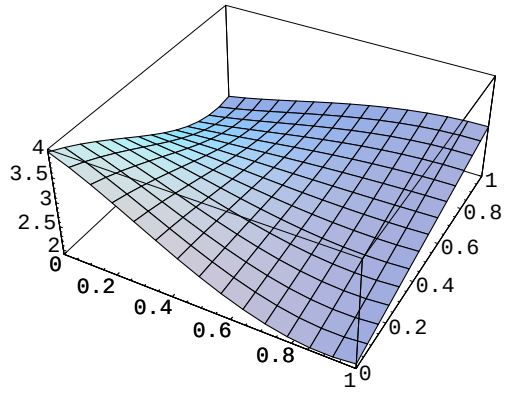


Figure 3: Graph of $P_1(x, y)$, where $\alpha_i = 2, \beta_j = 1$, $\lambda = \frac{1}{2}$.

From the central point value control equation Eq.(10), the value of central point can be control through modifying the parameters. For example, we take $\mu_{0,0} = \frac{1}{8}, \mu_{0,1} = \mu_{1,1} = \frac{1}{4}, \mu_{1,0} = \frac{3}{8}$, then $M = 2.5$. It is easy to prove that $\alpha_i = 1.25, \beta_j = 0.5$ satisfy Eq.(10). For those parameters, denote the interpolation function for the given interpolation data by $P_2(x, y)$. Figure 4 shows the graph of the interpolation surface $P_2(x, y)$. It can be to test that $P_2(\frac{1}{2}, \frac{1}{2}) = 2.5$.

Furthermore, take $\mu_{0,0} = 0.1, \mu_{0,1} = 0.25, \mu_{1,0} = 0.3$, $\mu_{1,1} = 0.35$, then $M = 2.55$. $\alpha_i = 0.5, \beta_j = 7.75$ satisfy Eq.(10). For those parameters, denote the interpolation function by $P_3(x, y)$. Figure 5 shows the graph of the interpolation surface $P_3(x, y)$. It can be to derive that $P_3(\frac{1}{2}, \frac{1}{2}) = 2.55$.

8. Concluding remarks

- This paper gives the explicit expression of a weighted bivariate blending rational interpolation; The interpolation depends only on the function values of the interpolated function. There are some parameters in the interpolation functions: $\alpha_{i,j}$ and $\beta_{i,j}$. $\alpha_{i,j}$ are for univariate interpolation on the x-direction as described by (1); $\beta_{i,j}$ are for univariate

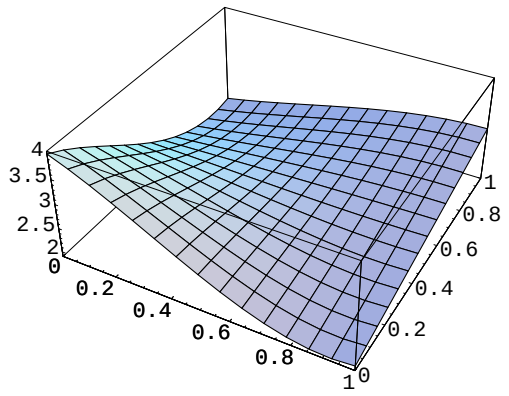


Figure 4: Graph of $P_2(x, y)$, where $\alpha_i = 1.25, \beta_j = 0.5$, $\lambda = \frac{1}{2}$.

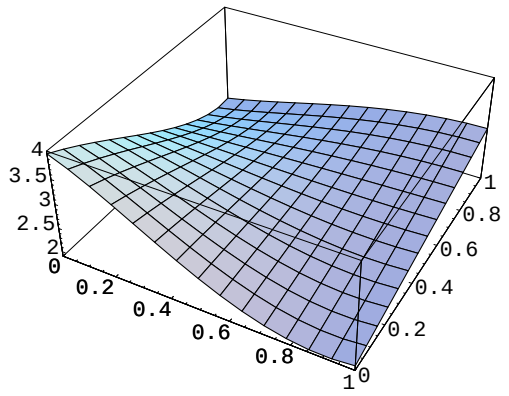


Figure 5: Graph of $P_3(x, y)$, where $\alpha_i = 0.5, \beta_j = 7.75$, $\lambda = \frac{1}{2}$.

interpolation on y-direction as described by (3). More important is that the smoothness property of the interpolation does not depend on the parameters $\alpha_{i,j}$ and $\beta_{i,j}$, this means that the interpolation function is C^1-continuous in the whole interpolating region $[x_1, x_n; y_1, y_m]$ for any parameters $\alpha_{i,j}$ and $\beta_{i,j}$. Positive parameters $\alpha_{i,j}$ and $\beta_{i,j}$ can be freely selected according to the needs of practical design.

- For each pitch of the interpolating surface, the value of the interpolation function depends on the interpolating data. The integral weights coefficients of the interpolation defined in (5) describe the "weight" of the interpolating knots in the interpolation function. Properties 2–4

shows the magnitude of the integral weights coefficients values, it could help us understand the interpolation based only on the function values deeply. And the bounded property of the interpolation is obtained.

- The point control method of the interpolation is developed in the paper. For any given real number M, from (9), there are varied expressions, so manifold control scheme can be derived similar to the process for central point value control problem.

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On asymmetric Hermitian and skew-Hermitian splitting iteration methods for weakly nonlinear systems

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Abstract: For the large sparse system of weakly nonlinear equations, based on separable property and strong dominance between the linear and the nonlinear terms and on the asymmetric Hermitian and skew-Hermitian splitting (AHSS) of the coefficient matrix, we establish two nonlinear composite iteration schemes, called Picard-AHSS and nonlinear AHSS-like methods. The advantage of these methods is that they only need to solve the linear sub-systems of constant coefficient matrices. In addition, these methods can also take full advantage of the dominance of Hermitian part. Therefore, computational workloads and computer storage may be saved in actual implementations. Theoretical analysis show that these new iteration methods are local convergent under suitable conditions. Numerical results show that both Picard-AHSS and nonlinear AHSS-like iteration methods are feasible and effective for the large scale system of weakly nonlinear equations, especially when the Hermitian part of the coefficient matrix is dominant.

Key words: weakly nonlinear systems, AHSS iteration method, HSS iteration method, inner/outer iteration scheme,
2000 MR Subject Classification: 65F10, 65F50, 65T50, 65Y05

1. Introduction

The system of weakly nonlinear equations of the form

$$Ax = \phi(x), \text{or } F(x) := Ax - \phi(x) = 0, \tag{1.1}$$

often arise in many areas of scientific computing and engineering applications, and in particular in discretions of certain nonlinear partial differential equations [3, 4, 10, 11, 24, 27] and in collocation approximations of nonlinear integral equations [25] and in saddle point problems from image processing [6, 13], where $A \in \mathbb{C}^{n\times n}$ is a large, sparse, positive real matrix, $\phi : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable function. Here, the system of nonlinear equations (1.1) is said to be weakly nonlinear if the linear term Ax is strongly dominant over the nonlinear term $\phi(x)$ in certain norm.

For general systems of nonlinear equations $F(x) = 0$, the Newton iteration method is the most popular, classic and important method

$$F'(x^{(k)})\Delta x^{(k)} = -F(x^{(k)}),\ x^{(k+1)} = x^{(k)} + \Delta x^{(k)},$$

where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable function. This method is attractive for its local convergence properties: if $x^{(0)}$ is a sufficiently good initial guess of x^*, then the convergence q-superlinearly to x^*, and F' is usually Lipschitz continuous at x^* and the convergence is q-quadratic [14, 24]. However, at each iteration step, the Newton method requires not only the explicit form of the Jacobian matrix but also the exact solution of the corresponding Newton

equations, which are very costly and complicated in actual application [2, 15]. In order to overcome these disadvantages and improve the efficiency of the Newton iteration method, we can simplify or avoid computation of the Jacobian matrix, reduce the cost of the function evaluation and solve Newton equations efficiently. Many variants in terms of approximate, quasi-update, inner/outer or inexact Newton methods have been established and analyzed. See [1, 3, 4, 12].

Since a general purpose method may be not always efficient for solving a special structure problem. For the system of weakly nonlinear equations (1.1), Bai [3] established a class of sequential two-stage iteration methods by taking into account concrete properties of the involved matrix and mapping. And based on the matrix multi-splitting technique, Bai [5] presented efficient parallel generalizations of the sequential two-stage iteration methods. Furthermore, Bai and Huang [9] proposed asynchronous multi-splitting two-stage iteration methods. These asynchronous methods have the potentials of converging much faster than their synchronous counterparts in [5], especially, when there is load imbalance. Moreover based on the linear and the nonlinear terms Ax and $\phi(x)$ are well separated and the former is strongly dominant over the latter, Bai and Yang [12] present the Picard-HSS and the nonlinear HSS-like iteration methods, and Zhu and Zhang [28] present the Picard-CSCS and the nonlinear CSCS-like iteration methods for the special case of the coefficient matrix A is Teoplitz matrix. Theoretical analysis and numerical results show that these iteration methods are feasible and effective in some cases.

In this paper, to make iteration methods initially introduced in [12] have more attractive and based on the dominance of the Hermitian part of the coefficient matrix, We establish two classes of nonlinear composite splitting iteration schemes, called Picard-AHSS and nonlinear AHSS-like iteration methods respectively, for solving the large scale system of weakly nonlinear equations (1.1). Compared with the Newton iteration method, The new iteration methods have the same advantage as the Picard-HSS and the nonlinear HSS-like iteration methods in [12]. But the new iteration methods could take advantage of the dominance of the Hermitian part of the coefficient matrix, thus computational workloads may be further saved.

The organization of this paper is as follows: In Section 2 we review the HSS and AHSS iteration methods for the non-Hermitian positive definite linear systems, In Section 3, we establish the Picard-AHSS and nonlinear AHSS-like iteration methods and discuss their convergence properties. Numerical examples are given in Section 4. Finally, in Section 5 we draw a brief conclusions.

2. The HSS and AHSS iteration methods

When the nonlinear mapping $\phi : \mathbb{D} \in \mathbb{C}^n \to \mathbb{C}^n$ is a constant vector, i.e., $\phi(x) = b$, the system of weakly nonlinear equations (1.1) reduces to the system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \text{ and } x, b \in \mathbb{C}^n. \tag{2.1}$$

To solve linear system (2.1) iteratively, efficient splitting of the coefficient matrix A are required [16, 22, 23, 26]. Since a matrix A naturally possesses a Hermitian and skew-Hermitian (HS) splitting $A = H + S$,

$$H = \frac{1}{2}(A + A^*) \text{ and } S = \frac{1}{2}(A - A^*). \tag{2.2}$$

Bai [7] present the HSS iteration method and have proved that for any positive α the HSS method converges unconditionally to the unique solution of the non-Hermitian positive definite system of linear equations (2.1). This method gains people's attention [7, 8, 19, 20].

The HSS iteration method. ([7]) Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(k+1)}$ for $k = 0, 1, 2, \cdots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2.3}$$

where α is a given positive constant and I denotes the identity matrix.

Based on HS splitting, Li, etc. [21] present a asymmetric Hermitian and skew-Hermitian splitting (AHSS) iteration method, and studied the convergence property. This methods is described as follows.

The AHSS iteration method. ([21]) Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(k+1)}$ for $k = 0, 1, 2, \cdots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\beta I + S)x^{(k+1)} = (\beta I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2.4}$$

where α is a nonnegative constant, β is a positive constant, and I denotes the identity matrix.

Just like the HSS method (2.3), the AHSS iteration alternates between the Hermitian part H and the skew-Hermitian part S of the matrix A. When $\alpha = \beta$, the AHSS iteration method reduces to the HSS iteration method.

3. The new iteration methods

3.1. The Picard-AHSS iteration method

The linearization is a traditional method for solving nonlinear equations. Based on the separability and strong dominance between the linear term Ax and the nonlinear term $\phi(x)$, we can use the Picard iteration method

$$Ax^{(k+1)} = \phi(x^{(k)}),\ k = 0, 1, 2, \cdots$$

to solve the system of weakly nonlinear equations (1.1). At each iteration step, we need to solve a linear system.

Apply the HSS or AHSS iteration method to solve linear systems, and the next iterate $x^{(k+1)}$ may be approximately computed. This naturally leads to the inexact Picard iteration methods, called Picard-HSS and Picard-AHSS iteration methods. The former is established by Bai and Yang in reference [12]. The Picard-AHSS iteration method is algorithmically described as follows.

The Picard-AHSS iteration method. Let $\phi : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be a continuously differentiable function and $A \in \mathbb{C}^{n\times n}$ be a positive defined matrix, with $H = \frac{1}{2}(A + A^*)$, $S = \frac{1}{2}(A - A^*)$ be the Hermitian and skew-Hermitian parts of A, respectively. Given an initial guess $x^{(0)} \in \mathbb{D}$ and a sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, compute $x^{(k+1)}$ for $k = 0, 1, 2, \cdots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

(a) $x^{(k,0)} := x^{(k)}$;

(b) For $l = 0, 1, 2, \cdots, l_k - 1$, solve the following linear systems to obtain $x^{(k,l+1)}$:

$$\begin{cases} (\alpha I + H)x^{(k,l+\frac{1}{2})} = (\alpha I - S)x^{(k,l)} + \phi(x^{(k)}), \\ (\beta I + S)x^{(k,l+1)} = (\beta I - H)x^{(k,l+\frac{1}{2})} + \phi(x^{(k)}), \end{cases} \tag{3.1}$$

where α is a nonnegative constant, β is a positive constant, and I denotes the identity matrix.

(c) $x^{(k+1)} := x^{(k,l_k)}$;

Now, we consider the local convergence of the Picard-AHSS iteration method.

Based on the iteration scheme of Picard-AHSS, the $(k+1)$-th Picard-AHSS iterate $x^{(k+1)}$ can be express as

$$x^{(k+1)} = T(\alpha,\beta)^{l_k}x^{(k)} + \sum_{j=0}^{l_k-1} T(\alpha,\beta)^j G(\alpha,\beta)\phi(x^{(k)}), \quad k = 0, 1, 2, \cdots, \tag{3.2}$$

where $T(\alpha,\beta) = (\beta I + S)^{-1}(\beta I - H)(\alpha I + H)^{-1}(\alpha I - S)$ and $G(\alpha,\beta) = (\alpha+\beta)(\beta I + S)^{-1}(\alpha I + H)^{-1}$.

Suppose $x^* \in \mathbb{D}$ is a solution of the system of weakly nonlinear equations (1.1), then

$$x^* = T(\alpha,\beta)^{l_k}x^* + \sum_{j=0}^{l_k-1} T(\alpha,\beta)^j G(\alpha,\beta)\phi(x^*). \tag{3.3}$$

Thus

$$x^{(k+1)} - x^* = T(\alpha,\beta)^{l_k}(x^{(k)} - x^*) + \sum_{j=0}^{l_k-1} T(\alpha,\beta)^j G(\alpha,\beta)(\phi(x^{(k)}) - \phi(x^*)).$$

By utilizing the R-convergence and similar arguments of Theorem 3.1 in [12], we obtain the following local convergence theory for the Picard-AHSS iteration method.

Theorem 3.1. *Let $\phi : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$ of a point $x^* \in \mathbb{D}$ at which $\phi'(x)$ is continuous and $F(x^*) := Ax^* - \phi(x^*) = 0$. Suppose $H = \frac{1}{2}(A + A^*)$, $S = \frac{1}{2}(A - A^*)$ are the Hermitian and the skew-Hermitian parts of the matrix A, respectively. Denote by*

$$\mu(\alpha,\beta) = \|(\beta I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\beta I - S)\|, \quad \omega = \|A^{-1}\phi'(x^*)\|.$$

Then there exists an open neighborhood $\mathbb{N} \subset \mathbb{N}_0$ of x^* such that for any $x^{(0)} \in \mathbb{N}$ and any sequence of positive integers $l_k, k = 0, 1, 2, \cdots$, the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by the Picard-AHSS iteration method is well-defined and convergent to x^*, provided $l_0 \geq \lfloor \ln(\frac{1-\omega}{1+\omega}) / \ln(\mu(\alpha,\beta)) \rfloor$ (where $\lfloor\cdot\rfloor$ is used to denote the smallest integer no less than the corresponding real number). Moreover, it holds that

$$\limsup_{k\to\infty} \|x^{(k)} - x^*\|^{\frac{1}{k}} \leq \omega + (1+\omega)\mu(\alpha,\beta)^{l_0}, \quad l_0 = \liminf_{k\to\infty} l_k;$$

in particular, if $\lim_{k\to\infty} l_k = \infty$, then the rate of convergence is R-linear, with the R-factor being at most ω, i.e.,

$$\limsup_{k\to\infty} \|x^{(k)} - x^*\|^{\frac{1}{k}} \leq \omega.$$

Proof. The proof uses arguments similar to those in proof of the convergence Theorem of the Picard-HSS iteration method [12]. □

Theorem 3.1 shows that the convergence rate of the Picard-AHSS iteration is essentially determined by the quantities ω and $\mu(\alpha,\beta)$. Usually, small ω and $\mu(\alpha,\beta)$ will lead to fast convergence of the Picard-HSS iteration.

3.2. The nonlinear AHSS-like iteration method

Since the Picard-AHSS and the Picard-HSS iteration methods share same drawback, that is the numbers of the inner iteration steps l_k, $k = 0, 1, 2, \cdots$ are often problem dependent and difficult to be determined in actual computations, we propose the following nonlinear AHSS-like iteration method to overcome above disadvantages.

The AHSS-like iteration method. Let $\phi : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be a continuously differentiable function and $A \in \mathbb{C}^{n\times n}$ be a positive defined matrix, with $A = H + S$, H and S be the Hermitian and skew-Hermitian parts of A, respectively. Given an initial guess $x^{(0)} \in \mathbb{D}$ and a sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, compute $x^{(k+1)}$ for $k = 0, 1, 2, \cdots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

$$\begin{cases}(\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + \phi(x^{(k)}),\\ (\beta I + S)x^{(k+1)} = (\beta I - H)x^{(k+\frac{1}{2})} + \phi(x^{(k+\frac{1}{2})}),\end{cases} \tag{3.4}$$

where α is a nonnegative constant, β is a positive constant, and I denotes the identity matrix.

In the following we deduce the convergence property for the nonlinear AHSS-like iteration method. We define

$$\begin{cases}U(x) = (\alpha I + H)^{-1}((\alpha I - S)x + \phi(x)),\\ V(x) = (\beta I + S)^{-1}((\beta I - H)x + \phi(x))\end{cases} \tag{3.5}$$

and

$$\psi(x) = V \circ U(x) := V(U(x)).$$

Then the nonlinear AHSS-like iteration scheme can be equivalently expressed as

$$x^{(k+1)} = \psi(x^{(k)}), k = 0, 1, 2, \cdots .$$

Suppose $x^* \in \mathbb{D}$ is a solution of the system of weakly nonlinear equations (1.1), we can easily verify the following fact by using the derivative chain rule

$$\begin{aligned}\psi'(x^*) &= V'(x^*)U'(x^*)\\ &= (\beta I + S)^{-1}(\beta I - H + \phi'(x^*))\,(\alpha I + H)^{-1}(\alpha I - S + \phi'(x^*)).\end{aligned}$$

By making use of the Ostrowski Theorem, i.e., Theorem 10.1.3 in [24], we know that if $\rho(\psi'(x^*)) < 1$, then x^* is a point of attraction of the nonlinear AHSS-like iteration. Then we can obtain the following local convergence theory for the AHSS-like iteration method.

Theorem 3.2. *Assume that $\phi : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is F-differentiable at a point $x^* \in \mathbb{D}$ such that $Ax^* = \phi(x^*)$. Suppose $A = H + S$, where H and S are the Hermitian and the skew-Hermitian parts of the matrix A, respectively. Denote by*

$$T(\alpha,\beta;x^*) = (\beta I + S)^{-1}(\alpha I - H + \phi'(x^*))\,(\alpha I + H)^{-1}(\beta I - S + \phi'(x^*)).$$

If $\rho(T(\alpha,\beta;x^*)) < 1$, then $x^* \in \mathbb{D}$ is a point of attraction of the nonlinear AHSS-like iteration.

Now we give a sufficient condition for guaranteeing the validity of $\rho(T(\alpha,\beta;x^*)) < 1$.

Corollary 3.3. *Assume that the conditions of Theorem 3.2 are satisfied. Denote by*

$$\delta = \max\{\|\phi'(x^*)(\beta I + S)^{-1}\|,\ \|\phi'(x^*)(\alpha I + H)^{-1}\|\},$$

$$a = \max_{\sigma_i\in\sigma(S)} \frac{\sqrt{\alpha^2-\sigma_i^2}}{\sqrt{\beta^2+\sigma_i^2}}, \quad b = \max_{\lambda_i\in\lambda(H)} |\frac{\beta-\lambda_i}{\alpha+\lambda_i}|,$$

where $\lambda(H)$ is the spectral set of H and $\sigma(S)$ is the singular-value set of S. Then $\rho(T(\alpha,\beta;x^*)) < 1$ holds, provided

$$\delta < \frac{2(1-ab)}{(a+b)+\sqrt{(a-b)^2+4}} \tag{3.6}$$

Proof. By straightforward computations we have

$$\begin{aligned}(\beta I + S)T(\alpha,\beta;x^*)(\beta I + S)^{-1} &= (\beta I + S)T(\alpha,\beta)(\beta I + S)^{-1} + (\beta I - H)(\alpha I + H)^{-1}\phi'(x^*)(\beta I + S)^{-1}\\ &+\phi'(x^*)(\alpha I + H)^{-1}(\alpha I - S)(\beta I + S)^{-1} + \phi'(x^*)(\alpha I + H^{-1})\phi'(x^*)(\beta I + S)^{-1},\end{aligned}$$

here

$$T(\alpha,\beta) = (\beta I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\beta I - S)$$

and for any given parameter α, if

$$\frac{\alpha\lambda_{\max}}{2\alpha+\lambda_{\max}} < \beta \le \alpha + 2\lambda_{\min},$$

then

$$\begin{aligned}\|T(\alpha,\beta)\| &\le \|(\beta I - S)(\alpha I + S)^{-1}\|\,\|(\alpha I - H)(\alpha I + H)^{-1}\|\\ &\le \max_{\sigma_i\in\sigma(S)} \frac{\sqrt{\alpha^2+\sigma_i^2}}{\sqrt{\beta^2+\sigma_i^2}} \max_{\lambda_i\in\lambda(H)} |\frac{\beta-\lambda_i}{\alpha-\lambda_i}|\\ &= ab < 1.\end{aligned}$$

See reference [21] for details. Hence,

$$\begin{aligned}\|T(\alpha,\beta;x^*)\| =&\|(\beta I + S)T(\alpha,\beta;x^*)(\beta I + S)^{-1}\|\\ \le&\|(\beta I + S)T(\alpha,\beta)(\beta I + S)^{-1}\| + \|(\beta I - H)(\alpha I + H)^{-1}\phi'(x^*)(\beta I + S)^{-1}\|\\ &+ \|\phi'(x^*)(\alpha I + H)^{-1}(\alpha I - S)(\beta I + S)^{-1}\| + \|\phi'(x^*)(\alpha I + H)^{-1}\phi'(x^*)(\beta I + S)^{-1}\|\\ \le&\|T(\alpha,\beta)\| + \|(\beta I - H)(\alpha I + H)^{-1}\|\,\|\phi'(x^*)(\beta I + S)^{-1}\|\\ &+ \|\phi'(x^*)(\alpha I + H)^{-1}\|\,\|(\alpha I - S)(\beta I + S)^{-1}\| + \|\phi'(x^*)(\alpha I + H)^{-1}\|\,\|\phi'(x^*)(\beta I + S)^{-1}\|\\ \le&ab + (a+b)\delta + \delta^2.\end{aligned}$$

Now, under the condition (3.6), we have $\rho(T(\alpha,\beta;x^*)) \le \|T(\alpha,\beta;x^*)\| < 1$. □

Corollary 3.3 shows that $\rho(T(\alpha,\beta;x^*)) < 1$ is valid if $\phi'(x^*)$ is reasonably small compared with the matrix A.

4. Numerical results

In this section, we present some numerical experiments to illustrate the effectiveness of our methods, and to further show the advantage of the Picard-AHSS and the nonlinear AHSS-like method over the Picard-HSS, the nonlinear HSS-like iteration methods, as well as the Newton-GMRES (5) iteration methods. The problem under consideration is the classical three-dimensional convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = \sin(u + 1), (x, y, z) \in \Omega \\ u(x, y, z) = 0, \qquad\qquad (x, y, z) \in \partial\Omega \end{cases} \tag{4.1}$$

on the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$, and q is a positive constant coefficient used to measure the magnitude of the convective terms. By applying the seven-point finite difference and assuming the numbers (N) of grid points in all three directions are the same, we obtain a system of weakly nonlinear equations of the form (1.1), where $n = N^3$, and for details to [7, 17, 18]. Different q and N result in different matrix A.

In our numerical experiments, we use the zeros vector as the initial guess, and stop the iteration as soon as

$$\frac{\|F(x^{(k)})\|_2}{\|F(x^{(0)})\|_2} \leq 10^{-6}.$$

Meanwhile, the stopping criterions for the inner iterations of the Picard-HSS and the Picard-AHSS methods are set to be

$$\frac{\|F'(x^{(k)})s^{(k,l_k)} + F(x^{(k)})\|_2}{\|F(x^{(k)})\|_2} \leq \eta_k.$$

where l_k is the number of the inner iteration steps and η_k is the prescribed tolerance for controlling the accuracy of the inner iterations at the k-th outer iterate. If η_k is fixed for all k, then it is simply denoted by η. The two sub-systems of linear equations are solved in the way if $ax = b$, then $x = a^{-1}b$.

The Picard-AHSS and the nonlinear AHSS-like methods are compared with the Picard-HSS and the nonlinear HSS-like methods for different problem sizes N ($N = 4, 8$) and tolerance $\eta = 0.1, 0.01, 0.001$, from aspects of numbers of outer, inner and total iteration steps (denoted as $\mathrm{IT_{out}}$, $\mathrm{IT_{int}}$ and IT, respectively) and total CPU time (denoted as CPU).

In our implementations, we adopt the optimal parameters $\alpha^* = \sqrt{\lambda_{\min}\lambda_{\max}}$ given in [7] for the Picard-HSS and the nonlinear HSS-like methods, and adopt the optimal parameters $\alpha^* = 0, \beta^* = 2\lambda_{\min}\lambda_{\max}/(\lambda_{\min} + \lambda_{\max})$ given in [21] for the Picard-AHSS and the nonlinear AHSS-like iteration methods. Although they only minimize the bound of iteration matrix, not the spectral radius of the iteration matrix. Admittedly, the optimal parameters α^* and β^* are crucial for guaranteeing fast convergence speeds of these parameter-dependent iteration methods, but they are often problem-dependent and are generally difficult to be determined [12].

In Table 1-2, we list the numerical results corresponding to four choices of the q, i.e., $q = 0, 10, 100, 1000$. From these tables, we see that all experimented methods (i.e., the Picard-HSS, the nonlinear HSS-like, the Picard-AHSS, and the nonlinear AHSS-like methods) can successfully produce approximate solutions to the system of weakly nonlinear equations for all of the matrices dimension (i.e., $n = 4^3$ and $n = 8^3$).

When the tolerance η for controlling the accuracy of the inner iterations is increasing, both number of inner iteration steps (and the amount iteration steps) and the amount of CPU time of

Table 1: The numbers of iteration and the total times for $N = 4$, $\alpha_{\mathrm{HSS}} = 3.5267$, $\alpha_{\mathrm{AHSS}} = 0$, $\beta_{\mathrm{AHSS}} = 2.0729$

	Iteration		q=0	q=10	q=100	q=1000
$\eta = 10^{-3}$	Picard-HSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	3
		$\mathrm{IT}_{\mathrm{int}}$	11	11	9	8
		IT	44	44	36	4
		CPU	0.0042	0.0028	0.0023	0.0019
	Picard-AHSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	3
		$\mathrm{IT}_{\mathrm{int}}$	1	2	6.752	0.6667
		IT	4	8	27	62
		CPU	0.0025	0.0017	0.0023	0.0043
	Newton-GMRES	$\mathrm{IT}_{\mathrm{out}}$	2	2	2	2
		$\mathrm{IT}_{\mathrm{int}}$	4	5	5	13
		IT	8	10	10	26
		CPU	0.0033	0.003	0.0029	0.0074
$\eta = 10^{-2}$	Picard-HSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	3
		$\mathrm{IT}_{\mathrm{int}}$	7	7	6	5.6667
		IT	28	28	24	17
		CPU	0.005	0.0022	0.0019	0.0026
	Picard-AHSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	3
		$\mathrm{IT}_{\mathrm{int}}$	1	2	4.75	14.3333
		IT	4	8	19	43
		CPU	0.0021	0.0015	0.002	0.0029
	Newton-GMRES	$\mathrm{IT}_{\mathrm{out}}$	3	3	3	3
		$\mathrm{IT}_{\mathrm{int}}$	3.3333	6.3333	4.6667	20
		IT	10	19	14	60
		CPU	0.0021	0.0036	0.0035	0.0088
$\eta = 10^{-1}$	Picard-HSS	$\mathrm{IT}_{\mathrm{out}}$	6	6	6	5
		$\mathrm{IT}_{\mathrm{int}}$	4	4	3	3
		IT	24	24	18	15
		CPU	0.0049	0.0024	0.002	0.0021
	Picard-AHSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	6	6
		$\mathrm{IT}_{\mathrm{int}}$	1	1	2.8333	7.6667
		IT	4	4	17	46
		CPU	0.0013	0.0014	0.002	0.0032
	Newton-GMRES	$\mathrm{IT}_{\mathrm{out}}$	3	5	5	6
		$\mathrm{IT}_{\mathrm{int}}$	3	3.8	4	6.3333
		IT	9	19	20	38
		CPU	0.0035	0.0038	0.0038	0.0091
	like-HSS	IT	21	42	59	75
		CPU	0.0049	0.0071	0.009	0.011
	like-AHSS	IT	3	7	22	66
		CPU	0.0012	0.0024	0.0056	0.0101

Table 2: The numbers of iteration and the total times for $N = 8$, $\alpha_{\mathrm{HSS}} = 2.0521$, $\alpha_{\mathrm{AHSS}} = 0$, $\beta_{\mathrm{AHSS}} = 0.7019$

	Iteration		q=0	q=10	q=100	q=1000
$\eta = 10^{-3}$	Picard-HSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	3
		$\mathrm{IT}_{\mathrm{int}}$	19.75	19.75	19	12.6667
		IT	79	79	76	38
		CPU	0.4203	0.4106	0.4051	0.3167
	Picard-AHSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	3
		$\mathrm{IT}_{\mathrm{int}}$	1	2	5	49.3333
		IT	4	8	20	148
		CPU	0.2071	0.1961	0.2377	0.5446
	Newton-GMRES	$\mathrm{IT}_{\mathrm{out}}$	2	2	2	2
		$\mathrm{IT}_{\mathrm{int}}$	15.5	11	17.5	14.5
		IT	31	22	35	29
		CPU	0.054	0.0593	0.0578	0.0529
$\eta = 10^{-2}$	Picard-HSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	4
		$\mathrm{IT}_{\mathrm{int}}$	13	13	13	9
		IT	52	52	52	36
		CPU	0.3515	0.3338	0.3485	0.3022
	Picard-AHSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	4	4
		$\mathrm{IT}_{\mathrm{int}}$	1	2	3.25	34.75
		IT	4	8	13	139
		CPU	0.2014	0.2003	0.2162	0.5061
	Newton-GMRES	$\mathrm{IT}_{\mathrm{out}}$	3	3	3	3
		$\mathrm{IT}_{\mathrm{int}}$	8.6667	13.3333	12	11.6667
		IT	26	40	36	35
		CPU	0.0612	0.0713	0.0732	0.0612
$\eta = 10^{-1}$	Picard-HSS	$\mathrm{IT}_{\mathrm{out}}$	7	7	6	6
		$\mathrm{IT}_{\mathrm{int}}$	6.8571	6.8571	6.8333	4.1667
		IT	48	48	41	25
		CPU	0.3609	0.3356	0.3136	0.2776
	Picard-AHSS	$\mathrm{IT}_{\mathrm{out}}$	4	4	5	6
		$\mathrm{IT}_{\mathrm{int}}$	1	1	2.2	17.5
		IT	4	4	11	105
		CPU	0.2171	0.2015	0.2047	0.424
	Newton-GMRES	$\mathrm{IT}_{\mathrm{out}}$	6	6	6	6
		$\mathrm{IT}_{\mathrm{int}}$	4.1667	3.8333	7	5.6667
		IT	25	23	42	34
		CPU	0.0847	0.087	0.098	0.0919
	like-HSS	IT	40	80	118	141
		CPU	0.3377	0.6802	0.9965	1.2689
	like-AHSS	IT	3	7	17	119
		CPU	0.2081	0.4174	0.6305	1.0609

all iteration methods are decreasing, and the numbers of outer iteration steps are fixed or slightly increasing. But the Picard-AHSS iteration method is better than the Picard-HSS iteration method.

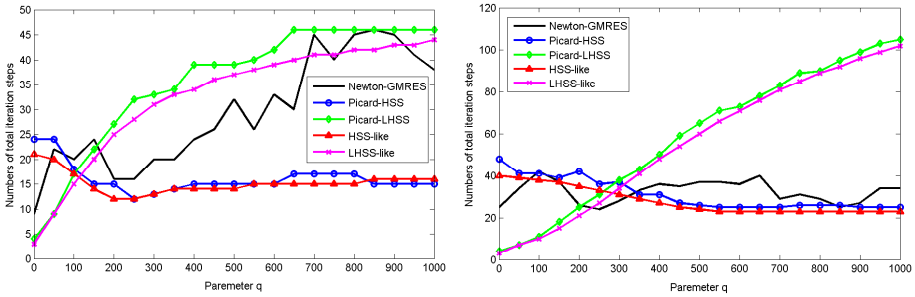


Figure 1: The numbers of total iteration steps with different parameter q.

From Table 1-2 and Figure 1, we observe that with the parameter q increase, the number of outer iteration steps are fixed and decreasing slightly, and the number of inner iteration steps and the total iteration steps of the Picard-HSS iterations are decreasing and those of the Picard-AHSS iterations are increasing. The total time have the same results. we find that the Picard-AHSS and the nonlinear AHSS-like methods are very effective when q is not not large.

When N is increasing, the numbers of outer iteration steps are fix and increasing slightly, but the numbers of inner iteration steps are increasing quickly, so the total iteration steps are increasing. And the amount of total CPU time of all iteration methods are increasing quickly.

Clearly, when the q is not large, in terms of iteration step, the nonlinear AHSS-like method and the Picard-AHSS perform better than the nonlinear HSS-like and the Picard-HSS iteration methods. In terms of CPU time, the situation is almost the same. Therefore, the nonlinear AHSS-like method and the Picard-AHSS method are the winners for solving this test problem. But when q is very large, the result is reverse.

5. Concluding remark

We discuss how to solve the large scale weakly nonlinear systems with dominated Hermitian part by the Picard-AHSS and the nonlinear AHSS-like iteration methods. Since the iteration methods are based on separable property of the linear and the nonlinear terms and on the asymmetric Hermitian and skew-Hermitian splitting (AHSS) of the involved coefficient matrix, these new iteration methods can be only applied to the systems of weakly nonlinear equations whose linear terms are strongly dominant and the skew-Hermitian part of the coefficient matrix is less dominant.

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Some aspects of Lorentz-Karamata spaces

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ABSTRACT. Let G be a locally $\sigma-$compact abelian group with Haar measure μ. In this small note, we introduce some simple properties of Lorentz-Karamata spaces $L_{p,q;b}(G)$ and also characterize the space of multipliers from $L^1(G)$ to Lorentz-Karamata spaces.

1. Introduction

A new generalization of Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and generalized Lorentz-Zygmund spaces was studied by D.E.Edmunds, R.Kerman and L.Pick in [8]. By using the Karamata theory which is studied in [3], they introduced Lorentz-Karamata spaces $L_{p,q;b}(\Omega)$ and compared quasinorms on these spaces. Also J.S.Neves studied $L_{p,q;b}(R,\mu)$ spaces in [12] where $p,q\in(0,\infty]$, b is a slowly varying function on $[1,\infty)$ and (R,μ) is a measure space. These spaces cover the generalized Lorentz-Zygmund spaces $L_{p,q;\alpha_1,\ldots\alpha_m}(R)$ (introduced in [7]), Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(R)$ (introduced in [1]), Zygmund spaces $L^{p}(\log L)^{\alpha}(R)$ (introduced in [2,17]), Lorentz spaces $L^{p,q}(R)$ and Lebesgue spaces $L^{p}(R)$ under convenient choices of slowly varying functions and parameters p, q.

In [6, 12], it is proved that $L_{p,q;b}(R,\mu)$ spaces endowed with a convenient norm and they are rearrangment-invariant Banach function spaces with associate spaces $L_{p',q';b^{-1}}(R,\mu)$ if (R,μ) is a resonant measure space, $p\in(1,\infty)$ and $q\in[1,\infty]$. Also it is showed that when $p\in(1,\infty)$ and $q\in[1,\infty)$, $L_{p,q;b}(R,\mu)$ spaces have absolutely continuous norm.

2. Preliminaries on $L_{p,q;b}(G)$ spaces

In this section, we will give some definitions and properties of slowly varying functions to establish Lorentz-Karamata spaces. Throughtout this paper, certain well-known terms such as rearrangement-invariant Banach space, absolutely continuous norm, Segal algebra, Fourier transform, homogeneous Banach space etc. will be used frequently in the sequel. We will not give their definitions, but one can refer to [2, 6, 15,16] and references therein.

Let G and $\widehat{G}$ be locally compact abelian groups in Pontryagin duality with Haar measures μ and η respectively, such that Plancherel theorem holds. $C_c(G)$ will denote the space of all continuous, complex-valued functions on G with compact

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support. For any two non-negative expressions A and B, the symbol $A \precsim B$ means that $A \leq cB$, for some positive constant c independent of the variables in the expressions A and B. If $A \precsim B$ and $B \precsim A$, we write $A \approx B$ and say that A and B are equivalent.

DEFINITION 2.1. *The family of all extended scalar valued (real or complex) μ-measurable functions on X will be denoted by $M(X,\mu)$ and $M_0(X,\mu)$ will stand for the subset of $M(X,\mu)$ consisting of all those functions which are finite $\mu-a.e.$*

DEFINITION 2.2. *Let f be a measurable function defined on a measure space (X,μ). We assume that the functions f are finite valued almost everywhere and for $y>0$*

$$\mu\{x \in X : \ |f(x)| > y\} < \infty.$$

Then the distribution function λ_f of f is defined by

$$\lambda_f(y) = \mu\{x \in X : \ |f(x)| > y\}. \tag{2.1}$$

The nonnegative rearrangement of f is given by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t \ \}, \ t \geq 0 \tag{2.2}$$

where $\inf \phi = \infty$. Also the average(maximal) function of f on $(0,\infty)$ is given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds. \tag{2.3}$$

Note that $\lambda_f(\cdot), f^*(\cdot)$ and $f^{**}(\cdot)$ are non-increasing and right continuous functions. Also, it is obvious that if X has a finite measure, then λ_f is bounded above by $\mu(X)$ and so $f^*(t) = 0$ for all $t \geq \mu(X)$.

A positive measurable function L, defined on some neighborhood of infinity, is said to be slowly varying if, for every $s>0$,

$$\frac{L(st)}{L(t)} \to 1 \quad (t \to +\infty). \tag{2.4}$$

These functions were introduced by Karamata [11], (see also [3],[14]). Another definition of slowly varying functions can be found in [8] such as:

DEFINITION 2.3. *A positive and Lebesgue measurable function b is said to be slowly varying (s.v.) on $[1,\infty)$ in the sense of Karamata if, for each $\varepsilon > 0$, $t^{\varepsilon}b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon}b(t)$ is equivalent to a non-increasing function on $[1,\infty)$.*

The detailed study of Karamata theory, properties and examples of s.v. functions can be found in [3,6,7,11] and [17,Chap.V, p.186]. Let $m \in \mathbb{N}$ and $\alpha = (\alpha_1,...,\alpha_m) \in \mathbb{R}^m$. If we denote by ϑ_α^m the real function defined by

$$\vartheta_\alpha^m(t) = \prod_{i=1}^m l_i^{\alpha_i}(t) \quad \text{for all } t \in (0,\infty),$$

where $l_1,...,l_m$ are positive functions defined on $(0,\infty)$ by

$$l_1(t) = 1 + |\log t|, \ l_i(t) = 1 + \log l_{i-1}(t), \ i \geq 2$$

then the following functions are s.v. on $[1,\infty)$:

(1) $b(t) = \vartheta_\alpha^m(t)$ with $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$;
(2) $b(t) = \exp(\log^\alpha t)$ with $0 < \alpha < 1$;
(3) $b(t) = \exp(l_m^\alpha(t))$ with $0 < \alpha < 1$, $m \in \mathbb{N}$;
(4) $b(t) = l_m(t)$ with $m \in \mathbb{N}$.

Given a s.v. function b on $[1,\infty)$, we denote by γ_b the positive function defined by

$$\gamma_b(t) = b\left(\max\left\{t, \frac{1}{t}\right\}\right) \quad \text{for all } t > 0.$$

It is known that any s.v. function b on $(0,\infty)$ is equivalent to a s.v. continuous function $\widetilde{b}$ on $(0,\infty)$. Consequently, without loss of generality, we assume that all s.v. functions in question are continuous functions in $(0,\infty)$ [9].

DEFINITION 2.4. *Let $p,q \in (0,\infty]$ and let b be a s.v. function on $[1,\infty)$. Lorentz-Karamata space $L_{p,q;b}(G)$ is defined to be the set of all functions $f \in M_0(G,\mu)$ such that*

$$\|f\|^*_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(0,\infty)} \tag{2.5}$$

is finite. Here $\|\cdot\|_{q;(0,\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0,\infty)$.

Let $0 < p,q \leq \infty$ and b be a s.v. function on $[1,\infty)$. Let us introduce the functional $\|f\|_{p,q;b}$ defined by

$$\|f\|_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^{**}(t) \right\|_{q;(0,\infty)}; \tag{2.6}$$

this is identical with that defined in (2.5) except that f^* is replaced by f^{**}. However, when $p=\infty$, $L_{p,q;b}(G)$ spaces are different from the trivial spaces if and only if $\left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \right\|_{q;(0,\infty)} < \infty$. It is easy to see that $L_{p,q;b}(G)$ spaces endowed with a convenient norm (2.6), are rearrangement-invariant Banach function spaces and have absolutely continuous norm when $p \in (1,\infty)$ and $q \in [1,\infty)$.

It is clear that, for $0 < p < \infty$, $L_{p,q;b}(G)$ spaces contain the characteristic function of every measurable subset of G with finite measure and hence, by linearity, every $\mu-$simple function. From the definition of $\|\cdot\|^*_{p,q;b}$, it follows that if $f \in L_{p,q;b}(G)$ and $p,q \in (0,\infty)$, then the function $\lambda_f(y)$ is finite valued. In this case, with a little thought, it is possible to construct a sequence of (simple) functions which satisfy Lemma 1.1 in [4]. Therefore, if we use the same method as employed in the proof of Proposition 2.4 in [10], we can show that Lebesgue dominated convergence theorem holds and so the set of simple functions is dense in Lorentz-Karamata spaces for $p \in (1,\infty)$ and $q \in [1,\infty)$. Also, we can see the density of $C_c(G)$ since μ is a Haar measure.

It follows from [12] that $\|f\|^*_{p,q;b} \leq \|f\|_{p,q;b} \precsim \|f\|^*_{p,q;b}$ for all $f \in M_0(G,\mu)$ where $1 < p \leq \infty$, $1 \leq q \leq \infty$ and b is a s.v. function on $[1,\infty)$. In particular, $L_{p,q;b}(G)$ spaces consist of all those functions f for which $\|f\|_{p,q;b}$ is finite. Since the function $f \to f^{**}$ is subadditive, it is obvious that $\|\cdot\|_{p,q;b}$ is a norm if $q \geq 1$.

For more information on Lorentz-Karamata spaces, one can refer to [2, 6, 8, 9, 12] and references therein.

3. Some results for Lorentz-Karamata spaces

The following proposition can be proved by using the techniques done in [15].

PROPOSITION 3.1. *Let f be a scalar valued, measurable functions on (G,μ). If we define the function $L_s f(t) = f(t-s)$ for any $s \in G$, then we have the following:*

(1) $\lambda_{L_s f}(y) = \lambda_f(y)$ *for all* $y \geq 0$,
(2) $(L_s f)^*(t) = f^*(t)$ *for all* $t \geq 0$ *and* $(L_s f)^{**}(t) = f^{**}(t)$ *for all* $t > 0$,
(3) *If* $p, q \in (0, \infty)$, *then* $\|L_s f\|^*_{p,q;b} = \|f\|^*_{p,q;b}$, $\|L_s f\|_{p,q;b} = \|f\|_{p,q;b}$.

PROPOSITION 3.2. *For any* $f \in L_{p,q;b}(G)$, $1 < p < \infty$ *and* $1 \leq q < \infty$, *the function* $s \to L_s f$ *is continuous from* G *into* $L_{p,q;b}(G)$.

PROOF. Since the set of simple functions is dense in $L_{p,q;b}(G)$, it is sufficient to show that for any simple function f, the mapping $s \to L_s f$ is continuous. Let $f = \sum_{i=1}^n k_i \chi_{E_i}$ where χ_E is the characteristic function of E. Then, we can find that $L_s f = \sum_{i=1}^n k_i \chi_{E_i+s}$. Now, we have

$$\left|\chi_{E_i+s} - \chi_{E_i}\right|(t) = \begin{cases} 1, & t \in (E_i + s) \triangle E_i \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

and

$$\lambda_{\chi_{E_i+s} - \chi_{E_i}}(y) = \begin{cases} \mu\left((E_i + s) \triangle E_i\right), & t < 1 \\ 0, & t \geq 1 \end{cases} \tag{3.2}$$

where $\triangle$ denotes the symmetric difference of sets. Therefore, we write

$$\left(\chi_{E_i+s} - \chi_{E_i}\right)^*(t) = \begin{cases} 1, & t < \mu\left((E_i + s) \triangle E_i\right) \\ 0, & t \geq \mu\left((E_i + s) \triangle E_i\right) \end{cases} \tag{3.3}$$

and

$$\begin{aligned} \left(\left\|\chi_{E_i+s} - \chi_{E_i}\right\|^*_{p,q;b}\right)^q &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \left(\chi_{E_i+s} - \chi_{E_i}\right)^*(t) \right\|^q_{q;(0,\infty)} \\ &= \int_0^\infty \left| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \left(\chi_{E_i+s} - \chi_{E_i}\right)^*(t) \right|^q dt \\ &= \int_0^{\mu((E_i+s)\triangle E_i)} t^{\frac{q}{p}-1} \gamma_b^q(t)\, dt \\ &= \int_0^{\mu((E_i+s)\triangle E_i)} t^{\frac{q}{p}-1} \gamma_{b^q}(t)\, dt \\ &\approx \sup_{0<t<\mu((E_i+s)\triangle E_i)} t^{\frac{q}{p}-1} \gamma_{b^q}(t) \\ &\approx \mu\left((E_i + s) \triangle E_i\right)^{\frac{q}{p}-1} \gamma_{b^q}\left(\mu\left((E_i + s) \triangle E_i\right)\right) \end{aligned} \tag{3.4}$$

by [12, Lemma 3.1(iv) and (v)]. With the absolutely continuity property of the norm, we get

$$\|L_s f - f\|^*_{p,q;b} \to 0 \text{ as } s \to 0 \text{ and similarly } \|L_s f - f\|_{p,q;b} \to 0 \text{ as } s \to 0.$$

□

In [4], a convolution operator T is defined and necessary conditions for convolution of two simple functions are found. By the help of O'Neil Theorem (see [13], p.133), convolution theorems for Lorentz spaces are also established in [4]. Now we will give a proposition about convolution in $L_{p,q;b}(G)$ spaces.

PROPOSITION 3.3. *Let* T *be a convolution operator like defined in* [4, *Definition* 2.1] *and* $h = T(f, g)$. T *can be uniquely extended so that if* $f \in L_{p,q;b}(G)$, $1 < p, q < \infty$ *and* $g \in L^1(G)$, *then* $h \in L_{p,s;b}(G)$, *where* $q \leq s$. *Moreover* $\|h\|_{p,q;b} \precsim \|f\|_{p,q;b} \|g\|_1$.

PROOF. Since $q \leq s$ implies $\|h\|_{p,s;b} \leq C \|h\|_{p,q;b}$ by [12, Theorem 3.2], it is sufficient to assume $q = s$. Let f and g be simple functions. By Lemma 2.2 and Theorem 2.4 in [4], we know that

$$\begin{aligned} h^{**}(t) &\precsim tf^{**}(t) g^{**}(t) + \int_t^\infty f^*(u) g^*(u) du \\ &\precsim tf^{**}(t) g^{**}(t) + f^{**}(t) \int_t^\infty g^*(u) du \\ &= tf^{**}(t) \frac{1}{t} \int_0^t g^*(u) du + f^{**}(t) \int_t^\infty g^*(u) du \\ &= f^{**}(t) \int_0^\infty g^*(u) du \end{aligned}$$

and then by [10],

$$h^{**}(t) \precsim f^{**}(t) \|g\|_1 .$$

Thus, we get the result with the definition of $\|\cdot\|_{p,q;b}$ and the density of the simple functions in $L^1(G)$ and $L_{p,q;b}(G)$ spaces. □

LEMMA 3.1. *There is an approximate identity $\{a_\alpha\}_{\alpha \in I}$ of $L^1(G)$ such that $\|a_\alpha\|_1 = 1$ for each $\alpha \in I$ and $f * a_\alpha \to f$ for all $f \in L_{p,q;b}(G)$ where $1 < p, q < \infty$.*

PROOF. The proof follows from Proposition 3.2 by simple functions. Let $\{U_\alpha\}_{\alpha \in I}$ be a decreasing neighbourhood system at the origin of G and a_α be non-negative continuous function with support in U_α such that $\int_G a_\alpha(x) d\mu(x) = 1$ for each $\alpha \in I$. If g is a simple function, then we have

$$\begin{aligned} \|g * a_\alpha - g\|_{p,q;b} &\leq \int_G \|L_s g - g\|_{p,q;b} a_\alpha(s) d\mu(s) \\ &\leq \sup_{s \in U_\alpha} \|L_s g - g\|_{p,q;b} \end{aligned}$$

and $\|g * a_\alpha - g\|_{p,q;b} \to 0$ by Proposition 3.2 and absolutely continuity of the norm. Since the set of simple functions is dense in $L_{p,q;b}(G)$ spaces, we get the result. □

Let $A_{p,q;b}(G)$ be the subspace of $L^1(G)$ with Fourier transforms in $L_{p,q;b}\left(\widehat{G}\right)$ i.e.

$$A_{p,q;b}(G) = \left\{ f \in L^1(G) : \widehat{f} \in L_{p,q;b}\left(\widehat{G}\right) \right\}$$

provided that each function in $L_{p,q;b}\left(\widehat{G}\right)$ is locally integrable. For every $f \in A_{p,q;b}(G)$, we can supply a norm in $A_{p,q;b}(G)$ by

$$\|f\|_{A_{p,q;b}} = \|f\|_1 + \left\|\widehat{f}\right\|_{p,q;b} .$$

Since the spaces $A_{p,q;b}(G)$ are generalization of the spaces $A_{p,q}(G)$ which are examined in [16], we will not give a detailed study of $A_{p,q;b}(G)$ spaces. Also, the following theorem can be proved easily.

THEOREM 3.1. *(i) There is an approximate identity $\{e_\alpha\}$ of $A_{p,q;b}(G)$ for $1 < p, q < \infty$, which is a bounded approximate identity of $L^1(G)$ such that $\|e_\alpha\|_1 \leq 1$ and $\widehat{e_\alpha}$ have compact support for all α.*

(ii) $\left(A_{p,q;b}(G), \|\cdot\|_{A_{p,q;b}}\right)$ *is a Segal Algebra, i.e.* $\left(A_{p,q;b}(G), \|\cdot\|_{A_{p,q;b}}\right)$ *is a homogeneous Banach space,* $\left(A_{p,q;b}(G), \|\cdot\|_{A_{p,q;b}}\right)$ *is a Banach algebra with its norm* $\|\cdot\|_{A_{p,q;b}} \geq \|\cdot\|_1$ *and* $\left(A_{p,q;b}(G), \|\cdot\|_{A_{p,q;b}}\right)$ *is a dense subspace of* $L^1(G)$.
(iii) Since $L^1(G)$ *and* $L_{p,q;b}(G)$ *are strongly character invariant,* $A_{p,q;b}(G)$ *is strongly character invariant and the maps* $f \to M_t f$, $t \to M_t f$ *are continuous where* $M_t f(x) = \langle x, t\rangle f(x)$ *for all* $f \in A_{p,q;b}(G)$, $x \in G$ *and* $t \in \widehat{G}$.
(iv) $A_{p,q;b}(G)$ *and* $L_{p,q;b}(G)$ *are essential Banach* $L^1(G)-$*modules.*
(v) $A_{1;1;1}(G) \subset A_{p,q;b}(G)$ *for all* $1 \leq p < \infty$, $1 \leq q \leq \infty$.
(vi) The Fourier transforms $\widehat{A_1(G)}$ *and* $\widehat{A_{p,q;b}(G)}$ *are dense in* $L_{p,q;b}\left(\widehat{G}\right)$ *for* $1 \leq p, q < \infty$.

Let A be a commutative normed algebra and B a (two-sided) A-module normed linear space. We denote by $M(A, B)$ the set of all bounded linear mappings

$$T : A \to B \text{ such that } T(ab) = aT(b) \text{ for any } a, b \in A.$$

Each element of $M(A, B)$ is called the multiplier of A to B.

THEOREM 3.2. *The multipliers space* $M\left(L^1(G), L_{p,q;b}(G)\right)$ *is isometric isomorphic to* $L_{p,q;b}(G)$ *for* $1 < p, q < \infty$. *Morever for any* $T \in M\left(L^1(G), L_{p,q;b}(G)\right)$, *there is a unique* f *in* $L_{p,q;b}(G)$ *such that* $T(g) = f * g$ *for every* $g \in L^1(G)$.

PROOF. By [6, Corollary 3.4.44], we know that the dual pair is of the form

$$\langle f, g\rangle = \int_G f(x)\, g(x)\, d\mu(x), \quad g \in L_{p,q;b}(G), \; f \in L_{p',q';1/b}(G).$$

Morever, for every $\psi \in \left(L_{p',q';1/b}(G)\right)^*$ there is a unique $g \in L_{p,q;b}(G)$ such that

$$\psi(f) = \int_G f g d\mu \text{ for every } f \in L_{p',q';1/b}(G).$$

Now, if $a \in L^1(G)$ and $f \in L_{p',q';1/b}(G)$, then we can define $a \otimes f = \widetilde{a} * f$ where $\widetilde{a}$ is the reflexive function of a. So, it follows from Proposition 3.3 that $L_{p',q';1/b}(G)$ is a Banach $L^1(G)-$module under the operation $\otimes$. Also, it is easy to define for $f \in L_{p',q';1/b}(G)$ that

$$a \otimes \psi(f) = \psi(a \otimes f) \tag{3.5}$$

for every $a \in L^1(G)$, $\psi \in \left(L_{p',q';1/b}(G)\right)^*$. Then $\left(L_{p',q';1/b}(G)\right)^*$ is also a Banach $L^1(G)-$module under the operation $\otimes$. Since there is an approximate identity $\{a_\alpha\}_{\alpha \in I}$ of $L^1(G)$ such that $\|a_\alpha\|_1 = 1$ and

$$\|a_\alpha \otimes f - f\|_{p',q';1/b} = \left\|a_\alpha * \widetilde{f} - \widetilde{f}\right\|_{p',q';1/b} \to 0.$$

It follows that

$$M\left(L^1(G), \left(L_{p',q';1/b}(G)\right)^*\right) \cong \left(L_{p',q';1/b}(G)\right)^*$$

and hence

$$M\left(L^1(G), L_{p,q;b}(G)\right) = L_{p,q;b}(G)$$

for $1 < p, q < \infty$. Also we have to show that when $\psi \in \left(L_{p',q';1/b}(G)\right)^*$ corresponds to $g \in L_{p,q;b}(G)$, $a \otimes \psi$ in $\left(L_{p',q';1/b}(G)\right)^*$ corresponds to $a * g$ in $L_{p,q;b}(G)$ for every $a \in L^1(G)$. In fact, we have

$$a \otimes \psi(f) = \psi(a \otimes f) = \int_G (\widetilde{a} * f)\, g d\mu = \int_G f\,(a * g)\, d\mu$$

for every $f \in L_{p',q';1/b}(G)$. Therefore for each $T' \in M\left(L^1(G), \left(L_{p',q';1/b}(G)\right)^*\right)$ there is a unique $\psi \in \left(L_{p',q';1/b}(G)\right)^*$ such that

$$T'(a) = a \otimes \psi \quad \text{for every } a \in L^1(G).$$

This implies that for every $T \in M\left(L^1(G), L_{p,q;b}(G)\right)$, there is a unique $g \in L_{p,q;b}(G)$ such that

$$T(a) = a * g \quad \text{for every } a \in L^1(G).$$

Thus, we get the result. □

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INTEGRAL-TYPE OPERATORS BETWEEN WEIGHTED BERGMAN SPACES ON THE UNIT DISK

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Abstract. The boundedness and compactness of the integral-type operator

$$I_g^{(n)}f(z)=\int_0^z f^{(n)}(\zeta)g(\zeta)d\zeta$$

where g is a holomorphic function on the unit disk and $n\in\mathbb{N}_0$, between weighted Bergman spaces are characterized.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$. Let $g\in H(\mathbb{D})$ and $n\in\mathbb{N}_0$. We define an integral-type operator denoted by $I_g^{(n)}$ as follows

$$I_g^{(n)}f(z)=\int_0^z f^{(n)}(\zeta)g(\zeta)d\zeta,\quad f\in H(\mathbb{D}). \tag{1}$$

For $n=0$ and $g(z)=h'(z)$, operator (1) is reduced to the operator introduced in [13]. For $n=1$ is obtained a counterpart of the operator. These operators and their n-dimensional extensions acting among spaces of holomorphic functions have been studied a lot recently, see, for example, [1]-[35] and the references therein.

Let $dA(z)=\frac{1}{\pi}dxdy=\frac{1}{\pi}rdrd\theta$ be the normalized area measure on $\mathbb{D}$, and $dA_\alpha(z)=(\alpha+1)(1-|z|^2)^\alpha dA(z)$, $\alpha>-1$, $z\in\mathbb{D}$. The weighted Bergman space $A^p_\alpha=A^p_\alpha(\mathbb{D})$, $p>0$, $\alpha>-1$ consists of all $f\in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha}=\int_{\mathbb{D}}|f(z)|^p dA_\alpha(z)<\infty.$$

When $p\geq 1$, the weighted Bergman space with the norm $\|\cdot\|_{A^p_\alpha}$ becomes a Banach space. If $p\in(0,1)$, it is a Frechet space with the translation invariant metric $d(f,g)=\|f-g\|^p_{A^p_\alpha}$. The corresponding Lebesgue space is denoted by $L^p_\alpha=L^p_\alpha(\mathbb{D})$.

The weighted-type space H^∞_α, $\alpha\geq 0$, consists of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{H^\infty_\alpha}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)|<\infty.$$

With the norm $\|\cdot\|_{H^\infty_\alpha}$ it is a Banach space. For $\alpha=0$ the space becomes the space of bounded holomorphic functions H^∞. The little weighted-type space, denoted by $H^\infty_{\alpha,0}$ is a subspace of H^∞_α consisting of all holomorphic functions f such that $\lim_{|z|\to 1}(1-|z|^2)^\alpha|f(z)|=0$.

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Here we characterize the boundedness and compactness of operator (1) between weighted Bergman spaces (for the case $p \in (0,1)$ we mean the metric boundedness and compactness). Our results extend one-dimensional results in [8].

Throughout the paper C denotes a positive constant not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \leq B \leq CA$.

2. Auxiliary results

Here we quote some auxiliary results which will be used in the proofs of the main results of this paper. The first one is the next classical result.

Lemma 1. *Let $f \in H(\mathbb{D})$. Then $f \in A^p_\alpha$ if and only if $f^{(n)}(z)(1-|z|^2)^n \in L^p_\alpha$, and*

$$\|f\|_{A^p_\alpha} \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)| + \left(\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{pn+\alpha} dA(z) \right)^{1/p}. \tag{2}$$

From (2) we see that if $f \in A^p_\alpha$, then $f^{(n)} \in A^p_{pn+\alpha}$ and $\|f^{(n)}\|_{A^p_{pn+\alpha}} \leq C\|f\|_{A^p_\alpha}$, and it is easily get that for every $z \in \mathbb{D}$ there is a $C > 0$ independent of f such that

$$|f^{(n)}(z)| \leq C \frac{\|f^{(n)}\|_{A^p_{pn+\alpha}}}{(1-|z|^2)^{(2+pn+\alpha)/p}} \leq C \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(2+pn+\alpha)/p}}. \tag{3}$$

The following criterion for compactness is proved in a standard way, see [14].

Lemma 2. *Let $0 < p, q < \infty$, $-1 < \alpha, \beta < \infty$, $g \in H(\mathbb{D})$ and $n \in \mathbb{N}_0$. Then $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ is compact if and only if for any bounded sequence $(f_m)_{m\in\mathbb{N}} \subset A^p_\alpha$ converging to zero on compacts of $\mathbb{D}$, we have $\|I^{(n)}_g f_m\|_{A^q_\beta} \to 0$ as $m \to \infty$.*

3. Main results

Here we prove our main results, which are incorporated in the following theorem.

Theorem 1. *Let $0 < p, q < \infty$, $-1 < \alpha,\ \beta < \infty$, $g \in H(\mathbb{D})$ and $n \in \mathbb{N}_0$. Then the following statements hold true.*

(a) If $p \leq q$, then $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ is bounded if and only if $g \in \mathfrak{M}$, where

$$\mathfrak{M} = \begin{cases} H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p}, & \text{if } n+(2+\alpha)/p \leq 1+(2+\beta)/q \\ \{0\}, & \text{if } n+(2+\alpha)/p > 1+(2+\beta)/q. \end{cases}$$

(b) If $p \leq q$, then $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ is compact if and only if $g \in \mathfrak{M}_0$, where

$$\mathfrak{M}_0 = \begin{cases} H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p,0}, & \text{if } n+(2+\alpha)/p < 1+(2+\beta)/q \\ \{0\}, & \text{if } n+(2+\alpha)/p \geq 1+(2+\beta)/q. \end{cases}$$

(c) If $p > q$, then the following statements are equivalent:

(i) $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ is bounded;

(ii) $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ is compact;

(iii) If $n+(1+\alpha)/p < 1+(1+\beta)/q$, then $g \in L^s_{s+\gamma-ns}$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$, $\gamma = (p\beta - q\alpha)/(p-q)$, while if $n+(1+\alpha)/p \geq 1+(1+\beta)/q$, then $g \equiv 0$.

Proof. (a) By Lemma 1, we have

$$\|f\|^q_{A^q_\beta} \asymp |f(0)|^q + \|f'\|^q_{A^q_{q+\beta}}.$$

Also $I^{(n)}_g f(0) = 0$ for every $f \in H(\mathbb{D})$ and $n \in \mathbb{N}_0$. Thus, we have

$$\|I^{(n)}_g f\|^q_{A^q_\beta} \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^q |g(z)|^q (1-|z|^2)^{\beta+q} dA(z). \tag{4}$$

First suppose that $n + (2+\alpha)/p \le 1 + (2+\beta)/q$ and $g \in H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p}$. Then by the Hölder inequality, (2), (3) and (4) we obtain

$$\begin{aligned}
\|I^{(n)}_g f\|^q_{A^q_\beta} &\le C\|g\|^q_{H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p}} \int_{\mathbb{D}} |f^{(n)}(z)|^q (1-|z|^2)^{(q/p)(2+\alpha)+nq-2} dA(z) \\
&\le C\|g\|^q_{H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p}} \|f\|^{q-p}_{A^p_\alpha} \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{\alpha+np} dA(z) \\
&\le C\|g\|^q_{H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p}} \|f\|^q_{A^p_\alpha},
\end{aligned} \tag{5}$$

from which the boundedness of $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ follows in this case.

If $n + (2+\alpha)/p > 1 + (2+\beta)/q$ and $g \equiv 0$ the boundedness of the operator trivially follows.

Conversely, suppose that $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ is bounded. Let

$$f_a(z) = \frac{(1-|a|^2)^{\frac{2+\alpha}{p}}}{(1-\overline{a}z)^{\frac{2(2+\alpha)}{p}}}, \quad a \in \mathbb{D}. \tag{6}$$

It is known that $\|f_a\|_{A^p_\alpha} = 1$, for each $a \in \mathbb{D}$. Using (2) and (3), we get

$$\begin{aligned}
\frac{|a|^n |g(a)|}{(1-|a|^2)^{n+(2+\alpha)/p}} &= C|(I^{(n)}_g f_a)'(a)| \le C \frac{\|(I^{(n)}_g f_a)'\|_{A^q_{q+\beta}}}{(1-|a|^2)^{1+(2+\beta)/q}} \\
&\le C \frac{\|I^{(n)}_g f_a\|_{A^q_\beta}}{(1-|a|^2)^{1+(2+\beta)/q}} \le C \frac{\|I^{(n)}_g\|_{A^p_\alpha \to A^q_\beta}}{(1-|a|^2)^{1+(2+\beta)/q}},
\end{aligned} \tag{7}$$

from which along with the maximum modulus theorem, for $n + (2+\alpha)/p > 1 + (2+\beta)/q$, it follows that $g \equiv 0$, while for $n + (2+\alpha)/p \le 1 + (2+\beta)/q$ it follows that $g \in H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p}$.

(b) First assume that $I^{(n)}_g : A^p_\alpha \to A^q_\beta$ is compact. By using the functions in (6) which are norm bounded and converge to zero uniformly on compacts of $\mathbb{D}$, as $|a| \to 1$, we get

$$\lim_{|a|\to 1} \|I^{(n)}_g f_a\|_{A^q_\beta} = 0.$$

From this, the second inequality in (7) and the maximum modulus theorem we easily get that $g \in \mathfrak{M}_0$.

Now suppose that $g \in \mathfrak{M}_0$. If $n + (2+\alpha)/p \ge 1 + (2+\beta)/q$ and $g \equiv 0$ then the result is obvious.

If $n + (2+\alpha)/p < 1 + (2+\beta)/q$, and $g \in H^\infty_{1-n+(2+\beta)/q-(2+\alpha)/p,0}$, then for every $\varepsilon > 0$, there is an $r \in (0,1)$ such that for $r < |z| < 1$

$$|g(z)|(1-|z|^2)^{1-n+(2+\beta)/q-(2+\alpha)/p} < \varepsilon.$$

Let $(f_m)_{m\in\mathbb{N}}$ be a bounded sequence in A^p_α, say by M, converging to zero uniformly on compacts of $\mathbb{D}$ as $m\to\infty$. We have

$$\begin{aligned}\|I_g^{(n)}f_m\|^q_{A^q_\beta} &\asymp \int_{\mathbb{D}}|f_m^{(n)}(z)|^q|g(z)|^q(1-|z|^2)^{\beta+q}dA(z)\\ &= \left(\int_{|z|\le r}+\int_{|z|>r}\right)|f_m^{(n)}(z)|^q|g(z)|^q(1-|z|^2)^{\beta+q}dA(z)\\ &= J_1(m)+J_2(m).\end{aligned}$$

By the Weierstrass theorem it follows that $f_m^{(n)}\to 0$ uniformly on compacts of $\mathbb{D}$, as $m\to\infty$. Hence for large enough m, say $m\ge m_0$, we have

$$|f_m^{(n)}(z)|<\varepsilon,\quad \text{for}\quad |z|\le r. \tag{8}$$

Using (8) we have that for $m\ge m_0$

$$J_1(m)\le\varepsilon^q\int_{|z|\le r}|g(z)|^q(1-|z|^2)^{\beta+q}dA(z)=\varepsilon K(r). \tag{9}$$

Similar to (5) is obtained

$$J_2(m)\le\varepsilon^q C\|f_m\|^{q-p}_{A^p_\alpha}\int_{\mathbb{D}}|f_m^{(n)}(z)|^p(1-|z|^2)^{\alpha+np}dA(z)\le\varepsilon^q CM^q. \tag{10}$$

From (9) and (10) and since ε is an arbitrary positive number it follows that $\|I_g^{(n)}f_m\|_{A^q_\beta}\to 0$ as $m\to\infty$. Hence by Lemma 2 the compactness of $I_g^{(n)}:A^p_\alpha\to A^q_\beta$ follows.

(c) $(i)\Rightarrow(iii)$. Suppose that $I_g^{(n)}:A^p_\alpha\to A^q_\beta$ is bounded. Following the lines of the proof of Theorem 1 in [12] it is not difficult to show that

$$B(g):=\int_{\mathbb{D}}\left(\frac{1}{(1-|z|^2)^{2+nq+\alpha q/p}}\int_{D(z,1/2)}|g(w)|^q(1-|w|^2)^{\beta+q}dA(w)\right)^{p/(p-q)}dA(z)$$

is finite, where

$$D(z,1/2)=\left\{w:\left|\frac{z-w}{1-\bar zw}\right|<\frac12\right\}$$

is the pseudohyperbolic disk centered at z with radius $1/2$.

By the subharmonicity, we have

$$|g(z)|^q\le\frac{1}{(1-|z|^2)^2}\int_{D(z,1/2)}|g(w)|^qdA(w),$$

from which along with the relation $1-|z|^2\asymp 1-|w|^2$ for $w\in D(z,1/2)$, we get

$$\|g\|^s_{L^s_{s+\gamma-ns}}=C\int_{\mathbb{D}}|g(w)|^s(1-|w|^2)^{s+\gamma-ns}dA(w)\le CB(g).$$

Thus $g\in L^s_{s+\gamma-ns}$. If $s+\gamma-ns\le -1\Leftrightarrow n+(1+\alpha)/p\ge 1+(1+\beta)/q$, we get $g\equiv 0$.

$(ii)\Rightarrow(i)$. This implication is obvious.

$(iii)\Rightarrow(ii)$. If $1+(1+\beta)/q\ge n+(1+\alpha)/p$ and $g\equiv 0$, then it is obvious that $I_g^{(n)}:A^p_\alpha\to A^q_\beta$ is compact. So let $1+(1+\beta)/q<n+(1+\alpha)/p$ and $(f_m)_{m\in\mathbb{N}}$ be a bounded sequence in A^p_α that converges to zero uniformly on compacts of $\mathbb{D}$ as $m\to\infty$. To show that $I_g^{(n)}:A^p_\alpha\to A^q_\beta$ is compact, by Lemma 2, it is enough to show that

$$\|I_g^{(n)}f_m\|^q_{A^q_\beta}\asymp\int_{\mathbb{D}}|f_m^{(n)}(z)|^q|g(z)|^q(1-|z|^2)^{\beta+q}dA(z)\to 0,\quad \text{as}\ \ m\to\infty.$$

Let $\varepsilon > 0$ be arbitrary. By the hypothesis there is an $r \in (0,1)$ such that

$$\int_{|z|>r} |g(z)|^s (1-|z|^2)^{s+\gamma-ns} dA(z) < \varepsilon^{p/(p-q)}.$$

By Hölder's inequality and (2) it follows that

$$\begin{aligned}
&\int_{|z|>r} |f_m^{(n)}(z)|^q |g(z)|^q (1-|z|^2)^{\beta+q} dA(z) \\
&= \int_{|z|>r} |f_m^{(n)}(z)|^q (1-|z|^2)^{nq+\alpha q/p} |g(z)|^q (1-|z|^2)^{\beta+q-nq-\alpha q/p} dA(z) \\
&\le \left(\int_{\mathbb{D}} |f_m^{(n)}(z)|^p (1-|z|^2)^{np+\alpha} dA(z) \right)^{q/p} \left(\int_{|z|>r} |g(z)|^s (1-|z|^2)^{s+\gamma-ns} dA(z) \right)^{(p-q)/p} \\
&\le \varepsilon C \|f_m\|_{A^p_\alpha}^q \le \varepsilon C M^q. \qquad (11)
\end{aligned}$$

Since (8) holds for large m, say $m \ge m_1$, then we have that

$$\begin{aligned}
\int_{|z|\le r} |f_m^{(n)}(z)|^q |g(z)|^q (1-|z|^2)^{\beta+q} dA(z) &\le \varepsilon^q \int_{|z|\le r} |g(z)|^q (1-|z|^2)^{\beta+q} dA(z) \\
&= \varepsilon^q K(r). \qquad (12)
\end{aligned}$$

From (11) and (12), we get $\|I_g^{(n)} f_m\|_{A^q_\beta} \to 0$ as $m \to \infty$, as claimed. □

Corollary 1. *Let $0 < p < \infty$, $-1 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $n \in \mathbb{N}_0$. Then the following statements hold.*

(a) $I_g^{(n)} : A^p_\alpha \to A^p_\alpha$ is bounded if and only if $g \in \mathfrak{N}$, where

$$\mathfrak{N} = \begin{cases} H^\infty_1, & \text{if } n = 0 \\ H^\infty, & \text{if } n = 1 \\ \{0\}, & \text{if } n > 1. \end{cases}$$

(b) $I_g^{(n)} : A^p_\alpha \to A^p_\alpha$ is compact if and only if $g \in H^\infty_{1,0}$, if $n = 0$, and $g \equiv 0$ if $n \in \mathbb{N}$.

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Random Iterative Algorithms for Nonlinear Mixed Family of Random Fuzzy and Crisp Operator Equation Couples in Fuzzy Normed Spaces[1]

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Abstract. The purpose of this paper is to introduce and study a new class of nonlinear mixed family of random fuzzy and crisp operator equation couples in fuzzy normed spaces based on the random version of the theory of (φ, ψ)-contractor due to Mihet. Further, some new random iterative algorithms for solving this kind of nonlinear operator equation couples in fuzzy normed spaces are constructed and the convergence of iterative sequences generated by the algorithms under joint orbitally complete conditions is proved. As applications, some new common fixed point theorems for a mixed family of fuzzy and crisp operators in fuzzy normed spaces are also given. The results presented in this paper improve and generalize the corresponding results of recent works.

Key Words and Phrases. Nonlinear mixed family of random fuzzy and crisp operator equation couple, (φ, ψ)-contractor, Joint orbitally complete condition, new random iterative algorithm, approximation and convergence.

AMS Subject Classification. 47H15, 54E70, 47S40.

1 Introduction

It is well known that fuzzy normed spaces are an important class of fuzzy metric spaces and offer an appropriate frame work for inexact measurements of an ordinary length in a linear space. The value of a fuzzy norm for a vector x is a fuzzy set on $[0, +\infty)$ rather than a number (see [1]). In [2], Goleț introduced a generalization of the concept of fuzzy normed spaces, which includes some earlier defined fuzzy normed spaces as special cases. Further, Goleț [2] pointed out: if we define on a linear space L the mapping $x \to F_x$, $F_x(t) = N(x, t)$ for $t \in [0, +\infty)$, then the triple $(L, F, \star)$ becomes a generalized probabilistic normed space (see [3], the random variable associated to the distribution function F_x can take the value ∞ with a probability greater than 0). If the following condition is satisfied: $\lim_{t\to+\infty} N(x, t) = 1$ for all $x \in L$, then the triple $(L, F, \star)$ is a probabilistic normed space. Conversely, if $(L, F, \star)$ is a probabilistic normed space, and we define $N(x, t) = F_x(t)$ then $(L, F, \star)$ becomes a fuzzy normed space. One can see a generality of fuzzy normed space. The specific for the both fuzzy and probabilistic norms are the distinct areas of applicability and the different interpretation ways. For more detail, see, for example, [4] and [5].

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On the other hand, since then, by using ψ-probabilistic contraction and a truthful probabilistic version of the Banach fixed point principle, some problems for nonlinear operators and common fixed point theorems have been introduced and studied by many authors (see, e.g., [6-14]). Sharma et al. [15] considered two nonfuzzy mappings and a sequence of fuzzy mappings to define a hybrid D-compatible condition. They also showed the existence of common fixed points under such condition, where the range of the one of the two nonfuzzy mappings is joint orbitally complete, and sketched a view of the present stage of the confrontation. As Sharma et al. [15] point out "instead of trying to modelize a not well enough known physiological system, it seemed rather preferable to examine what are the required mathematical conditions holding on mathematical systems that would be progressively adjusted closer and closer to known biological situations". Furthermore, by using a random version of the theory of contractor and the concept of probabilistic Ψ-contractor couple, Cho et al. [11] introduced and studied a system of nonlinear operator equations for a mixed family of fuzzy and crisp operators in probabilistic normed spaces, and also proved some new existence theorems of solutions of a new system of nonlinear operator equations for a mixed family of fuzzy and crisp operators and some new convergence results of sequences generated by the new iterative algorithms under joint orbitally complete conditions.

In 2007, Mihet [16] introduced the concept of probabilistic (φ, ψ)-contractor couple, and considered a class of set-valued nonlinear operator equations in Menger probabilistic normed spaces, under a larger class of t-norms. Based on the concept of probabilistic (φ, ψ)-contractor couple duo to Mihet [16], Lan et al. [17] introduced and studied a new class of nonlinear operator equation couples with a mixed family of fuzzy and crisp operator equations in Menger probabilistic normed spaces, and discussed the existence of solutions for the nonlinear operator equation couples and the convergence of iterative sequences generated by the algorithms under a larger class of t-norms and joint orbitally complete conditions.

Motivated and inspired by the above works, in this paper, by using a random version of the theory of (φ, ψ)-contractor duo to Mihet [16], we shall introduce and study the following new class of nonlinear mixed family of random fuzzy and crisp operator equation couples in fuzzy normed spaces:

Find a measurable operator $x : \Lambda \to \mathbb{X}$ such that for all $\lambda \in \Lambda$,

$$\begin{cases} S_{i\,\lambda,x(\lambda)}(u(\lambda)) \geq d(x(\lambda)), \\ S_{j\,\lambda,x(\lambda)}(u(\lambda)) \geq e(x(\lambda)), \end{cases} \tag{1.1}$$

where $d, e : \mathbb{X} \to (0, 1]$ are two nonlinear operators, $S_i, S_j : \Lambda \times \mathbb{X} \to W(\mathbb{Y})$ are two random fuzzy operators satisfying some conditions for some $i, j \in \mathbb{N}$, $(\Lambda, \mathcal{A})$ is a measure space, $\mathbb{X}$ is a separable real vector space and $W(\mathbb{Y})$ be a collection of all fuzzy sets over $\mathbb{Y}$.

Obviously, equations (1.1) are equivalent to the following nonlinear equations for set-valued operators:

$$\begin{cases} u(\lambda) \in S_i(\lambda, x(\lambda)), \\ u(\lambda) \in S_j(\lambda, x(\lambda)). \end{cases} \tag{1.2}$$

Moreover, (1.2) becomes to the following nonlinear operator equations:

$$\begin{cases} u(\lambda) = f(\lambda, x(\lambda)), \\ u(\lambda) = g(\lambda, x(\lambda)), \end{cases} \tag{1.3}$$

when $f, g : \Lambda \times \mathbb{X} \to \mathbb{X}$ are two random operators satisfied $S_i(\lambda, x(\lambda)) \subset f(\Lambda \times \mathbb{X})$ and $S_j(\lambda, x(\lambda)) \subset g(\Lambda \times \mathbb{X})$ for any $(\lambda, x) \in \Lambda \times \mathbb{X}$ and some $i, j \in \mathbb{N}$.

Without loss of generality, we can suppose that $u = \theta$. In fact, if $u \neq \theta$, for some $i, j \in \mathbb{N}$, putting $T_i, T_j : \Lambda \times \mathbb{X} \to CB(\mathbb{Y})$ are two set-valued operators with $T_i(x) = S_i(x) - u$ and $T_j(x) = S_j(x) - u$ for all $x \in \mathbb{X}$, then (1.2) is equivalent to $\begin{cases} \theta \in T_i(\lambda, x(\lambda)), \\ \theta \in T_j(\lambda, x(\lambda)). \end{cases}$ Thus, in order to discuss equations (1.1), we can turn to discuss the following equations:

$$\begin{cases} S_{i\,\lambda,x(\lambda)}(\theta) \geq d(\lambda, x(\lambda)), \\ S_{j\,\lambda,x(\lambda)}(\theta) \geq e(\lambda, x(\lambda)), \end{cases} \quad \text{or} \quad \begin{cases} \theta = f(\lambda, x(\lambda)), \\ \theta = g(\lambda, x(\lambda)), \end{cases} \tag{1.4}$$

under some conditions.

Further, we will construct some new random iterative algorithms for solving this kind of nonlinear operator equation couples in fuzzy normed spaces and prove the convergence of iterative sequences generated by the algorithms under joint orbitally complete conditions. As applications, we shall also prove some new common fixed point theorems for a mixed family of fuzzy and crisp operators in fuzzy normed spaces.

2 Preliminaries

Throughout this paper, we suppose that $(\Lambda, \mathcal{A})$ is a complete σ-finite measure space and $\mathbb{X}$ is a separable real vector space over the real or complex number field $\mathcal{K}$. We denote by $\mathcal{B}(\mathbb{X})$, $2^{\mathbb{X}}$ the class of Borel σ-fields in $\mathbb{X}$ and the family of all the nonempty subsets of $\mathbb{X}$, respectively. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ be the set of all natural numbers.

Definition 2.1. An operator $x : \Lambda \to \mathbb{X}$ is said to be measurable if, for any $B \in \mathcal{B}(\mathbb{X})$, $\{\lambda \in \Lambda : x(\lambda) \in B\} \in \mathcal{A}$.

Definition 2.2. An operator $F : \Lambda \times \mathbb{X} \to \mathbb{X}$ is called a random operator if for any $x \in \mathbb{X}$, $F(\lambda, x) = y(\lambda)$ is measurable. A random operator F is said to be continuous (resp. linear, bounded) if for any $\lambda \in \Lambda$, the operator $F(\lambda, \cdot) : \mathbb{X} \to \mathbb{X}$ is continuous (resp. linear, bounded).

Similarly, we can define a random operator $a : \Lambda \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$. We shall write $F_\lambda(x) = F(\lambda, x(\lambda))$ and $a_\lambda(x, y) = a(\lambda, x(\lambda), y(\lambda))$ for all $\lambda \in \Lambda$ and $x(\lambda)$, $y(\lambda) \in \mathbb{X}$. It is well known that a measurable operator is necessarily a random operator.

Definition 2.3. A set-valued operator $P : \Lambda \to 2^{\mathbb{X}}$ is said to be measurable if, for any $B \in \mathcal{B}(\mathbb{X})$, $P^{-1}(B) = \{\lambda \in \Lambda : P(\lambda) \cap B \neq \emptyset\} \in \mathcal{A}$.

Definition 2.4. An operator $u : \Lambda \to \mathbb{X}$ is called a measurable selection of a set-valued measurable operator $\Gamma : \Lambda \to 2^{\mathbb{X}}$ if u is measurable and for any $\lambda \in \Lambda$, $u(\lambda) \in \Gamma(\lambda)$.

Definition 2.5. A set-valued operator $Q : \Lambda \times \mathbb{X} \to 2^{\mathbb{X}}$ is called a random set-valued operator if, for any $x \in \mathbb{X}$, $Q(\cdot, x)$ is measurable.

In what follows, let $W(\mathbb{X})$ denote the collection of all fuzzy sets over $\mathbb{X}$. An operator $S : D \subset \mathbb{X} \to W(\mathbb{X})$ is called a fuzzy operator. For each $x \in \mathbb{X}$, $S(x)$ (denote it by S_x, in the sequel) is a fuzzy set on $\mathbb{X}$ and $S_x(y)$ is the membership function of y in S_x.

Definition 2.6. A fuzzy operator $T : \Lambda \times \mathbb{X} \to W(\mathbb{X})$ is a random fuzzy operator if, for any $x \in \mathbb{X}$, $T(\cdot, x) : \Lambda \to W(\mathbb{X})$ is a measurable fuzzy operator (denoted by $T_{\lambda,x}$, short down T).

Obviously, the random fuzzy operator includes set-valued operator, random set-valued operator and fuzzy operator as the special cases.

A fuzzy number $x : \mathbb{R} \to [0,1]$ is called convex if its α-level sets $[x]_\alpha$ (i.e., α-cut set of x) is a convex set in $\mathbb{R}$ for every $\alpha \in (0,1]$, where $[x]_\alpha = \{s : x(s) \geq \alpha\}$. A fuzzy number x is called normal if there exists $s_0 \in \mathbb{R}$ such that $x(s_0) = 1$. A fuzzy number x is called nonnegative if $x(s) = 0$ for all $s < 0$. The fuzzy number $\widetilde{0}$ is defined by $\widetilde{0}(s) = 1$ for $s = 0$ and $\widetilde{0}(s) = 0$ for $s \neq 0$. Let G be the set of all nonnegative upper semicontinuous normal convex fuzzy numbers. Obviously, if $x \in G$, then each of its α-level sets $[x]_\alpha$ is a closed interval $[a^\alpha, b^\alpha]$, where $a^\alpha \in \mathbb{R}^+$ and $b^\alpha \in \mathbb{R}^+ \cup \{+\infty\}$. When $b^\alpha = +\infty$, then $[a^\alpha, b^\alpha]$ means the interval $[a^\alpha, +\infty)$.

Definition 2.7. Let $\|\cdot\|$ be a mapping from real vector space $\mathbb{X}$ into G and the mappings l, r (respectively, left norm and right norm) : $[0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $l(0,0) = 0, r(1,1) = 1$. Denote $[\|x\|]_\alpha = [\|\|x\|\|_1^\alpha, \|\|x\|\|_2^\alpha]$, for all $x \in \mathbb{X}, \alpha \in (0,1]$. The quadruple $(\mathbb{X}, \|\cdot\|, l, r)$ is called a fuzzy normed space (briefly, an FN-space) and $\|\cdot\|$ a fuzzy norm, if the following conditions are satisfied: (FN-1) $\|x\| = \widetilde{0}$ if and only if $x = \theta$; (FN-2) $\|kx\| = |k| \cdot \|x\|$ for all $x \in \mathbb{X}, k \in \mathbb{R}$; (FN-3) for all $x, y \in \mathbb{X}$, $\|x+y\|(\nu+\mu) \geq l(\|x\|(\nu), \|y\|(\mu))$ whenever $\nu \leq \|\|x\|\|_1^1, \mu \leq \|\|y\|\|_1^1$ and $\nu + \mu \leq \|\|x+y\|\|_1^1$, and $\|x+y\|(\nu+\mu) \leq r(\|x\|(\nu), \|y\|(\mu))$ whenever $\nu \geq \|\|x\|\|_1^1, \mu \geq \|\|y\|\|_1^1$ and $\nu + \mu \geq \|\|x+y\|\|_1^1$.

Remark 2.1. ([12]) There is a little difference between the above definition and the definition of the fuzzy normed space given in [18]. In [18], the definition requires that for all $x \in \mathbb{X}, x \neq \theta$, there exists $\alpha_0 \in (0,1]$ independent of x such that $\|\|x\|\|_2^\alpha < \infty$ for all $\alpha \leq \alpha_0$ and $\inf_{\alpha \leq \alpha_0} \|\|x\|\|_1^\alpha > 0$.

Remark 2.2. ([19]) By induction, we can generalize the triangle inequality (FN-3) as the following: (FN-3)$'$ for any $n \in N$ and $x_1, \cdots, x_n \in \mathbb{X}$, $\|\sum_{i=1}^n x_i\|(\nu_1) \geq l(\|x_1\|(s_1), l(\|x_2\|(s_2), \cdots, l(\|x_{n-1}\|(s_{n-1}), \|x_n\|(s_n)) \cdots))$ whenever $s_i \leq \|\|x_i\|\|_1^1$, $i = 1, \cdots, n$ and for any $1 \leq k \leq n-1$, $\nu_k = \sum_{i=k}^n s_i \leq \|\|\sum_{i=k}^n x_i\|\|_1^1$, and $\|\sum_{i=1}^n x_i\|(\upsilon_1) \leq r(\|x_1\|(\tau_1), r(\|x_2\|(\tau_2), \cdots, r(\|x_{n-1}\|(\tau_{n-1}), \|x_n\|(\tau_n)) \cdots))$ whenever $\tau_i \geq \|\|x_i\|\|_1^1$, $i = 1, \cdots, n$ and for all $1 \leq k \leq n-1$, $\upsilon_k = \sum_{i=k}^n \tau_i \geq \|\|\sum_{i=k}^n x_i\|\|_1^1$.

Remark 2.3. ([1, 20]) Each classical normed space and each Menger probabilistic normed space can be considered as a fuzzy normed space.

By using (FN-3)$'$ and the same method as Theorem 2.1 in [12], we can obtain the following lemma.

Lemma 2.1. Let $(\mathbb{X}, \|\cdot\|, l, r)$ be an FN-space. If $\{r^n(s)\}$ is an equi-continuous sequence of mappings at $s = 0$, where $r^1(s) = s$, $\quad r^n(s) = r(s, r^{n-1}(s))$, $n = 2, 3, \cdots$, then for each $\alpha \in (0,1]$, there exists $\beta \in (0, \alpha]$ such that $\|\|\sum_{i=1}^n x_i\|\|_2^\alpha \leq \sum_{i=1}^n \|\|x_i\|\|_2^\beta$ for all $x_i \in \mathbb{X}$, $i = 1, 2, \cdots, n$.

Definition 2.8. Let $(\mathbb{X}, \|\cdot\|, l, r)$ be an FN-space. A sequence $\{x_n\}$ in $\mathbb{X}$ is said to

(i) converge to $x \in \mathbb{X}$ (often write $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$) if $\lim_{n\to+\infty} \|x_n - x\| = \widetilde{0}$, i.e., $\lim_{n\to+\infty} \|\|x_n - x\|\|_2^\alpha = 0$ for all $\alpha \in (0,1]$.

(ii) be a Cauchy sequence if $\lim_{m,n\to+\infty} \|x_m - x_n\| = \widetilde{0}$, i.e., $\lim_{m,n\to+\infty} \|\|x_m - x_n\|\|_2^\alpha = 0$ for all $\alpha \in (0,1]$.

An FN-space $(\mathbb{X}, \|\cdot\|, l, r)$ is called complete if every Cauchy sequence in $\mathbb{X}$ converges in $\mathbb{X}$. In the sequel, we always suppose that $(\mathbb{X}, \|\cdot\|, l, r)$ is an FN-space satisfying the following condition: $\lim_{t\to+\infty} \|x\|(t) = 0$ for all $x \in \mathbb{X}$.

Definition 2.9. A nonempty subset A in $(\mathbb{X}, \|\cdot\|, l, r)$ is said to be bounded if there exists an $M > 0$ such that $\sup_{x\in A} \|\|x\|\|_2^\alpha \leq M \quad \forall \alpha \in (0,1]$.

Definition 2.10. Let $(\mathbb{X}, \|\cdot\|, l, r)$ be an FN-space and $A, B \in W(\mathbb{X})$. For each $\alpha \in (0,1]$,

we define $|||A|||_2^\alpha$ and $\rho_\alpha(A,B)$ as follows:

$$|||A|||_2^\alpha = \inf_{a\in A} |||a|||_2^\alpha \tag{2.1}$$

and $\rho_\alpha(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}|||a-b|||_2^\alpha, \sup_{b\in B}\inf_{a\in A}|||a-b|||_2^\alpha\}$.

By the definition of $|||A|||_2^\alpha$ and $\rho_\alpha(A,B)$, it is easy to obtain the following result:

Lemma 2.2. Let $(\mathbb{X}, \|\cdot\|, l, r)$ be an FN-space which satisfies the following conditions: $\lim_{s\to 0^+} r(s,s) = 0$ and $\lim_{s\to\infty}\|x\|(s) = 0$ for all $x \in \mathbb{X}$. If $A, B \in 2^{\mathbb{X}}$, then: (i) $|||A|||_2^\alpha = 0$ for all $\alpha \in (0,1]$ if and only if $\theta \in A$; (ii) $|||kA|||_2^\alpha = |k|\cdot|||A|||_2^\alpha$ for all $k \in \mathbb{R}$; (iii) $\rho_\alpha(x+A, x+B) = \rho_\alpha(x-A, x-B) = \rho_\alpha(A,B)$ for all $x \in \mathbb{X}$.

In the following Ψ is the class of all functions $\psi : [0,+\infty) \to [0,+\infty)$ and Φ is the class of all functions $\varphi : (0,1] \to (0,1]$. $\overline{\Psi}$ is the class of all functions ψ in Ψ which are bijective, nondecreasing and such that $\sum_{n=1}^\infty \psi^n(t) < +\infty$, for all $t > 0$, where ψ^n denotes the n-th iteration of ψ. $\overline{\Phi}$ is the collection of all nondecreasing functions φ such that $\lim_{n\to+\infty}\varphi^n(t) = 1$ for all $t \in (0,1]$. It is easy to see that if $\psi \in \overline{\Psi}$ then $\psi(t) < t$ and $\lim_{n\to+\infty}(\psi^{-1})^n(t) = +\infty$ for all $t > 0$ and if $\varphi \in \overline{\Phi}$, then $\varphi(t) > t$ for $t \in (0,1]$.

Definition 2.11. Let $(\mathbb{X}, \|\cdot\|, l, r)$ and $(\mathbb{Y}, \|\cdot\|, l, r)$ be two FN-spaces, and $\Gamma_1, \Gamma_2 : \Lambda \times \mathbb{X} \to S(\mathbb{Y}, \mathbb{X})$ be two operators, where $S(\mathbb{Y}, \mathbb{X})$ denotes the set of all odd operators from $\mathbb{Y}$ to $\mathbb{X}$ (i.e., $T \in S(\mathbb{Y}, \mathbb{X})$ if and only if $T(-y) = -T(y)$ for all $y \in \mathbb{Y}$), and $\varphi \in \overline{\Phi}$, $\psi \in \overline{\Psi}$. Then (Γ_1, Γ_2) is said to be a (φ,ψ)-contractor couple of two fuzzy operators $P, Q : D \subset \mathbb{X} \to W(\mathbb{Y})$, if for all $\alpha \in (0,1]$,

$$\rho_{\varphi(\alpha)}(P(x+\Gamma_1(x)y), Q(x)+y) \le \psi(\max\{|||Q(x)|||_2^\alpha, |||y|||_2^\alpha, |||Q(x)+y|||_2^\alpha\}), \tag{2.2}$$
$$\forall x \in D(Q), y \in \{y \in \mathbb{Y} : x+\Gamma_1(x)y \in D(P)\},$$
$$\rho_{\varphi(\alpha)}(Q(x+\Gamma_2(x)y), P(x)+y) \le \psi(\max\{|||P(x)|||_2^\alpha, |||y|||_2^\alpha, |||P(x)+y|||_2^\alpha\}), \tag{2.3}$$
$$\forall x \in D(P), y \in \{y \in \mathbb{Y} : x+\Gamma_2(x)y \in D(Q)\}.$$

Remark 2.4. If $\varphi(t) = t$ for $t \in (0,1]$, (Γ_1, Γ_2) is simply called a ψ-contractor couple of P and Q (see [19]).

Now, we define an orbit for mixed operators (S_n, f, g) and a joint orbitally complete space as follows:

Definition 2.12. Let $f, g : \Lambda \times \mathbb{X} \to \mathbb{X}$ be two nonlinear operators and $\{S_{n\,\lambda}\}_{n=1}^\infty$ a sequence of random fuzzy operators from $\Lambda \times \mathbb{X}$ into $W(\mathbb{X})$. If, for some $x_0(\lambda) \in \mathbb{X}$, there exist sequences $\{x_n(\lambda)\}$ and $\{y_n(\lambda)\}$ in $\mathbb{X}$ such that

$$\begin{aligned} \{y_{2n+1}(\lambda)\} &= \{f_\lambda(x_{2n+1})\} \subset S_{2n+1\,\lambda}(x_{2n}), \\ \{y_{2n+2}(\lambda)\} &= \{g_\lambda(x_{2n+2})\} \subset S_{2n+2\,\lambda}(x_{2n+1}) \end{aligned} \tag{2.4}$$

for all $n = 0, 1, 2, \cdots$, then $\vartheta(S_{n\,\lambda}, f_\lambda, g_\lambda, x_0(\lambda)) = \{y_n(\lambda) : n \in \mathbb{N}\}$ is called an orbit for the mixed operators $(S_{n\,\lambda}, f_\lambda, g_\lambda)$ for all $\lambda \in \Lambda$.

Definition 2.13. $\mathbb{X}$ is called x_0-joint orbitally complete if every Cauchy sequence of each orbit at $x_0(\lambda)$ is convergent in $\mathbb{X}$ for all $\lambda \in \Lambda$.

Remark 2.5. ([15]) Clearly, if $\mathbb{X}$ is an any complete space and $x_0 \in \mathbb{X}$, then $\mathbb{X}$ is x_0-joint orbitally complete, while converse is not necessarily true.

3 Random Iterative Algorithms

In this section, we suggest and analyze a new class of iterative methods and construct some new random iterative algorithms for solving the problem (1.4).

Definition 3.1. Let $(\mathbb{X}, \|\cdot\|, l, r)$ and $(\mathbb{Y}, \|\cdot\|, l, r)$ be two FN-spaces. An operator $P : D(P) \subset \mathbb{X} \to W(\mathbb{Y})$ is said to be closed, if for any $x_n \in D(P)$ and $y_n \in P(x_n)$ we have $x \in D(P)$ and $y \in P(x)$ whenever $x_n \to x$ and $y_n \to y$.

In order to give our main results, we first list the following condition $(\mathcal{C}^*)$:

Let $P, Q : D \subset \mathbb{X} \to W(\mathbb{Y})$ be two fuzzy operators. For any given $x \in D(Q), y \in P(x)$ and $\hat{x} \in D(P), \hat{y} \in Q(\bar{x})$, there exist $z \in P(x - \Gamma_1(x)y)$ and $\hat{z} \in Q(\hat{x} - \Gamma_2(\hat{x})\hat{y})$, respectively, such that

$$|||z|||_2^{\varphi(\alpha)} \le \rho_{\varphi(\alpha)}(P(x - \Gamma_1(x)y), Q(x) - y),\ |||\hat{z}|||_2^{\varphi(\alpha)} \le \rho_{\varphi(\alpha)}(Q(\hat{x} - \Gamma_2(\hat{x})\hat{y}), P(\hat{x}) - \hat{y}).$$

Remark 3.1. If $P = Q$ is a set-valued operator and $\varphi(t) = t$ for $t \in (0, 1]$, then the condition $(\mathcal{C}^*)$ coincides with (iv) of Theorem 3.1 in [12].

By (2.1) and Lemma 2.2, we introduce the following algorithms for our main results:

Algorithm 3.1. Let $(\mathbb{X}, \|\cdot\|, l, r)$ and $(\mathbb{Y}, \|\cdot\|, l, r)$ be two FN-spaces, f, g be two operators from $\Lambda \times \mathbb{X}$ into $\mathbb{X}$, $\{S_{n\,\lambda}\}_{n=1}^{\infty}$ be a sequence of random fuzzy operators from $\Lambda \times \mathbb{X}$ into $W(\mathbb{X})$. Suppose that the condition $(\mathcal{C}^*)$ holds and for some $i, j \in \mathbb{N}$, the following conditions are satisfied: (i) $S_{i\,\lambda}(x) \subset f(\Lambda \times \mathbb{X})$ and $S_{j\,\lambda}(x) \subset g(\Lambda \times \mathbb{X})$ for all $\lambda \in \Lambda$ and $x \in \mathbb{X}$; (ii) $x + \Gamma_1(x)y \in D(S_{i\,\lambda})$ for all $x \in D(S_{j\,\lambda})$ and $y \in \mathbb{Y}$, $x + \Gamma_2(x)y \in D(S_{j\,\lambda})$ for all $x \in D(S_{i\,\lambda})$ and $y \in \mathbb{Y}$; (iii) (Γ_1, Γ_2) is a (φ, ψ)-contractor couple of $S_{i\,\lambda}$ and $S_{j\,\lambda}$. For all $\lambda \in \Lambda$, some $i, j \in \mathbb{N}$ and any given $x_0(\lambda) \in D(S_{j\,\lambda})$ and $y_0(\lambda) \in S_{j\,\lambda}(x_0)$, we define two sequences $\{x_n(\lambda)\}$ in $\mathbb{X}$ and $\{y_n(\lambda)\}$ in $\mathbb{Y}$ satisfying the following conditions:

$$\begin{aligned}
&x_{2n+1}(\lambda) = (1 - \sigma_{2n})x_{2n}(\lambda) + \sigma_{2n}(x_{2n}(\lambda) - \Gamma_1(x_{2n}(\lambda))y_{2n}(\lambda)),\\
&x_{2n+2}(\lambda) = (1 - \sigma_{2n+1})x_{2n+1}(\lambda) + \sigma_{2n+1}(x_{2n+1}(\lambda) - \Gamma_2(x_{2n+1}(\lambda))y_{2n+1}(\lambda)),\\
&y_{2n}(\lambda) \in S_{j\,\lambda}(x_{2n}), \quad y_{2n+1}(\lambda) \in S_{i\,\lambda}(x_{2n+1}),\\
&|||y_n|||_2^{\varphi(\alpha)} \le \psi^n(|||y_0(\lambda)|||_2^{\alpha}), \quad \forall \alpha \in (0, 1] \quad \text{and some } i, j \in \mathbb{N},
\end{aligned} \tag{3.1}$$

where $\{\sigma_n\}$ is a real monotone decreasing sequence in $(0,\ 1]$ and $\sigma_n \to \sigma \in (0, 1]$ as $n \to +\infty$.

Algorithm 3.2. Let $(\mathbb{X}, \|\cdot\|, l, r)$ and $(\mathbb{Y}, \|\cdot\|, l, r)$ be two FN-spaces and A, $V : \Lambda \times \mathbb{X} \to \mathbb{X}$ be two nonlinear operators. Let $\{S_{n\,\lambda}\}_{n=1}^{\infty}$, Γ_1, Γ_2, φ and ψ be the same as in Algorithm 3.1. Suppose that the conditions $(\mathcal{C}^*)$, and (ii) and (iii) in Algorithm 3.1 hold. If

(I) for all $\lambda \in \Lambda$, some $i, j \in \mathbb{N}$ and $x \in \mathbb{X}$, $S_{i\,\lambda}(x) \subset \mathbb{X} - A(\Lambda \times \mathbb{X})$ and $S_{j\,\lambda}(x) \subset \mathbb{X} - V(\Lambda \times \mathbb{X})$,

then, for some $i, j \in \mathbb{N}$ and any $x_0(\lambda) \in D(S_{j\,\lambda})$ and $y_0(\lambda) \in S_{j\,\lambda}(x_0)$, we have two sequences $\{x_n(\lambda)\}$ in $\mathbb{X}$ and $\{y_n(\lambda)\}$ in $\mathbb{Y}$, respectively, defined as follows:

$$\begin{aligned}
x_{2n+1}(\lambda) &= x_{2n}(\lambda) - \Gamma_1(x_{2n}(\lambda))y_{2n}(\lambda),\\
x_{2n+2}(\lambda) &= x_{2n+1}(\lambda) - \Gamma_2(x_{2n+1}(\lambda))y_{2n+1}(\lambda),
\end{aligned}$$

where the sequence $\{y_n(\lambda)\}$ in $\mathbb{Y}$ is defined by (2.4).

4 Approximation and Convergence

In this section, we first prove the existence of solutions of the equation couple (1.4) and the convergence of iterative sequences generated by Algorithm 3.1.

Theorem 4.1. Let $(\mathbb{X}, \|\cdot\|, l, r)$ and $(\mathbb{Y}, \|\cdot\|, l, r)$ be two complete FN-spaces and f, g, $\{S_{n\lambda}\}_{n=1}^{\infty}$, Γ_1, Γ_2, φ and ψ be the same as in Algorithm 3.1. Suppose that the condition $(\mathcal{C}^*)$, the conditions (i)-(iii) in Algorithm 3.1 and the following conditions hold: (iv) in FN-space $(\mathbb{X}, \|\cdot\|, l, r)$, $\lim_{a\to 0^+} r(a,a) = 0$, $\{r^n(t)\}$ is equi-continuous at $t = 0$ and $\lim_{t\to+\infty} \|x\|(t) = 0$ for all $x \in \mathbb{X}$; (v) $g(\Lambda \times \mathbb{X})$ is x_0-joint orbitally complete for some $x_0(\lambda) \in \mathbb{X}$; (vi) $S_{n\lambda}$ is closed for any $n \in \mathbb{N}$ and all $\lambda \in \Lambda$, and there exists a constant $M > 0$ such that for all $\alpha \in (0,1]$ and any constant $\omega > \tau > 0$,

$$|||\omega\Gamma_1(x)y|||_2^{\varphi(\alpha)} \le \tau M|||y|||_2^{\varphi(\alpha)}, \qquad \forall x \in D(S_{j\lambda}), \quad y \in \mathbb{Y}, \tag{4.1}$$

$$|||\omega\Gamma_2(x)y|||_2^{\varphi(\alpha)} \le \tau M|||y|||_2^{\varphi(\alpha)}, \qquad \forall x \in D(S_{i\lambda}), \quad y \in \mathbb{Y}. \tag{4.2}$$

Then the nonlinear operator equation couple (1.4) has a solution $x^*(\lambda)$ such that $\{f_\lambda(x^*)\} = \{g_\lambda(x^*)\} \subset \bigcap_{i=1}^{\infty} S_{i\lambda}(x^*)$ for all $\lambda \in \Lambda$. Further,

$$x_n(\lambda) \to x^*(\lambda), \quad y_n(\lambda) \to \theta \quad \text{as} \quad n \to \infty, \quad \forall \lambda \in \Lambda,$$

where $\{x_n(\lambda)\}$ in $\mathbb{X}$ and $\{y_n(\lambda)\}$ in $\mathbb{Y}$ are two sequences generated by Algorithm 3.1.

Proof. By (3.2), (4.1) and (4.2), for all $\alpha \in (0,1]$ we get

$$|||x_{2n+1}(\lambda) - x_{2n}(\lambda)|||_2^{\varphi(\alpha)} \le \sigma M|||y_{2n}(\lambda)|||_2^{\varphi(\alpha)} \le \sigma M\psi^{2n}(|||y_0(\lambda)|||_2^{\alpha}),$$
$$|||x_{2n+2}(\lambda) - x_{2n+1}(\lambda)|||_2^{\varphi(\alpha)} \le \sigma M|||y_{2n+1}(\lambda)|||_2^{\varphi(\alpha)} \le \sigma M\psi^{2n+1}(|||y_0(\lambda)|||_2^{\alpha}),$$

and so $|||x_{n+1} - x_n|||_2^{\varphi(\alpha)} \le \sigma M\psi^n(|||y_0(\lambda)|||_2^{\alpha})$ for all $\alpha \in (0,1]$. Since $\{r^n(t)\}$ is equi-continuous at $t = 0$, for each $\alpha \in (0,1]$, Lemma 2.11 implies that there exists a constant $\beta \in (0,\alpha]$ such that for all $m, n \in \mathbb{N}$ $(m > n)$,

$$|||x_m(\lambda) - x_n(\lambda)|||_2^{\varphi(\alpha)} \le \sum_{i=n}^{m-1} |||x_{i+1}(\lambda) - x_i(\lambda)|||_2^{\varphi(\beta)} \le \sum_{i=n}^{m-1} \sigma M\psi^i(|||y_0(\lambda)|||_2^{\beta}). \tag{4.3}$$

By the definition of $\psi \in \overline{\Psi}$, we know $\sum_{i=0}^{\infty} \psi^i(|||y_0(\lambda)|||_2^{\beta}) < +\infty$. Thus, for given $\epsilon > 0$, there exists $\mathbb{N}_0 \in \mathbb{N}$ such that

$$\sum_{i=n}^{m-1} \psi^i(|||y_0(\lambda)|||_2^{\beta}) < \frac{\epsilon}{\sigma M}, \quad \forall m > n \ge \mathbb{N}_0. \tag{4.4}$$

From (4.3), (4.4) and $\varphi : (0,1] \to (0,1]$, it follows that for all $\alpha \in (0,1]$

$$\varphi(\alpha) \in (0,1] \quad \text{and} \quad |||x_n(\lambda) - x_m(\lambda)|||_2^{\varphi(\alpha)} < \epsilon, \quad \forall m > n \ge \mathbb{N}_0$$

and so $\{x_n(\lambda)\}$ is a Cauchy sequence in $\mathbb{X}$ for all $\lambda \in \Lambda$. By the completeness of $\mathbb{X}$, we can suppose that $x_n(\lambda) \to x^*(\lambda) \in \mathbb{X}$ for all $\lambda \in \Lambda$. Moreover, it follows from (3.2) that $y_n(\lambda) \to \theta$ as $n \to +\infty$ for all $\lambda \in \Lambda$. Since $y_{2n}(\lambda) \in S_{j\lambda}(x_{2n}(\lambda)), y_{2n+1}(\lambda) \in S_{i\lambda}(x_{2n+1}(\lambda))$

and $S_{i\lambda}$, $S_{j\lambda}$ are closed for some $i, j \in \mathbb{N}$ and all $\lambda \in \Lambda$, it follows from (3.2) and the assumption (i) in Algorithm 3.1 that for some $i, j \in \mathbb{N}$,

$$\theta \in S_{i\lambda}(x^*), \quad \theta \in S_{j,\lambda}(x^*), \quad \forall \lambda \in \Lambda,$$

i.e., $x^*(\lambda)$ is a solution of (1.4) and $\{f_\lambda(x^*)\} = \{g_\lambda(x^*)\} \subset \bigcap_{i=1}^{\infty} S_{i\lambda}(x^*), \quad \forall \lambda \in \Lambda$. This completes the proof. □

As the applications of Theorem 4.1, we give the following results.

Theorem 4.2. Let $(\mathbb{X}, \|\cdot\|_*)$ be a line metric space, and $f, g : \Lambda \times \mathbb{X} \to \mathbb{X}$ be two nonlinear operators. Suppose that $\{S_{n\lambda}\}_{n=1}^{\infty}$, Γ_1, Γ_2, φ and ψ are the same as in Theorem 4.1 and satisfy the condition $(\mathcal{C}^*)$, the conditions (i)-(iii) in Algorithm 3.1 and the conditions (v) and (vi) in Theorem 4.1 hold. Then there exists $z(\lambda) \in \mathbb{X}$ such that $\theta = f_\lambda(z) = g_\lambda(z)$ and $\{f_\lambda(z)\} = \{g_\lambda(z)\} \subset \bigcap_{i=1}^{\infty} S_{i\lambda}(z)$ for all $\lambda \in \Lambda$. Moreover, $x_n(\lambda) \to z(\lambda)$ and $y_n(\lambda) \to \theta$ as $n \to +\infty$ for all $\lambda \in \Lambda$, where $\{x_n(\lambda)\}$ in $\mathbb{X}$ and $\{y_n(\lambda)\}$ in $\mathbb{Y}$ are two sequences generated by Algorithm 3.1.

Proof. Let $\|x\|(t) = \widetilde{0}(t - \|x\|_*)$. Then $(\mathbb{X}, \|\cdot\|, \min, \max)$ is an FN-space induced by $(\mathbb{X}, \|\cdot\|_*)$ and the all conditions of Theorem 4.1 hold. Hence, the conclusions follow directly from Theorem 4.1. □

Theorem 4.3. Assume that $(\mathbb{X}, \|\cdot\|, l, r)$ and $(\mathbb{Y}, \|\cdot\|, l, r)$ are two complete FN-spaces, and A, V, $\{S_{n\lambda}\}_{n=1}^{\infty}$, Γ_1, Γ_2, φ and ψ are the same as in Algorithm 3.2 and satisfy the condition $(\mathcal{C}^*)$. Further, suppose that the conditions (ii) and (iii) in Algorithm 3.1 and (I) in Algorithm 3.2 hold. If the following additional conditions are satisfied: (II) in FN-space $(\mathbb{X}, \|\cdot\|, l, r)$, $\lim_{a\to 0^+} r(a, a) = 0$, $\{r^n(t)\}$ is equi-continuous at $t = 0$ and $\lim_{t\to+\infty} \|x\|(t) = 0$ for all $x \in \mathbb{X}$; (III) $\mathbb{X} - V(\Lambda \times \mathbb{X})$ is x_0-joint orbitally complete for some $x_0(\lambda) \in \mathbb{X}$; (IV) $S_{n\lambda}$ is closed for any $n \in \mathbb{N}$ and all $\lambda \in \Lambda$, and there exists a constant $M > 0$ such that for all $\alpha \in (0, 1]$ and some $i, j \in \mathbb{N}$,

$$\||\Gamma_1(x)y\||_2^{\varphi(\alpha)} \le M\||y\||_2^{\varphi(\alpha)}, \quad \forall x \in D(S_{j\lambda}), \quad y \in \mathbb{Y},$$
$$\||\Gamma_2(x)y\||_2^{\varphi(\alpha)} \le M\||y\||_2^{\varphi(\alpha)}, \quad \forall x \in D(S_{i\lambda}), \quad y \in \mathbb{Y},$$

then there exists $x_*(\lambda) \in \mathbb{X}$ such that $x_*(\lambda) = A_\lambda(x_*) = V_\lambda(x_*)$ and $\{x_*(\lambda)\} \subset \bigcap_{i=1}^{\infty} S_{i\lambda}(x^*)$, for all $\lambda \in \Lambda$. Furthermore, $x_n(\lambda) \to x^*(\lambda), y_n(\lambda) \to \theta$ as $n \to \infty$ for all $\lambda \in \Lambda$, where $\{x_n(\lambda)\}$ in $\mathbb{X}$ and $\{y_n(\lambda)\}$ in $\mathbb{Y}$ are two sequences generated by Algorithm 3.2.

Proof. Let $f_\lambda(x) = x(\lambda) - A_\lambda(x)$ and $g_\lambda(x) = x(\lambda) - V_\lambda(x)$ for all $\lambda \in \Lambda, x \in \mathbb{X}$. It is easy to see that all the conditions of Theorem 4.1 are satisfied. Therefore, the conclusion of Theorem 4.3 follows from Theorem 4.1 immediately. □

Remark 4.1. Similarly, we can obtain the corresponding conclusions if the FN-spaces in Theorem 4.3 are replaced by the classical normed space, probabilistic normed space and the Menger probabilistic normed space, respectively. Therefore, the results presented in this paper improve and generalize corresponding results of [11-17] and [19].

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Non-Polynomial Spline Method for Fractional Diffusion Equation

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Abstract

The one-dimensional fractional diffusion equation is studied systematically using the non-polynomial spline method. The Caputo fractional derivative is used for formulation. An example is solved to assess the accuracy of the method. The numerical results are obtained for different values (n) of equation. An effective and easy-to-use method for solving such equations is needed.

Keywords: Fractional diffusion equation, Caputo fractional derivative, non-polynomial spline.

1. Introduction

Fractional diffusion equations have attracted during the last few decades for modelling many physical and chemical processes and in engineering. Many authors have presented the existence and approximations of the solutions to one-dimensional fractional diffusion equation. In [1] two-step Adomian decomposition method is used analytical solution for the space fractional diffusion equation. Mingrong Cui [2] proposed high-order compact finite difference scheme and analysis the condition for stability. Finite difference method is presented for this problem and some examples are given in [3]. Also in [5] a class of initial-boundary value fractional diffusion equations with variable coefficients on a finite domain are examined using numerical method and analysis of stability, consistency and convergence. The analytical solutions of the space fractional diffusion equations are presented by

modified decomposition method [6]. Ray examined the analytical solutions of the space fractional diffusion equations by two-step Adomian decomposition method [7]. In this paper, we consider one-dimensional fractional diffusion equation :

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + q(x,t), \tag{1}$$

with initial condition $u(x,0) = f(x)$, $0 \leq x \leq 1$ and boundary conditions $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$, $t \geq 0$ where $d(x)$ represents the diffusion coefficent and $q(x,t)$ the source/sink function. Sources provide energy or material to the system where sinks absorb energy or material. Eq. (1) becomes the classical diffusion equation for $\alpha = 2$. It models a superdiffuse flow for $1 < \alpha < 2$ and a classical advective flow for $\alpha = 1$ [5].

In this paper, non-polynomial spline method is considered for numerical solution of one-dimensional fractional diffusion equation. Many authors have used spline method to solve different problems. In [8], parabolic equation is solved using non-polynomial cubic spline method.
The paper has been organized as follows: In section 2, Caputo fractional derivative is given briefly. In section 3, non-polynomial cubic spline method is investigated. Analysis of the method is given in section 4. In section 5 an example is given, a conclusion is given in section 6.

2. Caputo Fractional Derivative

There are various kind of fractional derivatives that widely used ones are the Grunwald-Letnikov, the Riemann-Liouville and the Caputo fractional derivatives. Caputo fractional derivative is a regularization in the time origin for the Riemann-Liouville derivative [9,10]. A nice comparison of these definitions from the view point of their applications in physics and engineering can be found in [11]. In this study, we use the Caputo fractional derivative that is defined as follow [12]:

$$D^{\alpha}_{*x} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt \tag{2}$$

for $m-1 < \alpha \leq m$ and $m \in N$.

3. Non-Polynomial Spline Method

We divide the interval $[a, b]$ into n equal subintervals using the grid points $x_i = a + ih$, $i = 0, 1, 2, ..., n$, with $a = x_0$, $x_n = b$, $h = (b - a)/n$ where n is an arbitrary positive integer.

Let $u(x)$ be the exact solution and u_i be an approximation to $u(x_i)$ obtained by the non-polynomial cubic $S_i(x)$ passing through the points (x_i, u_i) and (x_{i+1}, u_{i+1}), we do not only require that $S_i(x)$ satisfies interpolatory conditions at x_i and x_{i+1}, but also the continuity of first derivative at the common nodes (x_i, u_i) are fulfilled. We write $S_i(x)$ in the form:

$$S_i(x) = a_i + b_i(x - x_i) + c_i sin\tau(x - x_i) + d_i cos\tau(x - x_i), \quad i = 0, 1, ..., n - 1 \quad (3)$$

where a_i, b_i, c_i and d_i are constants and τ is a free parameter.

A non-polynomial function $S(x)$ of class $C^2[a, b]$ interpolates $u(x)$ at the grid points x_i, $\quad i = 0, 1, 2, ..., n$, depends on a parameter τ, and reduces to ordinary cubic spline $S(x)$ in $[a, b]$ as $\tau \to 0$.

To derive expression for the coefficients of Eq. (3) in term of u_i, u_{i+1}, M_i and M_{i+1}, we first define:

$$S_i(x_i) = u_i, \quad S_i(x_{i+1}) = u_{i+1}, \quad S''(x_i) = M_i, \quad S''(x_{i+1}) = M_{i+1}. \quad (4)$$

From algebraic manipulation, we get the following expression:

$$a_i = u_i + \frac{Mi}{\tau^2},\ b_i = \frac{u_{i+1}-u_i}{h} + \frac{M_{i+1}-M_i}{\tau\theta},\ c_i = \frac{M_i cos\theta - M_{i+1}}{\tau^2 sin\theta},\ d_i = -\frac{M_i}{\tau^2},$$

where θ=τh and $i = 0, 1, 2, ..., n - 1$.

Using the continuity of the first derivative at (x_i, u_i), that is $S'_{i-1}(x_i) = S'_i(x_i)$ we obtain the following relations for i=1, ..., $n - 1$.

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = (1/h^2)(u_{i+1} - 2u_i + u_{i-1}) \quad (5)$$

where $\alpha = (-1/\theta^2 + 1/\theta \sin\theta)$, $\beta = (1/\theta^2 - \cos\theta/\theta \sin\theta)$ and $\theta = \tau h$.

The method is fourth-order convergent if $1-2\alpha-2\beta = 0$ and $\alpha = 1/12$ [8].

4. Analysis 0f The Method

To illustrate the application of the non-polynomial spline method developed in the previous section, we consider the one-dimensional fractional diffusion equation that is given in Eq.(1). At the grid point (x_i, u_i), the proposed equation may be discretized by using Caputo fractional derivative

$$\frac{u_i - f_i}{k} = d_i \left[\frac{1}{\Gamma(m-\alpha)} \int_0^{x_i} (x_i - \mu)^{m-\alpha-1} u_i^{(m)}(\mu) d\mu \right] + q_i, \tag{6}$$

where $d_i = d(x_i), \quad f_i = f(x_i) = u_{i-1}$ and $q_i = q(x_i, t_j)$.

In this paper, we take $\alpha = 1.8$, so the previous discretization will be as follow:

$$\frac{u_i - f_i}{k} = \frac{d_i}{\Gamma(0.2)} I + q_i, \tag{7}$$

where $I = \int_0^{x_i} (x_i - \mu)^{-0.8} u_i''(\mu) d\mu$.

Applying integration by parts two times, we obtain

$$I = x_i^{-0.8} u_i'(0) - 0.8 x_i^{-1.8} u_i(0) + \frac{36}{25} I_1, \tag{8}$$

where $I_1 = \int_0^{x_i} (x_i - \mu)^{-2.8} u_i(\mu) d\mu$.

Expanding $u_i(\mu)$ in Taylor series about a point $\mu = x_i$ and then substituting I_1 in Eq.(8) and then I in Eq. (7), we obtain

$$\frac{u_i - f_i}{k} = \frac{d_i}{\Gamma(0.2)} [x_i^{-0.8} u_i'(0) - 0.8 x_i^{-1.8} u_i(0) + \frac{36}{25} (u_i(x_i) \int_0^{x_i} (x_i - \mu)^{-2.8} d\mu$$

$$+ u_i'(x_i) \int_0^{x_i} (x_i - \mu)^{-1.8} d\mu + \frac{u_i''(x_i)}{2} \int_0^{x_i} (x_i - \mu)^{-0.8} d\mu)] + q_i. \tag{9}$$

Substituting $M_i = u_i^{''}(x_i)$ and $N_i = u_i^{'}(0)$ in Eq. (9) we get

$$\frac{u_i - f_i}{k} = \frac{d_i}{\Gamma(0.2)}[x_i^{-0.8}N_i - 0.8x_i^{-1.8}u_i(0) + \frac{36}{25}(u_i(x_i)\int_0^{x_i}(x_i-\mu)^{-2.8}d\mu$$

$$+u_i^{'}(x_i)\int_0^{x_i}(x_i-\mu)^{-1.8}d\mu + \frac{M_i}{2}\int_0^{x_i}(x_i-\mu)^{-0.8}d\mu)] + q_i. \quad (10)$$

Solving Eq. (10) for M_i, we get

$$M_i = \frac{1}{C_i}\left(G_iu_i - G_if_i - \frac{25}{18}x_i^{-0.8}N_i + \frac{10}{9}x_i^{-1.8}u_i(0) - 2D_iu_i - 2E_iu_i^{'} - G_iq_i\right) \quad (11)$$

where $C_i = \int_0^{x_i}(x_i-\mu)^{-0.8}d\mu$, $D_i = \int_0^{x_i}(x_i-\mu)^{-2.8}d\mu$, $E_i = \int_0^{x_i}(x_i-\mu)^{-1.8}d\mu$ and $G_i = \frac{18kd_i}{25\Gamma(0.2)}$.

The following approximations for the first-order derivative of u in Eq. (11) can be used

$$u_i^{'} \cong \frac{u_{i+1} - u_{i-1}}{2h}, u_{i+1}^{'} \cong \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}, u_{i-1}^{'} \cong \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h}, \quad (12)$$

So Eq. (11) becomes

$$M_i = \frac{1}{C_i}\left(G_iu_i - G_if_i - \frac{25}{18}x_i^{-0.8}N_i + \frac{10}{9}x_i^{-1.8}u_i(0) - 2D_iu_i - 2E_i\frac{u_{i+1} - u_{i-1}}{2h} - G_iq_i\right). \quad (13)$$

M_{i+1} and M_{i-1} can be obtained easily from M_i putting $i+1$ and $i-1$ respectively instead of i. Substituting this equalities in Eq. (5), we find the following $(n-1)$ linear algebraic equations in the $(n+1)$ unknowns for $i = 0, 1, ..., n$:

$$\left[\frac{\alpha}{C_{i+1}}\left(G_{i+1}-2D_{i+1}-\frac{3}{h}E_{i+1}\right)+\frac{2\beta E_i}{C_i h}+\frac{\alpha E_{i-1}}{C_{i-1}h}-\frac{1}{h^2}\right]u_{i+1}+$$
$$\left[\frac{4\alpha E_{i+1}}{C_{i+1}h}+\frac{2\beta}{C_i}(G_i-2D_i)-\frac{4\alpha E_{i-1}}{C_{i-1}h}+\frac{2}{h^2}\right]u_i+$$
$$\left[\frac{\alpha}{C_{i-1}}\left(G_{i-1}-2D_{i-1}-\frac{3}{h}E_{i-1}\right)+\frac{2\beta E_i}{C_i h}-\frac{\alpha E_{i+1}}{C_{i+1}h}-\frac{1}{h^2}\right]u_{i-1}=$$
$$\left(\frac{\alpha G_{i+1}f_{i+1}}{C_{i+1}}+\frac{\beta G_i f_i}{C_i}+\frac{\alpha G_{i-1}f_{i-1}}{C_{i-1}}\right)+\left(\frac{25\alpha N_{i+1}}{18C_{i+1}x_{i+1}^{0.8}}+\frac{25\beta N_i}{9C_i x_i^{0.8}}+\frac{25\alpha N_{i-1}}{18C_{i-1}x_{i-1}^{0.8}}\right)$$
$$-\left(\frac{10\alpha u_{i+1}(0)}{9C_{i+1}x_{i+1}^{1.8}}+\frac{20\beta u_i(0)}{9C_i x_i^{1.8}}+\frac{10\alpha u_{i-1}(0)}{9C_{i-1}x_{i-1}^{1.8}}\right)+\left(\frac{\alpha G_{i+1}q_{i+1}}{C_{i+1}}+\frac{2\beta G_i q_i}{C_i}+\frac{\alpha G_{i-1}q_{i-1}}{C_{i-1}}\right) \quad (14)$$

We need two more equations. The two end conditions can be derivated as follows:

$$u(0,t)=g(t), \quad u(1,t)=h(t) \quad (15)$$

Substituting

$$\begin{aligned}
c1 &= \frac{\alpha}{C_{i+1}}\left(G_{i+1}-2D_{i+1}-\frac{3}{h}E_{i+1}\right)+\frac{2\beta E_i}{C_i h}+\frac{\alpha E_{i-1}}{C_{i-1}h}-\frac{1}{h^2}\\
c2 &= \frac{4\alpha E_{i+1}}{C_{i+1}h}+\frac{2\beta}{C_i}(G_i-2D_i)-\frac{4\alpha E_{i-1}}{C_{i-1}h}+\frac{2}{h^2}\\
c3 &= \frac{\alpha}{C_{i-1}}\left(G_{i-1}-2D_{i-1}-\frac{3}{h}E_{i-1}\right)+\frac{2\beta E_i}{C_i h}-\frac{\alpha E_{i+1}}{C_{i+1}h}-\frac{1}{h^2}
\end{aligned}$$

The method is described in matrix form in the following way for the equations above:

$$A=\begin{bmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 0\\
c1 & c2 & c3 & 0 & \dots & 0 & 0\\
0 & c1 & c2 & c3 & 0 & \dots & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & \dots & 0 & 0 & c1 & c2 & c3\\
\cdot & \cdot & \cdot & \cdot & 0 & 0 & 1
\end{bmatrix}$$

$$B = \begin{bmatrix} \underbrace{\left(\frac{\alpha G_0 f_0}{C_0} + \frac{\beta G_1 f_1}{C_1} + \frac{\alpha G_2 f_2}{C_2}\right) + \left(\frac{25\alpha N_0}{18C_0 x_0^{0.8}} + \frac{25\beta N_1}{9C_1 x_1^{0.8}} + \frac{25\alpha N_2}{18C_2 x_2^{0.8}}\right) - \left(\frac{10\alpha u_0(0)}{9C_0 x_0^{1.8}} + \frac{20\beta u_1(0)}{9C_1 x_1^{1.8}} + \frac{10\alpha u_2(0)}{9C_2 x_2^{1.8}}\right) + \left(\frac{\alpha G_0 q_0}{C_0} + \frac{2\beta G_1 q_1}{C_1} + \frac{\alpha G_2 q}{C_2}\right.}_{g(t)} \\ \left(\frac{\alpha G_1 f_1}{C_1} + \frac{\beta G_2 f_2}{C_2} + \frac{\alpha G_3 f_3}{C_3}\right) + \left(\frac{25\alpha N_1}{18C_1 x_1^{0.8}} + \frac{25\beta N_2}{9C_2 x_2^{0.8}} + \frac{25\alpha N_3}{18C_3 x_3^{0.8}}\right) - \left(\frac{10\alpha u_1(0)}{9C_1 x_1^{1.8}} + \frac{20\beta u_2(0)}{9C_2 x_2^{1.8}} + \frac{10\alpha u_3(0)}{9C_3 x_3^{1.8}}\right) + \left(\frac{\alpha G_1 q_1}{C_1} + \frac{2\beta G_2 q_2}{C_2} + \frac{\alpha G_3 q}{C_3}\right. \\ \vdots \\ \underbrace{\left(\frac{\alpha G_{n-2} f_{n-2}}{C_{n-2}} + \frac{\beta G_{n-1} f_{n-1}}{C_{n-1}} + \frac{\alpha G_n f_n}{C_n}\right) + \left(\frac{25\alpha N_{n-2}}{18C_{n-2} x_{n-2}^{0.8}} + \frac{25\beta N_{n-1}}{9C_{n-1} x_{n-1}^{0.8}} + \frac{25\alpha N_n}{18C_n x_n^{0.8}}\right) - \ldots}_{h(t)} \end{bmatrix}$$

$$U = [u_0, u_1, ..., u_n]' \tag{16}$$

Finally the approximate solution is obtained by solving the fractional diffusion equation using Matlab 7.0.1.

$$AU = B. \tag{17}$$

5. A Numerical Example

In this section, to illustrate our methods we have solved fractional diffusion equation. All computations are done by using MATLAB 7.0.1.

Consider the fractional diffusion problem (1) with the diffusion coefficient $d(x) = \frac{\Gamma(2.2)x^{2.8}}{6}$, the source/sink function $q(x,t) = -(1+x)e^{-t}x^3$, subject to initial condition $u(x,0) = x^3$, for $0 \le x \le 1$ and boundary conditions $u(0,t) = 0$, $u(1,t) = e^{-t}$ for $t \ge 0$. The exact solution of this problem for $\alpha = 2$ is $u(x,t) = e^{-t}x^3$. The observed maximum absolute errors for various values of n for the fraction order 1.8 are given in Table 1.

Table 1: The maximum absolute errors, $k = 0.01$

n	Spline method
11	0.1762
21	0.1663
41	0.1481
61	0.1307
121	0.0859

6. Conclusion

In this paper non-polynomial cubic spline method is applied for the numerical solution of one-dimentional fractional diffusion equation. Caputo fractional derivative is used for fractional derivative term. We have concluded that numerical results converge to the exact solution when h goes to zero. Use of non-polynomial splines have shown that it is an applicable method for determining the diffusion.

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